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Minisuperspace canonical quantization of the Reissner-Nordström black hole via conditional symmetries

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We use the conditional symmetry approach to study the r evolution of a minisuperspace spherically symmetric model both at the classical and the quantum level. After integration of the coordinates t, θ and ϕ in the gravitational plus electromagnetic action the configuration space dependent dynamical variables turn out to correspond to the r-dependent metric functions and the electrostatic field. In the context of the formalism for constrained systems (Dirac-Bergmann, Arnowitt-Deser-Misner) with respect to the radial coordinate r, we set up a pointlike reparametrization invariant Lagrangian. It is seen that, in the constant potential parametrization of the lapse, the corresponding minisuperspace is a Lorentzian three-dimensional flat manifold which obviously admits six Killing vector fields plus a homothetic one. The weakly vanishing r Hamiltonian guarantees that the phase space quantities associated to the six Killing fields are linear holonomic integrals of motion. The homothetic field provides one more rheonomic integral of motion. These seven integrals are shown to comprise the entire classical solution space, i.e. the space-time of a Reissner-Nordström black hole, the r-reparametrization invariance since one dependent variable remains unfixed, and the two quadratic relations satisfied by the integration constants. We then quantize the model using as supplementary conditions acting on the wave function, the quantum analogues of the various subalgebras of the classical conditional symmetries. We find that, as a semiclassical analysis shows, in all but one allowed case the ensuing solutions to the Wheeler-DeWitt equation exhibit a good correlation with the classical regime. In the remaining case, the emerging semiclassical geometry is a four-dimensional homogeneous space-time, thus exhibiting no curvature singularity.

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I. INTRODUCTION

Symmetry considerations have acquired a very prominent role in all branches of theoretical physics. This is probably due to the fact that all conservation laws in physics are the result of some kind of symmetry in the corresponding physical system. In this sense a symmetry is a kind of variation of the Lagrangian of a dynamical system that leaves the equations of motion invariant. One of the most important types of such symmetries which has lots of applications in classical mechanics and quantum field theory is the well-known Noether symmetry. Mathematically, the famous Noether theorem states that a vector field X is a symmetry for a given dynamical system if the Lie derivative of its Lagrangian along this vector field vanishes $\mathcal{L}_X L = 0$ [1,2]. The first application, to the best of our knowledge, of this criterion in constrained systems is given in [3]. Under this condition the vector field X generates the conserved currents from which the integrals of motion can be obtained (see [4–10] for the applications of the Noether symmetry approach in various cosmological models and black hole physics). More generally, the symmetries of a Riemannian space may also be represented by a vector field X which satisfies an equation of the form $\mathcal{L}_X \mathbf{A} = \mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are some geometric objects [11]. For instance, in a Riemannian space with metric $\mathcal{G}_{\mu\nu}$,

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X is a conformal Killing vector if $\mathbf{A} = \mathcal{G}_{\mu\nu}$ and $\mathbf{B} = \phi(x^\alpha)\mathcal{G}_{\mu\nu}$. In the case where $\phi(x^\alpha) = 0$ the vector X is known as a Killing vector and when $\phi(x^\alpha)$ is a nonvanishing constant X is a homothetic vector. There are also other kinds of such symmetries that we will not mention here but a classification of them can be found in [12].

In the canonical formulation of general relativity the space of all Riemannian three-dimensional metrics and matter fields on the spatial hypersurfaces form an infinite-dimensional space, the so-called superspace, which is the basic configuration space of quantum gravity. However, in cosmology due to the many symmetries of the underlying cosmological models the infinite degrees of freedom of the corresponding superspace are truncated to a finite number and thus a particular minisuperspace model is achieved. It is easy to show that the evolution of such a system, when the equations of motion are obtained from an action principle, can be produced by a Lagrangian of the following form:

$$L = \frac{1}{2n} G_{\alpha\beta}(q)\dot{q}^\alpha \dot{q}^\beta - nV(q), \tag{1.1}$$

where q^α and n are the dependent dynamical variables and the lapse function representing the coordinates of the minisuperspace with metric $G_{\alpha\beta}(q)$, $V(q)$ is a potential function and an overdot indicates derivation with respect to some independent dynamical parameter. Since the dynamics of the system in this formalism resembles the motion of a point particle with coordinates q^α in a Riemannian space with metric $G_{\alpha\beta}$, many interesting features may occur when this space has some symmetries. In particular, one can define a conditional symmetry generated by a vector field ξ which is a simultaneous conformal Killing vector field of the metric $G^{\alpha\beta}(q)$ and the potential function $V(q)$, that is [13]

$$\mathcal{L}_\xi G^{\alpha\beta} = \phi(q) G^{\alpha\beta}, \qquad \mathcal{L}_\xi V(q) = \phi(q)V(q). \tag{1.2}$$

As noted above, each symmetry corresponds to a phase-space quantity representing an integral of motion. In [13], it is shown that the integrals of motion resulting from (1.2) can be written as

$$Q_I = \xi_I^\alpha p_\alpha, \tag{1.3}$$

where $p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}$ is the momentum conjugate to q^α. In order to pass to the quantum theory associated with these models, one should note that the variation of (1.1) with respect to n yields

$$\frac{1}{2n^2} G_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta + V(q) = 0 \tag{1.4}$$

which, being the zero-energy condition, leads to the Hamiltonian constraint

$$H = n\left[\frac{1}{2} G^{\alpha\beta} p_\alpha p_\beta + V(q)\right] = n\mathcal{H} = 0. \tag{1.5}$$

Therefore, following the canonical quantization method, this Hamiltonian gives rise to the Wheeler-DeWitt (WDW) equation $\hat{\mathcal{H}}\Psi(q) = 0$, where $\Psi(q)$ is the wave function of the quantized system and $\hat{\mathcal{H}}$ should be written in a suitable operator form. Now, it is easy to see that the Poisson brackets of (1.3) with the Hamiltonian vanish weakly on the constrained surface. In the lapse parametrization $n = \frac{N}{V}$, where the potential is constant, the aforementioned Poisson brackets vanish identically. The quantum counterpart of this statement is that the operator forms of (1.3) and the scaled Hamiltonian commute with each other which means that $\hat{Q}_I$ and $\hat{\mathcal{H}}$ have simultaneous eigenfunctions. In summary, the quantum counterpart of the theory with the aforesaid symmetry can be described by the following equations (more details are presented in the following sections):

$$\hat{\mathcal{H}}\Psi(q) = 0, \qquad \hat{Q}_I\Psi(q) = \kappa_I \Psi(q), \tag{1.6}$$

where κ_I are the eigenvalues of Q_I.

In this paper we study the behavior of a static, spherically symmetric space-time in the framework of the presence of conditional symmetries in minisuperspace constrained systems. The phase-space variables turn out to correspond to the r-dependent metric functions and to an electrostatic field with which the action of the model is augmented.

In Sec. II we follow [13–15] and construct a minisuperspace Lagrangian, in the form of (1.1), using the canonical decomposition along the radial coordinate r which now plays the role of a dynamical variable. We then deal with some considerations on this minisuperspace constrained system possessing conditional symmetries and by passing to the Hamiltonian formalism we reveal six conditional symmetries and a rheonomic integral of motion. Under these conditions we show that the classical solution of such a system can be identified with the space-time of a Reissner-Nordström (RN) black hole [16,17] (for higher dimensions see [18]).

In Sec. III we consider the quantization of the system in which we adopt the quantum analogues of the linear integrals of motion as supplementary conditions imposed on the wave function, the latter also satisfying, of course, the Wheeler-DeWitt quantum constraint. To see how we can recover the classical solutions from the quantum wave functions, we present a semiclassical analysis of the model above described in Sec. IV. The curious and interesting situation of the vanishing quantum potential is investigated and fully explained in Sec. V. Finally, some concluding remarks are included in the discussion.

II. CLASSICAL FORMULATION AND CONDITIONAL SYMMETRIES

The general form of a static, spherically symmetric line element is

$$ds^2 = -a^2(r)dt^2 + n^2(r)dr^2 + b^2(r)(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.1}$$

with $n(r)$ playing the role of the r-lapse function, while $a(r)$ and $b(r)$ are the "dynamical" dependent variables in the r foliation. In order to acquire the RN solution we need to consider an electrostatic field minimally coupled to gravity. Thus, the full action is written as

$$\begin{aligned} S_{g+\mathrm{em}} &= \int \mathcal{L}_{\mathrm{GR}} d^4x + \int \mathcal{L}_{\mathrm{EM}} d^4x \\ &= \int \sqrt{-g} R d^4x - \int \sqrt{-g} F_{\mu\nu} F^{\mu\nu} d^4x, \end{aligned} \tag{2.2}$$

where $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$ is the antisymmetric electromagnetic tensor and A_μ is the potential with $A_0 = f(r)$ and $A_1 = A_2 = A_3 = 0$. In (2.2) we have chosen the units $c = 1$, $G = \frac{1}{4\pi}$. The variation of this action with respect to the space-time metric $g_{\mu\nu}$ leads to Einstein's field equations,

$$E_{\mu\nu} = 2T_{\mu\nu}, \tag{2.3}$$

where $E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu} = F_{\mu\kappa} F_\nu{}^\kappa - \frac{1}{4} g_{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}$ is the stress-energy tensor associated with the electromagnetic field.

The variation of action (2.2) with respect to the field A_μ leads to the equations of motion,

$$F^{\mu\nu}{}_{;\mu} = 0, \tag{2.4}$$

which together with the consistency conditions

$$F_{\mu\nu;\kappa} + F_{\kappa\mu;\nu} + F_{\nu\kappa;\mu} \equiv 0, \tag{2.5}$$

comprise the complete set of Maxwell's equations in the absence of electromagnetic sources ($J^\mu = 0$). Of course, (2.5) is identically satisfied due to the defining form adopted for the field strength tensor $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$. As for (2.4), it is satisfied by virtue of the assumption of staticity and spherical symmetry, i.e. by the form of both the line element (2.1) and of $A_\mu (= (f(r), 0, 0, 0))$: One must first solve algebraically the reduced constraint equation $E_{rr} = 2T_{rr}$ for $n(r)$. The substitution of this solution into the rest of equation (2.3) yields a system that can be solved algebraically for $a''(r)$ and $f''(r)$. Final substitution of all three relations into the left-hand side of the reduced equation (2.4) results in the zero four-vector $F^{\mu\nu}{}_{;\nu}$. Thus, the adopted symmetry assumption is seen to be compatible with action (2.2), i.e. the assumption of absence of electromagnetic sources.

Apart from the field theory approach, one can be led to effectively the same equations of motion by integration of the redundant degrees of freedom in action (2.2), i.e. integrating over t, θ and ϕ and ignoring a multiplicative (infinite) constant. All system information is then contained in a reduced, pointlike action $S = \int L(a, b, f, a', b', f', n) dr$ with the following Lagrange function:

$$L = \frac{1}{2n}\left(8ba'b' + 4ab'^2 + 4\frac{b^2}{a} f'^2\right) + 2na, \tag{2.6}$$

where $'$ denotes differentiation with respect to the spatial coordinate r. It is easy to verify that the Euler-Lagrange equations ensuing from (2.6) are equivalent to the reduced Einstein's equations obtained by the substitution of the line element (2.1) and $A_\mu = (f(r), 0, 0, 0)$ in (2.3). The Lagrangian (2.6) belongs to a particular form of singular Lagrangians: $L = \frac{1}{2n} G_{\mu\nu} q'^\mu q'^\nu + nV(q)$. If one uses the freedom to reparametrize the lapse, then (2.6) can be brought to a form in which the potential V is constant. In our case we choose to set $n = \frac{N}{2a}$, which leads to

$$L = \frac{1}{2N}(16aba'b' + 8a^2b'^2 + 8b^2f'^2) + N, \tag{2.7}$$

or, in a more concise form, $L = \frac{1}{2N} \bar{G}_{\mu\nu} q'^\mu q'^\nu + N$ with $q'^\mu = (a', b', f')$ and

$$\bar{G}_{\mu\nu} = \begin{pmatrix} 0 & 8ab & 0 \\ 8ab & 8a^2 & 0 \\ 0 & 0 & 8b^2 \end{pmatrix}. \tag{2.8}$$

As shown in [13], it is in this particular lapse parametrization that the conditional symmetries of the phase space, as defined in [19], become Killing vector fields of the supermetric (2.8) in the configuration space. As it can be straightforwardly verified, the above given metric $\bar{G}_{\mu\nu}$ is flat and admits the following six Killing vectors:

$$\begin{aligned} &\xi_1 = \partial_f, \qquad \xi_2 = \frac{1}{2ab}\partial_a, \qquad \xi_3 = \frac{f}{2ab}\partial_a + \frac{1}{2b}\partial_f, \\ &\xi_4 = -a\partial_a + b\partial_b - f\partial_f, \qquad \xi_5 = -\frac{a^2+f^2}{2ab}\partial_a + \partial_b - \frac{f}{b}\partial_f, \\ &\xi_6 = -af\partial_a + bf\partial_b - \frac{a^2+f^2}{2}\partial_f. \end{aligned} \tag{2.9}$$

These form an algebra under the Lie bracket; the non-vanishing structure constants of this algebra are

$$C^2_{31}=-C^2_{13}=C^1_{14}=-C^1_{41}=C^4_{61}=-C^4_{16}=1,$$
$$C^2_{24}=-C^2_{42}=C^3_{26}=-C^3_{62}=C^5_{45}=-C^5_{54}=C^6_{46}=-C^6_{64}=1,$$
$$C^3_{15}=-C^3_{51}=2,\quad C^5_{63}=-C^5_{36}=\frac{1}{2}.$$

Additionally, the supermetric $\bar{G}_{\mu\nu}$ exhibits a homothetic symmetry ($\pounds_{\xi_h}\bar{G}_{\mu\nu}=\bar{G}_{\mu\nu}$) generated by

$$\xi_h=\frac{a}{4}\frac{\partial}{\partial a}+\frac{b}{4}\frac{\partial}{\partial b}+\frac{f}{4}\frac{\partial}{\partial f}, \tag{2.10}$$

which will be used in order to completely integrate the system of the Euler-Lagrange equations:

$$(\mathrm{EL})_N := -\frac{\partial L}{\partial N}, \tag{2.11a}$$

$$(\mathrm{EL})_{q^i} := \frac{d}{dr}\left(\frac{\partial L}{\partial \dot{q}^i}\right)-\frac{\partial L}{\partial q^i}. \tag{2.11b}$$

Let us now turn to the Hamiltonian formulation; invoking the usual definition of the momenta

$$p_N := \frac{\partial L}{\partial N'}=0, \tag{2.12a}$$

$$p_a := \frac{\partial L}{\partial a'}=\frac{8abb'}{N}, \tag{2.12b}$$

$$p_b := \frac{\partial L}{\partial b'}=\frac{8a(ba'+ab')}{N}, \tag{2.12c}$$

$$p_f := \frac{\partial L}{\partial f'}=\frac{8b^2f'}{N}, \tag{2.12d}$$

and following Dirac's algorithm [20], we acquire one first class primary constraint $p_N\approx 0$, the Hamiltonian

$$H=N\mathcal{H}=N\left(-\frac{p_a^2}{16b^2}+\frac{p_a p_b}{8ab}+\frac{p_f^2}{16b^2}-1\right), \tag{2.13}$$

and the first class secondary constraint $\{p_N,H\}\approx 0 \Rightarrow \mathcal{H}\approx 0$. If we associate the phase-space quantities $Q_I := \xi^\mu_I p_\mu$ with the six Killing vector fields (2.9), we are provided with six linear integrals of motion,

$$Q_1=p_f,\quad Q_2=\frac{1}{2ab}p_a,\quad Q_3=\frac{f}{2ab}p_a+\frac{1}{2b}p_f,$$
$$Q_4=-ap_a+bp_b-fp_f,\quad Q_5=-\frac{a^2+f^2}{2ab}p_a+p_b-\frac{f}{b}p_f,$$
$$Q_6=-afp_a+bfp_b-\frac{a^2+f^2}{2}p_f, \tag{2.14}$$

which form a Poisson bracket algebra with the previously mentioned structure constants. As also stated in [13], under the given lapse parametrization (in which the potential is constant) the Poisson brackets of the Q_I's with the Hamiltonian H are exactly equal to zero and not just weakly vanishing, $\{Q_I,H\}=0$, for $I=1,\ldots,6$. Moreover, since $\mathcal{H}\approx 0$, the constancy of the potential part is carried over to the quadratic in the momenta kinetic term, leading inevitably the latter to become a Casimir invariant of the Lie algebra formed by the Q_I's. In the case we are studying this is

$$Q_C=\frac{1}{4}(Q_2Q_5+Q_3^2)=\mathcal{H}+1. \tag{2.15}$$

As it is known, the integrals of motion, Q_I's, become constants, say κ_I's, on the solution space. However, these are not the only existing integrals of motion. As shown in [21], in principle, all conformal Killing vectors of the supermetric define rheonomic integrals of motion. For example, the relation $\pounds_\xi\bar{G}_{\mu\nu}=\omega\bar{G}_{\mu\nu}$ implies that if we define the phase-space quantity $Q_\xi=\xi^\mu p_\mu$, then

$$\frac{dQ_\xi}{dr}=\{Q_\xi,H\}=\omega(q)\frac{N}{2}\bar{G}^{\mu\nu}p_\mu p_\nu=\omega(q)N \tag{2.16}$$

holds. The latter equality is valid since $\mathcal{H}=\frac{1}{2}\bar{G}^{\mu\nu}p_\mu p_\nu-1\approx 0$. Thus, by integration over r the above equation is turned into the rheonomic integral,

$$Q_\xi-\int\omega(q(r))N\mathrm{d}r=\mathrm{const.} \tag{2.17}$$

For $\omega=0$, there is no explicit r dependence and the corresponding integrals are just the Q_I's generated by the six Killing vector fields. In the case of a nonvanishing ω, the usefulness of (2.17) is limited, since one needs to know *a priori* the trajectories $q(r)$ that solve the Euler-Lagrange equation (2.11). Nevertheless, for the homothetic Killing field the previous problem is circumvented since $\omega=$ constant; another choice would be to pick up a particular conformal Killing vector field and properly gauge fix the lapse, i.e. choose $N=\frac{1}{\omega}$. In what follows we will use the homothetic vector field ξ_h and avoid any gauge fixing of the lapse N. We thus write the following seven relations, that are valid on the solution space:

$$Q_I=\kappa_I,\qquad I=1,\ldots,6 \tag{2.18a}$$

$$Q_h - \int N\mathrm{d}r = c_h \Rightarrow \frac{1}{4}(a p_a + b p_b + f p_f) - \int N\mathrm{d}r = c_h \tag{2.18b}$$

with κ_I's and c_h being constants. It is quite interesting that the above relations completely determine the entire classical solution space along with the two relations quadratic in the κ_I's emanating from the two Casimir invariants of the algebra. Indeed, after substitution of (2.12), if we choose to algebraically solve the system of equations consisting of (2.18a) for $I = 1, \ldots, 5$ and (2.18b) with respect to $a(r)$, $a'(r)$, $f(r)$, $f'(r)$, $\int N\mathrm{d}r$ and $N(r)$, we obtain the relations

$$a = \pm \frac{\sqrt{-4b(\kappa_1\kappa_3 + \kappa_2\kappa_4) + 4b^2(\kappa_2\kappa_5 + \kappa_3^2) + \kappa_1^2}}{2\kappa_2 b}, \tag{2.19a}$$

$$a' = \mp \frac{b'(\kappa_1^2 - 2b(\kappa_1\kappa_3 + \kappa_2\kappa_4))}{2\kappa_2 b^2\sqrt{-4b(\kappa_1\kappa_3 + \kappa_2\kappa_4) + 4b^2(\kappa_2\kappa_5 + \kappa_3^2) + \kappa_1^2}}, \tag{2.19b}$$

$$f = \frac{\kappa_3}{\kappa_2} - \frac{\kappa_1}{2\kappa_2 b}, \tag{2.19c}$$

$$f' = \frac{\kappa_1 b'}{2\kappa_2 b^2}, \tag{2.19d}$$

$$\int N dr = \frac{4b(\kappa_2\kappa_5 + \kappa_3^2) - (4c_h\kappa_2 + 2\kappa_1\kappa_3 + 3\kappa_2\kappa_4)}{4\kappa_2} \tag{2.19e}$$

$$N = \frac{4b'}{\kappa_2}, \tag{2.19f}$$

with b remaining an arbitrary function of r. The consistency conditions $a' = \frac{da}{dr}$ and $f' = \frac{df}{dr}$ are identically satisfied, while $N = \frac{d}{dr}\int N\mathrm{d}r$ leads to the requirement

$$\kappa_2\kappa_5 + \kappa_3^2 = 4, \tag{2.20}$$

which is valid due to the Casimir invariant (2.15), since the Hamiltonian (also known as the quadratic constraint) is zero. Additionally, and somewhat unexpectedly, if one substitutes (2.19) into the equation we have not used, i.e. $Q_6 = \kappa_6$, one is led to the following relation between constants:

$$\kappa_1\kappa_5 + 2\kappa_2\kappa_6 - 2\kappa_3\kappa_4 = 0. \tag{2.21}$$

This relation is also valid on the solution space, because of the existence of the second Casimir invariant,

$$\tilde{Q}_C = Q_1 Q_5 + 2Q_2 Q_6 - 2Q_3 Q_4. \tag{2.22}$$

If the form of the Q_I's (2.14) is substituted into $\tilde{Q}_C$ we find that it vanishes identically, irrespectively of the classical solution. Therefore, Eq. (2.21) is retrieved on the solution space.

It is an easy task to check that (2.19) together with (2.20) is the solution of the equations of motion (2.11). By a convenient reparametrization of the constants κ_I [four of which are arbitrary because of the requirements (2.20) and (2.21)]

$$\begin{aligned} &\kappa_1 = -4Q, \qquad \kappa_2 = \frac{2}{c}, \qquad \kappa_3 = \frac{2c_3}{c}, \\ &\kappa_4 = 4\ \mathrm{cm} + c_3 Q, \qquad \kappa_5 = 2c - \frac{2c_3^2}{c}, \\ &\kappa_6 = 2(Q(c^2 + c_3^2) + 2cc_3 m), \end{aligned} \tag{2.23}$$

the corresponding space-time line element in (2.1) turns out to be

$$\begin{aligned} ds^2 = &-c^2\left(1 - \frac{2m}{b(r)} + \frac{Q^2}{b^2(r)}\right)dt^2 \\ &+ \left(1 - \frac{2m}{b(r)} + \frac{Q^2}{b^2(r)}\right)^{-1} db^2(r) + b^2(r)d\theta^2 \\ &+ b^2(r)\sin^2\theta d\phi^2 \end{aligned} \tag{2.24}$$

which, of course, is the well-known RN metric involving only two essential parameters: the mass m and the charge Q; c is absorbable by a rescaling of the time coordinate, i.e. $t \to \frac{t}{c}$,

$$\begin{aligned} ds^2_{\mathrm{RN}} = &-\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 \\ &+ r^2 d\theta^2 + r^2\sin^2\theta d\phi^2. \end{aligned} \tag{2.25}$$

The solution is valid in the region $r > 0$ and exhibits a curvature singularity at $r = 0$. At a physical level, one can interpret the resulting geometry outside the singularity as a consequence of the existence of a motionless point particle carrying a charge Q and a mass M residing at the origin $r = 0$. It is to be noted that one could start from this physical setup by including appropriate terms with Dirac-delta functions in the right-hand side of (2.3) and (2.4). Then, by restricting attention to the region outside the support of the delta function, the equations to be solved would be identical to those here considered; thus the same classical solution would be reached.

Some remarks are in order:

(i) κ_1, and therefore Q_1, is the only linear integral of motion depending solely on an essential constant.

(ii) The quantities Q_2, Q_3 and Q_5 incorporate the nonessential constants c and c_3. Therefore, these can be claimed to be completely gauged fixable, since one can utilize the arbitrariness of c and c_3 to change their values.

(iii) Q_4 and Q_6 depend on both essential and nonessential constants, but still their values are gauge dependent.

(iv) The value of the Casimir invariant, Q_C, on the solution space is

$$\frac{1}{4}(\kappa_2\kappa_5 + \kappa_3^2) = 1, \tag{2.26}$$

as expected by (2.15), since the Hamiltonian constraint $\mathcal{H}$ is weakly zero.

We can add here that the constant c can be set equal to one but not zero. On the other hand, c_3 can be taken equal to zero since it is absorbed additively. Moreover, c_3 is connected to the gauge freedom of the electrostatic scalar potential, since by using (2.23) one can see that $f(r) = c_3 - \frac{cQ}{b(r)}$.

By setting $c = 1$, and $c_3 = 0$ the values of the six κ_I's become

$$\kappa_1 = -4Q, \qquad \kappa_2 = 2, \qquad \kappa_3 = 0,$$
$$\kappa_4 = 4m, \qquad \kappa_5 = 2, \qquad \kappa_6 = 2Q \tag{2.27}$$

which are the values one would obtain if solution (2.25) had been taken as the starting point for the computation of the linear integrals of motion.

III. QUANTIZATION THROUGH SYMMETRIES

The identification of the linear integrals of motion as physical quantities leads to the need of expressing them as operators. The algebra defined by these operators has to match the classical Lie algebra and, moreover, one has to determine which of them can be applied at the same time on the wave function together with the constraints mentioned in the previous section. These issues have been clearly addressed in [13]. In a quick view, we start with the usual definition of the momenta ($\hbar = 1$) as operators,

$$p_\alpha \rightarrow \hat{p}_\alpha := -\mathbb{i}\frac{\partial}{\partial q^\alpha}, \tag{3.1}$$

where q^α is any one of a, b, f, N. After that, the quantum analogues of the conditional symmetries Q_I are expressed in the most general form of a linear Hermitian (under an arbitrary measure μ) differential operator of the first order:

$$\hat{Q}_I := -\frac{\mathbb{i}}{2\mu}(\mu\xi_I^\alpha\partial_\alpha + \partial_\alpha\mu\xi_I^\alpha). \tag{3.2}$$

It has been proved in [13] that operators $\hat{Q}_I$ defined as in (3.2) satisfy the same algebra as do the classical quantities Q_I, i.e. $[\hat{Q}_I, \hat{Q}_J]F = C_{IJ}^K\hat{Q}_K F$ for any function F for which the action of the operators is well defined. It is noteworthy that this happens for any arbitrary measure $\mu(a, b, f)$.

Apart from the primary constraint

$$\hat{p}_N = -\mathbb{i}\frac{\partial}{\partial N}\Psi = 0 \Rightarrow \Psi = \Psi(a, b, f), \tag{3.3}$$

the main operator one has to apply is the quantum analogue of the Hamiltonian constraint or, equivalently in the particular lapse parametrization, of the Casimir invariant ($\hat{Q}_C$), since

$$\hat{\mathcal{H}}\Psi = (\hat{Q}_C - 1)\Psi = 0. \tag{3.4}$$

In order to fix the kinetic part of the Hamiltonian operator we demand Hermiticity under the same measure μ; we thus have [22]

$$\hat{\mathcal{H}}_c\Psi = \left[-\frac{1}{2\mu}\partial_\alpha(\mu\bar{G}^{\alpha\beta}\partial_\beta) - 1\right]\Psi = 0. \tag{3.5}$$

The addition of a term proportional to the Ricci scalar of the supermetric $\bar{G}_{\alpha\beta}$ is not needed since the superspace is flat. In what follows we will, invoking a sense of naturality, choose the measure μ to be equal to $\sqrt{|\det\bar{G}_{\alpha\beta}|} = 16\sqrt{2}ab^2$. This choice ensures that the derivative part of the quadratic constraint operator becomes the Laplace-Beltrami operator which is also scalar under general configuration space transformations. Further, it also renders the linear operators (3.2) pure derivations, i.e. it makes them have the derivatives acting on the far right since $(\mu\xi_I^\beta)_{;\beta}$ vanishes for every $I = 1, \ldots, 6$.

Apart from $\hat{\mathcal{H}}$, we also have at our disposal the conditional symmetries. They too can act on the wave function and provide the connection to the solution space of the classical theory. The wave function of the system is to be realized as an eigenstate of those physical quantities that can be measured together:

$$\hat{Q}_I\Psi = \kappa_I\Psi, \tag{3.6}$$

for all the subsets of Q_I's for which the structure constants of the subalgebra they form, satisfy the integrability conditions

$$C_{JK}^I\kappa_I = 0. \tag{3.7}$$

Equation (3.7) has been proven as an integrability condition in [13,23], and gives a selection rule for determining those operators which can be applied at the same time on the wave function. The results of the use of (3.7) can be summarized, according to the various subalgebras, as follows:

(1) For the entire algebra and for all five- and four-dimensional subalgebras (3.7) is not valid.
(2) For the non-Abelian three-dimensional subalgebra $\{Q_1, Q_4, Q_6\}$, the integrability condition (3.7) implies that all the corresponding κ_I, $I = 1, 4, 6$ must be zero (since the algebra is semisimple). For a generic configuration this is unacceptable in view of the fact that, for instance, κ_1 corresponds to the essential constant Q.
(3) For the three non-Abelian two-dimensional subalgebras $\{Q_2, Q_4\}$, $\{Q_4, Q_5\}$ and $\{Q_4, Q_6\}$, the results of the application of (3.7) are similar to the previous case. For the first of them (3.7) implies that $\kappa_2 = 0$, a condition that cannot be met in view of $\kappa_2 = \frac{2}{c}$ [see (2.23)]. For the other two κ_5 or κ_6 respectively must be zero, a fact implying a kind of gauge fixing for the constants c and c_3, hence restricting the generality.

We are thus led to consider the following Abelian subalgebras:
(1) the three-dimensional subalgebra made up by Q_2, Q_3, and Q_5
(2) the two-dimensional subalgebras:
  (a) Q_1, Q_2
  (b) Q_2, Q_3
  (c) Q_2, Q_5
  (d) Q_3, Q_4
  (e) Q_3, Q_5
  (f) Q_5, Q_6.

Of course, there are also six one-dimensional subalgebras but these cannot be considered on account of the existence of *two* essential constants needed to describe the underlying geometry. Cases (2a), (2d) and (2f) of the two-dimensional subalgebras are of particular interest, since they involve integrals that are connected with essential constants (those are Q_1, Q_4, and Q_6). Let us proceed with the examination of each case.

A. The three-dimensional subalgebra and the marginal cases (2b), (2c) and (2e)

In considering the three-dimensional Abelian subalgebra spanned by Q_2, Q_3 and Q_5, and with the choice of measure $\mu = 16\sqrt{2}ab^2$, the given ξ_I's in (2.9) and definitions (3.2), we obtain the following set of differential equations:

$$\hat{Q}_2\Psi = \kappa_2\Psi \Rightarrow \frac{\mathbb{i}}{2ab}\partial_a\Psi + \kappa_2\Psi = 0, \qquad (3.8a)$$

$$\hat{Q}_3\Psi = \kappa_3\Psi \Rightarrow \mathbb{i}\left(\frac{f}{2ab}\partial_a\Psi + \frac{1}{2b}\partial_f\Psi\right) + \kappa_3\Psi = 0, \qquad (3.8b)$$

$$\hat{Q}_5\Psi = \kappa_5\Psi \Rightarrow \mathbb{i}\left[\left(\frac{a^2+f^2}{2ab}\right)\partial_a\Psi - \partial_b\Psi + \frac{f}{b}\partial_f\Psi\right] - \kappa_5\Psi = 0, \qquad (3.8c)$$

together with the Hamiltonian constraint

$$\hat{\mathcal{H}}\Psi = \frac{1}{8b}\left[\frac{1}{2b}(\partial_{aa}\Psi - \partial_{ff}\Psi) - \frac{1}{a}\partial_{ab}\Psi\right] - \Psi = 0. \qquad (3.9)$$

By solving successively from (3.8a) to (3.8c), the dependence of $\Psi(a, b, f)$ on its arguments is completely determined:

$$\Psi = \lambda e^{\mathbb{i}b(\kappa_2(a^2-f^2)+2\kappa_3 f+\kappa_5)}, \qquad (3.10)$$

with λ being an arbitrary constant. By substituting solution (3.10) into (3.9) we get

$$\kappa_2\kappa_5 + \kappa_3^2 - 4 = 0, \qquad (3.11)$$

which is an identity in view of (2.26).

The state of the system described by (3.10) resembles the situation that arose in [13] for the case of Schwarzschild geometry. There too, the enforcement of the maximal Abelian subgroup led to a plane wave solution. Furthermore, that algebra was also spanned by integrals of motion which had no connection to essential constants of the underlying geometry.

If we now choose to consider the two-dimensional cases that are made up from Q_2, Q_3 and Q_5, namely (2b), (2c) and (2e), we are led to essentially the same solution for Ψ.

(i) The set of Eqs. (3.8a), (3.8b) and (3.9) leads to a solution that differs from (3.10) by a phase $\frac{\kappa_2\kappa_5+\kappa_3^2-4}{\kappa_2}$ which, however, is zero due to (2.26).
(ii) If we now consider equations (3.8a), (3.8c) and (3.9), we end up with the following wave function:

$$\Psi_{25} = \lambda_1 e^{2bf\sqrt{\kappa_2\kappa_5-4}} e^{\mathbb{i}b(\kappa_2(a^2-f^2)+\kappa_5)} + \lambda_2 e^{-2bf\sqrt{\kappa_2\kappa_5-4}} e^{\mathbb{i}b(\kappa_2(a^2-f^2)+\kappa_5)}, \qquad (3.12)$$

that seems quite different from (3.10). Nevertheless, as we have previously mentioned, there is a nonessential constant [the constant c_3 which refers to the freedom of the scalar potential $f(r)$] that can be set to zero by a gauge transformation. Then, κ_3 becomes zero and (2.26) leads to $\kappa_2\kappa_5 = 4$. Under this condition, (3.12) becomes (3.10).
(iii) Lastly, we take into account the set of Eqs. (3.8b), (3.8c) and (3.9). The common solution of this set is different from (3.10) by a phase $\frac{a^2-f^2}{\kappa_5}(\kappa_2\kappa_5 + \kappa_3^2 - 4)$, which again is zero because of (2.26).

So, as it is evident from the above considerations, all three cases are connected to each other, giving the same plane wave solution that emerges from the consideration of the maximal Abelian algebra.

B. The two-dimensional case (2a) (Q_1, Q_2)

This subalgebra contains Q_1 whose value on the solution space is proportional to the essential constant Q

($\kappa_1 = -4Q$), meaning that Q_1 is purely connected to a quantity referring to the geometry of space-time. We consider equation

$$\hat{Q}_1\Psi = \kappa_1\Psi \Rightarrow \mathbb{i}\partial_f\Psi + \kappa_1\Psi = 0, \tag{3.13}$$

together with (3.8a) and (3.9). The common solution for the given set of equations is

$$\Psi = \frac{\lambda}{\sqrt{b}}\exp\left(\mathbb{i}\frac{\kappa_1^2 + 4bf\kappa_1\kappa_2 + 4a^2b^2\kappa_2^2 + 16b^2}{4b\kappa_2}\right), \tag{3.14}$$

with λ being again an arbitrary constant. With this wave function we are led to a probability density

$$\mu\Psi^*\Psi \propto ab, \tag{3.15}$$

that encompasses only the two scale factors and is completely free of the variable f. The latter is only present in the phase of the wave function.

C. The two-dimensional case (2d) (Q_3, Q_4)

The linear integral Q_4 assumes the constant value $\kappa_4 = 4(cm + c_3Q)$ on the solution space. As we can see, it bears a connection mainly to m, since c_3 can be set equal to zero. However, its value, in contrast to the previous case, is somewhat gauge dependent due to the involvement of nonessential constants. In this case we use the equation

$$\hat{Q}_4\Psi = \kappa_4\Psi \Rightarrow \mathbb{i}(a\partial_a\Psi - b\partial_b\Psi + f\partial_f\Psi) - \kappa_4\Psi = 0, \tag{3.16}$$

as well as (3.8b) and the WDW equation (3.9). The integration of (3.16) leads to a solution of the form

$$\Psi(a,b,f) = a^{-\mathbb{i}\kappa_4}\psi_1\left(ab, \frac{f}{a}\right). \tag{3.17}$$

It is useful to use the new variables $u = ba$, $v = \frac{f}{a}$ and a, for which the imposition of Eq. (3.8b) on the previous wave function leads to

$$\mathbb{i}((v^2-1)\partial_v\psi_1 + uv\partial_u\psi_1) + (2\kappa_3u + \kappa_4v)\psi_1. \tag{3.18}$$

Even though κ_3 can be set equal to zero through a gauge transformation, we choose to carry it until the final result. The solution of (3.18) reads

$$\psi_1(u,v) = e^{2\mathbb{i}\kappa_3uv}u^{\mathbb{i}\kappa_4}\psi_2\left(\ln(u\sqrt{v^2-1})\right). \tag{3.19}$$

At this stage, a new change of variables is in order; setting $u = \frac{e^w}{\sqrt{v^2-1}}$ the WDW equation (3.9) becomes

$$\psi_2''(w) + 2\mathbb{i}\kappa_4\psi_2'(w) + 4e^{2w}(\kappa_3^2-4)\psi_2(w) = 0. \tag{3.20}$$

The general solution of this equation is

$$\psi_2(w) = e^{-\frac{1}{2}\kappa_4(\pi+2\mathbb{i}w)}\left[\lambda_1 I_{\mathbb{i}\kappa_4}\left(2e^w\sqrt{(\kappa_3^2-4)}\right) + \lambda_2 I_{-\mathbb{i}\kappa_4}\left(2e^w\sqrt{(\kappa_3^2-4)}\right)\right], \tag{3.21}$$

with λ_1, λ_2 being arbitrary constants while $I_\nu(x)$ is the modified Bessel function of the first kind. Thus, the final form of the wave function $\Psi(a,b,f)$ is

$$\Psi = (a^2-f^2)^{-\frac{1}{2}(\mathbb{i}\kappa_4)}e^{2\mathbb{i}bf\kappa_3}\left[\lambda_1 I_{\mathbb{i}\kappa_4}\left(2b\sqrt{a^2-f^2}\sqrt{(\kappa_3^2-4)}\right) + \lambda_2 I_{-\mathbb{i}\kappa_4}\left(2b\sqrt{a^2-f^2}\sqrt{(\kappa_3^2-4)}\right)\right]. \tag{3.22}$$

D. The two-dimensional case (2f) (Q_5, Q_6)

The constant value of Q_6 is $\kappa_6 = 2(2c_3cm + (c_3^2 + c^2)Q)$. Under the gauge conditions $c_3 = 0$ and $c = 1$, κ_6 equals to $2Q$. Our starting point is the differential equation,

$$\hat{Q}_6\Psi = \kappa_6\Psi \Rightarrow \mathbb{i}\left(af\partial_a\Psi - bf\partial_b\Psi + \frac{1}{2}(a^2+f^2)\partial_f\Psi\right) - \kappa_6\Psi = 0, \tag{3.23}$$

whose solution is

$$\Psi(a,b,f) = e^{\frac{-2\mathbb{i}\kappa_6 f}{a^2-f^2}}\psi_1\left(ab, \frac{f^2}{a} - a\right). \tag{3.24}$$

By defining as new variables $u = ab$ and $v = \frac{f^2}{a} - a$ and substituting the above form of Ψ in Eq. (3.8c) we get

$$\mathbb{i}v(v\partial_v\psi_1(u,v) - u\partial_u\psi_1(u,v)) + 2\kappa_5u\psi_1(u,v) = 0. \tag{3.25}$$

Its integration yields the function

$$\psi_1(u,v) = e^{-\mathbb{i}\frac{\kappa_5 u}{v}}\psi_2(uv). \tag{3.26}$$

At this stage we introduce the new variable $w = uv$. Subsequent substitution into the WDW equation (3.9) leads to

$$2\mathbb{i}\kappa_5 w^2\psi_2'(w) + (2\kappa_6^2 + (\mathbb{i}\kappa_5 - 8w)w)\psi_2(w) = 0, \tag{3.27}$$

admitting the solution

$$\psi_2(w) = \frac{\lambda}{\sqrt{w}}\exp\left(-\mathbb{i}\frac{\kappa_6^2 + 4w^2}{\kappa_5 w}\right). \tag{3.28}$$

The wave function is written in the original variables as

$$\Psi(a,b,f) = \frac{\lambda}{\sqrt{b(f^2-a^2)}}\exp\left(\frac{\mathbb{i}(4a^4b^2 + 4b^2f^4 + a^2b^2(-8f^2+\kappa_5^2) - 2bf\kappa_5\kappa_6 + \kappa_6^2)}{b(a^2-f^2)\kappa_5}\right), \tag{3.29}$$

and leads to a probability density

$$\mu\Psi^*\Psi \propto \frac{ab}{f^2-a^2}. \tag{3.30}$$

At this point, one could think that we have attained two different representations for the physical quantity Q: The first was the case (2a) with the use of $\hat{Q}_1$ and $\hat{Q}_2$, where classically $\kappa_1 = 4Q$. The second is this, with $\hat{Q}_5$ and $\hat{Q}_6$ (under gauge conditions $c = 1$, $c_3 = 0$, $\kappa_6 = 2Q$).

However, the wave function (3.29), under the transformation $(a,b,f) \to (\alpha,\beta,\varphi)$ with

$$a = \frac{\alpha}{\alpha^2-\phi^2}, \quad b = \beta(\phi^2-\alpha^2), \quad f = \frac{\phi}{\phi^2-\alpha^2}, \tag{3.31}$$

and κ_5, κ_6 expressed in the gauge $c = 1$, $c_3 = 0$, is turned into

$$\Psi(\alpha,\beta,\phi) = \frac{\lambda}{\sqrt{\beta}}\exp\left(\frac{-2\mathbb{i}((1+\alpha^2)\beta^2 - 2Q\beta\phi + Q^2)}{\beta}\right). \tag{3.32}$$

In the same gauge, the wave function (3.14) becomes

$$\Psi_{12}(a,b,f) = \frac{\lambda}{\sqrt{b}}\exp\left(\frac{-2\mathbb{i}((1+a^2)b^2 - 2Qbf + Q^2)}{b}\right). \tag{3.33}$$

These two wave functions assume the same functional form. What is important though is, that the very same transformation transforms the Killing vector of the supermetric ξ_6 into $\frac{1}{2}\xi_1$ in the new variables (the factor $\frac{1}{2}$ expresses the fact that $\kappa_1 = 4Q$ while $\kappa_6 = 2Q$ under the considered gauge).

IV. SEMICLASSICAL ANALYSIS

In this section we are going to present a semiclassical analysis of the problem reviewed in the previous sections. To accomplish this task, we examine a wave function of the form

$$\Psi(a,b,f) = \Omega(a,b,f)e^{\mathbb{i}S(a,b,f)}, \tag{4.1}$$

in the WDW equation (3.9). Here $\Omega(a,b,f)$ and $S(a,b,f)$ are some real functions representing the magnitude and the phase of the wave function, respectively. Upon using this expression for the wave function, the WDW equation leads to the continuity equation,

$$\frac{1}{16b^2}\left[2\left(\frac{\partial\Omega}{\partial a}\frac{\partial S}{\partial a} - \frac{\partial\Omega}{\partial f}\frac{\partial S}{\partial f}\right) + \Omega\left(\frac{\partial^2 S}{\partial a^2} - \frac{\partial^2 S}{\partial f^2}\right)\right] - \frac{1}{8ab}\left(\frac{\partial\Omega}{\partial a}\frac{\partial S}{\partial b} + \frac{\partial\Omega}{\partial b}\frac{\partial S}{\partial a} + \frac{\partial^2 S}{\partial a\partial b}\right) = 0, \tag{4.2}$$

and the modified Hamilton-Jacobi equation

$$-\frac{1}{16b^2}\left(\frac{\partial S}{\partial a}\right)^2 + \frac{1}{8ab}\frac{\partial S}{\partial a}\frac{\partial S}{\partial b} + \frac{1}{16b^2}\left(\frac{\partial S}{\partial f}\right)^2 - 1 + \mathcal{Q} = 0, \tag{4.3}$$

in which

$$\mathcal{Q} = \frac{1}{\Omega}\left[\frac{1}{16b^2}\left(\frac{\partial^2\Omega}{\partial a^2} - \frac{\partial^2\Omega}{\partial f^2}\right) - \frac{1}{8ab}\frac{\partial^2\Omega}{\partial a\partial b}\right] \tag{4.4}$$

is the quantum potential. A glance at Eq. (4.3) shows that it is of the form

$$\mathcal{H}\left(q^\mu, p_\mu = \frac{\partial S}{\partial q^\mu}\right) + \mathcal{Q} = 0, \tag{4.5}$$

where $\mathcal{H}$ is the Hamiltonian defined in (2.13), $q^\mu = (a,b,f)$ are the variables of the configuration space and $p_\mu = (p_a, p_b, p_f)$ are the momenta conjugate to q^μ given by (2.12b–2.12d). Therefore, in the semiclassical picture, the equations of motion can be written as

$$\begin{cases} \frac{8}{N}abb' = \frac{\partial S}{\partial a}, \\ \frac{8}{N}(aba' + a^2b') = \frac{\partial S}{\partial b}, \\ \frac{8}{N}b^2f' = \frac{\partial S}{\partial f}. \end{cases} \tag{4.6}$$

If the quantum potential (4.4) is nonzero, the solutions to the above system differ from the classical solutions by some correction terms coming from the quantum mechanical considerations; in the cases where the quantum potential is equal to zero, we expect that solving the system (4.6) will reproduce the pure classical solutions. In the following subsections we will deal with this issue with the help of the wave functions obtained in the previous section.

A. The three-dimensional subalgebra and the two-dimensional marginal cases

We start with the wave function (3.10) which, with the notation introduced in this section, yields

$$S(a,b,f)=b[\kappa_2(a^2-f^2)+2\kappa_3 f+\kappa_5],\qquad \Omega=\text{const.} \tag{4.7}$$

It is clear that the quantum potential is zero, hence nothing but the classical solutions may be retrieved by the semiclassical analysis. Indeed, in this case the system (4.6) takes the form

$$\begin{cases}\frac{8}{N}abb'=2\kappa_2 ab,\\ \frac{8}{N}(aba'+a^2b')=\kappa_2(a^2-f^2)+2\kappa_3 f+\kappa_5,\\ \frac{8}{N}b^2f'=2b(\kappa_3-\kappa_2 f).\end{cases} \tag{4.8}$$

To solve the above system of equations, let us for the moment assume $N=2$ (this assumption will be justified later) while we use the numerical values (2.27) for the κ_I's. Under these conditions, the first equation of (4.8) can be immediately integrated giving

$$b(r)=r, \tag{4.9}$$

in which we have ignored an additive integration constant. Using this result in the third equation of (4.8) we obtain

$$f(r)=\frac{C_1}{r}, \tag{4.10}$$

where C_1 is an integration constant. Now, upon insertion of these expressions for $b(r)$ and $f(r)$ in the second equation of (4.8) we arrive at the following differential equation for $a(r)$:

$$2ra(r)a'(r)+a^2(r)=1-\frac{C_1^2}{r^2}, \tag{4.11}$$

which admits the solution

$$a(r)=\left(1+\frac{C_2}{r}+\frac{C_1^2}{r^2}\right)^{1/2}, \tag{4.12}$$

where C_2 is another constant of integration. A simple calculation based on the above relations gives

$$2aba'b'+a^2b'^2+b^2f'^2=1, \tag{4.13}$$

which shows that the assumption $N=2$ is compatible with the expression (2.19) for the lapse function. Now, if we identify the integration constants with the charge and mass parameters as $C_1=Q$ and $C_2=-2m$, the line element (2.1) takes the form of a RN black hole (2.25), as expected in the case of vanishing quantum potential.

B. The two-dimensional subalgebra (Q_1,Q_2)

In this subalgebra the wave function is given by (3.14) for which again we have used the numerical values (2.27) for the κ_I's,

$$S(a,b,f)=\frac{2Q^2-4Qbf+2a^2b^2+2b^2}{b},$$
$$\Omega(a,b,f)=\frac{\lambda}{\sqrt{b}}. \tag{4.14}$$

From (4.4) it is seen that the quantum potential is again equal to zero. The equations of the system (4.6) become

$$\begin{cases}\frac{8}{N}abb'=4ab,\\ \frac{8}{N}(aba'+a^2b')=2+2a^2-\frac{2Q^2}{b^2},\\ \frac{8}{N}b^2f'=-4Q,\end{cases} \tag{4.15}$$

which, again after choosing $N=2$, can be easily integrated providing the result

$$b(r)=r,\qquad f(r)=\frac{Q}{r}+C_1,\qquad a(r)=\left(1+\frac{C_2}{r}+\frac{Q^2}{r^2}\right)^{1/2}. \tag{4.16}$$

We see that the standard form (2.25) of the classical RN black hole solution can be recovered if one sets the integration constant $C_1=0$ and identifies the integration constant C_2 with the mass parameter as $C_2=-2m$. It seems appropriate to mention that the solutions (4.9), (4.10) and (4.12) of the three-dimensional subalgebra do not contain any of the particular values of the essential parameters of the RN black hole, but Q and m appear as integration constants after solving the system. However, in the solutions (4.16) the charge parameter enters directly into the space-time geometry (not as an integration constant) while the mass parameter is still an integration constant. This is a reflection of the fact that none of the constant values $(\kappa_2,\kappa_3,\kappa_5)$ of the quantities (Q_2,Q_3,Q_5) which span the three-dimensional subalgebra depends on the essential constants, while in the two-dimensional case (Q_1,Q_2), the constant κ_1 is indeed essential.

C. The two-dimensional subalgebra (Q_3, Q_4)

In this case, the expression (3.22) gives the wave function in terms of the Bessel functions. However, since the Bessel functions can be written as a superposition of the Hankel functions, we write the wave function as

$$\Psi(a,b,f) = (a^2-f^2)^{-2im}[c_1 H^{(1)}_{4im}(4b\sqrt{a^2-f^2}) + c_2 H^{(2)}_{4im}(4b\sqrt{a^2-f^2})], \quad (4.17)$$

where $H^{(1),(2)}_\nu(z)$ are the Hankel functions of the first and second kind, respectively, and we have used the numerical values (2.27) for κ_I's. In the classical limit, i.e. for large values of r, we have $b(r)\sim r$, $a(r)\sim 1$ and $f(r)\sim 0$. Under these conditions the argument of the aforesaid Hankel functions takes a large value and therefore, in view of the asymptotical behavior of the Hankel functions which is $H^{(1),(2)}_\nu(z)\sim z^{-1/2}e^{\pm i[z-(2\nu+1)\pi/4]}$, we can infer the following form of the wave function in the semi-classical approximation:

$$\Psi(a,b,f) \sim \frac{1}{\sqrt{b}(a^2-f^2)^{1/4}}(a^2-f^2)^{-2im}e^{4ib\sqrt{a^2-f^2}}. \quad (4.18)$$

Hence, comparing this expression with (4.1) we get

$$\Omega(a,b,f) \sim \frac{1}{\sqrt{b}(a^2-f^2)^{1/4}}, \quad (4.19)$$

and

$$S(a,b,f) = -2m\ln(a^2-f^2) + 4b\sqrt{a^2-f^2}. \quad (4.20)$$

From (4.19) and with the help of (4.4) one obtains the quantum potential,

$$\mathcal{Q}(a,b,f) = -\frac{1}{64b^2(a^2-f^2)}, \quad (4.21)$$

thereby observing that, unlike the previous subsection, its value is not equal to zero. Therefore, due to quantum effects, some modifications are expected to appear upon solving the system of equations (4.6). Using the expression (4.20) this system takes the form

$$\begin{cases} \frac{8}{N}abb' = \frac{4ab}{\sqrt{a^2-f^2}} - \frac{4ma}{a^2-f^2}, \\ \frac{8}{N}(aba'+a^2b') = 4\sqrt{a^2-f^2}, \\ \frac{8}{N}b^2f' = -\frac{4bf}{\sqrt{a^2-f^2}} + \frac{4mf}{a^2-f^2}. \end{cases} \quad (4.22)$$

If, as before, we choose the gauge $N=2$, the first and the third equations of the above system give $f'/f = -b'/b$ which can be immediately integrated to obtain

$$f(r) = \frac{Q}{b(r)}, \quad (4.23)$$

where Q is an integration constant. With this relation at hand, after some algebra with the first and the second equations of (4.22), we get

$$\begin{cases} \frac{a'}{a} = -\frac{Q^2}{a^2b^2\sqrt{a^2b^2-Q^2}} + \frac{m}{a^2b^2-Q^2}, \\ \frac{b'}{b} = \frac{1}{\sqrt{a^2b^2-Q^2}} - \frac{m}{a^2b^2-Q^2}, \end{cases} \quad (4.24)$$

which gives rise to

$$(ab)' = \frac{\sqrt{a^2b^2-Q^2}}{ab}, \quad (4.25)$$

from which we obtain

$$a^2b^2 = r^2 + Q^2, \quad (4.26)$$

where a constant of integration has been set equal to zero. Now, with a straightforward calculation based on the system (4.24) and (4.23) we find

$$a(r) = e^{-m/r}\left(1+\frac{Q^2}{r^2}\right)^{1/2}, \quad b(r) = re^{m/r}, \quad f(r) = \frac{Q}{r}e^{-m/r}. \quad (4.27)$$

Again we see that the essential constant m enters, in this case, directly into the space-time metric while the essential matter parameter Q appears as an integration constant. The solutions (4.27) tend asymptotically to the RN line element (2.25), however, unlike the RN solution, this one does not exhibit a horizonlike singularity. Now, let us see what happens in the limit of small r. In this limit the argument of the Bessel functions in the wave function (3.22) is small. According to the behavior $z^\nu(\lambda_1+\lambda_2 z^2+O(z^4))$ for the Bessel function with a small argument, the wave function takes the form

$$\Psi(a,b,f) = [\lambda_1 + \lambda_2 b^2(a^2-f^2)]e^{4im\ln b}, \quad (4.28)$$

which, with the notation of (4.1), gives

$$\Omega(a,b,f) = [\lambda_1 + \lambda_2 b^2(a^2-f^2)], \quad (4.29)$$

and

$$S(a,b,f) = 4m\ln b. \quad (4.30)$$

Expression (4.29) yields a nonzero quantum potential of the form

$$\mathcal{Q}(a,b,f) = -\frac{\lambda_2}{4[\lambda_1 + \lambda_2 b^2(a^2-f^2)]} \quad (4.31)$$

while with (4.30) the system (4.6) admits the solution

$$a(r)=(2mr+a_0)^{1/2}, \quad b(r)=\beta(\text{const}), \quad f(r)=\text{const}, \tag{4.32}$$

with a_0 being an integration constant. This geometry describes a homogeneous space-time whose Riemann tensor has vanishing covariant derivative, and thus all its higher derivative curvature scalars are zero. The Ricci scalar is found to be $\frac{2}{\beta^2}$ while all other curvature scalars are monomials of $\frac{2}{\beta^2}$ or zero. The classical curvature singularity at $r=0$ is thus replaced by an innocuous coordinate singularity, while the mass and the electric charge are merged into the constant β uniquely describing the curvature of the emerging semiclassical geometry.

At this point some further clarifications concerning the interpretation of the above fact are in order. If someone wished to attribute a meaning of singularity avoidance to it, one should have at one's disposal an appropriate Hilbert space for the physical states of the model (solutions to the Hamiltonian constraint), with a corresponding measure which makes these states normalizable, and a set of observables defined as linear operators on this space. Since such a construction has not been carried out in this work, we deter from adopting such a term.

D. The two-dimensional subalgebra (Q_5,Q_6)

According to the wave function (3.29) we have

$$S(a,b,f)=\frac{2a^4b^2+2b^2f^4+2a^2b^2(1-2f^2)-4Qbf+2Q^2}{b(a^2-f^2)},$$
$$\Omega(a,b,f)=\frac{\lambda}{\sqrt{b(f^2-a^2)}}, \tag{4.33}$$

in which we have used again the numerical values (2.27) for the κ_I's. A simple calculation based on the relation (4.4) shows that $\mathcal{Q}=0$, i.e. the quantum potential vanishes in this case as well. Also, the system of equation (4.6) takes the form

$$\begin{cases} \frac{8}{N}abb' = \frac{4a[a^4b^2-2a^2b^2f^2+b^2f^2(f^2-1)+2Qbf-Q^2]}{b(a^2-f^2)^2}, \\ \frac{8}{N}(aba'+a^2b') = \frac{2[a^4b^2+b^2f^4+a^2b^2(1-2f^2)-Q^2]}{b^2(a^2-f^2)}, \\ \frac{8}{N}b^2f' = \frac{4[-a^4b^2f+a^2b(bf+2bf^3-Q)+f(-b^2f^4-Qbf+Q^2)]}{b(a^2-f^2)^2}. \end{cases} \tag{4.34}$$

Because of the vanishing quantum potential, we expect that the classical solutions satisfy the above equations. Indeed, a combination of the first and third equations of the above system gives

$$(bf)'=\frac{bf-Q}{b(a^2-f^2)}, \tag{4.35}$$

in which we have chosen again the gauge $N=2$. If, for the moment, we assume $(bf)'=0$, the above equation yields $bf=Q$. This condition is satisfied by the classical solutions

$$b(r)=r, \qquad f(r)=\frac{Q}{r}, \tag{4.36}$$

whereby, using them in the second equation of (4.34), we get

$$2raa'+a^2=1-\frac{Q^2}{r^2}, \tag{4.37}$$

with the following solution for $a(r)$:

$$a(r)=\left(1+\frac{A}{r}+\frac{Q^2}{r^2}\right)^{1/2}. \tag{4.38}$$

It is seen that after identifying the integration constant A with the mass parameter as $A=-2m$, we obtain the standard form of the RN black hole line element (2.25).

V. EXPLANATION OF THE VANISHING OF THE QUANTUM POTENTIAL

As it has become evident in the previous section, the quantum potential $\mathcal{Q}$ is different from zero only in the case where $\hat{Q}_3$ and $\hat{Q}_4$ are imposed as "simultaneous" eigenoperators. In all other cases, the quantum potential becomes zero. Since this vanishing can be considered as a proof for a kind of consistency (since the semiclassical solutions coincide with the classical ones), we are going, in this section, to give an algebraic explanation for it.

Let us start with the eigenvalue problem

$$\hat{Q}_I\Psi=\kappa_I\Psi \Rightarrow \hat{Q}_I(\Omega e^{\mathbb{i}S})=\kappa_I\Omega e^{\mathbb{i}S}$$
$$\Rightarrow \hat{Q}_I\Omega+\mathbb{i}\Omega\hat{Q}_IS=\kappa_I\Omega. \tag{5.1}$$

Due to the form of Q_I (3.2), (5.1) can be split into a real and an imaginary part,

$$\mathbb{i}\hat{Q}_IS=\kappa_I \tag{5.2}$$

and

$$\hat{Q}_I\Omega=0, \tag{5.3}$$

respectively.

The quantum potential is just

$$\mathcal{Q}=\frac{1}{\Omega}\Box\Omega=\frac{1}{\Omega}\hat{Q}_c\Omega=\frac{1}{\Omega}(\hat{Q}_3^2+\hat{Q}_2\hat{Q}_5)\Omega, \tag{5.4}$$

where the last equation holds due to (3.4), (3.5) (which are a consequence of both the constant potential parametrization and the measure which allows the linear operators to have the derivatives on the far right). Thus, the first case of the Abelian 3 d subalgebra is clear: the Laplacian is zero because (5.3) holds for each and every element of the algebra ($I = 2, 3, 5$), leading to a vanishing $\mathcal{Q}$.

For the 2d subalgebras:

(1) ($\hat{Q}_1$, $\hat{Q}_2$).—It must hold that

$$\hat{Q}_1\Omega = 0 \quad \text{and} \quad \hat{Q}_2\Omega = 0. \tag{5.5}$$

Thus, the quantum potential $\mathcal{Q}$ becomes (since $\hat{Q}_2\hat{Q}_5 = \hat{Q}_5\hat{Q}_2$)

$$\mathcal{Q} = \frac{1}{\Omega}\hat{Q}_3^2\Omega. \tag{5.6}$$

By definition (3.2) and the choice of measure ($\mu = \sqrt{\bar{G}}$), the $\hat{Q}_I$'s have all derivations on the far right. Moreover, by virtue of (2.14), we can see that Q_3 can be written as a linear combination (with functions) of Q_1 and Q_2, therefore dictating

$$\hat{Q}_3 = f\hat{Q}_2 + \frac{1}{2b}\hat{Q}_1. \tag{5.7}$$

The latter relation means that also $\hat{Q}_3\Omega = 0$ and, as a result, again $\mathcal{Q} = 0$.

(2) ($\hat{Q}_2$, $\hat{Q}_3$).—This case is straightforward: By assumption

$$\hat{Q}_2\Omega = 0 \quad \text{and} \quad \hat{Q}_3\Omega = 0, \tag{5.8}$$

which implies $\hat{Q}_c\Omega = 0$, thereby securing the vanishing of $\mathcal{Q}$.

(3) ($\hat{Q}_2$, $\hat{Q}_5$).—In this case, one is left with $\mathcal{Q} = \frac{1}{\Omega}\hat{Q}_3^2\Omega$ and, apparently, a linear combination cannot be used [i.e. $Q_3 \neq A(q)Q_2 + B(q)Q_5$]. Nevertheless, the situation can be resolved by invoking the existence of the second Casimir invariant $\tilde{Q}_C$ [Eq. (2.22)] of the six-dimensional algebra [which, thankfully, is identically zero in the differential representation corresponding to (2.14), otherwise there would be two quadratic constraints]:
Equation (5.3) holds for $I = 2$ and $I = 5$, i.e.

$$\hat{Q}_2\Omega = 0 \quad \text{and} \quad \hat{Q}_5\Omega = 0, \tag{5.9}$$

additionally (2.22) can, demanding Hermiticity and bearing in mind that $[\hat{Q}_3, \hat{Q}_4] = 0$, be written in operator form as

$$\hat{Q}_2\hat{Q}_6 + \hat{Q}_6\hat{Q}_2 + \frac{1}{2}(\hat{Q}_1\hat{Q}_5 + \hat{Q}_5\hat{Q}_1) - 2\hat{Q}_4\hat{Q}_3 \equiv 0, \tag{5.10}$$

which, acting upon Ω yields [by virtue of (5.9)]

$$\hat{Q}_2\hat{Q}_6\Omega + \frac{1}{2}\hat{Q}_5\hat{Q}_1\Omega - 2\hat{Q}_4\hat{Q}_3\Omega = 0. \tag{5.11}$$

Due to the algebra satisfied by the Q_I's (in particular $[\hat{Q}_2, \hat{Q}_6] = \hat{Q}_3$, $[\hat{Q}_1, \hat{Q}_5] = 2\hat{Q}_3$) one can bring $\hat{Q}_2$ and $\hat{Q}_5$ to the far right and thus (5.11) reduces to

$$\hat{Q}_4\hat{Q}_3\Omega = 0. \tag{5.12}$$

At this stage, it is easy to check that

$$\hat{Q}_4 = (f^2 - a^2)b\hat{Q}_2 + b\hat{Q}_5, \tag{5.13}$$

which means that also

$$\hat{Q}_4\Omega = 0. \tag{5.14}$$

Thus, relations (5.12) and (5.14) imply that $\hat{Q}_3\Omega = \mathbb{i}\lambda\Omega$, with $\lambda \in \mathbb{R}$ since $\hat{Q}_3\Omega$ is imaginary and Ω is real.
Let us now see what is the action of $\hat{Q}_3$ on the full wave function Ψ:

$$\hat{Q}_3\Psi = \mathbb{i}\lambda\Psi + \mathbb{i}\Psi\hat{Q}_3 S. \tag{5.15}$$

We also calculate [using (5.15)]

$$\hat{Q}_3^2\Psi = -\lambda^2\Psi - 2\lambda\Psi\hat{Q}_3 S - \Psi(\hat{Q}_3 S)^2 + \mathbb{i}\hat{Q}_3^2 S. \tag{5.16}$$

The quadratic constraint on the wave function is

$$\hat{Q}_3^2\Psi + \hat{Q}_2\hat{Q}_5\Psi - 4\Psi = 0 \tag{5.17}$$

(the order of $\hat{Q}_2$, $\hat{Q}_5$ is irrelevant since they commute). By substitution of (5.16) into (5.17) we get

$$-2\lambda\Psi\hat{Q}_3 S - \Psi(\hat{Q}_3 S)^2 + \mathbb{i}\hat{Q}_3^2 S + (\kappa_2\kappa_5 - 4 - \lambda^2)\Psi = 0. \tag{5.18}$$

If we break (5.18) into a real and an imaginary part, we get

$$(\hat{Q}_3 S)^2 + 2\lambda\hat{Q}_3 S + \lambda^2 + 4 - \kappa_2\kappa_5 = 0 \quad \text{and} \tag{5.19}$$

$$\hat{Q}_3^2 S = 0, \tag{5.20}$$

respectively. Equation (5.20) indicates that $\hat{Q}_3 S$ is a constant and therefore (5.20) is satisfied identically. The trinomial (5.19) has the solution

$$\hat{Q}_3 S = -\lambda \pm \mathbb{i}\sqrt{4-\kappa_2\kappa_5}. \tag{5.21}$$

Under this, Eq. (5.15) becomes

$$\hat{Q}_3\Psi = \pm\sqrt{4-\kappa_2\kappa_5}\Psi = \kappa_3\Psi. \tag{5.22}$$

So, Ψ is an eigenfunction of $\hat{Q}_3$ and (5.3) must hold also for $I=3$, implying that $\hat{Q}_3\Omega = 0$ and therefore $\mathcal{Q}=0$.

(4) $(\hat{Q}_3, \hat{Q}_5)$.—This is an easy case, since $\mathcal{Q}$ becomes zero immediately by $\hat{Q}_3\Omega = \hat{Q}_5\Omega = 0$.

(5) $(\hat{Q}_5, \hat{Q}_6)$.—Here $\hat{Q}_5\Omega = \hat{Q}_6\Omega = 0$, which means that $\mathcal{Q} = \frac{1}{\Omega}\hat{Q}_3^2\Omega$. But, Q_3 can be written as

$$\hat{Q}_3 = \frac{f}{a^2-f^2}\hat{Q}_5 + \frac{1}{b(f^2-a^2)}\hat{Q}_6, \tag{5.23}$$

which leads to $\hat{Q}_3\Omega = 0$ and, consequently, to $\mathcal{Q}=0$.

VI. DISCUSSION

We have investigated the classical and quantum aspects of a reparametrization invariant minisuperspace action which describes the coupled Einstein-Maxwell system under the assumption of spherical symmetry. We would like to emphasize that our assumption of staticity is somewhat redundant, since it is implied by spherical symmetry in conjunction with the field equations (see e.g. [24]). This is known as the generalized Birkhoff's theorem for electrovacuum. It states that solution (2.25) is unique, and is valid outside any spherically symmetric, charged matter distribution. Of course inside the aforesaid distribution the equations of motion must be accordingly modified and thus its solution is not (2.25) (see e.g. [25]).

At the classical level of our analysis, the independent dynamical variable is the radial coordinate r while the two unknown functions $a(r)$, $b(r)$ appearing in the general spherically symmetric line element, span, along with the electromagnetic potential variable $A_\mu = (f(r),0,0,0)$, the configuration space of the (in principle) dynamical dependent variables. The way the r-lapse function $n(r)$ enters the Lagrangian (2.6) and the line element (2.1) makes manifest the invariance of the action under arbitrary parametrizations $r=h(\tilde{r})$. One can thus be led to the unique lapse parametrization $n(r)=\frac{N(r)}{2a}$ in which the potential $V(q)$ becomes constant, see (2.7). The corresponding supermetric (2.8) describes a Minkowskian configuration space manifold and admits the six Killing vector fields (2.9). With their help we can, in the appropriate phase space, define the conditional symmetries (2.14) which have a vanishing Poisson bracket with the Hamiltonian (2.13) and are thus constant on the constraint surface $\mathcal{H}\approx 0$ (2.18a). The existence of the homothetic vector (2.10) provides us with another rheonomic integral of motion (2.18b). It is noteworthy and interesting that their counterparts in the velocity phase space completely describe the classical solution space as well as the two quadratic relations (2.20) and (2.21) corresponding to the two existing Casimir invariants (2.15) and (2.22) of the algebra spanned by the six Q_I's. Indeed, using (2.19) and the consistency relation $N=\frac{d}{dr}\int N dr$, we algebraically (i.e. without ever solving the corresponding differential equations) acquire the classical Reissner-Nordström solution (2.24), the quadratic relations (2.20), (2.21) and the reparametrization invariance since $b(r)$ remains undefined. Thus, we have the general solution of the Einstein-Maxwell equations purely in terms of the symmetries of the corresponding minisuperspace action.

At the quantum level, we demand Hermiticity under the unique natural measure $\mu=\sqrt{G}$ in order to turn the conditional symmetries Q_I and the Hamiltonian constraint $\mathcal{H}$ into operators (3.2), (3.5). In order to determine which of the linear operators can be considered, we use the integrability condition (3.7) which implies that only the elements of certain subalgebras can be simultaneously applied on the wave function $\Psi(a,b,f)$. We thus arrive at four distinct families of quantum states (see the corresponding subsections of Sec. III). Due to the well-known problems of interpretation of the wave function, we turn, in Sec. IV, to the semiclassical approximation in order to get a glimpse at the fate of the classical singularity. We thus arrive at the conclusion that the semiclassical equations of motion corresponding to the asymptotic limit of the wave function (4.17) (derived from the subalgebra $\hat{Q}_3$, $\hat{Q}_4$) give rise to respective semiclassical geometries that contain no curvature or horizonlike singularity (for similar results in the context of loop quantum cosmology see [26]). A very interesting occurrence is the vanishing of the quantum potential $\mathcal{Q}$ in the other three cases, a fact that leads to the semiclassical equations of motion giving rise to the classical solution space. On the one hand, this is a negative feature since it prohibits us from gaining some quantum information at the semiclassical level; on the other hand, it can also be considered as showing the consistency of the quantum theory in consideration, and thus as a positive occurrence. It is thus interesting to examine the reason for this vanishing of the quantum potential. This is done in Sec. V: The main reason is that the form of the wave function $\Psi=\Omega e^{\mathbb{i}S}$ dictates that whenever a first order linear operator is applied as $\hat{Q}\Psi=\kappa\Psi$, the condition on Ω is homogeneous, $\hat{Q}\Omega=0$. This, in conjunction with the two Casimir invariants and the particular form of the operators, fully explains the vanishing of the quantum potential $\mathcal{Q}$.

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