

University of Texas Rio Grande Valley

ScholarWorks @ UTRGV

---

Mathematical and Statistical Sciences Faculty  
Publications and Presentations

College of Sciences

---

7-27-2016

## Riemann-Hilbert approach for the FQXL model: A generalized Camassa-Holm equation with cubic and quadratic nonlinearity

Zhen Wang

*Dalian University of Technology*

Zhijun Qiao

*The University of Texas Rio Grande Valley, zhijun.qiao@utrgv.edu*

Follow this and additional works at: [https://scholarworks.utrgv.edu/mss\\_fac](https://scholarworks.utrgv.edu/mss_fac)



Part of the [Mathematics Commons](#)

---

### Recommended Citation

Wang, Zhen, and Zhijun Qiao. 2016. "Riemann-Hilbert Approach for the FQXL Model: A Generalized Camassa-Holm Equation with Cubic and Quadratic Nonlinearity." *Journal of Mathematical Physics* 57 (7): 073505. <https://doi.org/10.1063/1.4959232>.

This Article is brought to you for free and open access by the College of Sciences at ScholarWorks @ UTRGV. It has been accepted for inclusion in Mathematical and Statistical Sciences Faculty Publications and Presentations by an authorized administrator of ScholarWorks @ UTRGV. For more information, please contact [justin.white@utrgv.edu](mailto:justin.white@utrgv.edu), [william.flores01@utrgv.edu](mailto:william.flores01@utrgv.edu).

# Riemann-Hilbert approach for the FQXL model: A generalized Camassa-Holm equation with cubic and quadratic nonlinearity

Zhen Wang<sup>1,a)</sup> and Zhijun Qiao<sup>2,a)</sup>

<sup>1</sup>*School of Mathematics Science, Dalian University of Technology, Dalian 116085, China*

<sup>2</sup>*School of Mathematical and Statistical Sciences, The University of Texas-Rio Grande Valley, 1201 West University Drive, Edinburg, Texas 78541, USA*

(Received 9 September 2015; accepted 10 July 2016; published online 27 July 2016)

In this paper, the inverse scattering transform associated with a Riemann-Hilbert problem is formulated for the FQXL model: a generalized Camassa-Holm equation  $m_t = \frac{1}{2}k_1[m(u^2 - u_x^2)]_x + \frac{1}{2}k_2(2mu_x + m_xu)$ ,  $m = u - u_{xx}$ , which was originally included in the work of Fokas [Physica D **87**, 145 (1995)] and was recently shown to be integrable in the sense of Lax pair, bi-Hamilton structure, and conservation laws by Qiao, Xia, and Li [e-print arXiv:1205.2028v2 (2012)]. We have discussed the following properties: direct scattering problems and Jost solutions, asymptotical and analytical behavior of Jost solutions, the scattering equations in a Riemann-Hilbert problem, and the multi-soliton solutions of the FQXL model. Then, one-soliton and two-soliton solutions are presented in a parametric form as a special case of multi-soliton solutions. *Published by AIP Publishing.* [<http://dx.doi.org/10.1063/1.4959232>]

## I. INTRODUCTION

Integrable systems and soliton theory play essential roles in understanding nonlinear phenomena to search for conservation laws, Hamiltonian structures, Lax pair, and multiple nonlinear wave interactions. Particularly, comprehending the long-time asymptotic of soliton solutions is crucial. In general, the typical asymptotic behavior is that any fast decaying profile is decomposed into some solitons with vanishing in dispersive part. The Korteweg-de Vries (KdV) equation was the first integrable example to possess such feature observed in numerical scheme.<sup>1</sup> Since then, much progress has been made for asymptotic formulas over the last thirty years. The most effective analytical method for the long-time asymptotics is the steepest descent method in the paper by Deift and Zhou.<sup>2,3</sup> Some earlier related works were done by Manakov<sup>4</sup> and Its.<sup>5</sup> Another remarkable progress in the analysis of integrable system and soliton equation was the nonlinearization approach by Cao.<sup>6</sup> More research on this method can be found in the well-known book (Ref. 7).

This method has already been applied to a number of integrable systems and soliton equations, but there are some exceptions. One of them is the well-known Camassa-Holm (CH)<sup>8</sup> peakon equation recently studied by Eckhardt and Teschl.<sup>9</sup> Holm and Ivanov gave its  $N$  soliton solution by the inverse scattering transformation method.<sup>10</sup> The other notable system is the Degasperis-Procesi (DP) equation<sup>11</sup> and the corresponding inverse scattering transform and Riemann-Hilbert problem were done by Constantin, Ivanov, and Lenells.<sup>12</sup> The CH and DP equations belong to a nonlinear quadratic peakon system, called the  $b$ -family:  $m_t + m_xu + bmu_x = 0$ ,  $m = u - u_{xx}$  by Holm and Staley,<sup>13</sup> and both equations have algebro-geometric solutions.<sup>14,15</sup> Constantin *et al.*<sup>16,17</sup> used the inverse scattering method to study the CH equation and give its soliton solutions. However, the integrable cubic peakon systems<sup>18–21</sup> have not been investigated yet for the Riemann-Hilbert problem.

<sup>a)</sup>Authors to whom correspondence should be addressed. Electronic addresses: [wangzhen@dlut.edu.cn](mailto:wangzhen@dlut.edu.cn) and [zhijun.qiao@utrgv.edu](mailto:zhijun.qiao@utrgv.edu)

This paper is contributed to the Riemann-Hilbert approach for the following generalized Camassa-Holm peakon system with both cubic and quadratic nonlinearity:

$$m_t = \frac{1}{2}k_1[m(u^2 - u_x^2)]_x + \frac{1}{2}k_2(2mu_x + m_xu), \quad m = u - u_{xx}, \tag{1}$$

which was originally included in the work of Fokas<sup>22</sup> as a special case, and was recently shown integrable in the sense of Lax pair, bi-Hamilton structure, and conservation laws by Qiao, Xia, and Li.<sup>23</sup> For our convenience in description, let us simply call equation (1) the FQXL model.

When  $k_1 = 0, k_2 = -2$ , the FQXL equation gives the celebrated Camassa-Holm equation,<sup>8</sup> while letting  $k_1 = -2$  and  $k_2 = 0$  yields the Fokas-Olver-Rosenau-Qiao (FORQ) equation, which is also known as the modified Camassa-Holm (MCH) equation in the literature. The FORQ/MCH equation was discovered independently by Fokas<sup>22</sup> and Olver and Rosenau,<sup>24</sup> and also it was derived by Fuchsstermer,<sup>25</sup> Qiao,<sup>18</sup> and Novikov.<sup>21</sup> The Cauchy problem of the FQXL equation with analytic initial data was studied in Ref. 26, where a nonlocal Cauchy-Kovalevsky theorem was proved. Furthermore, the FQXL model was already proven admitting classical solitons, peakons, kinks, and multi-soliton solutions in Ref. 23, where in particular the weak-kinks and the kink-peakon interactional solutions were proposed for the first time.

The FQXL model's Lax pair is<sup>23</sup>

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix}_x = U \begin{pmatrix} \psi \\ \phi \end{pmatrix}, U = \frac{1}{2} \begin{pmatrix} -1 & \lambda m \\ -k_1 \lambda m - k_2 \lambda & 1 \end{pmatrix}, \tag{2}$$

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix}_t = V \begin{pmatrix} \psi \\ \phi \end{pmatrix}, V = -\frac{1}{2} \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{3}$$

where

$$\begin{aligned} A &= \frac{1}{\lambda^2} + \frac{1}{2}k_1(u^2 - u_x^2) + \frac{k_2}{2}(u - u_x), \\ B &= -\frac{(u - u_x)}{\lambda} - \frac{1}{2}\lambda m[k_1(u^2 - u_x^2) + k_2u], \\ C &= \frac{[k_1(u + u_x) + k_2]}{\lambda} + \frac{1}{2}\lambda[k_1^2m(u^2 - u_x^2) + k_1k_2(mu + u^2 - u_x^2) + k_2^2u]. \end{aligned}$$

The compatibility condition of Lax pair generates

$$U_t - V_x + [U, V] = 0, \tag{4}$$

which is exactly FQXL equation (1).

For the spectral problem, we can rewrite it in the following component form:

$$2\psi_x = -\psi + \lambda m \phi, \tag{5}$$

$$2\phi_x = -(k_1 \lambda m + k_2 \lambda)\psi + \phi. \tag{6}$$

Under the change of local coordinates

$$\partial_y = \frac{2}{m} \partial_x, \phi = \frac{1}{\lambda} \left[ \psi_y + \frac{1}{m} \psi \right], \tag{7}$$

we are able to get a second order spectral problem in terms of variable  $y$

$$\psi_{yy} + \left[ \left( \frac{1}{m} \right)_y - \frac{1}{m^2} \right] \psi = -\lambda^2 \frac{k_1 m + k_2}{m} \psi. \tag{8}$$

This spectral problem has an energy dependent potential, and the spectral parameter  $\lambda$  and the potential function  $m$  are present together in a product and the power of spectral parameter  $\lambda$  is quadratic. This kind of spectral problems also occur in the inverse scattering transform of Kaup-Boussinesq equation.

The whole paper is organized as follows. Direct scattering problems and Jost solutions are provided in Section II. Asymptotical and analytical properties of Jost solutions are discussed in Section III, and we reformulate the scattering equations in a Riemann-Hilbert problem. In Section IV, we set up

scattering data, discrete spectrum, and the time evolution, and analytical and parametric solutions for the Riemann-Hilbert problem are presented. In Section VI, multi-soliton solutions of the FQXL model are studied in the case of reflectionless potentials. In Section VII, we obtain one-soliton and two-soliton solutions in a parametric form as a special case of multi-soliton solutions.

## II. SCATTERING PROBLEM OF THE FQXL MODEL

Let us consider the case of asymptotic behavior under the following limit:

$$\lim_{|x| \rightarrow \pm\infty} m(x, t) = m_0 \tag{9}$$

where  $m_0$  is a non-zero constant. Since function  $m(y, t)$  is included in the denominator in equation Eq. (8),  $m_0 = 0$  is not allowed here. Actually, in this paper we require  $k_1 k_2 > 0$ , and  $m(x, 0) > 0$  for  $x \in R$ .

Let us consider the asymptotic behavior of function  $m(x, t)$ , then the spectral problem (8) is degenerated as follows:

$$\psi_{yy} = - \left[ \lambda^2 \left( k_1 + \frac{k_2}{m_0} \right) - \frac{1}{m_0^2} \right] \psi = -k^2 \psi, \tag{10}$$

where  $k^2 = \left[ \lambda^2 \left( k_1 + \frac{k_2}{m_0} \right) - \frac{1}{m_0^2} \right]$ . Thus, the spectral parameter  $\lambda$  can be expressed below,

$$\lambda^2 = \frac{k^2 m_0^2 + 1}{k_1 m_0^2 + k_2 m_0}. \tag{11}$$

If  $k$  is real, then we have  $\lambda^2 > \frac{1}{m_0(k_1 m_0 + k_2)}$ ,  $k_1 > 0, k_2 > 0$ , and the solution to Eq. (10) is oscillated. So, the continuous spectrum is located on the real line in the complex  $k$ - plane. Solving Eq. (11) for  $\lambda$  leads to

$$\lambda = \sigma k \sqrt{\frac{m_0}{k_1 m_0 + k_2} + \frac{1}{m_0 k^2 (k_1 m_0 + k_2)}}, \tag{12}$$

where  $\sigma = \pm 1$  and  $k \neq 0$ . An approximation of  $\lambda$  for large  $|k|$  yields

$$\lambda(k, \sigma) = \sigma \sqrt{\frac{m_0}{k_1 m_0 + k_2}} \left( k + \frac{1}{2m_0^2 k} - \frac{1}{8m_0^4 k^3} + O\left(\frac{1}{k^5}\right) \right). \tag{13}$$

From Eq. (12), one can see that  $\lambda$  is symmetric with respect to pair  $(k, \sigma)$  i.e.,

$$\lambda(k, \sigma) = \lambda(-k, -\sigma). \tag{14}$$

By the standard procedure of inverse scattering method, for real  $k \neq 0$  i.e., for real  $\lambda \neq \pm \frac{1}{\sqrt{k_1 m_0^2 + k_2 m_0}}$ , a solution basis can be determined through following asymptotic behaviors when  $y \rightarrow \infty$ :

$$\phi_1(y, k) = e^{-iky} + o(1), y \rightarrow \infty, \tag{15}$$

$$\phi_2(y, k) = e^{iky} + o(1), y \rightarrow \infty, \tag{16}$$

and when  $y \rightarrow -\infty$ , a second basis is similarly given

$$\psi_1(y, k) = e^{-iky} + o(1), y \rightarrow -\infty, \tag{17}$$

$$\psi_2(y, k) = e^{iky} + o(1), y \rightarrow -\infty. \tag{18}$$

Because the spectral parameter  $\lambda$  depends on both  $k$  and  $\sigma$ , we use  $\phi_1(y, k, \sigma), \phi_2(y, k, \sigma), \psi_1(y, k, \sigma), \psi_2(y, k, \sigma)$  to denote the Jost solutions of Eq. (10).

Since the potential function  $m$  and parameter  $k$  are real, we have

$$\psi_1(y, k, \sigma) = \overline{\psi_2(y, -k, -\sigma)}, \tag{19}$$

$$\phi_1(y, k, \sigma) = \overline{\phi_2(y, -k, -\sigma)}. \tag{20}$$

Apparently, for all real  $k \neq 0$ , if  $\phi(y, k, \sigma)$  is a solution of spectral problem (10), so is  $\phi(y, -k, -\sigma)$ , due to  $\phi(y, k, \sigma)$  and  $\phi(y, -k, -\sigma)$  sharing the same value of  $\lambda$ . With the aid of (14), one easily sees

$$\phi_1(y, k, \sigma) = \phi_2(y, -k, -\sigma), \psi_1(y, k, \sigma) = \psi_2(y, -k, -\sigma). \quad (21)$$

These solutions are called the Jost solutions to Equation (8).

Based on the above properties, let us adopt  $\psi(y, k, \sigma)$  and  $\bar{\psi}(y, k, \sigma)$  as a set of solution basis of spectral problem (8) instead of  $\psi_1(y, k, \sigma)$  and  $\psi_2(y, k, \sigma)$ . Similarly, we can use  $\phi(y, k, \sigma)$  and its conjugate  $\bar{\phi}(y, k, \sigma)$  to replace  $\phi_1(y, k, \sigma)$  and  $\phi_2(y, k, \sigma)$  to form another solution basis.

Because Eq. (10) is a second order linear ordinary differential equation, each of the two bases can be expressed as a linear combination of the other basis, namely,

$$\psi(y, k, \sigma) = a(k, \sigma)\phi(y, k, \sigma) + b(k, \sigma)\bar{\phi}(y, k, \sigma), \quad (22)$$

$$\bar{\psi}(y, k, \sigma) = c(k, \sigma)\phi(y, k, \sigma) + d(k, \sigma)\bar{\phi}(y, k, \sigma). \quad (23)$$

One can readily get  $c(k, \sigma) = \bar{b}(k, \sigma)$  and  $d(k, \sigma) = \bar{a}(k, \sigma)$  by taking the conjugate of the first equation on both sides and comparing with the second equation. Eqs. (22) and (23) can be cast into the following vector form:

$$\begin{pmatrix} \psi(y, k, \sigma) \\ \bar{\psi}(y, k, \sigma) \end{pmatrix} = \begin{pmatrix} a(k, \sigma) & b(k, \sigma) \\ \bar{b}(k, \sigma) & \bar{a}(k, \sigma) \end{pmatrix} \begin{pmatrix} \phi(y, k, \sigma) \\ \bar{\phi}(y, k, \sigma) \end{pmatrix}, \quad (24)$$

where the coefficients matrix is defined as the scattering matrix  $T(k, \sigma)$ ,

$$T(k, \sigma) = \begin{pmatrix} a(k, \sigma) & b(k, \sigma) \\ \bar{b}(k, \sigma) & \bar{a}(k, \sigma) \end{pmatrix}. \quad (25)$$

The Wronskian bilinear form  $W(f_1, f_2) = f_1 f_{2y} - f_2 f_{1y}$  of any two independent solutions does not depend on  $y$ . Therefore by the asymptotic conditions shown in Eqs. (15)–(18), we have

$$W(\phi(y, k, \sigma), \bar{\phi}(y, k, \sigma)) = W(\psi(y, k, \sigma), \bar{\psi}(y, k, \sigma)) = 2ik. \quad (26)$$

Furthermore, from Eq. (22), we have

$$a(k, \sigma) = \frac{1}{2ik} W[\psi(y, k, \sigma), \bar{\phi}(y, k, \sigma)] \quad (27)$$

and

$$b(k, \sigma) = \frac{1}{2ik} W[\phi(y, k, \sigma), \psi(y, k, \sigma)]. \quad (28)$$

Eqs. (22), (23), and (26) imply

$$\det(T(k, \sigma)) = |a(k, \sigma)|^2 - |b(k, \sigma)|^2 = 1. \quad (29)$$

Denoting the reflection coefficient by

$$\mathcal{R}(k, \sigma) = b(k, \sigma)/a(k, \sigma) \quad (30)$$

and the transmission coefficient by

$$\mathcal{T}(k, \sigma) = 1/a(k, \sigma), \quad (31)$$

then we may deduce the following unitary form:

$$|\mathcal{R}(k, \sigma)|^2 + |\mathcal{T}(k, \sigma)|^2 = 1. \quad (32)$$

One may only care  $\mathcal{R}(k, \sigma)$  and  $\mathcal{T}(k, \sigma)$  on the half line  $k > 0$  due to the following facts:

$$\mathcal{R}(-k, -\sigma) = \bar{\mathcal{R}}(k, \sigma), \mathcal{T}(-k, -\sigma) = \bar{\mathcal{T}}(k, \sigma). \quad (33)$$

Actually, Eqs. (22) and (23) imply that

$$\psi(y, -k, -\sigma) = a(-k, -\sigma)\phi(y, -k, -\sigma) + b(-k, -\sigma)\bar{\phi}(y, -k, -\sigma) \quad (34)$$

and

$$\bar{\psi}(y, k, \sigma) = a(-k, -\sigma)\bar{\phi}(y, -k, -\sigma) + b(-k, -\sigma)\phi(y, -k, -\sigma), \tag{35}$$

respectively. Comparing with the second equation in Eq. (24), one can get

$$\bar{a}(k, \sigma) = a(-k, -\sigma), \quad \bar{b}(k, \sigma) = b(-k, -\sigma), \tag{36}$$

which yield Eq. (33).

### III. ANALYTICAL PROPERTY AND RIEMANN HILBERT PROBLEM

In this section, we discuss the asymptotic behavior of  $a(k)$  and the Jost solutions for large  $|k|$ . The solution of (8) can be assumed in the following form:

$$\psi(y, k, \sigma) = \exp\left(-iky + \int_{-\infty}^y \chi(z, k, \sigma) dz\right), \tag{37}$$

where  $\chi(y, k, \sigma)$  satisfies

$$\chi_y + \chi^2 - 2ik\chi - k^2 = \frac{1}{m^2} + \left(\frac{1}{m}\right)_y - \frac{m_0^2 k^2 + 1}{m_0(k_1 m_0 + k_2)} \left(k_1 + \frac{k_2}{m}\right). \tag{38}$$

By Eq. (22) and the asymptotic properties of  $\phi(y, k, \sigma)$  showing in Eqs. (15) and (16), we may have

$$\psi(y, k, \sigma) = a(k, \sigma)e^{-iky} + b(k, \sigma)e^{iky} + o(1), \tag{39}$$

when  $\text{Im } k > 0$  and  $y \rightarrow +\infty$ . In fact, since  $e^{iky}$  vanishes when  $y \rightarrow +\infty$  under the condition of  $\text{Im } k > 0$ , Eq. (39) implies the following result:

$$\psi(y, k, \sigma)e^{iky} = a(k, \sigma),$$

and therefore

$$\ln a(k, \sigma) = \int_{-\infty}^{\infty} \chi(x, k, \sigma) dx, \quad \text{Im } k > 0. \tag{40}$$

Due to  $a(k, \sigma)$  independent of  $t$ , the expression  $\int_{-\infty}^{\infty} \chi(y, k, \sigma) dx$  represents integral of motion for all  $k$ . Thus, Eq. (38) admits a solution with the asymptotic expansion in terms of  $\frac{1}{k}$

$$\chi(y, k, \sigma) = p_1 k + p_0 + \sum_{n=1}^{\infty} \frac{p_{-n}}{k^n}, \tag{41}$$

where  $p_1, p_0, p_{-n}$  can be obtained through Equation (38). The first term  $p_1$  satisfies the following quadratic equation:

$$p_1^2 - 2ip_1 - 1 = -\frac{m_0}{(k_1 m_0 + k_2)} \left(k_1 + \frac{k_2}{m}\right), \tag{42}$$

which is solved with solutions

$$p_1 = i \left(1 \pm \sqrt{\frac{m_0}{k_1 m_0 + k_2} \left[k_1 + \frac{k_2}{m}\right]}\right). \tag{43}$$

Since  $\int_{-\infty}^{+\infty} p_1 dy$  represents integral of motion, we should choose minus in Eq. (43) in order to make certain that integral motion is finite. For the next order of  $k$ , we can get the expression of  $p_0$  and  $p_{-1}$  in terms of  $p_1$ ,

$$p_0 = \frac{p_1 y}{2(i - p_1)}, \tag{44}$$

$$p_{-1} = -\frac{m_y + m^2 p_0 y + m^2 p_0 - 1}{2m^2(p_1 - i)} + \frac{k_1 + k_2/m}{2(m_0^2 k_1 + m_0 k_2)(p_1 - i)}. \tag{45}$$

In addition,  $p_{-n}$  ( $n > 1$ ) can recursively be determined through solving Eq. (38) and given by

$$p_{-n} = \frac{p_{-(n-1),y} + \sum_{j=-(n-1)}^0 p_j p_{-(n-1)-j}}{2(i - p_1)}. \tag{46}$$

Substituting Eq. (41) into Eq. (40) yields

$$\ln a(k, \sigma) = -i\alpha k + \sum_{n=1}^{\infty} \frac{I_{-n}}{k^n}, \tag{47}$$

where  $\alpha$  is a positive constant determined by

$$\alpha = \int_{-\infty}^{\infty} \left( \sqrt{\frac{m_0}{k_1 m_0 + k_2} \left( k_1 + \frac{k_2}{m} \right)} - 1 \right) dy, \tag{48}$$

and

$$I_{-n} = \int_{-\infty}^{\infty} p_{-n} dy, n \geq 0,$$

are conserved quantities, and particularly

$$I_0 = \int_{-\infty}^{+\infty} p_0 dy = \int_{-\infty}^{+\infty} \frac{p_1 y}{2(i - p_1)} dy = -\ln(i - p_1) \Big|_{-\infty}^{+\infty} = 0.$$

Apparently, the asymptotics of  $a(k, \sigma)$  for  $\text{Im } k > 0$  and  $|k| \rightarrow \infty$  obeys  $a(k, \sigma) \rightarrow e^{-i\alpha k}$ , which implies

$$a(k, \sigma) e^{i\alpha k} \rightarrow 1, \quad |k| \rightarrow \infty, \quad \text{and} \quad \text{Im } k > 0. \tag{49}$$

Let us now consider the asymptotics of the Jost solutions. One can check that the asymptotics of  $\phi(y, k, \sigma)$  for  $|k| \rightarrow \infty$  has the form

$$\phi(y, k, \sigma) = e^{-iky+kG(y)} \eta(y, k) \eta(y, k) = X_0(y) + \frac{X_1(y)}{k} + \frac{X_2(y)}{k^2} + \dots, \tag{50}$$

where  $G(y) \rightarrow 0$  and  $\eta(y, k) \rightarrow 1$  for  $y \rightarrow \infty$  due to Eq. (16). Substituting Eq. (50) into Eq. (8) leads to the following expression:

$$\phi(y, k, \sigma) = e^{-iky-ik \int_y^{\infty} \left( 1 - \sqrt{\frac{m_0}{k_1 m_0 + k_2} \left( k_1 + \frac{k_2}{m} \right)} \right) dy} \left[ X_0(y) + \frac{X_1(y)}{k} + \dots \right]. \tag{51}$$

where  $X_0(y) = \frac{(k_1+k_2/m_0)^{\frac{1}{4}}}{(k_1+k_2/m)^{\frac{1}{4}}}$ , and all other  $X_j(y)$ ,  $j = 1, 2, \dots$  are finite.

Introducing the function

$$\xi(y) = \exp \left[ y - \int_y^{\infty} \left( \sqrt{\frac{m_0}{k_1 m_0 + k_2} \left( k_1 + \frac{k_2}{m} \right)} - 1 \right) dz \right], \tag{52}$$

then we may rewrite Eq. (51) as

$$\phi(y, k, \sigma) = [\xi(y)]^{-ik} \left[ \left( \frac{\xi(y)}{\xi'(y)} \right)^{1/2} + \frac{X_1(y)}{k} + \frac{X_2(y)}{k^2} + \dots \right]. \tag{53}$$

On the other hand, spectral problem (10) is able to be equivalent to the following integral equation:

$$\phi(y, k, \sigma) = e^{-iky} - \int_y^{+\infty} \frac{e^{ik(y-z)} - e^{-ik(y-z)}}{2ik} \phi(z, k, \sigma) M(z, t) dz, \tag{54}$$

where  $M(z, t) = \left[ \left( \frac{1}{m} \right)_z + \frac{1}{m^2} - \frac{1}{m_0^2} - \lambda^2 k_2 \left( \frac{1}{m} - \frac{1}{m_0} \right) \right]$ . If we take  $\chi(y, k, \sigma) = \psi(y, k, \sigma) e^{iky}$ , then in the lower half plane of  $k$  (i.e  $\text{Im } k < 0$ ) this equation is converted to

$$\chi(y, k, \sigma) = 1 - \int_y^{+\infty} \frac{e^{2ik(y-z)} - 1}{2ik} \chi(z, k, \sigma) M(z, t) dz. \tag{55}$$

Since  $z > y$  in Eq. (55), then the integrand of (55) is exponentially decreasing as well as bounded. Thus  $\chi(y, k, \sigma)$  is analytically continuous in the domain  $\text{Im } k < 0$ . More detailed explanation can be found in Ref. 2.

Notice that  $\int_y^\infty \left( \sqrt{\frac{m_0}{k_1 m_0 + k_2}} \left( k_1 + \frac{k_2}{m} \right) - 1 \right) dz$  is bounded for all values of  $y$ . In fact,

$$\begin{aligned} \left| \int_y^{+\infty} \left( \sqrt{\frac{m_0}{k_1 m_0 + k_2}} \left( k_1 + \frac{k_2}{m} \right) - 1 \right) dz \right| &= \left| \int_y^{-\infty} \frac{k_2 \left( \frac{1}{m} - \frac{1}{m_0} \right) dz}{(k_1 + k_2/m_0) \left( 1 + \sqrt{\frac{k_1 + k_2/m}{k_1 + k_2/m_0}} \right)} \right| \\ &\leq \left| \frac{k_2}{k_1 + k_2/m_0} \right| \int_{-\infty}^{\infty} \left| \frac{1}{m} - \frac{1}{m_0} \right| dz < \infty \\ &\leq \left| \frac{k_2}{k_1 + k_2/m_0} \right| \int_{-\infty}^{\infty} \left| \frac{m_0 - m}{m_0 m} \right| dz < \infty, \end{aligned}$$

where  $m = m(y, t)$  is a Schwartz class function. Therefore, the function

$$\underline{\phi}(y, k, \sigma) \equiv \phi(y, k, \sigma) [\xi(y)]^{ik} \tag{56}$$

is also analytic for  $\text{Im } k < 0$ .

Similarly,

$$\begin{aligned} \underline{\psi}(y, k, \sigma) &\equiv \psi(y, k, \sigma) \exp \left\{ ik \left[ y + \int_{-\infty}^y \left( \sqrt{\frac{k_1 + k_2/m}{k_1 + k_2/m_0}} - 1 \right) dz \right] \right\} \\ &= \left( \frac{\xi(y)}{\xi'(y)} \right)^{1/2} + \frac{\tilde{X}_1(y)}{k} + \frac{\tilde{X}_2(y)}{k^2} + \dots \end{aligned} \tag{57}$$

is analytic for  $\text{Im } k > 0$ .

Multiplying (22) by  $\xi(y)/a(k, \sigma)$  and using (48), (56), and (57), we arrive at

$$\frac{\underline{\psi}(y, k, \sigma)}{e^{ik\alpha} a(k, \sigma)} = \underline{\phi}(y, k, \sigma) + \mathcal{R}(k, \sigma) \bar{\phi}(y, k, \sigma) [\xi(y)]^{2ik}. \tag{58}$$

Therefore, the function  $\underline{\psi}(y, k, \sigma)/(e^{ik\alpha} a(k, \sigma))$  is analytic for  $\text{Im } k > 0$ , and  $\phi(y, k, \sigma)$  is analytic for  $\text{Im } k < 0$ . So, (58) can be regarded as an additive **Riemann-Hilbert Problem**—boundary value problem with a jump  $\mathcal{R}(k, \sigma) \bar{\phi}(y, k, \sigma) [\xi(y)]^{2ik}$  on the real line. It should be noted that  $\underline{\psi}(y, \bar{k}, \sigma)$  is analytic in the upper half-plane and  $\bar{\phi}(y, \bar{k}, \sigma)$  is analytic in the lower half-plane. Since  $a(k)$  can be expressed by  $\psi(y, k, \sigma)$  and  $\bar{\phi}(y, k, \sigma)$ , it is also analytic in the upper half-plane of  $k$ .

#### IV. SCATTERING DATA AND TIME EVOLUTION

##### A. Discrete spectrum

In this section, let us discuss discrete spectrum of the FQXL model. Suppose that  $k_0(\sigma) \in C_+$  is a zero of  $a(k, \sigma)$ , where  $C_+$  is the closed contour of complex  $k$ -plane. Then  $\psi(y, k_0, \sigma)$  and  $\phi(y, -k_0, -\sigma)$  are linearly dependent (27) due to

$$\psi(y, k_0, \sigma) = b_0(\sigma) \phi(y, -k_0, -\sigma). \tag{59}$$

This implies that  $\psi(y, k_0, \sigma)$  exponentially decays along  $y \rightarrow -\infty$  and  $y \rightarrow +\infty$ . Therefore,  $\psi(y, k_0, \sigma)$  is a well defined eigenfunction of the discrete spectrum with an eigenvalue  $k_0$ .

Let us now show that  $a(k, \sigma)$  has only simple zeros in the upper half complex  $k$ -plane. The dot above one letter will be used to denote the derivatives of the latter with respect to  $k$  at the point  $k_0$ . From Eq. (11), we have

$$\dot{\lambda} = \frac{k}{\dot{\lambda}(k_0)(k_1 + k_2/m_0)}$$



and

$$W_y[\psi(y, k, \sigma), \psi(y, k_n, \sigma)] = (\lambda^2 - \lambda_n^2)(k_1 + \frac{k_2}{m})\psi(y, k, \sigma)\psi(y, k_n, \sigma), \tag{60}$$

where  $\lambda_n$  denote the value of  $\lambda$  at the  $n$ th zeros point of  $a(k, \sigma)$ ,  $k = k_n$ , in other words,  $\lambda_n = \lambda(k_n)$ .

Replacing the spectral parameter  $\lambda$  by  $k$  in Eq. (11) yields

$$W_y[\psi(y, k, \sigma), \psi(y, k_n, \sigma)] = (k^2 - k_n^2) \frac{k_1 + k_2/m}{k_1 + k_2/m_0} \psi(y, k, \sigma)\psi(y, k_n, \sigma). \tag{61}$$

Differentiating the above equation with respect to  $k$  and setting  $k = k_n$ , we have

$$W_y[\dot{\psi}(y, k_n, \sigma), \psi(y, k_n, \sigma)] = 2k_n \frac{k_1 + k_2/m}{k_1 + k_2/m_0} \psi^2(y, k_n, \sigma), \tag{62}$$

which can be integrated as follows:

$$W[\dot{\psi}(y, k, \sigma), \psi(y, k_n, \sigma)] = 2k_n(k_1 + \frac{k_2}{m_0})^{-1} \int_{-\infty}^y (k_1 + \frac{k_2}{m})\psi^2(z, k_n, \sigma)dz. \tag{63}$$

Here the boundary condition at both infinities is applied. Similarly, we can obtain

$$W[\phi(y, k, \sigma), \phi(y, k_n, \sigma)] = 2k_n(k_1 + \frac{k_2}{m_0})^{-1} \int_y^{\infty} (k_1 + \frac{k_2}{m})\phi^2(z, k_n, \sigma)dz. \tag{64}$$

Thus, the derivative of  $a(k, \sigma)$  can be calculated with the aid of Eqs. (59) and (60)

$$\begin{aligned} \dot{a}(k_n) &= \frac{1}{2ik_n} \{W[\dot{\psi}(y, k_n, \sigma), \bar{\phi}(y, k_n, \sigma)] + W[\psi(y, k_n, \sigma), \dot{\bar{\phi}}(y, k_n, \sigma)]\} \\ &= -ib_n \left(k_1 + \frac{k_2}{m_0}\right)^{-1} \int_{-\infty}^{\infty} \left(k_1 + \frac{k_2}{m}\right) \psi^2(y, k_n, \sigma)dy. \end{aligned}$$

So, we have concluded that all the zeros of  $a(k, \sigma)$  are simple zeros when  $k_1 + \frac{k_2}{m} > 0$ .

Let us consider the function

$$a_1(k, \sigma) = e^{i\alpha k} \prod_{n=1}^N \frac{k + ik_n}{k - ik_n} a(k, \sigma), \tag{65}$$

which is analytical for  $\text{Im } k > 0$  without any zeros because  $a(k, \sigma)$  has simple zeros only at the points of the discrete spectrum  $ik_n$ .

Therefore  $\ln a_1(k)$  is also analytic in the upper half plane. By Eq. (49), we have  $\ln a_1(k, \sigma) \rightarrow 0$  for  $|k| \rightarrow \infty$ . Eq. (65) implies

$$|a_1(k, \sigma)| = |a(k, \sigma)|. \tag{66}$$

For real  $k$ , the Kramers-Kronig dispersion relation provides

$$\ln a_1(k, \sigma) = \ln |a(k, \sigma)| + i \arg a_1(k, \sigma) \tag{67}$$

with

$$\arg a_1(k, \sigma) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\ln |a(k', \sigma)|}{k' - k} dk', \tag{68}$$

where  $P$  denotes the principle value. By the Sohotski Plemelj formula, it can also be written as

$$\ln a_1(k, \sigma) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\ln |a(k', \sigma)|}{k' - k - i0} dk'. \tag{69}$$

The integration of the kind  $\int_{-\infty}^{+\infty} \frac{f(k')}{k' - k \pm i0} dk'$  could be regarded as  $\mp i f(k) + P \int_{-\infty}^{+\infty} \frac{f(k')}{k' - k} dk'$ , where  $P \int$  denote the principal values of the integral.

Thus, Eq. (65) has another form

$$\ln a(k, \sigma) = -i\alpha k + \sum_{n=1}^N \ln \frac{k - ik_n}{k + ik_n} + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\ln |a(k', \sigma)|}{k' - k - i0} dk'. \tag{70}$$

On the other hand, for  $\text{Im } k > 0$ , Cauchy theorem gives

$$\begin{aligned} \ln a_1(k, \sigma) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{\ln a_1(k', \sigma)}{k' - k} dk' \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln a_1(k', \sigma)}{k' - k} dk' + \frac{1}{2\pi i} \int_{\Gamma_c} \frac{\ln a_1(k', \sigma)}{k' - k} dk' \end{aligned} \tag{71}$$

where  $\Gamma$  and  $\Gamma_c$  denote the real line and the infinite semicircle in the upper half plane, respectively.

Due to  $\ln a_1(k, \sigma) \rightarrow 0$  for  $|k| \rightarrow \infty$ , the second part of integral in Eq. (71) on infinite semicircle  $\Gamma_c$  vanishes. Hence, we have

$$\begin{aligned} \ln a_1(k, \sigma) &= \frac{1}{2(\pi i)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\ln |a(k'', \sigma)|}{k'' - k' - i0} dk'' \frac{1}{k' - k} dk' \\ &= \frac{1}{2(\pi i)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{k' - k} \frac{1}{k'' - k' - i0} dk' \ln |a(k'', \sigma)| dk'' \\ &= \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\ln |a(k'', \sigma)|}{k'' - k} dk'', \end{aligned}$$

Apparently, Eq. (65) can not only generate Eq. (70) but also implies

$$\ln a(k, \sigma) = -i\alpha k + \sum_{n=1}^N \ln \frac{k - ik_n}{k + ik_n} + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\ln |a(k', \sigma)|}{k' - k} dk'. \tag{72}$$

Due to  $\lambda(i/m_0, \sigma) = 0$  and Eq. (40), we have  $\chi(y, i/m_0, \sigma) = 0$  and  $\ln a(i/m_0, \sigma) = 0$ . Therefore Eq. (72) implies

$$\alpha = \sum_{n=1}^N \ln \left( \frac{1 + k_n m_0}{1 - k_n m_0} \right)^{m_0} + \frac{m_0^2}{\pi} \int_{-\infty}^{\infty} \frac{\ln |a(k', \sigma)|}{m_0 k' i + 1} dk'$$

in terms of scattering data. Similarly, for  $-\sigma$ , we have

$$\alpha = \sum_{n=1}^N \ln \left( \frac{1 + k_n m_0}{1 - k_n m_0} \right)^{m_0} + \frac{m_0^2}{\pi} \int_{-\infty}^{\infty} \frac{\ln |a(k', -\sigma)|}{m_0 k' i + 1} dk'.$$

Thus, combining the above two formulas yields

$$\alpha = \sum_{n=1}^N \ln \left( \frac{1 + k_n m_0}{1 - k_n m_0} \right)^{m_0} + \frac{m_0^2}{\pi} \int_0^{\infty} \frac{\ln |a(k', \sigma) a(k', -\sigma)|}{(m_0 k')^2 + 1} dk'. \tag{73}$$

### B. Time evolution of Jost functions for the FQXL model

In this section, we will know that the discrete spectrum in the upper half plane consists of finitely many points  $k_n = ik_n$ ,  $n = 1, \dots, N$ , which are the simple zeros of  $a(k, \sigma)$ , and each  $\kappa_n$  is real and  $0 < \kappa_n < 1/m_0$ .

**Eigenfunctions.** For a given real value  $\kappa_n$  ( $0 < \kappa_n < 1/m_0$ ), apparently from (13), the spectral problem (10) has two eigenvalues  $\lambda_n(\sigma) = \lambda(i\kappa_n, \sigma)$  and each of them is associated with two eigenfunctions  $\varphi^{(n)}(y, \sigma)$ . Let us rewrite the eigenfunctions as

$$\varphi^{(n)}(y, k, \sigma) \equiv \varphi(y, i\kappa_n, \sigma). \tag{74}$$

According to (17), (18), and (59), the asymptotical behavior of  $\varphi^{(n)}$  is described by

$$\varphi^{(n)}(y, k, \sigma) = e^{\kappa_n y} + o(e^{\kappa_n y}) \quad \text{for } y \rightarrow -\infty, \tag{75}$$

$$\varphi^{(n)}(y, k, \sigma) = b_n(\sigma) e^{-\kappa_n y} + o(e^{-\kappa_n y}) \quad \text{for } y \rightarrow \infty. \tag{76}$$

**Scattering data.** The following set:

$$\mathcal{S} \equiv \{\mathcal{R}(k, \sigma), (k > 0), \quad \kappa_n, \quad b_n(\sigma), \quad n = 1, \dots, N, \quad \sigma = \pm 1\} \tag{77}$$

is called the **scattering data**. Consider the spectral problem (10) as a usual form extensively studied in the literature. Under some initial conditions, one can construct the scattering data and analyze the analytical properties of asymptotical solution (i.e., Jost solution) and figure out the coefficient  $a(k)$  as usual.

In order to set up the time-dependence of the scattering data, we have to need the time evolution of the eigenfunction  $\psi(k, y)$ . The evolutionary scattering data can be determined by the time part (3) of Lax pair. With the aid of the coordinate transform (7), we can rewrite the spectral problem as

$$\psi_t = -\frac{A}{2}\psi - \frac{B}{2}\phi = -\frac{B}{\lambda m}\psi_x - \left(\frac{A}{2} + \frac{B}{2\lambda m}\right)\psi, \quad (78)$$

where  $A, B$ , and  $\alpha$  are the same as ones in Equation (4).

Through analyzing the asymptotical behavior of the time-dependence problem (3), we have

$$\begin{aligned} A &\rightarrow \frac{1}{\lambda^2} + \frac{1}{2}(k_1 m_0^2 + k_2 m_0), \\ B &\rightarrow -\frac{m_0}{\lambda} - \frac{\lambda m_0}{2}(k_1 m_0^2 + k_2 m_0), \end{aligned}$$

as  $x \rightarrow \infty$  (i.e.,  $y \rightarrow \infty$ ),  $\lim_{x \rightarrow \infty} m(x, t) = m_0$ . Therefore, when  $x \rightarrow \infty$ , Eq. (78) has the following asymptotical form:

$$\begin{aligned} \psi_t &\rightarrow \left[ \frac{1}{\lambda^2} + \frac{(k_1 m_0^2 + k_2 m_0)}{2} \right] \psi_x + \gamma \psi \\ &\rightarrow \frac{m_0^2}{2} \left[ \frac{k_1 m_0 + k_2}{k^2 m_0^2 + 1} + \frac{k_1 m_0 + k_2}{2} \right] \psi_y + \gamma \psi, \end{aligned} \quad (79)$$

where we have used the relation  $\psi_x = \frac{m_0}{2}\psi_y$ .

In the light of the theory of Jost solution, we have the following property:

$$\lim_{y \rightarrow \pm\infty} \phi(y, k, \sigma) e^{iky} = 1, \quad \lim_{y \rightarrow \pm\infty} \psi(y, k, \sigma) e^{iky} = 1. \quad (80)$$

Hence,

$$\psi(y, k, \sigma) = a(k, \sigma) \phi(y, k, \sigma) + b(k, \sigma) \bar{\phi}(y, k, \sigma), \quad k \in \mathbb{R}, \quad (81)$$

where  $\bar{\phi}(y, k, \sigma)$  is the conjugate of eigenfunction  $\phi(y, k, \sigma)$ . Due to

$$\psi \rightarrow a e^{-iky} + b e^{iky}, \quad y \rightarrow \infty,$$

by Eq. (79), we have

$$\begin{aligned} a_t &= \frac{m_0^2}{2} \left[ \frac{k_1 m_0 + k_2}{k^2 m_0^2 + 1} + \frac{k_1 m_0 + k_2}{2} \right] (-ika) + \gamma a, \\ b_t &= \frac{m_0^2}{2} \left[ \frac{k_1 m_0 + k_2}{k^2 m_0^2 + 1} + \frac{k_1 m_0 + k_2}{2} \right] (ikb) + \gamma b. \end{aligned}$$

Requiring  $a_t = 0$  leads to

$$\gamma = \frac{ikm_0^2}{2} \left[ \frac{k_1 m_0 + k_2}{k^2 m_0^2 + 1} + \frac{k_1 m_0 + k_2}{2} \right].$$

Thus, the time evolution of  $b(k, t, \sigma)$  is controlled by

$$b_t = ikm_0^2 \left[ \frac{k_1 m_0 + k_2}{k^2 m_0^2 + 1} + \frac{k_1 m_0 + k_2}{2} \right] b, \quad (82)$$

which admits the solution in the following form:

$$b = b(k, 0, \sigma) \exp \left( ikm_0^2 \left[ \frac{k_1 m_0 + k_2}{k^2 m_0^2 + 1} + \frac{k_1 m_0 + k_2}{2} \right] t \right) \quad (83)$$

where  $b(k, 0, \sigma)$  is the initial data of  $b(k, t, \sigma)$ . So, the scattering coefficient  $\mathfrak{R} = b/a$  is given by

$$\mathfrak{R}(k, t, \sigma) = \mathfrak{R}(k, 0, \sigma) \exp\left(ikm_0^2 \left[ \frac{k_1m_0 + k_2}{k^2m_0^2 + 1} + \frac{k_1m_0 + k_2}{2} \right] t\right).$$

In particular, let  $k = i\kappa_n$  be the discrete spectrum, then we have

$$R_n(t, \sigma) = \frac{b(i\kappa_n, \sigma)}{i\dot{a}(i\kappa_n, \sigma)} = R_n(0, \sigma) \exp(-f(\kappa_n)t), \tag{84}$$

where

$$f(\kappa_n) = \kappa_n m_0^2 \left[ \frac{k_1m_0 + k_2}{1 - \kappa_n^2 m_0^2} + \frac{k_1m_0 + k_2}{2} \right]. \tag{85}$$

**V. ANALYTICAL SOLUTIONS FOR RIEMANN-HILBERT PROBLEM**

For an arbitrary  $k$  from the lower half plane ( $\text{Im } k < 0$ ), by using the Residue Theorem with the aid of (48), (52), and (59), we can compute the integral

$$\begin{aligned} I &= \frac{1}{2\pi i} \oint_{C_+} \frac{\underline{\psi}(y, k', \sigma)}{e^{ik'\alpha} a(k', \sigma)} \frac{dk'}{k' - k} = \sum_{n=1}^N \frac{\underline{\psi}(y, i\kappa_n, \sigma)}{(i\kappa_n - k) e^{-\kappa_n \alpha} a'(i\kappa_n, \sigma)} \\ &= \sum_{n=1}^N \frac{b_n(i\kappa_n, \sigma) [\xi(x)]^{-2\kappa_n} \underline{\phi}(y, -i\kappa_n, -\sigma)}{a'(i\kappa_n, \sigma)(i\kappa_n - k)}, \end{aligned} \tag{86}$$

where  $C_+$  denotes the closed contour in the upper half plane as shown in Figure 1.

On the other hand, from Equation (58) we have

$$\begin{aligned} I &= \frac{1}{2\pi i} \oint_{C_+} \frac{\underline{\psi}(y, k', \sigma)}{e^{ik'\alpha} a(k', \sigma)} \frac{dk'}{k' - k} \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left( \underline{\phi}(y, k', \sigma) + \mathcal{R}(k', \sigma) \bar{\underline{\phi}}(y, k', \sigma) [\xi(y)]^{2ik'} \right) \frac{dk'}{k' - k} \\ &\quad + \frac{1}{2\pi i} \int_{\Gamma_+} \frac{\underline{\psi}(y, k', \sigma)}{e^{ik'\alpha} a(k', \sigma)} \frac{dk'}{k' - k}, \end{aligned} \tag{87}$$

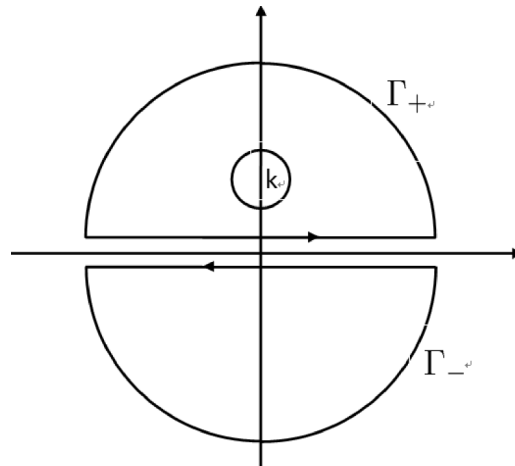


FIG. 1. The contours of  $\Gamma$ .

where  $\Gamma_+$  is the infinite semicircle in the upper half plane except the line along real axis as shown in Fig. 1. With the help of asymptotical behavior shown in (57) and the limiting of  $a(k, \sigma)$  shown in (49), one is able to straightforward get the integral over  $\Gamma_+(1/2)(\xi(y)/\xi'(y))^{1/2}$ .

Similarly, we can also compute

$$\begin{aligned} -\underline{\phi}(y, k, \sigma) &= \frac{1}{2\pi i} \oint_{C_-} \underline{\phi}(y, k', \sigma) \frac{dk'}{k' - k} \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \underline{\phi}(y, k', \sigma) \frac{dk'}{k' - k} + \frac{1}{2\pi i} \int_{\Gamma_-} \underline{\phi}(y, k', \sigma) \frac{dk'}{k' - k}, \end{aligned} \tag{88}$$

where  $C_-$  is the closed contour in the lower half plane and  $\Gamma_-$  is the infinite semicircle in the lower half plane as shown in Fig. 1. The integral over  $\Gamma_-$  can analogously be computed and equals to  $-(1/2)(\xi(y)/\xi'(y))^{1/2}$  through considering the asymptotic behavior (53) for  $|k| \rightarrow \infty$  and the definition given in (56).

Let us now use Equations (86)-(88) and then for  $\text{Im } k < 0$ , we have

$$\begin{aligned} \underline{\phi}(y, k, \sigma) &= \left(\frac{\xi(y)}{\xi'(y)}\right)^{1/2} + \int_{-\infty}^{\infty} \mathcal{R}(k', \sigma) \bar{\phi}(y, k', \sigma) [\xi(y)]^{2ik'} \frac{dk'}{k' - k} \\ &\quad + \sum_{n=1}^N \frac{b_n(i\kappa_n, \sigma) [\xi(y)]^{-2\kappa_n} \underline{\phi}(y, -i\kappa_n, -\sigma)}{a'(i\kappa_n, \sigma)(k - i\kappa_n)}. \end{aligned} \tag{89}$$

In particular, selecting  $k = -i\kappa_p, p = 1, \dots, N$  in Eq. (89) yields the following two expressions:

$$\begin{aligned} \underline{\phi}(y, -i\kappa_p, \sigma) &= \left(\frac{\xi(y)}{\xi'(y)}\right)^{1/2} + \int_{-\infty}^{\infty} \mathcal{R}(k', \sigma) \bar{\phi}(y, -k', -\sigma) [\xi(y)]^{2ik'} \frac{dk'}{k' + i\kappa_p} \\ &\quad + i \sum_{n=1}^N \frac{b_n(i\kappa_n, \sigma) [\xi(y)]^{-2\kappa_n} \underline{\phi}(y, -i\kappa_n, -\sigma)}{a'(i\kappa_n, \sigma)(\kappa_p + \kappa_n)} \end{aligned} \tag{90}$$

and

$$\begin{aligned} \underline{\phi}(y, -i\kappa_p, -\sigma) &= \left(\frac{\xi(y)}{\xi'(y)}\right)^{1/2} + \int_{-\infty}^{\infty} \mathcal{R}(k', -\sigma) \bar{\phi}(y, -k', \sigma) [\xi(y)]^{2ik'} \frac{dk'}{k' + i\kappa_p} \\ &\quad + i \sum_{n=1}^N \frac{b_n(i\kappa_n, -\sigma) [\xi(y)]^{-2\kappa_n} \underline{\phi}(y, -i\kappa_n, \sigma)}{a'(i\kappa_n, -\sigma)(\kappa_p + \kappa_n)}. \end{aligned} \tag{91}$$

Apparently, Eqs. (90) and (91) are a linear system of  $\bar{\phi}(y, -i\kappa_n, -\sigma)$  and  $\underline{\phi}(y, -i\kappa_n, \sigma)$  for  $p = 1, \dots, N$ . So, if  $\xi(y)$  is known, then we need to solve these equations for  $\bar{\phi}(y, k, \sigma)$  and  $\underline{\phi}(y, -i\kappa_n, \sigma)$ .

From Sec. II, we already know  $\lambda(-i/m_0) = 0$ . Since  $\phi(y, k, \sigma)$  does not depend on  $m$  when  $\lambda = 0$  and  $\phi(y, k, \sigma)$  is determined by its asymptotical behavior at  $-\infty$ , one can deduce  $\phi(y, -i/m_0, \sigma) = e^{-y/m_0}$ . Thus, as  $k = -i/m_0$ , Eq. (89) reads as

$$\begin{aligned} e^{-y/m_0} [\xi(y)]^{1/m_0} &= \left(\frac{\xi(y)}{\xi'(y)}\right)^{1/2} + \int_{-\infty}^{\infty} \mathcal{R}(k', \sigma) \bar{\phi}(y, k', \sigma) [\xi(y)]^{2ik'} \frac{dk'}{k' + i/m_0} \\ &\quad + i \sum_{n=1}^N \frac{b_n(i\kappa_n, \sigma) [\xi(y)]^{-2\kappa_n} \underline{\phi}(y, -i\kappa_n, -\sigma)}{a'(i\kappa_n, \sigma)(\kappa_n + 1/m_0)}. \end{aligned} \tag{92}$$

$\phi(y, k, \sigma)$  and  $\phi(y, -i\kappa_n, \sigma)$  are closely related through Eqs. (89)–(92). Since Equation (92) is a first order differential equation with respect to  $\xi$ , one can solve it for  $\xi(y)$ . Apparently, Equations (89)–(92) consist of a system of singular integral equations for  $\bar{\phi}(y, k, \sigma)$ ,  $\underline{\phi}(y, -i\kappa_n, \sigma)$  and  $\xi(y)$ . Because the time evolution of the scattering data is given by (84) and  $\xi(y, t)$  can implicitly be solved in terms of the scattering data, the evolution of the potential function is uniquely determined by the scattering data set  $\mathcal{S}$  in (77). Solving Equation (52) casts the expression of  $m(y, t)$  in terms of  $\xi(y, t)$

$$m(y, t) = \frac{k_2}{(k_1 + k_2/m_0)(\xi/\xi')^2 - k_1}. \tag{93}$$

In what follows, we introduce a new variable  $z$  by  $e^z = \xi(y, t)$ , and therefore  $y$  could be a function of  $z$  and  $t$  and is denoted by  $y = Y(z, t)$ . Then, we have

$$\frac{dz}{dy} = \frac{\xi'(y, t)}{\xi(y, t)}, \quad \frac{dy}{dz} = Y_z(z, t) = \frac{\xi(y, t)}{\xi'(y, t)},$$

and

$$m(y, t) = \frac{k_2}{(k_1 + k_2/m_0)Y_z^2(z, t) - k_1}. \quad (94)$$

## VI. REFLECTIONLESS POTENTIAL

In this section, we consider a simplified case of the inverse scattering problem for the FQXL model with the reflection coefficient  $\mathcal{R}(k', \sigma)$  vanishing for all real  $k$ , namely, the case of so-called reflectionless potential. The integration terms in Eqs. (89)-(91) will disappear and Eqs. (89)-(91) can be regarded as a normal linear system, which can be solved. In fact, this kind of potential corresponds to multi-soliton solutions of the FQXL model.

Eq. (84) gives the time evolution of  $R_n$ . Let  $\lambda_n(\sigma)$  denote the values of  $\lambda$  at zeros of  $a(k, \sigma)k = i\kappa_n$ . As we already observed, both  $\lambda_n(\sigma)$  and  $R_n(t, \sigma)$  are real. Let us define the  $N \times N$  matrix with its entries

$$M_{pq}(z, t, \sigma) = \delta_{pq} - \sum_{n=1}^N \frac{R_n(t, -\sigma)R_q(t, \sigma)e^{-2z(\kappa_n + \kappa_q)}}{(\kappa_p + \kappa_n)(\kappa_n + \kappa_q)}, \quad (95)$$

which are also real, where

$$\delta_{pq} = \begin{cases} 0, & p \neq q; \\ 1, & p = q. \end{cases}$$

Suppose  $M$  is invertible, then we may compute

$$A_q(z, t, \sigma) = \sum_{p=1}^N [M^{-1}(z, t, \sigma)]_{qp} - \sum_{p=1}^N \sum_{n=1}^N [M^{-1}(z, t, \sigma)]_{qp} \frac{R_n(t, \sigma)e^{-2z\kappa_n}}{\kappa_p + \kappa_n}. \quad (96)$$

Therefore, the solution of (90) and (91) is

$$\underline{\phi}(y, -i\kappa_n, \sigma) = X_0 A_q(z, t, \sigma), \quad (97)$$

where  $X_0(y) = \left(\frac{\xi(y, t)}{\xi'(y, t)}\right)^{\frac{1}{2}} = \frac{(k_1 + k_2/m_0)^{\frac{1}{4}}}{(k_1 + k_2/m)^{\frac{1}{4}}}$  deduced from Eq. (93) and Eq. (89) with the reflectionless potential  $m(y, t)$  gives

$$\underline{\phi}(y, k, \sigma) = X_0 \left[ 1 + \sum_{n=1}^N \frac{iR_n(t, \sigma)e^{-2\kappa_n z} A_n(z, t, -\sigma)}{k - i\kappa_n} \right]. \quad (98)$$

Also, Equation (92) can be converted into following expression with  $A_q(z, t, \sigma)$ :

$$e^{(-y+z)/m_0} = X_0 \left( 1 - \sum_{n=1}^N \frac{R_n(t, \sigma)e^{-2\kappa_n z} A_n(z, t, -\sigma)}{\kappa_n + 1/m_0} \right). \quad (99)$$

When  $k = i/m_0$ , we can get a similar formula for  $\underline{\psi}(x, k, \sigma)$

$$e^{(y-z)/m_0} = X_0 \left( 1 - \sum_{n=1}^N \frac{R_n(t, -\sigma)e^{-2\kappa_n z} A_n(z, t, \sigma)}{\kappa_n - 1/m_0} \right). \quad (100)$$

Eliminating  $X_0$  from Equations (99) and (100), we will arrive at

$$y \equiv Y(z, t) = z + \frac{m_0}{2} \ln \frac{f_-(z, t)}{f_+(z, t)}, \quad (101)$$

$$f_{\pm}(z, t) = 1 - \sum_{n=1}^N \frac{R_n(\pm\sigma)e^{-2\kappa_n z} A_n(z, t, \mp\sigma)}{\kappa_n \pm 1/m_0}, \tag{102}$$

which is the relation between  $y$  and  $z$  and also the implicit relation between  $y$  and  $\xi(y, t)$ .

So far, we give the implicitly formal solution shown in Equations (101) and (93) in terms of  $(y, t)$ . The time evolution of  $R_n$  used in the implicit solution is provided in (84), and  $Y(z, t)$  is also presented in terms of the scattering data. So, Equations (93), (101), and (102) present a parametric representation of the solution for the potential function  $m(y, t)$ . In order to get the solution in terms of  $x$ , one should recall transform Eq. (7). Combining Eq. (7) with Eq. (101) generates the following coordinate transform :

$$x = \int \frac{m(y, t)}{2} dy = \frac{1}{2} \int \frac{k_2 Y_z}{(k_1 + k_2/m_0)Y_z^2 - k_1} dz, \tag{103}$$

$$m(x, t) = \frac{k_2}{(k_1 + k_2/m_0)(\xi/\xi')^2 - k_1} = \frac{k_2}{(k_1 + k_2/m_0)Y_z^2(z, t) - k_1}, \tag{104}$$

where  $Y(z, t)$  is given by (101) and (102).

**A. One soliton solution**

Suppose  $-i\kappa_1$  is a single zero of  $a(k, \sigma)$ , then Eqs. (90) and (91) can be simplified as follows:

$$\underline{\phi}(y, -i\kappa_1, -\sigma) = X_0 - \frac{R_1(i\kappa_1, -\sigma)e^{-2\kappa_1 z} \underline{\phi}(y, -i\kappa_1, \sigma)}{2\kappa_1}, \tag{105}$$

$$\underline{\phi}(y, -i\kappa_1, \sigma) = X_0 - \frac{R_1(i\kappa_1, \sigma)e^{-2\kappa_1 z} \underline{\phi}(y, -i\kappa_1, -\sigma)}{2\kappa_1}, \tag{106}$$

which leads to the following explicit expression:

$$\underline{\phi}(y, -i\kappa_1, \sigma) = \frac{X_0[1 - E(\sigma, t)]}{1 - E(\sigma, t)E(-\sigma, t)}, \tag{107}$$

where  $E(\sigma, t) = \frac{1}{2\kappa_1} R_1(t, \sigma)e^{-2\kappa_1 z}$ . Meanwhile, Eq. (89) is converted to

$$\phi(y, k, \sigma) = X_0 \left[ 1 + \frac{2i\kappa_1}{k - i\kappa_1} \frac{E(\sigma, t)(1 - E(-\sigma, t))}{1 - E(\sigma, t)E(-\sigma, t)} \right]. \tag{108}$$

Apparently, taking  $k = \frac{i}{m_0}$  yields

$$e^{\frac{(y-z)}{m_0}} = X_0 \left[ 1 - \frac{2\kappa_1}{\kappa_1 - 1/m_0} \frac{E(-\sigma, t)[1 - E(\sigma, t)]}{1 - E(\sigma, t)E(-\sigma, t)} \right] \tag{109}$$

and taking  $k = -\frac{i}{m_0}$  yields

$$e^{-\frac{(y-z)}{m_0}} = X_0 \left[ 1 - \frac{2\kappa_1}{\kappa_1 + 1/m_0} \frac{E(\sigma, t)[1 - E(-\sigma, t)]}{1 - E(\sigma, t)E(-\sigma, t)} \right], \tag{110}$$

which we require to generate the following explicit formula for variables  $y$  and  $z$ :

$$y = z + \frac{m_0}{2} \ln \frac{\kappa_1 m_0 - 1 - 2\kappa_1 m_0 E(-\sigma, t) + (\kappa_1 m_0 + 1)E(\sigma, t)E(-\sigma, t)}{\kappa_1 m_0 + 1 - 2\kappa_1 m_0 E(\sigma, t) + (\kappa_1 m_0 - 1)E(\sigma, t)E(-\sigma, t)} \times \frac{\kappa_1 m_0 + 1}{\kappa_1 m_0 - 1}. \tag{111}$$

Hence, Eqs. (103) and (104) give an implicit expression solution to the FQXL model for  $k_2 \neq 0$  and  $k_1 + k_2/m > 0$ . Figure 2 shows the evolution of the soliton solution. It keeps its profile along time.

**B. Two soliton solution**

Apparently, the potential functions satisfy the following four equations:

$$\phi(-i\kappa_1, \sigma) = X_0 - E_1(\sigma)\phi(-i\kappa_1, -\sigma) - \frac{2\kappa_2}{\kappa_1 + \kappa_2} E_2(\sigma)\phi(-i\kappa_2, -\sigma) \tag{112}$$

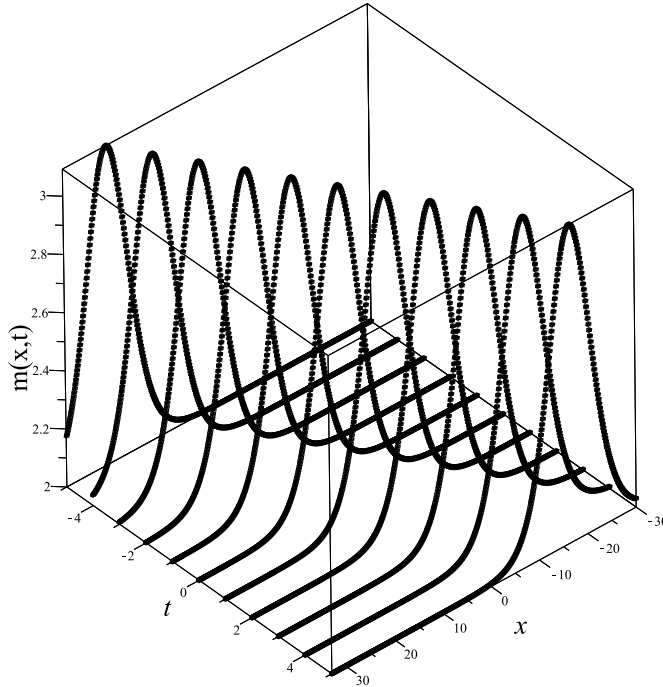


FIG. 2. The soliton evolution of time with  $m_0=2, \kappa=0.2, k_1=0.1, k_2=1$ .

$$\phi(-i\kappa_2, \sigma) = X_0 - \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_1(\sigma)\phi(-i\kappa_1, -\sigma) - E_2(\sigma)\phi(-i\kappa_2, -\sigma) \tag{113}$$

$$\phi(-i\kappa_1, -\sigma) = X_0 - E_1(-\sigma)\phi(-i\kappa_1, \sigma) - \frac{2\kappa_2}{\kappa_1 + \kappa_2} E_2(-\sigma)\phi(-i\kappa_2, \sigma) \tag{114}$$

$$\phi(-i\kappa_2, -\sigma) = X_0 - \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_1(-\sigma)\phi(-i\kappa_1, \sigma) - E_2(-\sigma)\phi(-i\kappa_2, \sigma), \tag{115}$$

where  $E_i(\pm\sigma) = \frac{R_i(0, \pm\sigma)}{2\kappa_i} e^{-2\kappa_i z - f(\kappa_i)t}, i = 1, 2$ . From these equations, we can solve  $\phi(-i\kappa_1, -\sigma), \phi(-i\kappa_2, -\sigma), \phi(-i\kappa_1, \sigma),$  and  $\phi(-i\kappa_2, \sigma)$  in terms of  $E_1(\sigma), E_1(-\sigma), E_2(\sigma),$  and  $E_2(-\sigma)$  as follows:

$$\phi(-i\kappa_1, -\sigma) = \frac{\begin{vmatrix} X_0 & \frac{2\kappa_2}{\kappa_1 + \kappa_2} E_2(\sigma) & 1 & 0 \\ X_0 & E_2(\sigma) & 0 & 1 \\ X_0 & 0 & E_1(-\sigma) & \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_2(-\sigma) \\ X_0 & 1 & \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_1(-\sigma) & E_2(-\sigma) \end{vmatrix}}{\begin{vmatrix} E_1(\sigma) & \frac{2\kappa_2}{\kappa_1 + \kappa_2} E_2(\sigma) & 1 & 0 \\ \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_1(\sigma) & E_2(\sigma) & 0 & 1 \\ 1 & 0 & E_1(-\sigma) & \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_2(-\sigma) \\ 0 & 1 & \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_1(-\sigma) & E_2(-\sigma) \end{vmatrix}}, \tag{116}$$



$$\phi(-i\kappa_2, -\sigma) = \frac{\begin{vmatrix} E_1(\sigma) & X_0 & 1 & 0 \\ \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_1(\sigma) & X_0 & 0 & 1 \\ 1 & X_0 & E_1(-\sigma) & \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_2(-\sigma) \\ 0 & X_0 & \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_1(-\sigma) & E_2(-\sigma) \end{vmatrix}}{\begin{vmatrix} E_1(\sigma) & \frac{2\kappa_2}{\kappa_1 + \kappa_2} E_2(\sigma) & 1 & 0 \\ \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_1(\sigma) & E_2(\sigma) & 0 & 1 \\ 1 & 0 & E_1(-\sigma) & \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_2(-\sigma) \\ 0 & 1 & \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_1(-\sigma) & E_2(-\sigma) \end{vmatrix}}, \quad (117)$$

$$\phi(-i\kappa_1, \sigma) = \frac{\begin{vmatrix} E_1(\sigma) & \frac{2\kappa_2}{\kappa_1 + \kappa_2} E_2(\sigma) & X_0 & 0 \\ \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_1(\sigma) & E_2(\sigma) & X_0 & 1 \\ 1 & 0 & X_0 & \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_2(-\sigma) \\ 0 & 1 & X_0 & E_2(-\sigma) \end{vmatrix}}{\begin{vmatrix} E_1(\sigma) & \frac{2\kappa_2}{\kappa_1 + \kappa_2} E_2(\sigma) & 1 & 0 \\ \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_1(\sigma) & E_2(\sigma) & 0 & 1 \\ 1 & 0 & E_1(-\sigma) & \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_2(-\sigma) \\ 0 & 1 & \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_1(-\sigma) & E_2(-\sigma) \end{vmatrix}}, \quad (118)$$

$$\phi(-i\kappa_2, \sigma) = \frac{\begin{vmatrix} E_1(\sigma) & \frac{2\kappa_2}{\kappa_1 + \kappa_2} E_2(\sigma) & 1 & X_0 \\ \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_1(\sigma) & E_2(\sigma) & 0 & X_0 \\ 1 & 0 & E_1(-\sigma) & X_0 \\ 0 & 1 & \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_1(-\sigma) & X_0 \end{vmatrix}}{\begin{vmatrix} E_1(\sigma) & \frac{2\kappa_2}{\kappa_1 + \kappa_2} E_2(\sigma) & 1 & 0 \\ \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_1(\sigma) & E_2(\sigma) & 0 & 1 \\ 1 & 0 & E_1(-\sigma) & \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_2(-\sigma) \\ 0 & 1 & \frac{2\kappa_1}{\kappa_1 + \kappa_2} E_1(-\sigma) & E_2(-\sigma) \end{vmatrix}}. \quad (119)$$

When  $k = -\frac{i}{m_0}$ , we have

$$e^{-\frac{y-z}{m_0}} = X_0 - \frac{2\kappa_1}{\kappa_1 + 1/m_0} E_1(\sigma) \phi(-i\kappa_1, -\sigma) - \frac{2\kappa_2}{\kappa_2 + 1/m_0} E_2(\sigma) \phi(-i\kappa_2, -\sigma). \quad (120)$$

When  $k = \frac{i}{m_0}$ , we obtain

$$e^{\frac{y-z}{m_0}} = X_0 - \frac{2\kappa_1}{\kappa_1 - 1/m_0} E_1(-\sigma) \phi(-i\kappa_1, \sigma) - \frac{2\kappa_2}{\kappa_2 - 1/m_0} E_2(-\sigma) \phi(-i\kappa_2, \sigma), \quad (121)$$

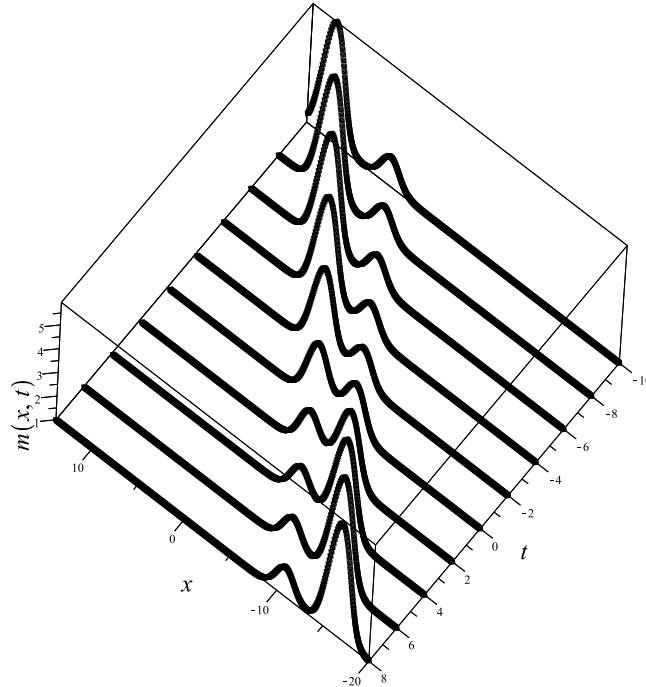


FIG. 3. Two soliton solution with  $k_1 = 0.5, k_2 = 2, \kappa_1 = 0.5, \kappa_2 = 0.65, m_0 = 1$ .

$$y = Y(z, t) = z + \frac{m_0}{2} \ln \left( \frac{1 - \frac{2\kappa_1}{\kappa_1 - 1/m_0} E_1(-\sigma) \phi(-i\kappa_1, \sigma)/X_0 - \frac{2\kappa_2}{\kappa_2 - 1/m_0} E_2(-\sigma) \phi(-i\kappa_2, \sigma)/X_0}{1 - \frac{2\kappa_1}{\kappa_1 + 1/m_0} E_1(\sigma) \phi(-i\kappa_1, -\sigma)/X_0 - \frac{2\kappa_2}{\kappa_2 + 1/m_0} E_2(\sigma) \phi(-i\kappa_2, -\sigma)/X_0} \right), \quad (122)$$

where  $\phi(-i\kappa_1, -\sigma)$ ,  $\phi(-i\kappa_2, -\sigma)$ ,  $\phi(-i\kappa_1, \sigma)$ , and  $\phi(-i\kappa_2, \sigma)$  are given in Equations (116)–(119), respectively.

Figure 3 shows the two soliton interactions of the FQXL model. The bigger soliton has a faster speed and overtakes the smaller one. The phase shift is also obvious in Figure 3 from the time line  $t = 0$  and  $t = 2$ .

## VII. CONCLUSION

In this paper, we apply the inverse scattering method to the FQXL model. Through using the Riemann-Hilbert problem, we obtain the exact solutions for the cubic FQXL peakon model in a general formula. The one-soliton and two-soliton solutions for the potential  $m(x, t)$  are graphed in Figures 2 and 3. The soliton solutions for the potential  $u(x, t)$  can be obtained through solving  $u - u_{xx} = m(x, t)$  with the Green function.

## ACKNOWLEDGMENTS

This work is partially supported by the National Natural Science Foundation of China (Grant Nos. 51579040, 51522902, and 51379033). The author Qiao thanks the Haitian Scholar Plan of Dalian University of Technology to support this collaborative work and also thanks the support by the 111 Project of China (No. B16002).

<sup>1</sup> N. J. Zabusky and M. D. Kruskal, *Phys. Rev. Lett.* **15**, 240 (1965).

<sup>2</sup> P. Deift and E. Trubowitz, *Commun. Pure Appl. Math.* **32**, 121 (1979).

<sup>3</sup> P. Deift and X. Zhou, *Ann. Math.* **137**, 295 (1993).

<sup>4</sup> S. V. Manakov, *Sov. Phys. JETP* **38**, 693 (1974).

- <sup>5</sup> A. R. Its, *Sov. Math. Dokl.* **24**, 452 (1981).
- <sup>6</sup> C. W. Cao, *Sci. Chin. A* **33**, 528 (1990).
- <sup>7</sup> P. Deift, A. R. Its, and X. Zhou, *Long-Time Asymptotics for Integrable Nonlinear Wave Equations*, Important Developments in Soliton Theory (Springer, Berlin, 1993).
- <sup>8</sup> R. Camassa and D. D. Holm, *Phys. Rev. Lett.* **71**, 1661 (1993).
- <sup>9</sup> J. Eckhardt and G. Teschl, *Nonlinearity* **29**, 1036 (2016).
- <sup>10</sup> D. D. Holm and R. I. Ivanov, *Inverse Probl.* **27**, 045013 (2011).
- <sup>11</sup> A. Degasperis and M. Procesi, in *Asymptotic Integrability Symmetry and Perturbation Theory*, edited by A. Degasperis and G. Gaeta (World Scientific, Singapore, 1999).
- <sup>12</sup> A. Constantina, R. Ivanov, and J. Lenells, *Nonlinearity* **23**, 2559 (2010).
- <sup>13</sup> D. D. Holm and M. F. Staley, *Phys. Lett. A* **308**, 437 (2003).
- <sup>14</sup> Z. J. Qiao, *Commun. Math. Phys.* **239**, 309 (2003).
- <sup>15</sup> Y. Hou, P. Zhao, E. G. Fan, and Z. J. Qiao, *SIAM J. Math. Anal.* **45**, 1216 (2013).
- <sup>16</sup> A. Constantin and J. Lenells, *Phys. Lett. A* **308**, 432 (2003).
- <sup>17</sup> A. Constantin, V. Gerdjikov, and R. Ivanov, *Inverse Probl.* **22**, 2197 (2006).
- <sup>18</sup> Z. J. Qiao, *J. Math. Phys.* **47**, 112701 (2006); **48**, 082701 (2007).
- <sup>19</sup> Z. J. Qiao and X. Q. Li, *Theor. Math. Phys.* **167**, 584 (2011).
- <sup>20</sup> A. N. W. Hone and J. P. Wang, *J. Phys. A: Math. Theor.* **41**, 372002 (2008).
- <sup>21</sup> V. Novikov, *J. Phys. A: Math. Theor.* **42**, 342002 (2009).
- <sup>22</sup> A. S. Fokas, *Physica D* **87**, 145 (1995).
- <sup>23</sup> Z. J. Qiao, B. Q. Xia, and J. B. Li, "Integrable system with peakon, weak kink, and kink-peakon interactional solutions," e-print [arXiv:1205.2028v2](https://arxiv.org/abs/1205.2028v2) (2012).
- <sup>24</sup> P. Olver and P. Rosenau, *Phys. Rev. E* **53**, 1900 (1996).
- <sup>25</sup> B. Fuchssteiner, *Physica D* **95**, 229 (1996).
- <sup>26</sup> R. Barostichi, A. Himonas, and G. Petronilho, *J. Funct. Anal.* **270**, 330 (2016).