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Zhijun Qiao
The University of Texas Rio Grande Valley, zhijun.qiao@utrgv.edu

Cewen Cao
Zhengzhou University

Walter Strampp
Universitat-GH Kassel

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Category of nonlinear evolution equations, algebraic structure, and r-matrix

Zhijun Qiao[a]
Theoretical Division and Center for Nonlinear Sciences, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, and Institute of Mathematics, Fudan University, Shanghai 200433, People's Republic of China

Cewen Cao[b]
Department of Mathematics, Zhengzhou University, Zhengzhou 450052, People's Republic of China

Walter Strampp[c]
Fachbereich 17—Mathematik/Informatik, Universität-GH Kassel, Heinrich-Plett-Str. 40, D-34109 Kassel, Germany

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In this paper we deal with the category of nonlinear evolution equations (NLEEs) associated with the spectral problem and provide an approach for constructing their algebraic structure and r-matrix. First we introduce the category of NLEEs, which is composed of various positive order and negative order hierarchies of NLEEs both integrable and nonintegrable. The whole category of NLEEs possesses a generalized Lax representation. Next, we present two different Lie algebraic structures of the Lax operator: one of them is universal in the category, i.e., independent of the hierarchy, while the other one is nonuniversal in the hierarchy, i.e., dependent on the underlying hierarchy. Moreover, we find that two kinds of adjoint maps are r-matrices under the algebraic structures. In particular, the Virasoro algebraic structures without a central extension of isospectral and nonisospectral Lax operators can be viewed as reductions of our algebraic structure. Finally, we give several concrete examples to illustrate our methods. Particularly, the Burgers' category is linearized when the generator, which generates the category, is chosen to be independent of the potential function. Furthermore, an isospectral negative order hierarchy in the Burgers' category is solved with its general solution. Additionally, in the KdV category we find an interesting fact: the Harry–Dym hierarchy is contained in this category as well as the well-known Harry–Dym equation is included in a positive order KdV hierarchy.
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I. INTRODUCTION

The integrability study of nonlinear evolution equations has been an attractive topic in soliton theory and nonlinear phenomenon. Calogero[1] proposed the C-integrable (namely, linearizable by an appropriate change of variables) and S-integrable (namely, integrable via some spectral transform technique) terminology for dealing with nonlinear partial diiferetial equations (PDEs). Many nonlinear PDEs were shown C-integrable and S-integrable.[2] Mikhailov, Shabat and Sokolov[3] discussed some classes of nonlinear C-integrable and S-integrable PDEs through using the symmetry approach. Flaschka, Newell and Tabor[4] considered in detail the Painleve analysis process for both ODEs and PDEs and investigated its test for integrable equations.

[a]Electronic mail: qiao@cnls.lanl.gov; qiaozj@hotmail.com
[b]Electronic mail: cwcao@public2.zz.ha.cn
[c]Electronic mail: strampp@hrz.uni-kassel.de

On the other hand, the r-matrix method is also an important part in classical and quantum integrable systems.[5] The classical r-matrix has been first introduced by Sklyanin in Refs. 6 and 7 as the limit of its quantum counterpart. Subsequently, Drinfeld used this to introduce a new geometric notion, that of a Poisson Lie group.[8] Following Drinfeld's ideas[8] Semenov–Tian–Shansky showed that the concept of a classical r-matrix leads to an algebraic construction of integrable systems generalizing the AKS scheme. In terms of the r-matrix[9] an effective view of the multi-Hamiltonian property of such equations can be presented. In addition, it gives a general explanation of the dressing transformations used for obtaining solutions in terms of group factorizations.[10] In Ref. 11 Jimbo constructed explicit solutions of the quantum YB equation for the generalized Toda system and moreover obtained many beautiful results[12–14] by using the r-matrix method.

For the study of an algebraic structure of integrable evolution equations, there has also been a discussion in the literature. For example, the well-known W-algebra was constructed by Orlov and Schulman through using the vertex operator.[15] The KP system was also found to have this kind of W-algebraic structure by Dickey,[16] which includes the Virasoro algebra as its subalgebra. The W-algebra played an important role in the so-called second Poinsson structure.[16] For this, the most important thing is to find the generators of W-algebra. All these facts were only for the case of integrable hierarchies. How about the case for both integrable and nonintegrable hierarchies? In this paper we will deal with this problem through introducing the category of nonlinear evolution equations (NLEEs). The category of NLEEs develops the positive order to the negative order hierarchies for both the integrable and the nonintegrable cases. In particular, the positive and the negative order integrable hierarchies will be generated by the recursion operator, its inverse, and some kernel elements from the pair of Lenard's operators. Mikhailov, Shabat and Sokolov[3] extended the integrable equations by employing the symmetry procedure and discussed the classifications for the integrable hierarchies. All of their results were for C-integrable and S-integrable cases. In this paper, we will discuss the case for both integrable and nonintegrable hierarchies and will not interfere with the existence of symmetries. Here, we point out that throughout this paper: ''integrable'' means the sense of Lax, namely, the PDE admits isospectral (i.e., $\lambda_t=0$) or usual nonisospectral (i.e., $\lambda_t=a\lambda^n$, $n\in\mathbb{Z}$, $a\in\mathbb{R}/\mathbb{C}$) Lax form; otherwise, we say the PDE is nonintegrable in the sense of the Lax form.

Our purpose in the present paper is to give an approach to the category of nonlinear evolution equations directly from a spectral problem and to connect the r-matrix to the category of NLEEs. The whole paper is organized as follows. In the next section we first introduce the notation of the category of NLEEs, which is composed of various positive and negative order hierarchies of both integrable and nonintegrable NLEEs, and then we give the generalized Lax representation (GLR). In Secs. III and IV we, respectively, present two different Lie algebraic structures of the Lax operator. One structure is produced independently of the hierarchy in the category while the other holds only within one hierarchy. Moreover, by using these algebraic structures we find that two kinds of adjoint maps result in r-matrices for the NLEEs. In Sec. V, it is pointed out that the well known Virasoro algebraic structures (without the central extension) of isospectral and nonisospectral Lax operators are obtained as reductions of our algebraic structure. Finally, in Sec. VI the examples of several continuous spectral problems are given to illustrate our methods. Particularly, the Burgers' category is linearized when the generator, which generates the category, is chosen to be independent of the potential function. Furthermore, an isospectral negative order hierarchy in the Burgers' category is solved with its general solution. Additionally, in the KdV category we find an interesting fact: the Harry–Dym hierarchy is contained in this category as well as the well-known Harry-Dym equation is included in a positive order KdV hierarchy.

Before displaying our main results, let us first give some necessary notations:

$$x\in R^l,\quad t\in R,\quad u=(u_1,\ldots,u_m)^T\in S^m(R^l,R)=\overbrace{S(R^l,R)\times\cdots\times S(R^l,R)}^{m},$$

$$u_i=u_i(x,t)\in S(R^l,R),\quad i=1,2,\ldots,m,$$

for arbitrarily fixed t, $S(R^l,R)$ stands for the Schwartz function space on R^l. $\mathcal{B}$ denotes all complex (or real) value functions $P(x,t,u)$ of the class C^∞ with respect to x, t, and of the class C^∞ in Gateaux's sense with respect to u. $\mathcal{B}^N=\{(P_1,\ldots,P_N)^T|P_i\in\mathcal{B}\}$, $\mathcal{V}^N$ stands for all linear operators $\phi=\phi(x,t,u)$: $\mathcal{B}^N\rightarrow\mathcal{B}^N$ which are of the class C^∞ with respect to x, t, and of the class C^∞ in Gateaux's sense with respect to u.

The Gateaux derivate of vector function $X\in\mathcal{B}^n$ in the direction $Y\in\mathcal{B}^m$ is defined by

$$X_*(Y)=\frac{d}{d\epsilon}\bigg|_{\epsilon=0}X(u+\epsilon Y). \tag{1.1}$$

For the two arbitrary vector fields X, $Y\in\mathcal{B}^m$, define the following operation:

$$[X,\ Y]=X_*(Y)-Y_*(X). \tag{1.2}$$

Then, $\mathcal{B}^m$ composes a Lie algebra about the above multiplication operation.[17] For the operator $\phi\in\mathcal{V}^N$, its Gateaux derivate operator $\phi_*:\mathcal{B}^m\rightarrow\mathcal{V}^N$ in the direction ξ is defined as follows:

$$\phi_*(\xi)=\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\phi(u+\epsilon\xi),\quad \xi\in\mathcal{B}^m. \tag{1.3}$$

If not otherwise stated, the spectral operators $L=L(u)$ [or the spectral operators $L=L(u,\ \lambda)$ with the spectral parameter λ] considered in this paper are denoted by $L\in\mathcal{V}^N$, and we always assume that $L_*:\mathcal{B}^m\rightarrow\mathcal{V}^N$ is an injective homomorphism. An operator H acting on a function f is denoted by $H\cdot f$. I stands for the $N\times N$ unit operator.

II. CATEGORY OF NLEEs AND GENERALIZED LAX REPRESENTATION (GLR)

In this section, a procedure for constructing the category of NLEEs and generalized Lax representations are presented, and, moreover, it is shown how to construct the $L-A-B$ triple representation[18] for a given nonlinear quation.

Let us start from a general $N\times N$ spectral problem:

$$L\cdot\psi=\lambda\psi,\quad L\in\mathcal{V}^N, \tag{2.1}$$

where λ is a spectral parameter, $\psi\in\mathcal{B}^N$. Denote the functional gradient of spectral parameter λ with regard to the potential vector u by $\delta\lambda/\delta u=(\delta\lambda/\delta u_1,\cdots,\delta\lambda/\delta u_m)^T$. Tu and Cao, respectively, gave some discussions about the calculations of the functional gradient in Ref. 19 and Ref. 20. Strampp ever studied recursion operators, spectral problems, and Bäcklund transformations by introducing a relation between recursion operators and eigenvalue functions.[21,22] Thus, we define the Lenard operators as follows:

Definition 2.1: If there exists a pair of $m\times m$ operators $K=K(u)$, $J=J(u)$: $S^m(R^l,\ R)\rightarrow S^m(R^,R)$ such that

$$K\cdot\frac{\delta\lambda}{\delta u}=\lambda^cJ\cdot\frac{\delta\lambda}{\delta u}, \tag{2.2}$$

then K, J are called a pair of Lenard operators of (2.1), and (2.2) is called the Lenard spectral problem of (2.1). Here the constant c is definitely chosen by the concrete form of (2.1).

In many cases, there exist (but not unique) the pair of Lenard's operators satisfying (2.2), and frequently both of them are Hamiltonian operators. For instance, for the KdV–Schrödinger spectral problem $\psi_{xx}+u\psi=\lambda\psi$, $\delta\lambda/\delta u=\psi^2$, only choosing $K=-\frac{1}{4}\partial^3-\frac{1}{2}(u\partial+\partial u)$, $J=\partial=\partial/\partial x$, we have $K\cdot\ \delta\lambda/\delta u=\lambda J\cdot\ \delta\lambda/\delta u$. Eq. (2.2) plays an important role in the nonlinearization theory and the construction of completely integrable finite-dimensional systems.[23]

Let $M=(m_{ij})_{N\times N}$, $\tilde{M}=(\tilde{m}_{ij})_{N\times N}$ be the arbitrarily given $1+l$-dimensional [i.e., independent variables $(x,\ t)\in R^l\times R,\ l\geqslant 1$] linear $N\times N$ matrix operators. Then we have the following definitions.

Definition 2.2: $G_0\in S^m(R^l,\ R)$, $G_{-1}\in S^m(R^l,\ R)$ *are, respectively, called the positive order and the negative order generators, if they, respectively, satisfy the operator equations,*

$$L_*(J\cdot G_0)=M, \tag{2.3}$$

$$L_*(K\cdot G_{-1})=\tilde{M}. \tag{2.4}$$

Denote the solution sets of (2.3) and (2.4) by $\mathcal{N}_J(M)$ and $\mathcal{N}_K(\tilde{M})$, respectively. In general, they are not empty.

Definition 2.3: Let $\mathcal{N}_J(M)\neq\varnothing$, $\mathcal{N}_K(\tilde{M})\neq\varnothing$ *and choose* $G_0\in\mathcal{N}_J(M)$, $G_{-1}\in\mathcal{N}_K(\tilde{M})$. *Write the recursion operator* $\mathcal{L}=J^{-1}K$. *The sequence* $\{G_j\}_{j=-\infty}^{\infty}\subseteq S^m(R^l,R)$ *recursively determined by*

$$G_j=\begin{cases}\mathcal{L}^j\cdot G_0, & j\geqslant 0,\\ \mathcal{L}^{j+1}\cdot G_{-1}, & j<0,\end{cases} \tag{2.5}$$

is called the Lenard's sequence of (2.1); the set of the following nonlinear equations:

$$u_t=X_m(u,G_0,G_{-1}),\ \ m\in Z, \tag{2.6}$$

produced by the vector field

$$X_m(u,G_0,G_{-1})\triangleq J\cdot G_m,\ \ m\in Z, \tag{2.7}$$

is called the category of nonlinear evolution equations of (2.1). The subset of the equations (2.6) obtained for $m\geqslant 0$ *is called the positive order category while the subset obtained for* $m<0$ *is called the negative order category*.

Apparently, the positive and the negative order generators G_0, G_{-1} depend on the choice of matrix operators M, $\tilde{M}$, thus the category (2.6) is composed of various hierarchies (both integrable and nonintegrable) of NLEEs which are generated according to the choice of operators M, $\tilde{M}$.

For example, with $M\equiv 0$ (i.e., $G_0\in\mathrm{Ker}\ J$), the hierarchy in the positive order category of (2.6) just reads as the isospectral hierarchy of evolution equations;[24] with $\tilde{M}\equiv 0$ (i.e., $G_{-1}\in\mathrm{Ker}\,K$), the hierarchy in the negative order category of (2.6) is exactly the second isospectral hierarchy of evolution equations studied in Ref. 25. Additionally, the negative order generator G_{-1} can be considered to produce finite-dimensional constrained Hamiltonian systems.[26] Obviously, the negative order category of (2.6) is generated with the help of the inverse recursion operator $\mathcal{L}$. Strampp and Oevel gave the inverse recursion operator in an explicit form for the nonlinear derivative Schrödinger equation.[27] In 1991 we suggested the commutator representations for the negative order hierarchy of isospectral NLEEs.[28] Afterwards, we[29] further found that the same spectral problem can generate two different hierarchies of integrable NLEEs: one is the usual higher order (i.e., positive order) hierarchy of NLEEs, the other is the negative order hierarchy of NLEEs. All these equations have the Lax representations.[29] Here we study the generalized case, i.e., the category of NLEEs.

With $M=I$ or $\tilde{M}=I$, under the basic condition $\mathcal{N}_J(I)\neq\varnothing$ or $\mathcal{N}_K(I)\neq\varnothing$, Eq. (2.6) actually gives the positive and the negative order hierarchies of nonisospectral evolution equations, which can be obtained from the following, Theorem 2.2. Thus, by the arbitrariness of M and $\tilde{M}$, Eq. (2.6) unifies together all possible hierarchies of evolution equations associated with the spectral problem (2.1). Due to this fact, Eq. (2.6) is named ‘‘the category of nonlinear evolution equations.’’

Theorem 2.1: *Let* $M=(m_{ij})_{N\times N}$, $\tilde{M}=(\tilde{m}_{ij})_{N\times N}$ *be two arbitrarily given* $N\times N$ *linear matrix operators,* $\mathcal{N}_J(M)\neq\emptyset$, *and* $\mathcal{N}_K(\tilde{M})\neq\emptyset$. *Suppose that for* $G=(G^{[1]},\ldots,G^{[m]})^T\in S^m(R^l,R)$ *and* $\alpha,\beta\in Z$ *the operator equation,*

$$[V,L]=L_*(K\cdot G)L^{\beta}-L_*(J\cdot G)L^{\alpha}, \tag{2.8}$$

possesses a solution $V=V(G)$; *then the vector field* $X_m=X_m(u,G_0,G_{-1})$ *satisfy*

$$L_*(X_m)=[W_m,L]+\bar{M}L^{m\eta},\quad m\in Z,\quad \bar{M}=\begin{cases}M, & m\geqslant 0,\\ \tilde{M}, & m<0,\end{cases} \tag{2.9}$$

where $\eta=\alpha-\beta$ *and the operator* W_m *is given by*

$$W_m=\sum V(G_j)L^{(m-j)\eta-\alpha},\quad \sum=\begin{cases}\sum_{j=0}^{m-1}, & m>0,\\ 0, & m=0,\\ -\sum_{j=m}^{-1}, & m<0.\end{cases} \tag{2.10}$$

Here G_j *are determined by (2.5), and* L^{-1} *is the inverse of* L, *i.e.,* $LL^{-1}=L^{-1}L=I$, *and* $[\cdot,\cdot]$ *denotes the usual commutator.*

Proof: For $m=0$, it is obvious. For $m>0$,

$$\begin{aligned}[W_m,L]&=\sum_{j=0}^{m-1}[V(G_j),L]L^{(m-j)\eta-\alpha}\\ &=\sum_{j=0}^{m-1}\{L_*(K\cdot G_j)L^{(m-j-1)\eta}-L_*(J\cdot G_j)L^{(m-j)\eta}\}\\ &=\sum_{j=0}^{m-1}\{L_*(J\cdot G_{j+1})L^{(m-j-1)\eta}-L_*(J\cdot G_j)L^{(m-j)\eta}\}=L_*(X_m)-L_*(J\cdot G_0)L^{m\eta}\\ &=L_*(X_m)-ML^{m\eta}.\end{aligned}$$

For $m<0$, the proof is similar. ■

Remark 2.1: The structure equation (2.8) of commutator representations is a natural generalization of the structure equation $[V,L]=L_*(K\cdot G)-L_*(J\cdot G)L$ presented by Cao Cewen.[30]

Remark 2.2: The choice of constants $\alpha,\beta\in Z$ is determined by the concrete form of (2.1). In many cases,[29] $V=V(G)$ can be solved for the given L.

Theorem 2.2: *The category (2.6) of NLEEs has the following representation:*

$$L_t=[W_m,L]+\bar{M}L^{m\eta},\quad m\in Z,\quad \bar{M}=\begin{cases}M, & m>0,\\ \tilde{M}, & m<0.\end{cases} \tag{2.11}$$

Proof: For $m\geqslant 0$, because $L_*(u_t)=L_t$ and L_* is injective,

$$L_t=[W_m,L]+ML^{m\eta}\Leftrightarrow L_*(u_t-X_m)=0\Leftrightarrow u_t=X_m,$$

which completes the proof. ■

Definition 2.4: Equation (2.11) and W_m *are called the generalized Lax representations (GLR) and the generalized Lax-operator (GLO), respectively.*

Obviously, with $\overline{M}=0$ (i.e., $G_0\in \mathrm{Ker}\, J$, $G_{-1}\in \mathrm{Ker}\, K$), Eq. (2.11) reduces the standard (i.e., isospectral case: $\lambda_t=0$) Lax representations, and with $\overline{M}=I$ [of course $\mathcal{N}_J(I)\neq\emptyset$ and $\mathcal{N}_K(I)\neq\emptyset$ are needed], Eq. (2.11) reduces the nonisospectral (i.e., $\lambda_t=\lambda^{m\eta}$, $m\in Z$) Lax representations. For two special cases: the isospectral case (i.e. $M=\widetilde{M}=0$) and the nonisospectral case (i.e., $M=\widetilde{M}=I$), Ma[31] discussed the Lax operator algebras of the positive order (i.e., $m>0$) hierarchy of NLEEs. But a general framework has not been obtained for all integer $m\in Z$ and all linear matrix operators $M,\widetilde{M}$. In the following sections, we shall construct a general frame–generalized algebraic structure and furthermore present the r-matrix for the category of NLEEs.

Remark 2.3: Equation (2.11) admits the structure of $L-A-B$ representations of the category (2.6) in an explicit form. Thus, we give a constructive approach to the Manakov operator pair A, B in the $L-A-B$ triple representation.[18] In Ref. 32, we determined the range of the $L-A-B$ triple representation through defining the Lie quotient algebras.

Remark 2.4: Equation (2.11) contains both the integrable and the nonintegrable hierachies because of the multiple choices of $\overline{M}$. Therefore, our category of NLEEs are not included in the system of multi-component KP and its reduction.

Corollary 2.1: Assume that the potential vector function u is independent of t and the following condition holds:

$$\left[\sum_{i=-r}^{s} c_i W_i\, ,L\right]=-\overline{M}\sum_{i=-r}^{s} c_i L^{i\eta},$$

with constants c_i $(-r\leqslant i\leqslant s)$. Then u will satisfy the stationary system of the category (2.6):

$$\sum_{i=-r}^{s} c_i X_i(u)=0,\ \forall r,s\in Z^+.$$

We shall give several concrete examples in Sec. VI.

III. UNIVERSAL ALGEBRAIC STRUCTURE AND *r*-MATRIX

From (2.9), we have seen that for various linear matrix operators $M,\widetilde{M}$, the category (2.6) of NLEEs indeed yields different hierarchies of NLEEs. That means the hierarchy in the category (2.6) changes according to the choice of $M,\widetilde{M}$. In this section, we shall construct the algebraic structure and r-matrix which holds for all hierarchies of NLEEs in the category (2.6). Let us start from the following definition.

Definition 3.1: Suppose that for a spectral operator $L\in\mathcal{V}^N$ and an integer $n\in Z$ there exist pairs (A,M) of vector fields $X\in\mathcal{B}^m$ and operators $A,M\in\mathcal{V}^N$ with the property

$$[A,L]=L_*(X)-ML^n. \tag{3.1}$$

Then (A,M) is called a Manakov operator pair of L. The set of all Manakov operator pairs is denoted by $\mathcal{M}_L^n$. X is called the vector field corresponding to (A,M). The set of all vector fields X is denoted by $V(\mathcal{M}_L^n)$. The set of all triples (A,M,X) is denoted by $\mathcal{P}_L^n$.

As long as Eq. (2.8) has an operator solution for a given $L\in\mathcal{V}^N$, then by theorem 2.1 and Eq. (2.9) there exists a triple $(A,M,X)\in\mathcal{P}_L^n$ satisfying (3.1).

It is easy to prove the following proposition.

Proposition 3.1:

(1) *The vector field associated with each Manakov operator pair is unique*;
(2) *both $\mathcal{P}_L^n$ and $\mathcal{M}_L^n$ form linear spaces.*

Apparently, if there is A, $M\in\mathcal{V}^N$ for $X\in\mathcal{B}^m$ such that Eq. (3.1) holds, then $u_t=X$ possesses the GLR $L_t=[A,\ L]+ML^n$. It is not difficult to see that $\mathcal{P}_L^n$ and $\mathcal{P}_L^0$, $\mathcal{M}_L^n$ and $\mathcal{M}_L^0$ are equivalent, respectively, under the bijective map Φ: $\mathcal{P}_L^n \to \mathcal{P}_L^0$, defined by $(A, M, X) \mapsto (A, ML^n, X)$. So, in the following we simply consider $\mathcal{P}_L^0$, $\mathcal{M}_L^0$ and write $\mathcal{M}_L^0 = \mathcal{M}_L$, $\mathcal{P}_L^0 = \mathcal{P}_L$.

Definition 3.2: Let (A, M, X), $(B, N, Y) \in \mathcal{P}_L$. *In* $\mathcal{M}_L$, *define a binary operation as follows:*

$$(A, M) \odot (B, N) = (A \odot B, M \odot N), \tag{3.2}$$

where

$$A \odot B = A_*(Y) - B_*(X) + [A,B], \tag{3.3}$$

$$M \odot N = M_*(Y) - N_*(X) + [M,B] - [N,A]. \tag{3.4}$$

Obviously (3.2) is a skew-symmetric and bilinear operation.

Theorem 3.1: *Let* (A, M, X), $(B, N, Y) \in \mathcal{P}_L$, *then* $(A \odot B, M \odot N, [X, Y]) \in \mathcal{P}_L$, *and* $\mathcal{M}_L$ *form a Lie algebra under the operation (3.2).*

Proof: Since $(\mathcal{V}^N, [\cdot,\cdot])$ builds up a Lie algebra under the usual commutator operation, we have

$$\begin{aligned}[][A, B],L] &= [[L, B],A] - [[L, A], B] \\ &= [L_*(X) - M, B] - [L_*(Y) - N, A] \\ &= [L_*(X), B] - [L_*(Y), A] + [N, A] - [M, B].\end{aligned}$$

For arbitrary $L \in \mathcal{V}^N$, $X, Y \in \mathcal{B}^m$, we also have

$$(L_*(X))_*(Y) - (L_*(Y))_*(X) = L_*([X,Y]).$$

Thus,

$$\begin{aligned}[A \odot B, L] &= [A_*(Y) - B_*(X) + [A, B], L] \\ &= [A_*(Y), L] - [B_*(X), L] + [L_*(X), B] - [L_*(Y), A] + [N, A] - [M, B] \\ &= ([A, L])_*(Y) - ([B, L])_*(X) + [N, A] - [M, B] \\ &= (L_*(X))_*(Y) - (L_*(Y))_*(X) - M_*(Y) + N_*(X) + [N, A] - [M, B] \\ &= L_*([X, Y]) - M \odot N.\end{aligned}$$

That means $(A \odot B, M \odot N, [X, Y]) \in \mathcal{P}_L$.

Now, we shall prove the Jacobi identity. Choosing any $(A_i, M_i, X_i) \in \mathcal{P}_L$, $i = 1, 2, c3$, then we have

$$\begin{aligned}(A_1 \odot A_2) \odot A_3 + c.p. &= (A_{1*}(X_2) - A_{2*}(X_1) + [A_1, A_2]) \odot A_3 + c.p. \\ &= [[A_1, A_2], A_3] + c.p. \\ &= 0.\end{aligned}$$

Similarly, we can show the following equality:

$$(M_1 \odot M_2) \odot M_3 + c.p. = 0, \tag{$*$}$$

which completes the proof. ■

Corollary 3.1: The set of all vector fields $V(\mathcal{M}_L)$ *forms a Lie subalgebra of* $\mathcal{B}^m$ *with regard to the operation (1.2).*

Denote the vector fields of $(A,\ M)$ and $(B,\ N)$ by X and Y, respectively, then $u_t=X,\ u_t=Y$ represent the two *different* hierarchies of NLEEs, respectively, determined by M,N. Theorem 3.1 shows that there is universal algebraic structure for the *different* hierarchies of NLEEs, and if both $u_t=X$, and $u_t=Y$ $(X,\ Y\in\mathcal{B}^m)$ have GLR, then so does the new hierarchy of equations $u_t=[X,\ Y]$ produced by $X,\ Y$.

For the given spectral operator $L\in\mathcal{V}^N$, we now consider the following adjoint map:

$$\mathrm{ad}_L : A\mapsto M=\mathrm{ad}_L\, A=[L,\ A],\ \forall A\in\mathcal{V}^N. \tag{3.5}$$

Then according to the original definition of an r-matrix,[9] we have the following theorem.

Theorem 3.2: *The adjoint map* ad_L *is an* r*-matrix.*

Proof: For any $A,\ B\in\mathcal{V}^N$, write $M=\mathrm{ad}_L\, A,\ N=\mathrm{ad}_L\, B$. Then we have

$$[A,\ B]_{\mathrm{ad}_L}\triangleq[\mathrm{ad}_L A,\ B]+[A,\ \mathrm{ad}_L B]=[M,\ B]+[A,\ N]=M\odot N.$$

The last equality holds because the associated vector fields are obviously zero. And Eq. $(*)$ implies that $[A,B]_{\mathrm{ad}_L}$ satisfies the Jacobi identity. Thus the adjoint map ad_L is an r-matrix. ■

In the last section we shall illustrate that through giving several examples.

IV. NONUNIVERSAL ALGEBRAIC STRUCTURE AND *r*-MATRIX

For a given spectral operator $L\in\mathcal{V}^N$ and integer $n\in Z$, in the above section we discussed the Manakov operator pair (A,M), the universal Lie algebraic structure and the r-matrix available for *different* hierarchies of NLEEs. Now, for a given $N\times N$ matrix operator M and a spectral operator $L\in\mathcal{V}^N$, we study the operator algebra and r-matrix which can be attached only to the *underlying* hierarchy of NLEEs.

Let us first give some conventions in this section: (i) M is invertible; (ii) For a given $L\in\mathcal{V}^N$, $\mathcal{V}_L^N$ stands for all matrix operators $S:\mathcal{B}^N\rightarrow\mathcal{B}^N$ possessing the following form $S=\Sigma_{\alpha\in Z}P_\alpha(u)L^\alpha$, $P_\alpha(u)\in\mathcal{B}$, where $\Sigma_{\alpha\in Z}$ is a finite sum. Next, we introduce the following definition.

Definition 4.1: Let $L\in\mathcal{V}^N$ *and* M *be a spectral operator and an* $N\times N$ *matrix operator, respectively. If there exist a vector field* $X\in\mathcal{B}^m$ *and operators* $A,P\in\mathcal{V}_L^N$ *such that*

$$[A,\ L]+MP=L_*(X), \tag{4.1}$$

then $(A,\ P)$ *is said to be an* LM *operator pair of* L. *The set of all such pairs is denoted by* $\mathcal{L}_L^M$. X *is called the vector field of* $(A,\ P)$ *associated with* LM. *The set of all associated vector fields is denoted by* $V(\mathcal{L}_L^M)$. *Furthermore, we denote the set of all triples* $(A,\ P,\ X)$ *by* $\mathcal{R}_L^M$.

For a given $L\in\mathcal{V}^N$ and an $N\times N$ matrix operator M or $\tilde{M}$ theorem 2.1 and Eq. (2.8) assure that there exists a triple $(A,\ P,\ X)\in\mathcal{R}_L^M$ satisfying (4.1). Definition 4.1 directly leads to the following proposition.

Proposition 4.1:

(1) *The vector field associated with each* LM *operator pair is unique.*
(2) *Both* $\mathcal{L}_L^M$ *and* $\mathcal{R}_L^M$ *are linear spaces.*

If for given operators $L,\ M$ there exist $A,\ P\in\mathcal{V}_L^N$ such that (4.1) holds, then obviously the evolution equation $u_t=X$ has the following representation [also called generalized Lax representation (GLR)]:

$$L_t=[A,\ L]+MP. \tag{4.2}$$

Now, we define a binary operation in $\mathcal{L}_L^M$.

Definition 4.2: Let $(A,\ P),(B,\ Q)\in\mathcal{L}_L^M,X,Y\in V(\mathcal{L}_L^M)$, *respectively, be the vector fields of* $(A,\ P),(B,\ Q)$. *Declare a binary operation,*

$$(A,P)\ominus(B,Q)=(A\ominus B,P\ominus Q), \tag{4.3}$$

through

$$A\ominus B=A_*(Y)-B_*(X)+[A,B], \tag{4.4}$$

$$\begin{aligned}P\ominus Q=&P_*(Y)-Q_*(X)+[A,Q]-[B,P]\\&+M^{-1}(M_*(Y)-[B,M])P-M^{-1}(M_*(X)-[A,M])Q.\end{aligned} \tag{4.5}$$

Proposition 4.2:

(1) *Equation (4.3) is a skew-symmetric, bilinear binary operation.*
(2) $\mathcal{V}_L^N$ *is closed under the operations (4.4) and (4.5).*

Proof: The proof follows directly from Definition 4.2. ■

Theorem 4.1: *Let* $(A,\ P,\ X),(B,\ Q,\ Y)\in\mathcal{R}_L^M$, *then* $(A\ominus B,\ P\ominus Q,\ [X,Y])\in\mathcal{R}_L^M$, *where* $[X,\ Y]$ *is defined by (1.2). Thus under the operation (4.3)* $\mathcal{L}_L^M$ *forms an algebra, and* $(V(\mathcal{L}_L^M),\ [\cdot,\cdot])$ *composes a Lie subalgebra of* $\mathcal{B}^m$.

Proof: Because $(A,\ P,\ X),(B,\ Q,\ Y)\in\mathcal{R}_L^M$, and

$$[\![A,\ B],\ L]\!]=[\![L,\ B],\ A]\!]-[\![L,\ A],\ B]\!]=[L_*(X),\ B]-[L_*(Y),\ A]+[MQ,\ A]-[MP,\ B],$$

we have

$$\begin{aligned}[A\ominus B,\ L]&=[A_*(Y),\ L]-[B_*(X),\ L]+[\![A,\ B],\ L]\!]\\&=(L_*(X))_*(Y)-(L_*(Y))_*(X)-(MP)_*(Y)+(MQ)_*(X)+[MQ,A]-[MP,\ B]\\&=L_*([X,\ Y])-M(P\ominus N),\end{aligned}$$

which completes the proof. ■

For a given spectral operator L and an $N\times N$ matrix operator M, denote the vector fields of $(A,\ P),(B,\ Q)$ by $X,\ Y$, respectively. Then from Sec. II we know $u_t=X,\ u_t=Y$ are the two different NLEEs in the *same* hierarchy. Theorem 4.1 reveals that there exists an algebraic structure available for all equations in the *same* hierarchy. And if $u_t=X,\ u_t=Y$ $(X,\ Y\in\mathcal{B}^m)$ have the GLR (4.2); then the evolution equation $u_t=[X,\ Y]$ is still in the same hierarchy, and possesses the GLR (4.2), too.

Remark 4.1: In general, $\mathcal{L}_L^M$ is not forming a Lie algebra under the operation (4.3), because the Jacobi identity cannot be guaranteed. Nevertheless, the subset $S_L^M\subset\mathcal{L}_L^M$, considered below, is an exception.

Set $S_L^M=\{(A,\ P)\in\mathcal{V}_L^N\times\mathcal{V}_L^N|\ P=M^{-1}\,\mathrm{ad}_L A\}$; then S_L^M is corresponding to the stationary system $X(u)=0$ of evolution equation $u_t=X(u)$.

Theorem 4.2: *For all* $(A,\ P)\in S_L^M$, *define a map* $r^M{:}A\mapsto P=M^{-1}\,\mathrm{ad}_L A$. *The map* r^M *is an* r*-matrix under the operation (4.5) iff* $M=aI,\ a\neq0,a\in R$.

Proof: For any $(A,\ P),(B,\ Q)\in S_L^M$, define

$$[A,\ B]_{r^M}\triangleq[r^M(A),\ B]+[A,\ r^M(B)].$$

Then

$$[A,\ B]_{r^M}=[P,\ B]+[A,\ Q]=P\ominus Q\Leftrightarrow M=aI,\ \ a\neq0,\ \ a\in R,$$

i.e., the map r^M is an r-matrix $\Leftrightarrow M=aI,\ a\neq0,\ a\in R$. ■

Since M and $\widetilde{M}$ can be fixed arbitrarily we have found two algebraic operator structures, namely a universal one being independent of the hierarchy in the category and a nonuniversal one depending on the underlying hierarchy. In addition, in this procedure we have found two kinds of adjoint maps being r-matrices.

The two algebraic structures are associated with the category of NLEEs (2.11) which includes both the integrable and the nonintegrable cases (see Remark 2.4). Therefore, here our algebraic structures are not contained in any W-algebras which are usually suitable for the integrable hierarchy such as the KP, etc.

In the next section, we shall give two reductions of the algebraic structure and the related r-matrix.

V. TWO REDUCTIONS: VIRASORO ALGEBRA AND *r*-MATRIX OF ISOSPECTRAL AND NONISOSPECTRAL LAX OPERATOR

If we choose $M=0$ in Definition 4.1, then we have $[A,L]=L_*(X)$. That means A is an isospectral ($\lambda_t=0$) Lax operator. Set $[B,L]=L_*(Y)$; then the operation $A\ominus B$ defined by (4.4) forms an algebraic structure of the isospectral Lax operator, which just coincides with the result described in Ref. 31. In this case, the r-matrix is zero, i.e., $\mathrm{ad}_L A=0$, $\forall A\in\mathcal{V}^N$.

In this section, we always choose $M=\widetilde{M}=I$ and assume that the conditions of Theorem 2.1 hold. Then, by Theorem 2.1, we obtain

$$(W_m,\ L^{m\eta},\ \sigma_m)\in\mathcal{R}_L^I,\ m\in Z,$$

where W_m is expressed through (2.10), σ_m stands for the corresponding vector field. Therefore W_m is a sequence of nonisospectral ($\lambda_t=\lambda^{m\eta}, m\in Z$) Lax operators and this matches with choosing $A=W_m$, $P=L^{m\eta}$ $(m\in Z)$, $X=\sigma_m$ in (4.1). By Theorem 4.1 $\{(W_m,L^{m\eta}),m\in Z\}$ represents an algebra under the operation (4.3), which is called the nonisospectral Lax operator algebra of the spectral operator L. In the stationary case where $\sigma_i=\sigma_j=0$ the following holds.

Theorem 5.1: *A realization of the operations (4.5) and (4.4) on pairs* $(W_i,L^{i\eta})$, $(W_j,L^{j\eta})\in S_L^I$ *is given by*

$$L^{i\eta}\ominus L^{j\eta}=(|i|-|j|)L^{(i+j-1)\eta},\quad \forall i,j\in Z, \tag{5.1}$$

$$W_i\ominus W_j=(|i|-|j|)W_{i+j-1},\quad \forall i,j\in Z, \tag{5.2}$$

respectively.

Proof: For $(W_i,\ L^{i\eta}),(W_j,\ L^{j\eta})\in S_L^I$, we have

$$[W_i,\ L]=-L^{i\eta},\quad [W_j,\ L]=-L^{j\eta}.$$

Thus, in the case $i,j\geqslant 0$,

$$\begin{aligned}L^{i\eta}\ominus L^{j\eta}&=[L^{i\eta},\ W_j]-[L^{j\eta},\ W_i]\\&=\sum_{k=0}^{i-1}L^{(i-1-k)\eta}IL^{(k+j)\eta}-\sum_{k=0}^{j-1}L^{(j-1-k)\eta}IL^{(k+i)\eta}\\&=iL^{(i+j-1)\eta}-jL^{(i+j-1)\eta}=(i-j)L^{(i+j-1)\eta}.\end{aligned}$$

Similarly, Eq. (5.1) holds for the other three cases $i\geqslant 0$, $j\leqslant 0$; $i\leqslant 0$, $j\leqslant 0$; $i\leqslant 0$, $j\geqslant 0$.

Equation (5.2) can be directly obtained by (5.1) and Theorem 4.1. ∎

Corollary 5.1: If $M=\widetilde{M}=I$, under the operation (5.1) the map $r^I\colon W_i\mapsto L^{i\eta}$ is an r-matrix.

Proof: This can be directly derived from Theorem 5.1 and Theorem 4.3. ∎

Remark 5.1: Theorem 5.1 and Corollary 5.1 actually describe the Lie algebraic structure of the Lax operator for the stationary equation $\sigma_j=0$ $(j\in Z)$ and the r-matrix of a concrete form of an operation (4.5) and (4.4), respectively.

For the usual nonstationary vector field $\sigma_j\neq 0$ $(j\in Z)$ in the nonisospectral case, (5.1) and (5.2) do not hold. But, we have the following results.

Theorem 5.2: *Let* $(W_j, L^{j\eta})\in\mathcal{L}^I_L$, $j\in Z$; *then for any* $i,j\in Z$, L *satisfies the relation*

$$L^{i\eta}\ominus L^{j\eta}=(|i|-|j|)\,\eta L^{(i+j+1)\eta-1},\quad \forall i,\ j\in Z. \tag{5.3}$$

Proof: We give the proof only for the case of $i\geqslant 0$, $j\geqslant 0$. The other cases are shown analogously.

Let $(W_i, L^{i\eta}, \sigma_i)$, $(W_j, L^{j\eta}, \sigma_j)\in\mathcal{R}^I_L$; then we have

$$\begin{aligned}(L^{i\eta})_*(\sigma_j)&=\sum_{k=0}^{i-1}L^{(i-1-k)\eta}L^{\eta}_*(\sigma_j)L^{k\eta}\\&=\sum_{k=0}^{i-1}L^{(i-1-k)\eta}([W_j, L^{\eta}]+\eta L^{(j+1)\eta+\eta-1})L^{k\eta}\\&=[W_j, L^{i\eta}]+i\,\eta L^{(i+j+1)\eta-1},\end{aligned}$$

and

$$(L^{j\eta})_*(\sigma_i)=[W_i, L^{j\eta}]+j\,\eta L^{(i+j+1)\eta-1}.$$

So, by Eq. (4.5) and noticing $M=I$, we obtain

$$L^{i\eta}\ominus L^{j\eta}=(i-j)\,\eta L^{(i+j+1)\eta-1},\quad \forall i,\ \ j\in Z^+, \tag{5.4}$$

which is the desired result. ∎

Equations (5.1), (5.2), and (5.3) are three special Virasoro algebras, namely, without a central extension. Because here we do calculations based on our definitions of binary operations (4.4) and (4.5), they have no central extensions.

Remark 5.2: For the usual nonstationary vector field $\sigma_j\neq 0$ $(j\in Z)$ in the nonisospectral case the operation (5.1) does not always satisfy the Jacobi identity, (see Remark 4.1). Thus Corollary 5.1 does not hold in general.

Remark 5.3: A particular case of Theorem 5.2 is $\eta=1$. Then Eq. (5.3) becomes

$$L^{i}\ominus L^{j}=(|i|-|j|)L^{i+j},\quad \forall i,\ \ j\in Z, \tag{5.5}$$

which implies the following equations:

$$W_i\ominus W_j=(|i|-|j|)W_{i+j},\quad \forall i,\ \ j\in Z, \tag{5.6}$$

and

$$[\sigma_i,\sigma_j]=(|i|-|j|)\sigma_{i+j},\quad \forall i,\ \ j\in Z. \tag{5.7}$$

Theorem 5.2 reveals that under Eq. (5.5) for the same nonisospectral hierarchy the following holds: if $u_t=\sigma_m$ and $u_t=\sigma_n$, respectively, possess the nonisospectral Lax operators W_m and W_n, then $u_t=\sigma_{m+n}$ still possesses the nonisospectral Lax operator $1/(|m|-|n|)\,W_m\ominus W_n$, $\forall m,n\in Z$. Thus, the Virasoro operator algebras (without the central extension) for the nonisospectral hierarchy of NLEEs is reflected by Eqs. (5.5)–(5.7).

Remark 5.4: If we choose $M=0$ and $M=I$, respectively, then under the algebraic operation (3.3) we can also have the Virasoro algebra of the Lax operator for the isospectral hierarchy and the nonisospectral hierarchy, which is actually a special case of universal algebraic structure.

VI. SOME EXAMPLES

Through taking several examples, we illustrate our methods. For our convenience, we make the following conventions:

$$f^{(m)}=\begin{cases}\dfrac{\partial^m}{\partial x^m}f=f_{mx}, & m\geqslant 0,\\ \underbrace{\int\cdots\int}_{-m} f\,dx, & m<0,\end{cases}\qquad \sum=\begin{cases}\sum\limits_{j=0}^{m-1}, & m>0,\\ 0, & m=0,\\ -\sum\limits_{j=m}^{-1}, & m<0,\end{cases}$$

$f_t=\partial f/\partial t$, $f_{mxt}=\partial^{m+1}f/\partial t\,\partial x^m\,(m\geqslant 0)$, $\partial=\partial/\partial x$, ∂^{-1} is the inverse of ∂, i.e., $\partial\partial^{-1}=\partial^{-1}\partial=1$, $\partial^m f$ means the operator $\partial^m f$ acts on some function g, i.e., $\partial^m f\cdot g=\partial^m(fg)$, $m\in Z$. C_m^k stands for the combinatorial constants: $C_m^k=m(m-1)\cdots(m-k+1)/k!$, i an imaginary unit satisfying $i^2=-1$, and $I_{2\times 2}$ the 2×2 unit matrix.

In the spectral problems (6.1), (6.32) and (6.74) the function u stands for the potential function, and the potential functions in spectral problems (6.43) and (6.56) are denoted by q, r. In those spectral problems, λ is always assumed to be a spectral parameter. The domain of the spatial variable x is Ω which becomes equal to $(-\infty, +\infty)$ or $(0, T)$, while the domain of the time variable t is the positive time axis $R^+=\{t|t\in R,\ t\geqslant 0\}$. In the case $\Omega=(-\infty, +\infty)$ the decaying condition at infinity and in the case $\Omega=(0, T)$ the periodicity condition for the potential function, is imposed.

6.1: Consider the Burger's spectral problem:[33]

$$L\cdot y=\lambda y,\quad L=L(u)=\partial+u. \tag{6.1}$$

Choosing the recursion operator $\mathcal{L}=\partial+\partial u\partial^{-1}$ leads to

$$\mathcal{L}\cdot y_x=\lambda y_x. \tag{6.2}$$

Obviously, $L_*(\xi)=\xi,\forall\xi\in\mathcal{B}$, i.e., L_* is an identity operator. In this case, the Lenard's operators pair is chosen as $J=1$, and $K=\mathcal{L}$.

The Lenard recursive sequence $\{G_j\}_{j=-\infty}^{\infty}$ $(G_j=\mathcal{L}^j\cdot M,\ j\in Z)$ gives the Burgers category of NLEEs:

$$u_t=\mathcal{L}^m\cdot M=(e^{-u^{(-1)}}(e^{u^{(-1)}}M^{(-1)})^{(m)})_x,\quad m\in Z, \tag{6.3}$$

where $M\in\mathcal{B}$ is an arbitrarily given function, and $\mathcal{L}=\partial e^{-u^{(-1)}}\partial e^{u^{(-1)}}\partial^{-1}$, $\mathcal{L}^{-1}=\partial e^{-u^{(-1)}}\partial^{-1}e^{u^{(-1)}}\partial^{-1}$ which implies $\mathcal{L}^j=\partial e^{-u^{(-1)}}\partial^j e^{u^{(-1)}}\partial^{-1}$, $j\in Z$.

For an arbitrary $G\in\mathcal{B}$, the operator equation $[V, L]=L_*(\mathcal{L}\cdot G)-L_*(G)L$, which matches with choosing $\beta=0$, $\alpha=1$ in (2.8), has the following solution:

$$V=V(G)=-G+G^{(-1)}\partial. \tag{6.4}$$

Thus the category (6.3) possesses the generalized Lax representation (GLR),

$$L_t=[W_m, L]+ML^m,\quad m\in Z, \tag{6.5}$$

with $W_m=M^{(-1)}L^m-L^m\cdot M^{(-1)}$, $L^m=e^{-u^{(-1)}}\partial^m e^{u^{(-1)}}$, $m\in Z$.

The transformation $u=(\ln v)_x$ yields a simple form of Eq. (6.3):

$$v_t=(vM^{(-1)})^{(m)},\quad m\in Z, \tag{6.6}$$

which has the GLR $L_t=[W_m, L]+ML^m$ with $L=\partial+(\ln v)_x$, $W_m=v^{-1}(M^{(-1)}\partial^m v - (vM^{(-1)})^{(m)})$.

Apparently, if M is chosen to be independent of v $(v=e^{u^{(-1)}})$, then the category (6.3) is linearized. Thus, (6.3) includes many linearized hierarchies. Now, let us discuss reductions of the category (6.3) or (6.6).

A. Positive case $(m=0, 1, 2,\ldots)$

In this case, the Lax operator W_m can be written as

$$W_m=v^{-1}\sum_{k=1}^{m} C_m^k v^{(m-k)}(M^{(-1)}\partial^k-M^{(k-1)}). \tag{6.7}$$

(i) With $M=0$, $0^{(-1)}=1$, the positive order category of (6.3) reads as the well-known Burgers' hierarchy,

$$u_t=((\partial+u)^m\cdot 1)_x. \tag{6.8}$$

Particularly, with $m=2$ it becomes the Burger's equation $u_t=u_{xx}+2uu_x$ whose Lax operator is $W_2=\partial^2+2u\partial$ in the standard Lax representation $L_t=[W_2,L]$. This corresponds to the isospectral case: $\lambda_t=0$. According to Eq. (6.6), a simple but quite interesting fact is that under the transformation $u=v_x/v$ the whole Burgers' hierarchy (6.8) is linearized as

$$v_t=v_{mx},\ m=0,1,2,\ldots. \tag{6.9}$$

Equation (6.9) can be solved very easily and have the standard Lax pair $W_m=v^{-1}\Sigma_{k=1}^m C_m{}^k v^{(m-k)}\partial^k$ and $L=\partial+v_x/v$. In this way, the solutions of all equations in the Burgers' hierarchy (6.8) can be worked out.

(ii) With $M=a$, $a^{(-1)}=ax+f(t)$, $a\in R$, $f(t)\in C^\infty(R)$, the positive order category of (6.3) becomes the nonisospectral $(\lambda_t=a\lambda^m)$ Burgers' hierarchy,

$$u_t=((\partial+u)^m\cdot(ax+f(t)))_x. \tag{6.10}$$

A representative equation $(m=2)$ of Eq. (6.10) is

$$u_t=(ax+f(t))(u_{xx}+2uu_x)+3au_x+au^2, \tag{6.11}$$

possessing the GLR $L_t=[W_2,L]+aL^2$ with $W_2=(ax+f(t))(\partial^2+2u\partial)-2au$ and $L=\partial+u$. By virtue of $M=a$ and $u=(\ln v)_x$, Eq. (6.10) is linearized as

$$v_t=(ax+f(t))v_{mx}+mav_{(m-1)x}, \tag{6.12}$$

which can be solved. Equation (6.12) has the generalized Lax operator (GLO) $W_m=(ax+f(t))v^{-1}\Sigma_{k=1}^m C_m^k v^{(m-k)}\partial^k-mav^{-1}v^{(m-1)}$. Particularly, Eq. (6.11) has a linearization equation $(m=2)$,

$$v_t=(ax+f(t))v_{xx}+2av, \tag{6.13}$$

possessing the GLO $W_2=(ax+f(t))(\partial^2+2v^{-1}v_x\partial)-2av^{-1}v_x$. In a general case, M can be extended as $M=\Sigma_{j=0}^n c_j(t)x^j$, $c_j(t)\in C^\infty(R)$, which will be considered below.

(iii) With $M=\Sigma_{j=0}^n c_j(t)x^j$, $c_j(t)\in C^\infty(R)$, the positive order category of (6.3) reads as a nonisospectral $(\lambda_t=(\Sigma_{j=0}^n c_j(t)x^j)\lambda^m)$ hierarchy,

$$u_t=\left((\partial+u)^m\cdot\left(f(t)+\sum_{j=0}^n c_j(t)\frac{x^{j+1}}{j+1}\right)\right)_x, \tag{6.14}$$

where an arbitrary $f(t)\in C^\infty(R)$ is attached by virtue of integration with respect to x. Of course, Eq. (6.14) is easily linearized as

$$v_t=\frac{\partial^m}{\partial x^m}\left(vf(t)+v\sum_{j=0}^n\frac{c_j(t)}{j+1}x^{j+1}\right), \tag{6.15}$$

via $u=(\ln v)_x$. Equation (6.15) has the generalized Lax operator,

$$W_m=v^{-1}\sum_{k=1}^{m} C_m^k v^{(m-k)}(M^{(-1)}\partial^k-M^{(k-1)}),$$

with

$$M^{(-1)}=f(t)+\sum_{j=0}^{n}\frac{c_j(t)}{j+1}x^{j+1}.$$

(iv) With $M=(u^{-1})_x$, $M^{(-1)}=u^{-1}$, the positive order category of (6.3) reads as the following hierarchy of NLEEs:

$$u_t=((\partial+u)^m\cdot u^{-1})_x,\quad m=0,1,2\ldots. \tag{6.16}$$

A representative equation of (6.16) is

$$u_t=\left(\frac{1}{u}\right)_{xx}, \tag{6.17}$$

with the GLO $W_0=-(u^{-1})_x+u^{-1}\partial$.

B. Negative case $(m=-1,-2,\ldots)$

(i) With $M=0$, the generator $G_{-1}=\partial e^{-u^{(-1)}}\partial^{-1}e^{u^{(-1)}}\partial^{-1}\cdot 0$ is determined by the following two seed functions:

$$\bar{G}_{-1}=f(t)(e^{-u^{(-1)}})_x \tag{6.18}$$

and

$$\tilde{G}_{-1}=g(t)(e^{-u^{(-1)}}(e^{u^{(-1)}})^{(-1)})_x, \tag{6.19}$$

where $f(t),\ g(t)\in C^\infty(R)$ are two arbitrarily given functions. Apparently, the seed function (6.18) produces the following isospectral $(\lambda_t=0)$ negative order hierarchy of (6.3),

$$u_t=f(t)(e^{-u^{(-1)}}1^{(m)})_x,\ \ m<0,\ \ m\in Z, \tag{6.20}$$

i.e.,

$$u_t=f(t)e^{-u^{(-1)}}\sum_{k=0}^{-m-1} c_k\frac{x^{-m-k-2}(-m-k-1-xu)}{(-m-k-1)!},\ \ c_0=1, \tag{6.21}$$

where $c_k=c_k(t)\in C^\infty(R)$ $(-m-1\geqslant k\geqslant 1)$ is arbitrarily given. Thus although Eq. (6.20) is nonlinear, we have its general solution:

$$u(x,t)=\frac{\Sigma_{k=0}^{-m-2}c_k(t)\,\dfrac{x^{-m-k-2}}{(-m-k-2)!}\,\partial_t^{-1}f(t)+h'(x)}{\Sigma_{k=0}^{-m-1}c_k(t)\,\dfrac{x^{-m-k-1}}{(-m-k-1)!}\,\partial_t^{-1}f(t)+h(x)},\ \ \forall h(x),\ \ c_k(t)\in C^\infty(R), \tag{6.22}$$

where $\partial_t^{-1}f(t)=\int f(t)dt$, $c_0(t)=1$, $h'(x)=(d/dx)\,h(x)$. Of course, Eq. (6.21) has the standard Lax representation $L_t=[W_m,L]$ with $W_m=-f(t)e^{-u^{(-1)}}\Sigma_{k=0}^{-m-1}c_k(t)\ x^{-m-k-1}/(-m-k-1)!$.

On the other hand, the seed function (6.19) generates the following isospectral $(\lambda_t=0)$ negative order hierarchy of (6.3):

$$u_t=g(t)(e^{-u^{(-1)}}(e^{u^{(-1)}})^{(m)})_x,\ \ m<0,\ \ m\in Z, \tag{6.23}$$

which is a hierarchy of integro-differential equations and can be changed to the linear differential equations,

$$v_{-mxt}=g(t)v, \quad m<0, \quad m\in Z, \tag{6.24}$$

via the transformation $u=v^{-1}v_x$. The Lax operator W_m of (6.23) or (6.24) is $W_m=-g(t)e^{-u^{(-1)}}(e^{u^{(-1)}})^{(m)}$ or $W_m=-g(t)v^{-1}v^{(m)}$, $m<0$.

(ii) With $M=a$, $a^{(-1)}=ax+f(t)$, $a\in R$, $f(t)\in C^{\infty}(R)$, the negative order category of (6.3) through setting $u=v^{-1}v_x$ reads as the linear equations,

$$v_{-mxt}=(ax+f(t))v, \quad m<0, \quad m\in Z, \tag{6.25}$$

which corresponds to the nonisospectral case: $\lambda_t=a\lambda^m$, and has the GLO $W_m=v^{-1}(ax+f(t))\partial^m v-v^{-1}(v(ax+f(t)))^{(m)}$, $m<0$. For a general case, we have the following.

(iii) Setting $M=\Sigma_{j=0}^{n}c_j(t)x^j$ $(c_j(t)\in C^{\infty}(R))$ yields a negative order hierarchy of (6.3),

$$u_t=\left(e^{-u^{(-1)}}\partial^m e^{u^{(-1)}}\cdot\sum_{j=0}^{n}c_j(t)\frac{x^{j+1}}{j+1}\right)_x, \quad m<0, \quad m\in Z, \tag{6.26}$$

which corresponds to the nonisospectral case $\lambda_t=(\Sigma_{j=0}^{n}c_j(t)x^j)\lambda^m$, and can be linearized as

$$v_{-mxt}=v\sum_{j=0}^{n}c_j(t)\frac{x^{j+1}}{j+1}, \ m<0, \ m\in Z, \tag{6.27}$$

via $u=v^{-1}v_x$. Equation (6.27) has the Lax operator

$$W_m=v^{-1}\sum_{j=0}^{n}\frac{c_j(t)}{j+1}(x^{j+1}\,\partial^m v-(vx^{j+1})^{(m)}), \quad m<0.$$

(iv) With $M=(v/v_x)_x$, $\partial^{-1}M=v/v_x$, the associated negative order hierarchy of (6.3) is

$$v_{-mxt}=\frac{v^2}{v_x}, \quad m<0, \quad m\in Z, \tag{6.28}$$

which has a representative equation $(m=-1)$

$$v_x v_{xt}=v^2, \tag{6.29}$$

with the Lax operator $W_{-1}=(1/v_x)\,\partial^{-1}v-(1/v)\,(v^2/v_x)^{(-1)}$.

Through choosing different M, we still have other hierarchies of (6.3). Because of the arbitrariness of M, all results in Secs. III–V are valid for the Burgers' (B) spectral problem (6.1). Particularly, the r-matrix ad_L becomes

$$\mathrm{ad}_L^B: W_m\mapsto ML^m, \quad m\in Z, \tag{6.30}$$

where $W_m=M^{(-1)}L^m-L^m\cdot M^{(-1)}$, $L^m=e^{-u^{(-1)}}\partial^m e^{u^{(-1)}}$, $M\in\mathcal{B}$ is an arbitrarily given function. And the r-matrix r^M $(M=a\neq 0,\ a\in R)$ reads as

$$r_B^a: W_m\mapsto L^m, \quad m\in Z, \tag{6.31}$$

where $W_m=(ax+f(t))L^m-L^m\cdot(ax+f(t))$. Equations (6.30) and (6.31) generate the stationary B-categorical systems $(L^m\cdot M^{(-1)})_x=0$ and $(L^m\cdot(ax+f(t)))_x=0$, respectively.

We can also apply the above procedure to other spectral problems. Now, we list some main results as follows.

6.2: KdV case. The KdV–Schrödinger spectral problem,[34]

$$L\cdot y=\lambda y,\ L=L(u)=\partial^2+u, \tag{6.32}$$

has the following Lenard operator pair:

$$K=\tfrac{1}{4}\partial^3+\tfrac{1}{2}(\partial u+u\partial),\quad J=\partial. \tag{6.33}$$

Apparently, $L_*(\xi)=\xi,\forall\xi\in\mathcal{B}$. Setting $u=-\phi_{xx}/\phi$ yields the product-form of K and its inverse,

$$\begin{aligned}K&=\tfrac{1}{4}\phi^{-2}\,\partial\phi^2\,\partial\phi^2\,\partial\phi^{-2},\\ K^{-1}&=4\phi^2\,\partial^{-1}\phi^{-2}\,\partial^{-1}\phi^{-2}\,\partial^{-1}\phi^2.\end{aligned} \tag{6.34}$$

Let $M,\ \tilde{M}\in\mathcal{B}$ be two arbitrarily given functions. Then the positive order and negative order generators,

$$G_0=M^{(-1)};\ \ G_{-1}=K^{-1}\cdot\tilde{M}=4\phi^2\,\partial^{-1}\phi^{-2}\,\partial^{-1}\phi^{-2}\,\partial^{-1}\cdot(\phi^2\tilde{M}), \tag{6.35}$$

leads to the KdV category of NLEEs

$$u_t=J\cdot G_m,\ \ m\in Z,\ \ G_m=\begin{cases}\mathcal{L}^m\cdot G_0, & m\geqslant 0,\\ \mathcal{L}^{m+1}\cdot G_{-1}, & m<0,\end{cases} \tag{6.36}$$

where the recursion operator $\mathcal{L}$ is given by

$$\mathcal{L}=J^{-1}K=\tfrac{1}{4}\partial^2+\tfrac{1}{2}(u+\partial^{-1}u\partial)=\tfrac{1}{4}\partial^{-1}\phi^{-2}\,\partial\phi^2\,\partial\phi^2\,\partial\phi^{-2},$$

and its inverse is

$$\mathcal{L}^{-1}=4\phi^2\,\partial^{-1}\phi^{-2}\,\partial^{-1}\phi^{-2}\,\partial^{-1}\phi^2\,\partial.$$

For an arbitrary $G\in\mathcal{B}$, the operator equation $[V,L]=L_*(K\cdot G)-L_*(J\cdot G)L$ has the following operator solution:

$$V=V(G)=-\tfrac{1}{4}G_x+\tfrac{1}{2}G\partial, \tag{6.37}$$

which implies that the KdV category (6.36) possesses the GLR,

$$L_t=[W_m,\ L]+\bar{M}L^m,\ \ m\in Z,\ \bar{M}=\begin{cases}M, & m\geqslant 0,\\ \tilde{M}, & m<0,\end{cases} \tag{6.38}$$

with the GLO

$$W_m=\sum V(G_j)L^{m-j-1}. \tag{6.39}$$

Here $V(G_j)$ is determined by (6.37) with $G=G_j=\mathcal{L}^j\cdot G_0,\ j\geqslant 0$ or $G=G_j=\mathcal{L}^{j+1}\cdot G_{-1},\ j<0$, $L=\partial^2+u=\phi^{-1}\,\partial\phi^{-2}\,\partial\phi^{-2}$, and $L^{-1}=\phi^2\,\partial^{-1}\phi^2\,\partial^{-1}\phi$.

In particular, we are concerned with the following reduction.

(i) With $M=4(u^{-1/2})_x$, $G_0=M^{(-1)}=4u^{-1/2}$, the positive order category of (6.36) reads as the well-known Harry–Dym hierarchy,

$$u_t=J\mathcal{L}^m\cdot 4u^{-1/2},\ \ m=0,1,2,\ldots. \tag{6.40}$$

With $m=1$, Eq. (6.40) yields the Harry–Dym equation,

$$u_t=\left(\frac{1}{\sqrt{u}}\right)_{xxx}, \tag{6.41}$$

which has now the GLR $L_t=[W_0,L]+4(u^{-1/2})_x L$ with $W_0=-(u^{-1/2})_x+2u^{-1/2}\partial$, and apparently belongs to the KdV category (6.36); with $m=2$, Eq. (6.40) yields a higher order Harry–Dym equation,

$$u_t=\tfrac{1}{4}\left(\frac{1}{\sqrt{u}}\right)_{4x}+u\left(\frac{1}{\sqrt{u}}\right)_{3x}+\tfrac{1}{2}u_x\left(\frac{1}{\sqrt{u}}\right)_{xx} \tag{6.42}$$

possessing the GLO $W_1=2u^{-1/2}\partial^3-(u^{-1/2})_x\partial^2+\frac{1}{2}((u^{-1/2})_{xx}+4u^{-1/2})\partial-\frac{1}{4}(u^{-1/2})_{xxx}+u^{-1/2}u_x$.

So, we have obtained an interesting fact: *the Harry–Dym equation (6.41) can be included in the KdV category (6.36) with the generalized Lax operator*. Similar to the process of the Burgers' case, we can also have many reduced hierarchies both positive and negative from Eq. (6.36).

6.3: AKNS case. The ZS-AKNS spectral problem,[35,36]

$$L\cdot y=\lambda y,\quad L=L(q,r)=i\begin{pmatrix}\partial & -q\\ r & -\partial\end{pmatrix},\quad y=\begin{pmatrix}y_1\\ y_2\end{pmatrix}, \tag{6.43}$$

has its Lenard's operators pair,

$$K=\begin{pmatrix}q\,\partial^{-1}q & \frac{1}{2}\partial-q\,\partial^{-1}r\\ \frac{1}{2}\partial-r\,\partial^{-1}q & r\,\partial^{-1}r\end{pmatrix},\quad J=i\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{6.44}$$

Apparently,

$$L_*(\xi)=\begin{pmatrix}0 & -i\xi_1\\ i\xi_2 & 0\end{pmatrix},\quad \xi=(\xi_1,\xi_2)^T\in\mathcal{B}^2, \tag{6.45}$$

is an injective homomorphism.

Equation (6.44) gives the recursion operator

$$\mathcal{L}=J^{-1}K=\frac{1}{2}i\begin{pmatrix}-\partial+2r\,\partial^{-1}q & -2r\,\partial^{-1}r\\ 2q\,\partial^{-1}q & \partial-2q\,\partial^{-1}r\end{pmatrix}. \tag{6.46}$$

Choosing two functions θ, $\sigma\in C^\infty(R)$ satisfying $\theta_x=\frac{1}{2}\theta^2+r^{-1}r_x\theta-2qr$, $\sigma_x=\frac{1}{2}\sigma^2+q^{-1}q_x\sigma-2qr$, leads to the inverse of $\mathcal{L}$,

$$\mathcal{L}^{-1}=K^{-1}J=-2i\begin{pmatrix}-\mathcal{E}(\partial r^{-1}\,\partial r^{-1}-2qr^{-1}) & -2\mathcal{E}\\ 2\mathcal{F} & \mathcal{F}(\partial q^{-1}\,\partial q^{-1}-2rq^{-1})\end{pmatrix}, \tag{6.47}$$

where $\mathcal{E}$, $\mathcal{F}$ denote the following two operators:

$$\mathcal{E}=e^{-\theta^{(-1)}}\,\partial^{-1}e^{\theta^{(-1)}}\,r\,\partial^{-1}r\,e^{\theta^{(-1)}}\,\partial^{-1}e^{-\theta^{(-1)}},\quad \mathcal{F}=e^{-\sigma^{(-1)}}\,\partial^{-1}e^{\sigma^{(-1)}}\,q\,\partial^{-1}q\,e^{\sigma^{(-1)}}\,\partial^{-1}e^{-\sigma^{(-1)}}. \tag{6.48}$$

Let $A,B,C,D\in\mathcal{B}$ be four arbitrarily given functions; then iff

$$M=\begin{pmatrix}0 & -B\\ -A & 0\end{pmatrix},\quad \tilde{M}=\begin{pmatrix}0 & -D\\ -C & 0\end{pmatrix}, \tag{6.49}$$

the operator equations $L_*(J\cdot G_0)=M,\ L_*(K\cdot G_{-1})=\tilde{M}$ give the positive order and negative order generators (function vectors),

$$G_0=\begin{pmatrix}A\\B\end{pmatrix},\quad G_{-1}=-2i\begin{pmatrix}-\mathcal{E}\cdot(\partial r^{-1}\cdot(r^{-1}C)_x-2qr^{-1}C+2D)\\ \mathcal{F}\cdot(\partial q^{-1}\cdot(q^{-1}D)_x-2rq^{-1}D+2C)\end{pmatrix}, \tag{6.50}$$

which directly leads to the AKNS category of NLEEs:

$$\begin{pmatrix}q\\r\end{pmatrix}_t=\begin{cases}J\mathcal{L}^m\cdot(A,B)^T, & m=0,1,2,...,\\ J\mathcal{L}^m\cdot(C,D)^T, & m=-1,-2,\dots,\end{cases} \tag{6.51}$$

where $J,\ \mathcal{L}$ and $\mathcal{L}^{-1}$ are defined by (6.44), (6.46) and (6.47), respectively.

For an arbitrarily given $G=(G^{[1]},G^{[2]})^T\in\mathcal{B}^2$, the operator equation $[V,L]=L_*(K\cdot G)-L_*(J\cdot G)L$ has the solution

$$V=V(G)=\frac{1}{2}\begin{pmatrix}-(rG^{[2]}-qG^{[1]})^{(-1)} & G^{[2]}\\ G^{[1]} & (rG^{[2]}-qG^{[1]})^{(-1)}\end{pmatrix}, \tag{6.52}$$

which is obviously a function matrix. Thus, the AKNS category (6.51‘) has the GLR:

$$L_t=[W_m,L]+\overline{M}L^m,\quad m\in Z, \tag{6.53}$$

$$\overline{M}=\begin{cases}\begin{pmatrix}0 & B\\ A & 0\end{pmatrix}, & m\geqslant 0,\\ \begin{pmatrix}0 & D\\ C & 0\end{pmatrix}, & m<0,\end{cases}$$

with the GLO

$$W_m=\sum V(G_j)L^{m-j-1},\quad m\in Z. \tag{6.54}$$

Here $V(G_j)$ is given by (6.52) with $G=G_j=\mathcal{L}^j\cdot(A,B)^T,\ j\geqslant 0$ or $\mathcal{L}^j\cdot(C,D)^T,\ j<0$, L is defined by (6.43), and its inverse L^{-1} is determined by

$$L^{-1}=i\begin{pmatrix}\mathcal{S}\,\partial q^{-1} & -\mathcal{S}\\ -\mathcal{T} & \mathcal{T}\partial r^{-1}\end{pmatrix}, \tag{6.55}$$

with the operators $\mathcal{S}=e^{-\rho^{(-1)}}\partial^{-1}e^{2\rho^{(-1)}}q\,\partial^{-1}e^{-\rho^{(-1)}},\mathcal{T}=e^{-\mu^{(-1)}}\partial^{-1}e^{2\mu^{(-1)}}r\,\partial^{-1}e^{-\mu^{(-1)}}$, where ρ and μ are two functions satisfying $\rho_x=\rho^2+q^{-1}q_x\rho-qr,\ \mu_x=\mu^2+r^{-1}r_x\mu-qr$.

Here, we omit the reductions and the r-matrix representation of the AKNS category (6.51).

6.4: WKI (Wadati–Konno–Ichikowa) case. The WKI spectral problem,[37]

$$L\cdot y=\lambda y,\ L=L(q,r)=\frac{1}{1-qr}\begin{pmatrix}i & -q\\ -r & -i\end{pmatrix}\partial,\ y=\begin{pmatrix}y_1\\y_2\end{pmatrix}, \tag{6.56}$$

has the following Lenard's operators pair:

$$K=\frac{1}{2i}\begin{pmatrix}-\frac{1}{2}\partial^2\frac{q}{p}\partial^{-1}\frac{q}{p}\partial^2 & \partial^3+\frac{1}{2}\partial^2\frac{q}{p}\partial^{-1}\frac{r}{p}\partial^2\\ \partial^3+\frac{1}{2}\partial^2\frac{r}{p}\partial^{-1}\frac{q}{p}\partial^2 & -\frac{1}{2}\partial^2\frac{r}{p}\partial^{-1}\frac{r}{p}\partial^2\end{pmatrix}, \tag{6.57}$$

$$J=\begin{pmatrix} 0 & -\partial^2 \\ \partial^2 & 0 \end{pmatrix}, \quad p=\sqrt{1-qr}, \tag{6.58}$$

which yields the recursion operator $\mathcal{L}=J^{-1}K$

$$\mathcal{L}=\frac{1}{2i}\begin{pmatrix} \partial+\frac{r}{2p}\partial^{-1}\frac{q}{p}\partial^2 & -\frac{r}{2p}\partial^{-1}\frac{r}{p}\partial^2 \\ \frac{q}{2p}\partial^{-1}\frac{q}{p}\partial^2 & -\partial-\frac{q}{2p}\partial^{-1}\frac{r}{p}\partial^2 \end{pmatrix}. \tag{6.59}$$

Apparently, the Gateaux derivative operator $L_*(\xi)$ of the spectral operator L in the direction $\xi=(\xi_1,\xi_2)^T\in\mathcal{B}^2$ is

$$L_*(\xi)=\frac{1}{1-qr}\begin{pmatrix} q\xi_2 & -i\xi_1 \\ i\xi_2 & r\xi_1 \end{pmatrix}L, \tag{6.60}$$

which is an injective homomorphism.

Through lengthy calculations, one can obtain the invertible operators of L, J, K and $\mathcal{L}$:

$$L^{-1}=\begin{pmatrix} -i\partial^{-1} & \partial^{-1}q \\ \partial^{-1}r & i\partial^{-1} \end{pmatrix}, \tag{6.61}$$

$$J^{-1}=\begin{pmatrix} 0 & \partial^{-2} \\ -\partial^{-2} & 0 \end{pmatrix}, \tag{6.62}$$

$$K^{-1}=2i\begin{pmatrix} \frac{1}{2}\partial^{-1}r\,\partial^{-1}r\,\partial^{-1} & \partial^{-3}-\frac{1}{2}\partial^{-1}r\,\partial^{-1}q\,\partial^{-1} \\ \partial^{-3}-\frac{1}{2}\partial^{-1}q\,\partial^{-1}r\,\partial^{-1} & \frac{1}{2}\partial^{-1}q\,\partial^{-1}q\,\partial^{-1} \end{pmatrix}, \tag{6.63}$$

$$\mathcal{L}^{-1}=2i\begin{pmatrix} \partial^{-1}-\frac{1}{2}\partial^{-1}r\,\partial^{-1}q\,\partial & -\frac{1}{2}\partial^{-1}r\,\partial^{-1}r\,\partial \\ \frac{1}{2}\partial^{-1}q\,\partial^{-1}q\,\partial & -\partial^{-1}+\frac{1}{2}\partial^{-1}q\,\partial^{-1}r\,\partial \end{pmatrix}. \tag{6.64}$$

Let A, B, C, D be four arbitrarily given C^∞-functions; then iff

$$M=\frac{1}{1-qr}\begin{pmatrix} qA & iB \\ iA & -rB \end{pmatrix}L, \quad \widetilde{M}=\frac{1}{1-qr}\begin{pmatrix} qC & iD \\ iC & -rD \end{pmatrix}L, \tag{6.65}$$

the operator equations $L_*(J\cdot G_0)=M$, $L_*(K\cdot G_{-1})=\widetilde{M}$ have the following solutions:

$$G_0=\begin{pmatrix} A^{(-2)} \\ B^{(-2)} \end{pmatrix}, \tag{6.66}$$

$$G_{-1}=\begin{pmatrix} 2iC^{(-3)}-i\,\partial^{-1}r\,\partial^{-1}\cdot(rD^{(-1)}+qC^{(-1)}) \\ -2iD^{(-3)}+i\,\partial^{-1}q\,\partial^{-1}\cdot(rD^{(-1)}+qC^{(-1)}) \end{pmatrix}, \tag{6.67}$$

which directly yields the WKI category of NLEEs:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = J \cdot G_m, \quad m \in Z, \tag{6.68}$$

$$G_m = \begin{cases} \mathcal{L}^m \cdot (A^{(-2)}, B^{(-2)})^T, & m = 0,1,2,..., \\ \mathcal{L}^m \cdot (C^{(-2)}, D^{(-2)})^T, & m = -1,-2,..., \end{cases} \tag{6.69}$$

where J, $\mathcal{L}$ and $\mathcal{L}^{-1}$ are defined by (6.58), (6.59) and (6.64), respectively.

For any given $G = (G^{[1]}, G^{[2]})^T \in \mathcal{B}^2$, the equation $[V, L] = L_*(K \cdot G)L^{-1} - L_*(J \cdot G)$ has the following operator solution:

$$V = V(G) = \begin{pmatrix} 0 & \overline{B} \\ \overline{C} & 0 \end{pmatrix} + \overline{A} \begin{pmatrix} -i & q \\ r & i \end{pmatrix} L, \tag{6.70}$$

where $\overline{A}$, $\overline{B}$, $\overline{C}$ are the following three functions given by

$$\overline{A} = \overline{A}(G) = \frac{1}{2p} \left(\frac{q}{p} G_{xx}^{[1]} - \frac{r}{p} G_{xx}^{[2]} \right)^{(-1)}, \quad p = \sqrt{1 - qr},$$

$$\overline{B} = \overline{B}(G) = \frac{1}{4i} \left(2 G_{xx}^{[2]} - \partial \frac{q}{p} \cdot \left(\frac{q}{p} G_{xx}^{[1]} - \frac{r}{p} G_{xx}^{[2]} \right)^{(-1)} \right),$$

$$\overline{C} = \overline{C}(G) = \frac{1}{4i} \left(2 G_{xx}^{[1]} + \partial \frac{r}{p} \cdot \left(\frac{q}{p} G_{xx}^{[1]} - \frac{r}{p} G_{xx}^{[2]} \right)^{(-1)} \right).$$

Thus, the WKI category (6.68) has the GLR:

$$L_t = [W_m, L] + \overline{M} L^{m+1}, \quad m \in Z, \tag{6.71}$$

$$\overline{M} = \begin{cases} \dfrac{1}{1 - qr} \begin{pmatrix} qA & iB \\ iA & -rB \end{pmatrix}, & m \geqslant 0, \\ \dfrac{1}{1 - qr} \begin{pmatrix} qC & iD \\ iC & -rD \end{pmatrix}, & m < 0, \end{cases} \tag{6.72}$$

with the GLO

$$W_m = \sum V(G_j) L^{m-j}, \quad m \in Z. \tag{6.73}$$

Here L, L^{-1} and $V(G_j)$ are given by (6.56), (6.61) and (6.70) with $G = G_j$ defined by (6.69), respectively.

6.5: The following spectral problem:

$$L \cdot y = \lambda y, \quad L = L(u) = \frac{1}{u} \begin{pmatrix} i & 1 - u \\ 1 & -i \end{pmatrix} \partial, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{6.74}$$

yields its Lenard operators pair,

$$K = \partial^3, \quad J = -2(\partial u + u \partial).$$

The Gateaux derivative operator $L_*(\xi)$ of the spectral operator L in the direction $\xi \in \mathcal{B}$ is

$$L_*(\xi)=\frac{\xi}{u^2}\begin{pmatrix} -i & -1 \\ -1 & i \end{pmatrix}\partial=\frac{\xi}{u}\begin{pmatrix} 0 & -i \\ 0 & -1 \end{pmatrix}L. \tag{6.75}$$

Apparently, L_* is a homomorphism and $L_*(\xi)=0 \Leftrightarrow \xi=0$.

In the category derived from Eq. (6.75), we can obtain the Harry–Dym hierarchy as well as some new integrable equations. For example, the following nonlinear equation:

$$v_{xt_{-2}}=2vv_{xx}+v_x^2 \tag{6.76}$$

is a new integrable equation with many unknown physical properties. In fact, this equation is included in an isospectral $(\lambda_{t_{-2}}=0)$ negative order hierarchy of (6.74), and its standard Lax operator is

$$W_{-2}=-V(G_{-2})L^{-1}-V(G_{-1})L^{-3},$$

where $V(G_j)$ $(j=-2,-1)$ is given by

$$V=V(G)=G_{xx}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}+G_x\begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix}L+2G\begin{pmatrix} -i & u-1 \\ -1 & i \end{pmatrix}L^2, \tag{6.77}$$

with $G=G_{-2}=-v^{(-1)}$, $G_{-1}=\frac{1}{2}$, respectively, and L^{-1} is the inverse of L, given by

$$L^{-1}=\begin{pmatrix} -i\partial^{-1} & \partial^{-1}v_x-\partial^{-1} \\ -\partial^{-1} & i\partial^{-1} \end{pmatrix}. \tag{6.78}$$

We will give in detail some reductions for the latter four spectral problems in a later paper.

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