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THE ECONOMICS OF COUNTERFEITING

BY ELENA QUERCIOLI AND LONES SMITH[1]

We develop a strategic theory of counterfeiting as a multi-market large game. Bad guys choose whether to counterfeit, and what quality to produce. Opposing them is a continuum of good guys who select a costly verification effort. In equilibrium, counterfeiters produce better quality at higher notes, but verifiers try sufficiently harder that verification still improves. We develop a graphical framework for deducing comparative statics. Passed and counterfeiting rates vanish for low and high notes. Our predictions are consistent with time series and cross-sectional patterns in a unique data set assembled largely from the Secret Service.

KEYWORDS: Passed money, seized money, cat and mouse game, hot potato game, implicit markets, supermodular games.

1. INTRODUCTION

COUNTERFEITING IS A MAJOR ECONOMIC PROBLEM, called "the world's fastest growing crime wave" (Phillips (2005)). This paper explores the counterfeiting of financial documents such as money, checks, or money orders. The domestic losses from check fraud may well have exceeded \$20 billion in 2003.[2] About one in 10,000 U.S. dollar notes is counterfeit, with the domestic public losing \$80 million in 2011, more than doubling since 2003. The indirect costs of counterfeiting are much greater, since it forces a re-design of U.S. currency every 7–10 years. As well, many costs are borne by the public checking the authenticity of their money.[3]

When we write counterfeit money (or checks), we have in mind two manifestations of it. *Seized* money is confiscated before entering circulation. *Passed*

[1]The paper began in 2005 as "Counterfeit, \$\$\$" as a model just of the hot potato game; the cat and mouse game was developed while Lones visited the Cowles Foundation at Yale for Spring 2006. We have profited from the insights and/or data of Charles Bruce (Director, National Check Fraud Center), Pierre Duguay (Deputy Governor, Bank of Canada), Antti Heinonen (European Central Bank, Counterfeit Deterrence Chairman), Ruth Judson (Federal Reserve), John Mackenzie (counterfeit specialist, Bank of Canada), Stephen Morris, and Lorelei Pagano (former Special Agent, Secret Service), as well as comments at I.G.I.E.R. (Bocconi), the 2006 Bonn Matching Conference, the 2006 SED in Vancouver, the Cleveland Fed, Tulane, Michigan, the Bank of Canada, the 2007 NBER-NSF GE conference at Northwestern, the 2008 Midwest Theory Conference in Columbus, the 2011 Yale Summer Theory Conference, Maryland, Pittsburgh, Stanford, Georgetown, Wisconsin, the St. Louis Fed, Western, Melbourne, the Philadelphia Fed, Chicago, NYU, and Princeton. We thank the Editor and two referees for comments that reshaped the paper entirely. Lones thanks the NSF for funding (Grant 0550014).

[2]Data here are sketchy. This estimate owes to a widely cited Nilson Report (www.nilsonreport.com).

[3]Arguably, the \$500M budget of the Bureau of Printing and Engraving, and maybe \$1B of the Secret Service and Treasury budgets owe to anti-counterfeiting. Also, there is a large private sector industry.

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money is found at a later stage, and leads to losses by the public. Whereas counterfeit money in toto is a stock variable, seized and passed are flows. We have gathered an original data set mostly from the Secret Service on counterfeit U.S. money over time and across denominations. All seized and passed counterfeit currency in the United States must be handed over to the Secret Service, and so very good data are potentially available. Our data include all 5,594,062 seized and 8,541,972 passed counterfeit notes in the United States for the years 1995–2004, supplemented by aggregate data for 2005–2007 and older published data. We have organized our data using two measures—the *passed rate*, or passed over circulation, and something we call the *seized-passed ratio*. Seized money is a volatile series (see Figure 3 below), as it owes to random, maybe large, counterfeiting discoveries, and also consists purely of contemporaneous counterfeit money. By contrast, passed money has been found by thousands of individuals, and may also have long been circulating.

Counterfeiting induces two distinct linked conflicts: first, counterfeiters against verifiers and police, and second, verifiers against each other. The extant literature focuses on the police-counterfeiter conflict; but this can only explain seized money. To understand passed money—namely, the source of harm to the public—we must also consider the conflict among passers.

We assume that notes of a single denomination change hands every period. Some fake notes pass into circulation and a larger collateral game then emerges: Good guys unwittingly pass on fakes they acquire in an anonymous random matching exchange setting. We explore the steady-state of this stock-flow model. To wit, (a) the counterfeits produced balance those seized and passed, and (b) those passing into circulation are matched by those found. Hence, the *passing fraction* of new fakes into circulation is the ratio of passed to seized plus passed, and the passed rate is the counterfeiting rate times the discovery rate of fake money.

We next document some key counterfeiting facts that motivate our theory. We first consider counterfeiting across denominations (Facts 1–3), and then explore the time series picture (Facts 4–6).

FACT 1: *One plus the seized-passed ratio* (a) *rises in the note*, *but* (b) *far less than proportionately*.

This clear trend holds in the U.S. denominations \$1, \$5, ..., \$100 over the samples of millions of passed and seized notes, as well as in Canada's six paper notes.[4] Slopes in the log-log diagram of Figure 1, that is, elasticities, are positive but well below 1 (0.18, on average).

[4]For Canada, from 1980 to 2005, the counterfeit-passed ratios are 0.095, 0.145, 0.161, 0.184, 0.202, and 3.054 for (respectively) \$5, \$10, \$20, \$50, \$100, and \$1000. The \$1000 note was ended in 2000.

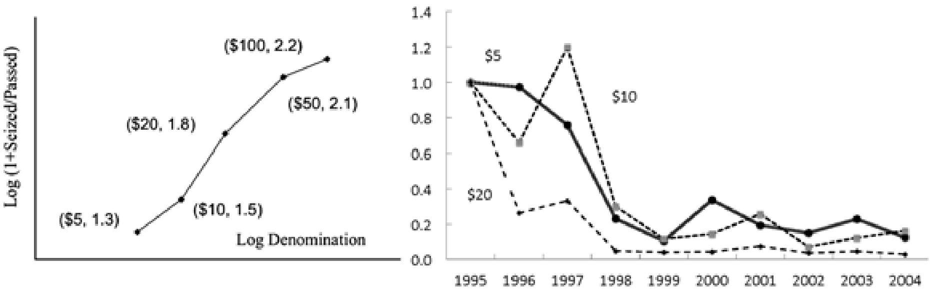

FIGURE 1.—Seized-passed ratio by denominations. At left is $1 + S/P$ for 1995–2008 for non–Colombian counterfeits in the United States (in a log-log plot). Data for the \$1 note in 1998 and 2005–2008 average 1.23. For this log-log graph, slopes are elasticities—positive and below 1. At right, we plot seized-passed ratios for the \$5–\$20 notes from 1995 to 2004, each normalized to 1 for 1995. The sample includes almost ten million passed and five million seized notes.

Since one plus the seized-passed ratio is the reciprocal of the passing fraction, it is a risk measure for criminals. But risk should rise with reward, and so passing higher notes should be a riskier endeavor, that is, one plus the seized-passed ratio should rise in the denomination. Our theory explains why this measure rises less than proportionately, as Fact 1 also claims.

FACT 2: *The passed rate* (a) *is tiny at* \$1; (b) *shoots up until* \$20; *and* (c) *falls at the highest notes*.

The passed rate is also a risk measure for transactors. As seen in the U.S. dollar and euro data, this risk naturally rises with reward at low notes (parts (a) and (b)). The left panel of Figure 2 plots the average fractions of passed

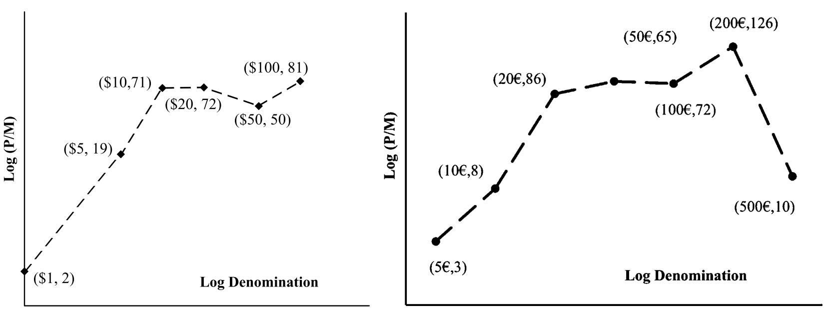

FIGURE 2.—Passed over circulation (P/M), dollar and euro. At left and right, respectively, are the average ratios of passed domestic counterfeit notes to the June circulation levels, for 1990–2007 (\$1 has some missing years), and for the euro, 2002–2013. Labels are $10^6 \times (P/M)$.

TABLE I

FRACTION OF NON-COLOMBIAN PASSED NOTES DIGITALLY PRODUCED, 1995–2004[a]

Note	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	Avg.
\$5	0.250	0.306	0.807	0.851	0.962	0.974	0.987	0.982	0.976	0.981	0.81
\$10	0.041	0.097	0.511	0.854	0.910	0.913	0.962	0.963	0.972	0.978	0.72
\$20	0.139	0.295	0.642	0.882	0.902	0.926	0.929	0.961	0.976	0.985	0.77
\$50	0.305	0.414	0.642	0.849	0.905	0.925	0.945	0.848	0.839	0.872	0.75
\$100	0.121	0.154	0.348	0.531	0.460	0.608	0.564	0.457	0.472	0.552	0.43

[a]Cheaper digital methods of production skyrocketed in 1995–1998, focused on lower notes.

notes.[5,6] The counterintuitive declining rate at high notes (part (c)) is not realized in the U.S. data. But the euro offers two higher value notes, and the passed rate of the 500 euro note is less than 10% that of the 200 euro note.

FACT 3: *Counterfeit quality rises in the note*.

Modern digitally produced counterfeits, using scanners and color printers, are cheaper to make. As Table I depicts, the fraction of such fakes falls in the note value. Judson and Porter (2003) found that 73.6% of passed \$100 bills were high quality *circulars*, but only 19.2% of \$50 bills, and less than 3% of all others. The "Supernote" (circular 14342) is the best quality counterfeit ever. North Korea made this \$100 note from bleached \$1 notes, with the intaglio printing process used by the Bureau of Engraving and Printing. Banks cannot detect it.

Turning to time series facts, there has been a sea change in both seized and passed money.

FACT 4: *Since* 1970, (a) *the seized-passed ratio is down* 90%, *and* (b) *the passed rate is up* 75%.

Historically, seized vastly exceeded passed; nowadays, most counterfeit money is passed, as the fraction passing into circulation has skyrocketed roughly from 10% to 80% (Figure 3). Specifically, starting in 1986, and then accelerating in 1995, the seized-passed ratio began to tumble. One brief time window 1995–1998 witnessed a stunning 80% drop in the seized-passed ratios for \$5, \$10, and \$20 notes (right panel of Figure 1). While the passed rates

[5]These ratios per million have averaged 1.96, 19.46, 71.21, 72.03, 49.94, 81.43, respectively.

[6]The common claim that the most counterfeited note domestically on an annualized basis is the \$20 is false over our time span. Accounting for the higher velocity of the \$20, on a per-transaction basis (the relevant measure for decision-making), the \$100 note is unambiguously the most counterfeited note.

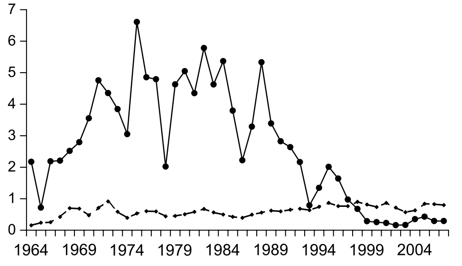

FIGURE 3.—U.S.A. passed and seized, 1964–2007. The solid line is seized money, and the dashed line passed money—per thousand dollars of circulation (all denominations). From 1970 to 1985, about 90% of counterfeit money was seized. The reverse holds (about 20%) for 2000–2007. The plummeting quotient of the solid and dashed lines is the seized-passed ratio.

have proven much more stable, they have still nearly doubled since 1967, and are up a third since 1990.

Fact 4 asserts that the risk measures for criminals and transactors have moved in opposite directions: The seized-passed ratio has plummeted, while the passed rate has shot up. We offer a technological explanation for these contrasting trends:

FACT 5: *The fraction of notes produced using inexpensive digital methods leapt up in* 1994–1998.

The introduction of digital means of counterfeiting was very rapid, focused in 1994–1998. This change is the major causal force behind our time series dynamics.

FACT 6: *Canada's introduction of color notes temporarily nearly stopped counterfeiting*.

Canada almost eliminated passed money as it colorized each note in the 1970s (Table II). This fact speaks to the critical importance of costly inattention, as color is easily perceived. We thus center our model on *endogenous verification*—a new assumption in the money literature. Our model confronts

TABLE II

"SCENES OF CANADA," MULTICOLORED SERIES, 1971–1976

Note	\$5	\$10	\$20	\$50	\$100
Passed rate peak year	1973	1971	1970	1975	1976
Colored notes introduction	12/72	11/71	6/70	5/75	5/76
% fall in passed rate	99	99	99	98	93

good guys with a risky choice: Since they might be purposefully or unwittingly handed a fake note, they trade off uncertain losses from counterfeiting against certain costs of greater scrutiny. Good guys expend effort screening out passed counterfeit money handed them; more effort yields a higher *verification rate*, or chance of catching fakes.

The verification rate is an endogenous quantity, reflecting the rival efforts of bad guys to fool victims, and of good guys to avoid being fooled. Higher quality fakes cost more, but better deceive good guys, and so pass more often. Our verification function has diminishing returns to effort and quality. With free entry into counterfeiting, profits must vanish; this determines the equilibrium effort. In particular, effort must rise in the note, and vanish at the lowest notes.

The collateral pairwise interaction of good guys is a game of strategic complements, since the more others verify, the more one should verify to protect oneself. Effort in the unique symmetric Nash equilibrium rises in the counterfeiting rate, vanishing if the rate does. We invert this relation, computing the equilibrium counterfeiting rate as a derived demand. Since effort vanishes approaching the lowest notes, so does the counterfeiting rate. The passed rate behaves similarly to counterfeit money. We use this to deduce Facts 2(a) and (b).

Next, turning to the optimization among counterfeiters, greater quality counterfeit notes frustrate verification efforts *at the margin*. By symmetry of a cross partial derivative, the counterfeiters' marginal returns to quality thus rise in effort. Since higher notes command higher verification efforts, they elicit a higher quality response; this yields Fact 3. Also, we argue that the rising quality ultimately depresses the passed rate at the highest notes (Fact 2(c)).

With constant quality, zero profits requires the passing fraction moves inversely to the note. A fake \$10 passes half as often as a fake \$5, and so one plus the seized-passed ratio has unit slope. But since costly quality optimally rises in the note, the passing fraction falls more slowly—yielding the less than unit slope in Fact 1(b). Theorem 2 argues that effort rises in the note proportionately faster than quality, raising the verification at higher notes (Fact 1(a)).

Theorem 2 explains the time series Fact 4 using the technological explanation Fact 5.

Relationship to the Literature. Existing counterfeiting work relies on a general equilibrium value of money, and so is unrelated.[7] In this paper, rather than assume that money is a priced asset, we have inferred the counterfeiting

[7]In Green and Weber (1996), only government agents can descry fake notes, whose stock is exogenous. Williamson (2002) assumes counterfeits of private bank notes are found with fixed chance; counterfeiting does not occur in most of his equilibria. Verification is random and exogenous in Nosal and Wallace (2007), who find no counterfeiting in equilibrium with a high counterfeiting cost. Li and Rocheteau (2011) later questioned this. In Banerjee and Maskin (1996), verification is either perfect or worthless for a good, but not a choice variable.

rate[8] via a new decision margin—costly variable intensity verification. If we *included general equilibrium effects*, they would add nothing to our predictions for passed and seized money—for they would only discount prices infinitesimally, given the 1 in 10,000 counterfeiting rates. At higher counterfeiting rates, or for counterfeit goods, costly verification and general equilibrium would both matter.

Modeling verification itself is not new: Williamson and Wright (1994) assumed transactors observe *fixed signals* of the authenticity of money *after* acquiring it. Our verification efforts are endogenous, and crucially occur *before* accepting money. Exogenous attention that does not respond to the payoff stakes cannot rationalize the facts of counterfeiting that we document.

The model and early analysis are in Sections 2–3. We derive theoretical predictions in Section 4, and compare them to the data in Section 5, including patterns in counterfeits found by Federal Reserve Banks that offer more evidence. The Appendix proves existence, uniqueness, and curve shapes.

2. THE MODEL

2.1. *The Pairwise Matching Model*

The story unfolds in periods $1, 2, 3, \ldots$. There are two types of risk neutral maximizing agents: a continuum unit mass of homogeneous *good guys*, and an infinitely elastic supply of homogeneous bad guys, who are potential counterfeiters. We focus on notes of fixed value $\Delta > 0$; genuine Δ notes are in fixed supply $M > 0$.

Money changes hands exogenously, from counterfeiters to good guys, among good guys, and between banks and good guys. Banks are a pass-through, returning notes to good guys. At the end of Section 5, we assume they swap a given fraction of notes with the Federal Reserve.

Each period, good guys with a note are randomly matched to those without a note, or to banks. And good guys without a note are randomly matched to good guys with a note, or to banks, or to bad guys. We explore a *steady-state*, with density measures of all transactions unchanged each period. But since counterfeiters—although rare—spend and never acquire notes, and we assume that bank withdrawals balance bank deposits, good guys with a note are on the long side of the market; they meet a trading partner with chance slightly less than 1.[9]

[8]In Knowles, Persico, and Todd (2001), a police search chance incentivizes a decision to carry drugs. By contrast, bad guys in our model have both an extensive margin (whether to counterfeiting), and an intensive one (quality). They have no analogue of our good guys, who respond both to quality and the counterfeiting rate.

[9]We do not model the rationing; we assume that good guys acquiring a note expect to soon spend it.

Good guys cannot distinguish other good guys from counterfeiters. They know that they may be handed a fake note. If they see a fake note, they reject it, and it becomes worthless *passed money*[10]—it is withdrawn from circulation, the passer losing its face value Δ. The *counterfeiting rate* $\kappa \in [0, 1)$ is the fraction of notes that are fake, from all transactors.[11]

Good guys expend *effort* $e \geq 0$ scrutinizing any note before accepting it. Real notes are never mistaken for counterfeits. A fake note is noticed with chance $v \in [0, 1]$, the *verification rate*. The *verification function* $v = V(e, q)$ intuitively rises in effort e and falls in *quality* $q > 0$. As quality is unobserved, good guys do not know the rate v, but infer it in equilibrium.

Good guys without notes next period become good guys with notes upon accepting a note. If matched, they go to a bank with chance $\beta \in (0, 1)$. Banks detect and confiscate fakes with fixed probability $\alpha \in (0, 1)$.[12] With chance $1 - \beta$, the good guy with a note meets a random transactor. So fake money is found in transactions at the *discovery rate* $\delta(v) = \beta\alpha + (1 - \beta)v$. The *passed rate* $p = \delta\kappa = P/M$ is the ratio of passed money to circulation for Δ notes.

Bad guys choose whether to enter, and if so, what quality q of notes to produce at *cost* $c(q)$. We assume a fixed expected production quantity $x \in (0, \infty)$.[13] Since verification efforts help the police, only an endogenous *passing fraction* $f(v) \leq 1 - v$ of production passes into circulation.[14] Intuitively, the first verifier catches a fraction v of notes, and police seize a share $1 - v - f(v)$. Criminals earn zero profits every period, net of legal penalty. As counterfeiters are invariably eventually caught,[15] and the stated penalty is constant across notes, we assume a *legal penalty* $\ell > 0$, that is, the expected punishment loss.

[10]Knowingly passing on fake currency is illegal by Title 18, Section 472 of the U.S. Criminal Code. We assume that no one engages in this crime of "uttering," seeking a "greater fool" to accept bad money.

[11]To wit, this is an average of a 100% counterfeiting rate from counterfeiters and a smaller counterfeiting rate from good guys, because at least one good guy has already verified circulating money.

[12]Bank tellers told us that they used protocols, and were not incentivized to look at higher notes more carefully. As evidence of $\alpha < 1$, ATMs dispense fake money (personal communication, John Mackenzie, Bank of Canada).

[13]Quantity is finite because the *marginal* distribution costs rise in output, as each passing attempt risks discovery: "If a counterfeiter goes out there and, you know, prints a million dollars, he's going to get caught right away because when you flood the market with that much fake currency, the Secret Service is going to be all over you very quickly."—Kersten (2005) [*All Things Considered*, July 23, 2005].

[14]The Secret Service also advises anyone receiving suspected counterfeit money: "Do not return it to the passer. Delay the passer if possible. Observe the passer's description."

[15]The Secret Service estimates that the conviction rate for counterfeiting arrests is close to 99%.

2.2. *Optimization and Equilibrium*

A verifier loses the value Δ of the note when three independent events simultaneously happen: (i) he is handed a fake note, and (ii) given such a fake note, his verifying efforts miss it, and (iii) given that he passes a fake note, the next agent catches it.[16] Good guys choose their effort e to minimize verification costs plus expected counterfeit losses next period,[17,18] taking as given the verification rate v and counterfeiting rate κ, that is,

$$e + \kappa\big(1 - V(e, q)\big)\delta(v)\Delta. \tag{1}$$

Since the model is in steady-state, we avoid time subscripts on all variables.

The *passing fraction* of fake notes into circulation is a smooth, falling function obeying $f(v) \le 1 - v$ and $f(0) > 0$. So perfect verification chokes off passing ($f(1) = 0$), and passing occurs if no one verifies. A counterfeiter cares about his quality because it lessens the verification rate. Counterfeiters maximize profits equal to expected revenues $f(v)x\Delta$ less costs $c(q) + \ell$, for an increasing convex cost function $c(q)$. Counterfeit profits thus equal

$$\Pi(e, q, \Delta) \equiv f\big(V(e, q)\big)x\Delta - c(q) - \ell. \tag{2}$$

Reflecting the decision margins of good and bad guys and the "rational expectations" verification rate, an *equilibrium* is a 4-tuple $(e^*, q^*, v^*, \kappa^*)$ such that: verification effort e^* minimizes costs (1) of good guys at the counterfeiting rate κ^*; quality q^* maximizes profits (2) given e^*; counterfeiting profits (2) vanish; and the verification rate is $v^* = V(e^*, q^*)$.

Solving four equilibrium equations in four unknowns is, in general, hard. But since the counterfeiters only supply notes, and never accept them, they do not care about the counterfeiting rate κ. So our equilibrium admits a block recursive structure, parsing into two anonymous pairwise-matching games: In the *cat and mouse game*, depicted in Figure 4 in Section 3.2, we solve for (v^*, q^*), using the bad guys' optimal quality q^* that maximizes profits (2) and selects a verification rate v^* for which they vanish. Next, we compute the verification rate via $v^* = V(e^*, q^*)$. Finally, in the *hot potato game*, we solve for κ^*: We use it inversely, finding the counterfeiting rate κ^* so that effort e^* solves the good guys' optimization (1) for the required q^* and rate v^*.

[16]As is the norm, we ignore technicalities of randomness and independence for a continuum of events, and assume simply that probabilities of individual events correspond to measures of aggregate events.

[17]While good guys without notes face a two-period optimization, we can simply reduce it to a static one. We assume that χ absorbs any discounting between periods in this simple optimization.

[18]If banks verify at a different rate than good guys, then the counterfeiting rate there will slightly differ. To simplify the analysis, we assume individuals use the same effort in all transactions.

2.3. *Model Assumptions*

We now make functional form assumptions that greatly simplify the analysis and offer discipline for the comparative statics predictions. We assume that V is homogeneous of degree zero, that is, doubling quality requires twice the effort to secure the same screening chance. Thus, an increasing *verification cost function* χ translates e and q into an endogenous rate v via the implicit relation $e = q\chi(v)$. So $V(e, q) \equiv \chi^{-1}(e/q)$ for all $e < q\chi(1)$, and $V(e, q) = 1$ for $e \geq q\chi(1)$. So verification is perfect for low qualities $q \leq e/\chi(1)$.

We assume that χ^{-1} obeys standard Inada conditions, and so its inverse χ obeys $\chi(0) = \chi'(0) = 0$ with $\chi' > 0$ and $\chi'' > 0$ for $v > 0$. When $v = V(e, q) < 1$, we may differentiate the identity $q\chi(V(e, q)) \equiv e$ to get $q\chi' V_q + \chi \equiv 0$ and $q\chi' V_e \equiv 1$. This yields the derivatives

$$V_e(e, q) = 1/q\chi'(v) > 0 \quad \text{and} \quad V_q(e, q) = -\chi(v)/q\chi'(v) < 0. \tag{3}$$

Twice differentiating $q\chi(V(e, q)) \equiv e$ yields $q\chi' V_{ee} + q\chi'' V_e^2 \equiv 0$, and so convexity of χ yields diminishing returns to effort: $q^2 V_{ee}(e, q) = -\chi''(v)/(\chi'(v))^3 < 0$. There are diminishing returns to quality in reducing verification when χ is strictly log-concave, so that $\chi''/\chi' < \chi'/\chi$:

$$q^2 V_{qq} = \frac{\chi}{\chi'} + \left(\frac{\chi}{\chi'}\right)^2 \left(\frac{\chi'}{\chi} - \frac{\chi''}{\chi'}\right) > 0 > \frac{\chi}{(\chi')^2}\left(\frac{\chi''}{\chi'} - \frac{\chi'}{\chi}\right) = q^2 V_{eq}. \tag{4}$$

We assume that $v\chi''(v)/\chi'(v) \geq 1$, and so $\chi'(v)/v$ is monotone, and the limit elasticity $\lim_{v\to 0} v\chi'(v)/\chi(v) \geq 2$ exists. Geometric functions $\chi(v) = v^B$ obey all assumptions if $B \geq 2$.

We assume that verification increasingly helps police, as captured by a convex passing fraction f. To limit this effect, f is strictly log-concave: $(\log f)'' = (f'/f)' < 0$, and therefore $f'(0) > -\infty$, with limit $f'(v)/f(v) \downarrow -\infty$ as $v \uparrow 1$. For example, if the police seize a constant fraction $\xi \in [0, 1)$ of fake notes, then $f(v) = (1 - \xi)(1 - v)$. Second, we jointly assume

$$\frac{vf''(v)}{f'(v)} + \frac{v\chi'(v)}{\chi(v)} \geq 1. \tag{5}$$

For instance, any quadratic passing fraction $f(v) = (1-v)(1-\gamma v)$ is monotone decreasing, convex, and log-concave when $0 \leq \gamma < 1$, and obeys $f(0) > 0 = f(1)$. With geometric costs $\chi(v) = v^B$, inequality (5) is slightly more restrictive, now requiring $\gamma \leq (2B - 1)/(2B + 1)$.

The human and physical capital cost $c(q)$ of the counterfeit quality q is smooth, with $c', c'' > 0$ for $q > 0$, $c(0) = 0$, and $c'(q) \to \infty$ as $q \uparrow \infty$. We assume a monotone cost of quality elasticity, and so a well-defined limit $\eta = \lim_{q\to 0} qc'(q)/c(q) \geq 2$:

$$(6) \qquad \left(\frac{qc'(q)}{c(q)}\right)' \geq 0.$$

3. EQUILIBRIUM DERIVATION

3.1. *The Hot Potato Game*

First, $e \leq q\chi(1)$, or else effort e is superfluous; therefore, the derivative V_e in (3) exists. Next, if $q = 0$, then perfect verification arises with negligible effort $e > 0$, and so we assume $q > 0$. Since $V_e(e, q) > 0$, the marginal product of effort in (1) rises in the verification rate v. One should examine a note more closely the more intensely it will be checked. So the minimizer $\hat{e}$ in (1) rises in v. Since benefits are linear in v, and χ is strictly convex with $\chi'(0) = 0$, any FOC solution with imperfect verification is a global minimum:

$$(7) \qquad 1 = \kappa V_e(\hat{e}, q)\delta(v)\Delta.$$

But identical good guys choose the same best response, that is, $\hat{v} = V(\hat{e}, q) = v$. As a product of weakly and strictly increasing functions, $\chi'(v)/\delta(v) = [\chi'(v)/v][v/\delta(v)]$ is increasing. The *derived counterfeiting demand curve* $v \mapsto \kappa$ slopes up, as fake notes are a bad, namely, the function

$$(8) \qquad \kappa(v, q) = \frac{q\chi'(v)}{\delta(v)\Delta} = \frac{\text{marginal verification cost}}{\text{discovery rate} \times \text{denomination}}.$$

So verification v is an equilibrium at quality q for the counterfeiting rate $\kappa(v, q)$. We depict the resulting *constant counterfeiting rate locus* $\bar{K}$, where (8) is fixed, in Figure 4. It is downward sloping because the derived counterfeit level (8) is increasing in verification and in quality.

3.2. *The Cat and Mouse Game*

Given free entry, expected profits (2) vanish. In (q, v)-space, this means:

$$(9) \qquad \Delta x f(v) - c(q) - \ell = 0.$$

Figure 4 depicts this *zero profit locus* $\bar{\Pi}$. It slopes down because a greater verification rate reduces expected revenue and so zero profits requires lower quality. It requires $\Delta > \underline{\Delta} \equiv \ell/(xf(0)) > 0$, for if not, counterfeiters lose money for any quality $q > 0$. All told, we deduce $e, q > 0$ in equilibrium. So $V(e, q)$ is smooth, and thus the quality FOC (10) holds:

$$(10) \qquad \Pi_q(e, q, \Delta) \equiv \Delta x f'\big(V(e, q)\big)V_q(e, q) - c'(q) = 0.$$

Absent police ($f(v) = 1 - v$), this *optimal quality locus* Q^* slopes up in Figure 4 since verification is submodular in effort and quality—as (4) implied: namely, a higher verification rate raises the marginal benefit of increasing quality.[19] We want to express (10) in (q, v)-space using equation (3):

$$-\Delta x f'(v)\frac{\chi(v)}{\chi'(v)} = qc'(q). \tag{11}$$

Taking logarithms in (9) and (11), we define $T(q) \equiv \log[c(q) + \ell]$ and $U(q) \equiv \log[qc'(q)]$ as well as $F(v) \equiv \log[xf(v)]$ and $G(v) \equiv \log[-xf'(v)\chi(v)/\chi'(v)]$. Then the $\bar{\Pi}$ and Q^* loci are

$$F(v) + \log \Delta = T(q) \quad \text{and} \quad G(v) + \log \Delta = U(q), \tag{12}$$

rephrasing (9) and (11) in additively separable forms. Now, if the passing fraction is linear $f(v) = 1 - v$, then Q^* is monotone since $U' > 0$ and $G' \geq 0$ by log-concavity of χ. More generally, Q^* globally slopes upward when the passing fraction is not too convex—namely, if $G'(v) > 0$, or equivalently:

$$\frac{f''}{f'} + \frac{\chi'}{\chi} - \frac{\chi''}{\chi'} > 0. \tag{13}$$

Since $\chi''(v) \geq v\chi'(v)$, inequality (13) is stronger than (5), and so Q^* need not slope up. But with geometric costs $\chi(v) = v^B$, inequality (13) reduces to $f''(v) < -f'(v)/v$, that is, Q^* slopes up if f is not too convex. Finally, even if inequality (13) fails, log-concavity of both f and χ usefully restrict slopes of Q^* and $\bar{\Pi}$ in equilibrium for our comparative statics:

$$G'(v) - F'(v) \equiv \frac{f''}{f'} - \frac{f'}{f} + \frac{\chi'}{\chi} - \frac{\chi''}{\chi'} > 0. \tag{14}$$

We now summarize the curves fixing equilibrium quality q and the verification rate v:

LEMMA 1—Slopes: (a) *The Q^* locus starts at $q = v = 0$, is initially flat, and hits $v = 1$ at some quality $q^\Delta < \infty$. If Q^* slopes down at an equilibrium, then it is steeper than $\bar{\Pi}$.* (b) *The locus $\bar{K}$ slopes down, is steeper than $\bar{\Pi}$, and less steep than Q^* whenever Q^* slopes down.*

[19] Given Topkis (1998), the maximization of profits (2) yields an implied falling map $e \mapsto q$ for a submodular passing fraction $f(V(e, q))$. By Rockafellar (1970), the composition of an increasing and convex function $g(v) = -f(v)$ with an increasing and supermodular one $W(e, -q) = V(e, q)$ is supermodular. So when the passing fraction f is convex enough to secure inequality (13), $f(V(e, q))$ is supermodular enough that Q^* slopes up.

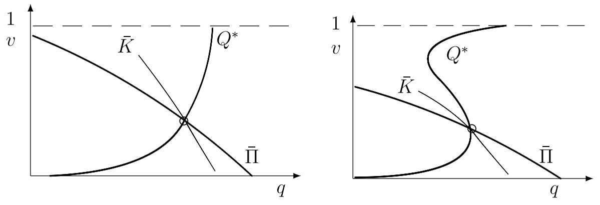

FIGURE 4.—Optimal quality, zero profit, and constant counterfeiting rate loci. This captures the equilibrium levels (v^*, q^*) solving the counterfeiters' zero profit condition (9) and first order condition (10), respectively, the $\bar{\Pi}$ and Q^* loci. Note that Q^* in (10) is steeper than $\bar{\Pi}$ in (9), if Q^* slopes down; Q^* and $\bar{\Pi}$ sandwich $\bar{K}$, where (8) is constant.

So quality vanishes when verification does, and also cannot explode near perfect verification. Figure 4 depicts the curves $\bar{K}$, Q^*, $\bar{\Pi}$, with the $\bar{K}$ locus sandwiched between $\bar{\Pi}$ and Q^* by (5).

We now analyze the equilibrium of the overall game: a pair (q^*, v^*) where Q^* and $\bar{\Pi}$ cross, then an effort $e^* = q^*\chi(v^*)$, and finally the counterfeiting rate κ^* solving (8). That $\kappa^* < 1$ is mathematically immaterial in the optimization (1), but is needed for a meaningful counterfeiting rate.

THEOREM 1—Existence and Uniqueness: *If* $\sqrt{3}xf(0)\chi'(1) < (1-\beta) \times c(1)^{1/\eta}\ell^{1-1/\eta}$, *then a counterfeiting equilibrium uniquely exists for each note* $\Delta > \underline{\Delta}$, *and fails to exist for* $\Delta \leq \underline{\Delta}$. *Verification is positive and imperfect* $(0 < v < 1)$, *and counterfeiting positive but bounded*, *with*

$$\kappa^* \leq \frac{\sqrt{3}xf(0)\chi'(1)}{(1-\beta)c(1)^{1/\eta}\ell^{1-1/\eta}}. \tag{15}$$

The maximum κ rate (15) is lower if counterfeiting is harder—higher unit quality costs $c(1)$ or legal costs ℓ, or lower production x or passing rate $f(0)$. The maximum rate falls with better verification—a higher bank chance β, or lower marginal verification costs $\chi'(1)$.

3.3. *Example*

Assume geometric costs $c(q) = q^A$ and $\chi(v) = v^B$ with $A, B \geq 2$, and $f(v) = 1 - v$ (i.e., no police). The zero profit and optimal quality equations (9) and (11) are

$$\Delta x(1-v) - q^A - \ell = 0 \quad \text{and} \quad Aq^A - \Delta xv/B = 0. \tag{16}$$

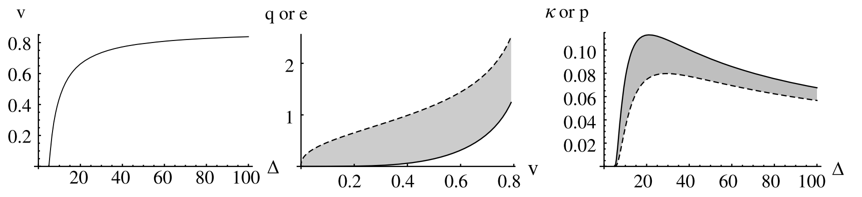

FIGURE 5.—Verification, effort, quality, and counterfeiting/passed rates in example in Section 3.3. Plots assume $A = 5$, $B = 3$, $x = 2$, $\ell = 10$, $\alpha = 4/5$, and $\beta = 1/4$. Left: The verification rate rises from $\underline{\Delta} = 5$ to $\bar{v} = 0.8$. Middle: Effort and quality (solid/dashed) and the ratio e/q rise from 0. Right: The counterfeiting/passed rate curves (solid/dashed) vanish at small and large notes.

Solving the zero profit condition in (16), verification vanishes for notes Δ approaching $\underline{\Delta} = \ell/x$. And as $\Delta \uparrow \infty$, the verification rate tends to $\bar{v} = AB/(1 + AB) < 1$, since[20]

$$q^A = (1 - \bar{v})x(\Delta - \underline{\Delta}) \quad \text{and} \quad v = \bar{v}(1 - \underline{\Delta}/\Delta). \tag{17}$$

So verification rises in the note Δ, but is forever imperfect. While effort $e = qv^B$ rises in Δ, quality rises much faster, and infinitely so initially as $B > 0$, as seen in Figure 5:

$$e = (1 - \bar{v})^{1/A} x^{1/A} \bar{v}^B \Delta^{-B} (\Delta - \underline{\Delta})^{B+1/A}. \tag{18}$$

Substitute the discovery rate $\delta(v) = \beta\alpha + (1 - \beta)v$, and the q and v expressions (17) into the equilibrium formula $\kappa = Bqv^{B-1}/(\delta(v)\Delta)$ from (8). This yields the next counterfeiting rate expression, unimodal in Δ, with $\kappa > 0$ positive on $(\underline{\Delta}, \infty)$, vanishing for both $\Delta \downarrow \underline{\Delta}$ and $\Delta \uparrow \infty$:

$$\kappa = \frac{B(1 - \bar{v})^{1/A} x^{1/A} \bar{v}^{B-1} \Delta^{1-B+1/A} (\Delta - \underline{\Delta})^{B-1+1/A}}{\beta\alpha\Delta + (1 - \beta)\bar{v}(\Delta - \underline{\Delta})}. \tag{19}$$

Figure 5 also depicts the like-shaped plot of the passed rate $p = \delta(v)\kappa$: It understates the counterfeiting rate, but the ratio p/κ rises in Δ, tending to $\bar{v} < 1$. Passed and counterfeiting rates vanish as $\Delta \downarrow \underline{\Delta}$ (since $B > 1 + 1/A$), and as $\Delta \uparrow \infty$, both vanish at the order $O(\Delta^{1/A-1})$.

[20]Given a strictly convex passing function $f(v) = (1 - v)(1 - \gamma v)$ (i.e., with police), a quadratic equation fixes the verification rate v. More police presence (higher γ) depresses ("crowds out") v, and elevates quality q.

4. EQUILIBRIUM COMPARATIVE STATICS

Equilibrium comparative statics analysis is possible largely by shifting curves in Figure 4.[21]

THEOREM 2: (a) *Assume legal costs rise. Then the verification effort and rate fall; counterfeit quality falls at low and high notes Δ, and always falls given* (13); *the counterfeiting rate falls.*

(b) *Assume that marginal verification costs fall. Then the verification effort and counterfeit quality fall, the verification rate rises, and the counterfeiting rate falls.*

(c) *If costs and marginal costs of counterfeiting fall, then the verification effort and counterfeit quality rise, the counterfeiting rate rises, and the verification rate falls if $c'(q)/c(q)$ also falls.*

(d) *The verification effort and rate and counterfeit quality rise in the note if $\Delta > \underline{\Delta}$. The verification effort and rate, and counterfeit quality all vanish as $\Delta \downarrow \underline{\Delta}$. The verification effort and quality explode as the note $\Delta \uparrow \infty$. The counterfeiting rate vanishes as $\Delta \downarrow \underline{\Delta}$ or $\Delta \uparrow \infty$.*

Part (a) attests that a greater legal penalty displaces verification effort, but still deters counterfeiting. In parts (b)–(c), we see that while lower verification costs are generally not formally equivalent to greater counterfeiting costs,[22] the sign of their effects on all variables is the same. Part (d) is consistent with the hump-shaped plot in Figure 5 (right panel) for the example in Section 3.3. In each case, we derive the effort comparative statics from the zero profit identity (9), and $v = V(e, q)$, and comparative statics of (v, q, κ) via the graphical apparatus.

PROOF OF THEOREM 2(a): Differentiate (9) in legal costs ℓ to get[23] $\Pi_q \dot{q} + \Pi_e \dot{e} + \Pi_\ell = 0$. Since quality is optimal, $\Pi_q \dot{q} = 0$—for $\dot{q} = 0$ if $q = 0$ in an open ball around ℓ, and otherwise $\Pi_q = 0$. Since $\Pi_e = \Delta f' V_e < 0$ and $\Pi_\ell = -1 < 0$, we have $\dot{e} < 0$: Effort falls when legal costs rise.

Legal costs shift the zero profit curve down at each quality, since verification effort must fall to avoid counterfeiter losses. Since the optimal quality locus Q^*

[21]We build on standard insights from supply and demand curve analysis. Notably, when both curves shift, verification (resp. quality) comparative statics reflect which curve shifts more horizontally (resp. vertically).

[22]An exact equivalence obtains in one *specific* case, since the model admits one functional degree of freedom: scaling verification costs $\hat{\chi} \equiv \chi/\nu$ is equivalent to inversely scaling the quality argument of counterfeiting costs to $\hat{c}(q) \equiv c(\nu q)$, since $q\hat{\chi}(v) \equiv (\nu q)\chi(v)$. Hence, $V(e/\nu, q) \equiv \chi^{-1}((e/\nu)/q) \equiv \chi^{-1}(e/(\nu q)) \equiv \hat{V}(e, \nu q)$.

[23]The notation $\dot{x}$ denotes the derivative of x in ℓ. Later, it denotes derivatives in any changing parameter.

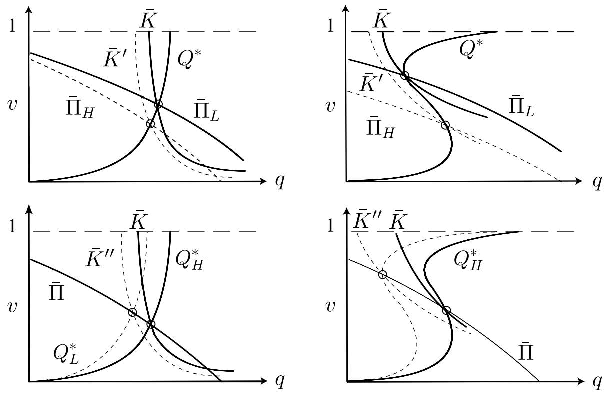

FIGURE 6.—Rising legal costs or falling verification costs: Theorem 2(a) and (b). Top: When legal costs rise, the zero profit curve $\bar{\Pi}$ shifts *left* ($\bar{\Pi}_L$ to $\bar{\Pi}_H$), while Q^* is unchanged. Verification worsens and quality falls if Q^* slopes up—that is, surely for low and high v. The counterfeiting rate falls, with $\bar{K}'$ left of $\bar{K}$. Bottom: If marginal verification costs χ' fall, the Q^* locus shifts left (Q^*_H to Q^*_L), and $\bar{\Pi}$ is fixed. Verification improves and quality falls. The counterfeiting rate falls: $\bar{K}''$ is left of $\bar{K}$, and each locus has a lower counterfeiting rate (8).

in (11) is unaffected by ℓ, its shape alone governs changes in (q, v). Verification falls, for either Q^* slopes up, or slopes down and is steeper than $\bar{\Pi}$. Finally, if Q^* is monotone, higher legal costs depress quality and the verification rate, thus lowering the counterfeiting rate—as in Figure 6 (top). *Q.E.D.*

PROOF OF THEOREM 2(b): Smoothly transform the old technology $t = 1$ into the new one ($t > 1$) with money more readily verified, via the parameterized verification cost function $\chi(v, t)$. Integration of the posited inequality $\chi_{vt}(v, t) < 0$ on $[1, v]$ yields $\chi_t(v, t) < 0$. Define $\mathcal{V}(e, q, t) = v$ iff $e = q\chi(v, t)$. First, the zero profit curve $\bar{\Pi}$ fixes how effort evolves. Differentiate its identity (9) in t, now written as:

$$\Delta x f\big(\mathcal{V}(e, q, t)\big) - c(q) - \ell = 0.$$

Its q derivatives cancel by (10). Then $\mathcal{V}_e \dot{e} + \mathcal{V}_t = 0$. Since $\mathcal{V}_e > 0$, effort e falls in t.

For every $t \geq 1$, implicitly define $\nu(v, t)$ by $\chi(v, t) \equiv \chi(\nu(v, t))$, so that $\nu_t(v, t) < 0 < \nu_v(v, t)$. Then $\chi(\nu(v, t))/\chi'(\nu(v, t))$ falls in t for each v, by log-concavity of χ. So Q^* in (11) shifts left in t for every v. As seen in Figure 6 (bottom), quality falls, and verification rises. The counterfeiting rate falls as (i) the locus $\bar{K}$ shifts down to $\bar{K}''$, and (ii) each locus has a lower counterfeiting rate (a lower marginal verification cost function χ' in (8)). *Q.E.D.*

PROOF OF THEOREM 2(c): Write $c(q,\tau)$ as a smooth function of τ, with $c_\tau, c_{q\tau} < 0$. From profits (9), $\Pi_q = 0$ implies $\Pi_\tau + \Pi_e \dot{e} = 0$. Since costs fall in τ, we have $\Pi_e < 0 < \Pi_\tau$, and so $\dot{e} > 0$.

Given $T_\tau, U_\tau < 0 < T_q, U_q$, when τ rises, $\bar{\Pi}$ and Q^* shift right, as q rises to maintain equality in (12). Since costs fall in τ, so does $c(q,\tau)/(c(q,\tau)+\ell)$. If c'/c also falls, then

$$U_\tau - T_\tau = \frac{d}{d\tau}\log\left(\frac{qc'(q,\tau)}{c(q,\tau)+\ell}\right) = \frac{d}{d\tau}\log\left(\frac{qc'(q,\tau)}{c(q,\tau)}\right) + \frac{d}{d\tau}\log\left(\frac{c(q,\tau)}{c(q,\tau)+\ell}\right) < 0. \tag{20}$$

Thus, as τ rises, Q^* shifts right more than $\bar{\Pi}$, lowering the verification rate (Figure 7, top).

Next, when $G' > 0$, the optimal quality locus Q^* slopes up, and quality rises when Q^* and $\bar{\Pi}$ shift right, as seen in Figure 7. But if $G' < 0$, then Q^* slopes down, and both $\bar{\Pi}$ and Q^* shift up: In this case, since Q^* is steeper than $\bar{\Pi}$ at an equilibrium by Lemma 1, quality rises in τ exactly when Q^* shifts up more than $\bar{\Pi}$ at each q. This occurs due to two re-enforcing effects. First, given (20), U shifts up more than T, for fixed q; thus, G must rise more than F. Next, since $F'(v) < G'(v) < 0$ by (14), this translates into a greater vertical shift in Q^* than $\bar{\Pi}$. At both left and right, the counterfeiting locus shifts up to $\bar{K}'$, and the counterfeiting rate rises. *Q.E.D.*

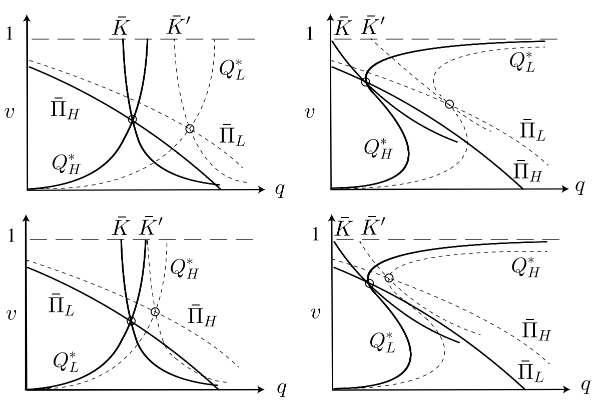

FIGURE 7.—Falling counterfeit costs or a falling note: Theorem 2(c) and (d) depicted. Top: The counterfeiting costs fall from H to L, pushing the locus Q^* right more than $\bar{\Pi}$, raising quality but worsening verification. The counterfeiting rate rises, since $\bar{K}$ shifts right to $\bar{K}'$. Bottom: When Δ rises from L to H, the locus $\bar{\Pi}$ shifts right more than Q^*, raising verification. Next, quality rises even when Q^* slopes down, since Q^* shifts up more than $\bar{\Pi}$ in this case. While $\bar{K}$ shifts right to $\bar{K}'$, the counterfeiting rate might fall, as Δ is higher in (8).

PROOF OF THEOREM 2(d): That $\dot{e} > 0$ follows by differentiating the zero profit identity for (2) in Δ, and using $\Pi_q = 0$:

$$\Pi_e \dot{e} + \Pi_\Delta = 0. \tag{21}$$

So $\dot{e} > 0$ since $\Pi_e < 0 < \Pi_\Delta$ by (2). Next, $\bar{\Pi}$ and Q^* shift right in (12) as Δ rises. The logic of (20) implies $U'(q) > T'(q)$, shifting $\bar{\Pi}$ right more than Q^*, raising verification. Next, if Q^* slopes up, quality rises: But when Q^* slopes down, quality rises since Q^* shifts up more than $\bar{\Pi}$ (Figure 7, bottom right), as $|F'(v)| > |G'(v)|$. Indeed, $F'(v) < G'(v)$ by (14) and $G'(v) < 0$.

As the note $\Delta \downarrow \underline{\Delta} = \ell/(xf(0))$, the $\bar{\Pi}$ locus (9) tends to the origin. So v and q vanish in equilibrium, where $\bar{\Pi}$ and Q^* cross, as does $e = q\chi(v)$. Next, as $\Delta \uparrow \infty$, the left side of (11) explodes as χ/χ' weakly rises in v, v weakly rises in Δ, and $-f'(v) \geq -f'(0) > 0$. So $qc'(q) \uparrow \infty$, and q explodes. So $e = q\chi(v)$ explodes as $\Delta \uparrow \infty$, for v is monotone in Δ.

Finally, we analyze the counterfeiting rate. While the $\bar{K}'$ locus at the higher denomination in Figure 7 (bottom) is right of the $\bar{K}$ locus, the counterfeiting rate (8) is also exogenously depressed by the higher note Δ. Thus, we must proceed analytically. First, the counterfeiting rate vanishes for low notes $\Delta \downarrow \underline{\Delta}$, since q and v vanish in (8), while the discovery rate obeys $\delta(v) \geq \beta\alpha > 0$. Next, assume $\Delta \uparrow \infty$. Substitute the optimal quality condition (10) into (8):

$$\kappa = \frac{q\chi'(v)}{\delta(v)\Delta} = \frac{q\chi'(v)}{\delta(v)} \frac{xf'(v)V_q(e,q)}{c'(q)} = \frac{-xf'(v)\chi(v)}{\delta(v)c'(q)}. \tag{22}$$

Since quality explodes, so too does marginal cost $c'(q)$. Now, $\chi(v) \leq \chi(1) < \infty$, and $-f'(1) \leq -f'(0) < \infty$ since f is convex. Hence, the counterfeiting rate vanishes: $\kappa \to 0$. *Q.E.D.*

5. PREDICTIONS ABOUT COUNTERFEIT MONEY

5.1. *Passed and Seized Money*

Fix $\Delta > \underline{\Delta}$. We relate our theory to the data using two steady-state approximations: first, new counterfeit production replenishes the outflow of seized and passed money, and second, new counterfeit production passing into circulation balances passed money outflows: $P = f(v)(S + P)$. Also using our zero profit equation (9), the *seized-passed ratio* S/P obeys

$$\frac{1}{1 + S/P} = f(v) = \frac{c(q) + \ell}{x\Delta} = \frac{\text{average costs}}{\text{denomination}}. \tag{23}$$

By Theorem 2(d), verification v rises in Δ, and so S/P rises in Δ (left equality of (23)). Next, look at the equality of the first and third terms in (23). With fixed quality, $1+S/P$ rises proportionately to Δ. But as quality rises in Δ, $1+S/P$ rises less than proportionately to Δ.

Define the *annualized passed rate* $p_a = P/M$, namely, the ratio of yearly passed money to circulation.[24] This is the passed rate p times the annual *velocity* $\mathcal{V}$, which varies by note.[25] By the formula (8) for the counterfeiting rate κ, the passed rate satisfies

$$\text{(24)} \qquad \frac{P/M}{\mathcal{V}} = \frac{p_a}{\mathcal{V}} = p = \delta\kappa = \frac{q\chi'(v)}{\Delta} = \frac{\text{marginal verification cost}}{\text{denomination}}.$$

Since the discovery rate $\delta(v)$ increases in the note Δ by Theorem 2(d), so does the ratio p/κ. So if the counterfeiting rate levels off, the passed rate must continue to rise—for example, it peaks at a higher note than the counterfeiting rate does in the example in Section 3.3.[26]

Theorem 2(a)–(d) has respective implications for seized and passed money:

COROLLARY 1: (a) *If legal costs rise, then the seized-passed ratio and the passed rate both fall.*

(b) *If marginal verification costs fall, then the seized-passed ratio rises.*

(c) *Assume that counterfeiting costs $c(q)$ and marginal counterfeiting costs $c'(q)$ fall. Then the seized-passed ratio falls—and the passed rate rises if $c'(q)/[c(q)+\ell]$ also does not fall.*

(d) *One plus the seized-passed ratio monotonically rises in $\Delta > \underline{\Delta}$, but does not rise in proportion to the note. The passed rate vanishes as $\Delta \downarrow \underline{\Delta}$ or $\Delta \uparrow \infty$.*

PROOFS: Parts (a)–(c) owe to Theorem 2(a)–(c) and the facts that the discovery rate δ and the seized-passed ratio S/P rise in the verification rate. While Theorem 2 made clear predictions about the unobserved counterfeiting rate, we can only conclude that the passed rate $\delta\kappa$ falls in part (a). For the counterfeiting rate moves opposite to the discovery rate in parts (b) and (c). For the exception in part (c), when $c'(q)/[c(q)+\ell]$ does not fall, the verification rate does not fall by inequality (20), and so the discovery rate does not fall.

[24]Circulation includes fake money, but the approximation $p_a \approx P/M = \delta(v)\kappa$ is accurate within 1% of 1%.

[25]Lower notes wear out faster, surely due to a higher velocity. Longevity estimates are 5.9, 4.9, 4.2, 7.7, 3.7, and 15 years, respectively, for \$1, …, and \$100 [www.newyorkfed.org/aboutthefed/fedpoint/fed01.html].

[26]Equations (23) and (24) can be used to estimate model parameters from our data. The "street price" of counterfeit notes is at most the average costs, and so at most $\Delta P[\Delta]/(S[\Delta]+P[\Delta])$. The implied U.S. street price ceilings for the \$5, \$10, \$20, \$50, and \$100 notes can be computed from Figure 1, to get \$3.37, \$5.95, \$9.30, \$19.20, \$35.70, respectively. We thank Pierre Duguay (Bank of Canada) for this insight; he said the predicted street prices are realistic. Next, we can likewise back out marginal verification costs from (24). They equal the passed rate times the denomination, peaking around 1/4 cent for the \$100 bill.

We have shown the first half of (d); finally, the passed rate vanishes when the counterfeiting rate does, as $\delta \leq 1$ always. *Q.E.D.*

We now revisit our motivational facts. Fact 1 ensues from the initial result in Corollary 1(d)–first, that the slope is positive, and second, that it is much less than 1. Thus, $1 + S/P$ rises less than proportionally to the note. For example, it does not even double moving from \$5 to \$100.

Consider Fact 2. If the velocity $\mathcal{V}$ is bounded, Corollary 1(d) implies that the annualized passed rate vanishes at the least notes. This is consistent with the U.S. passed rates in Figure 2 (left panel). The plot for the euro also illustrates our extra claim that the passed rate vanishes for very large notes—the passed rate of the 500 euro is less than 8% of the 200 euro.

Theorem 2(d) predicts Fact 3, that quality rises in the denomination. It also makes sense of the greater prevalence of lower quality digital counterfeiting at the lower notes (see Table I), as well as the fact that the highest quality "Supernote" was the \$100.

Facts 4 and 5 are best understood jointly. For if we assume that the main structural change in the 1990s was a reduction in the cost of producing fake notes (Fact 5), when digital technology lowered the non-Colombian costs of counterfeiting notes (Table I), then Corollary 1(c) predicts the crash in the seized-passed ratio and the rise in the passed rate. The last proviso part (c) of Fact 4 is relevant if the fixed costs of counterfeiting fall proportionately much more than marginal costs. This is an apt description of the digital technology change.

Finally, to see Fact 6, observe that Canada's 1970–1976 color note introduction massively raised the fixed costs of counterfeiting—namely, a rare novel color printing technology was required. This is a compelling case where a fall in verification costs is not equivalent to a rise in counterfeiting costs. This is addressed by the last proviso of (the reverse of) Corollary 1(c), which predicts a fall in the passed rate along with a rise in the seized-passed rate.

5.2. *Missed Counterfeit Money*

We conclude by turning our attention to one compelling additional piece of evidence for this model: Commercial banks transfer damaged or unneeded notes to the Federal Reserve Banks (FRB), who find \$5–\$10 million of fake money yearly. This offers a reverse test of our model, since passed money at the FRB *missed* earlier detection. Figure 8 depicts plots that fall from \$1 through \$50 for the three years with available data—first, the FRB share of all passed notes, and second, the ratio of the internal FRB passed rate and the passed rate. All told, *Federal Reserve Banks find proportionately fewer counterfeits as the note rises, until* \$100. These trends might be surprising, as the lowest notes are the poorest quality fakes (Fact 3), and so easiest for verifiers to catch before

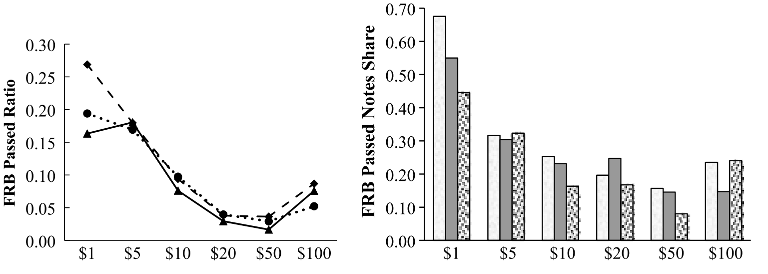

FIGURE 8.—Passed money at the FRB. At left, the ratios of internal FRB and passed rates in 1998 (dashed), 2002 (dotted), 2005 (solid). At right, the respective FRB passed note shares.

they hit the FRB.[27] We now argue that the endogenously vanishing verification rate at low notes rationalizes this.

A fake note lands at an FRB if four independent events transpire: it is fake, it is deposited into a bank, it is not found, and then it is transferred to an FRB. If the FRB has a perfect counterfeit detection, then the (internal) *FRB passed rate* is the counterfeit fraction of transferred notes:

$$\zeta = \frac{\text{fake notes hitting FRB}}{\text{total notes hitting FRB}} = \frac{\kappa\beta(1-\alpha)\phi}{\beta(1-\kappa)\phi + \kappa\beta(1-\alpha)\phi}$$
$$\approx \kappa(1-\alpha).$$

The approximation is accurate within $\kappa \approx 0.0001$, or 0.01%. While this depends on the unobserved counterfeiting rate, the *FRB passed ratio* $\zeta/p \approx (1-\alpha)/\delta$ does not. The discovery rate δ rises in the note Δ, since v does by Theorem 2(d). So our theory predicts a falling FRB passed ratio, matching Figure 8, except at the \$100 bill. Here, the simplifying assumption of constant bank verification chance α is most strained: If the bank detection chance of the (high quality) fake \$100 note is sufficiently lower, so that $\alpha[100] < \alpha[50]$, then the FRB ratio rises at \$100. The right panel of Figure 8 illustrates a striking related fact that the FRB share of counterfeit notes found also falls in the denomination, until the \$100.

APPENDIX: OMITTED PROOFS

A.1. *Optimal Quality and Zero Profit Curves: Proof of Lemma 1(a)*

The Q^* locus starts at $q = v = 0$ and is initially flat—since $-\infty < f'(0) < 0$ and the limit of $v\chi'(v)/\chi(v)$ as $v \downarrow 0$ finitely exists, whereas $v/q = -[v\chi'(v)/\chi(v)][c'(q)/\Delta x f'(v)] \to 0$ as $q, v \to 0$. Also, Q^* hits $v = 1$ at quality $q^\Delta < \infty$,

[27]See Table 6.1, 6.3, and 6.3 (resp.) in Treasury (2000, 2003, 2006), and Table 1 in Judson and Porter (2003).

where $q^{\Delta}c'(q^{\Delta}) = -\Delta f'(1)\chi(1)/\chi'(1) > 0$, for $1 - v \geq f(v) > 0$ and the convex passing fraction f implies a slope $f'(v) \geq -1$ as $v \uparrow 1$.

CLAIM 1—Strict SOC: *The second order condition at an optimum is strict*: $\Pi_{qq} < 0$.

PROOF: The SOC for maximizing $\Pi(e, q, \Delta)$ is locally necessary:

$$\Pi_{qq} = \Delta x f' V_{qq} + \Delta x f'' V_q^2 - c'' \leq 0. \tag{25}$$

The derivative of the quality first order condition (10) in the note Δ yields

$$0 = \Pi_{qq}\dot{q} + \Pi_{qe}\dot{e} + \Pi_{q\Delta}. \tag{26}$$

For a contradiction, assume $\Pi_{qq} = 0$. Then (21) and (26) yield different formulas for $1/\dot{e}$ at the optimum, which can be simplified using $\Pi_{qe} = \Delta(f'V_{qe} + f''V_eV_q)$ and $\Pi_{q\Delta} = f'V_q$:

$$1/\dot{e} = \frac{f'V_{qe} + f''V_eV_q}{f'V_q} = \frac{f'V_e}{f} \quad \Rightarrow \quad 0 < \frac{V_{qe}}{V_q} = \left(\frac{f'}{f} - \frac{f''}{f'}\right)V_e.$$

This is a contradiction, because $V_e > 0$ and $f'/f < f''/f'$ by strict log-concavity of f. *Q.E.D.*

CLAIM 2: *If Q^* slopes down at an equilibrium, then it is steeper than $\bar{\Pi}$.*

PROOF: We now argue that the SOC reduces to $G'(v)T'(q) > F'(v)U'(q)$. Since the respective slopes of the $\bar{\Pi}$ and Q^* curves are $T'(q)/F'(v)$ and $U'(q)/G'(v)$, this says that *if Q^* is negatively sloped, then it is absolutely steeper than $\bar{\Pi}$—in other words, $G'(v) < 0$ implies $T'(q)/F'(v) > U'(q)/G'(v)$*. Reformulating the SOC (25), we find

$$\begin{aligned} 0 > \Pi_{qq}(v, q, \Delta) &= c'\left[\frac{V_{qq}}{V_q} + \frac{f''}{f'}V_q\right] - c''(q) \\ &= c'\left[\frac{-1}{q}\left(1 + \frac{\chi}{\chi'}\left(\frac{\chi'}{\chi} - \frac{\chi''}{\chi'}\right)\right) - \frac{f''}{f'}\left(\frac{\chi}{q\chi'}\right)\right] - c''(q) \end{aligned} \tag{27}$$

by (9) and (3) and (4). Taking the quotient of (10) and (9), using $V_q = -\chi/(q\chi')$, we find

$$\frac{f'}{f} = -\frac{qc'(q)}{c(q) + \ell}\frac{\chi'}{\chi} \quad \Rightarrow \quad \frac{q\chi'}{\chi} = -F'(v)/T'(q). \tag{28}$$

That $G'(v)T'(q) > F'(v)U'(q)$ follows from (27) and (28), for they yield

$$\frac{\chi''}{\chi'} - \frac{\chi'}{\chi} + \frac{f'}{f} - \frac{f''}{f'} < \frac{\chi'}{\chi}\frac{qc''(q)}{c'(q)} + \frac{f'}{f} + \frac{\chi'}{\chi}$$
$$= \frac{q\chi'}{\chi}\left(\frac{qc''(q) + c'(q)}{qc'(q)} - \frac{c'(q)}{c(q) + \ell}\right)$$

and thus, (14) yields $F'(v) - G'(v) = -[F'(v)/T'(q)](U'(q) - T'(q))$, as desired. *Q.E.D.*

A.2. *Constant Counterfeiting Rate Curve Slope: Proof of Lemma 1(b)*

Differentiating the log of (8), the proportionate change in the counterfeiting rate is

$$\frac{d\kappa}{\kappa} = \frac{dq}{q} + \left(\frac{v\chi''(v)}{\chi'(v)} - \frac{v\delta'(v)}{\delta(v)}\right)\frac{dv}{v} - \frac{d\Delta}{\Delta}.$$

Holding κ and Δ fixed, the change in quality along the $\bar{K}$ locus obeys

$$\left.\frac{dq}{q}\right|_{\bar{K}} = \left(\frac{v\delta'(v)}{\delta(v)} - \frac{v\chi''(v)}{\chi'(v)}\right)\frac{dv}{v}. \tag{29}$$

Along the $\bar{\Pi}$ locus, the change in quality obeys

$$\left.\frac{dq}{q}\right|_{\bar{\Pi}} = \frac{\Delta xvf'(v)}{qc'(q)}\frac{dv}{v} = -\frac{v\chi'(v)}{\chi(v)}\frac{dv}{v} \tag{30}$$

after substituting (11). By log-concavity of χ, we see that (29) strictly exceeds (30). Thus, the slope of $\bar{K}$ exceeds that of $\bar{\Pi}$, but we now show that it is less than the slope of Q^*. This is clear when Q^* has positive slope. Indeed, log-differentiating (11):

$$\left(1 + \frac{qc''(q)}{c'(q)}\right)\left.\frac{dq}{q}\right|_{Q^*} = \left(\frac{vf''(v)}{f'(v)} + \frac{v\chi'(v)}{\chi(v)} - \frac{v\chi''(v)}{\chi'(v)}\right)\frac{dv}{v}.$$

When Q^* has negative slope, it is steeper than $\bar{K}$ (see (29)) since $c''(q)/c'(q) \geq 0$ and by (5):

$$\frac{vf''(v)}{f'(v)} + \frac{v\chi'(v)}{\chi(v)} \geq 1 > \frac{v\delta'(v)}{\delta(v)}.$$

Q.E.D.

A.3. *Existence and Uniqueness: Proof of Theorem 1*

We establish equilibrium in (v, q, e) space, and derive the counterfeiting rate κ from (8).

A.3.1. *Existence*

When $\Delta > \underline{\Delta}$, we find a solution to the zero profit and optimal quality equations (9) and (11) in Figure 4. Since $f' < 0 < c'$, (9) implicitly defines a continuous and decreasing function $q = Q_0(v)$. Given $c(Q_0(0)) = \Delta x f(0) - \ell > 0$ when $\Delta > \underline{\Delta}$, we have $Q_0(0) > 0$. Since $\Delta x f(0) > \ell$ and $f(1) = 0$, choose $\hat{v} < 1$ with $\Delta x f(\hat{v}) = \ell$. Then $Q_0(v) \to 0$ as $v \to \hat{v}$. By the Implicit Function Theorem (IFT), since $qc'(q)$ is strictly increasing, the quality FOC (11) implicitly defines a differentiable function $q = Q_1(v)$. Since the limit $v\chi'(v)/\chi(v)$ exists and is positive as $v \to 0$, both sides of (11) vanish, and so $Q_1(0) = 0$. Easily, (11) is positive at $v = \hat{v}$, and so $Q_1(\hat{v}) > 0$. Given $Q_1(0) = 0 < Q_0(0)$ and $Q_1(\hat{v}) > 0 = Q_0(\hat{v})$, the Intermediate Value Theorem yields $v \in (0, \hat{v})$ with $Q_0(v) = Q_1(v)$. But then $0 < v < 1$ and $0 < q = Q_1(v) = Q_0(v) < \infty$. So $\kappa > 0$ by (8). Since $Q_0(v), Q_1(v)$ are differentiable in Δ, so is $q[\Delta]$ and $v[\Delta]$. (Or, apply the IFT to the system (9) and (11).)

A.3.2. *Uniqueness*

Assume two equilibria (e_1, q_1) and (e_2, q_2) for a note Δ. If $q_1 = q_2$, then $e_1 = e_2$, as profits fall in effort. Assume $q_1 < q_2$. By a line integral of Π along the smooth optimal quality curve from (e_1, q_1) to (e_2, q_2), that is, $\mathcal{Q}^* = \{(e, q) : \Pi_q(e, q) = 0, q_1 \le q \le q_2\}$:

$$0 - 0 = \Pi(e_2, q_2) - \Pi(e_1, q_1) = \int_{\mathcal{Q}^*} (\Pi_e, \Pi_q) \cdot (de, dq) = \int_{e_1}^{e_2} \Pi_e \, de.$$

Since $\Pi_e < 0$, $e_1 = e_2$. Then $v_1 > v_2$, and so profits are higher at (e_2, q_2) than (e_1, q_1), a contradiction. (Also, $0 < v_i < 1$, since $\bar{\Pi}$ has positive intercepts and Q^* rises from the origin.)

A.3.3. *The Peak Counterfeiting Rate*

We proceed in three steps.

STEP 1: Modifying the counterfeiting rate formula (22) for zero profits (9), we find

$$\kappa(v) = -\frac{xf'(v)\chi(v)}{\delta(v)c'(q)} = \frac{xf(v)\chi'(v)}{\delta(v)} \frac{q}{c(q) + \ell} \le \frac{xf(0)\chi'(1)}{(1-\beta)c'(q)} \tag{31}$$

since $(c(q) + \ell)/q$ is minimized when $qc'(q) - c(q) = \ell$, where it equals the marginal cost $c'(q)$, and since $\delta(v) \ge (1 - \beta)v$ and $\chi'(v)/v$ is weakly increasing.

STEP 2—Lower Bound on Cost and Marginal Cost of Quality: Since $qc'(q)/c(q)$ is monotone by (6), $c'(q)/c(q) \ge \eta/q$ if $q > 0$. Integrating on $[1, q)$ yields $\log c(q) - \log c(1) \ge \log q^\eta$ if $q \ge 1$. So $c(q) \ge c(1)q^\eta$. Then $c'(q)/c(q) \ge \eta/q$ implies $c'(q) \ge c(1)\eta q^{\eta-1}$.

STEP 3—A Fixed Upper Bound for the Counterfeiting Rate: Define producer surplus $\pi(q) \equiv qc'(q) - c(q)$. Let $Q(\ell)$ be the quality yielding producer surplus $\pi(Q(\ell)) \equiv \ell$. Then

$$\ell = \pi\big(Q(\ell)\big) = Q(\ell)c'\big(Q(\ell)\big) - c\big(Q(\ell)\big) \geq c(1)\eta Q(\ell)^{\eta} - c(1)Q(\ell)^{\eta}$$

by the cost bounds in Step 2. This implies a lower bound that allows us to simplify (31):

$$\begin{aligned} c'\big(Q(\ell)\big) &> \frac{\pi\big(Q(\ell)\big)}{Q(\ell)} \geq \frac{\ell}{Q(\ell)} \geq \frac{\ell}{\big(\ell/c(1)(\eta+1)\big)^{1/\eta}} \\ &\geq c(1)^{1/\eta}\ell^{1-1/\eta}/\sqrt{3} \end{aligned}$$

since $(1+\eta)^{1/\eta}$ is monotone decreasing in $\eta > 1$, and we assumed $\eta \geq 2$. *Q.E.D.*

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