2017

An Introduction to Ricci Flow for Two-Dimensional Manifolds

Paul Bracken
The University of Texas Rio Grande Valley

An Introduction to Ricci Flow for Two-Dimensional Manifolds

Paul Bracken

Department of Mathematics,

University of Texas,

Edinburg, TX

78540

paul.bracken@utrgv.edu

Abstract

The study of differentiable manifolds is a deep an extensive area of mathematics. A technique such as the study of the Ricci flow turns out to be a very useful tool in this regard. This flow is an evolution of a Riemannian metric driven by a parabolic type of partial differential equation. It has attracted great interest recently due to its important achievements in geometry such as Perelman's proof of the geometrization conjecture and Brendle-Schoen's proof of the differentiable sphere theorem. It is the purpose here to give a comprehensive introduction to the Ricci flow on manifolds of dimension two which can be done in a reasonable fashion when the Euler characteristic is negative or zero. A brief introduction will be given to the case in which the Euler characteristic is positive.

Keywords: metric, tensor, manifold, Ricci flow, curvature, isometry, evolution equation

MSc: 53C44, 51H25, 58A05

1 Introduction

The Ricci flow and its relative the normalized Ricci flow are techniques designed to permit the metric of a Riemannian manifold (M^n, g) to evolve in terms of an evolution parameter [**1-4**]. The basic approach of the Ricci flow is to introduce a type of evolution for the Riemannian metric inspired by the classical heat equation. The Ricci flow is then an evolution of a Riemannian metric which is driven by a parabolic partial differential equation and has played a fundamental role in several important recent achievements in differential geometry, such as Perelman's proof of the geometrization conjecture and Brendle-Schoen's proof of the differentiable sphere theorem [**5-6**]. Of the many possible reasons for studying the Ricci flow, one is to resolve the geometrization conjecture of Thurston for closed 3-manifolds. The geometrization conjecture states that any closed 3-manifold can be canonically decomposed into pieces in such a way that each admits a unique homogeneous geometry, or a unique geometric structure. A geometric structure on an n-dimensional manifold may be regarded as a complete locally homogeneous Riemannian metric g. In general, it is not expected that a solution $(M^3, g(t))$ of the Ricci flow starting on an arbitrary closed 3-manifold will converge to a complete locally homogeneous metric. Topological and geometric properties of M^3 are deduced from the behavior of $g(t)$ [**7-8**].

The Ricci flow can be related to one of the great achievements of nineteenth-century mathematics, the Uniformization Theorem, first demonstrated by Poincaré. This theorem implies that every smooth surface admits an essentially unique conformal metric of constant curvature, or interpreted otherwise, as the statement that every 2-dimensional manifold admits a canonical geometry [**9**]. Thus it may be regarded as a classification of such manifolds into three families: those of constant positive, zero or negative curvature. There has been great interest in the subject especially due to the great progress of Perelman [**10-11**]. Also, recent interest in generalized Weierstrass representations which include classes of integrable evolutions might lead one to wonder which types of manifold of a given dimension can result from them and which cannot. The Ricci flow idea might be of use in the investigation of this area.

The objective here is to study extensively the Ricci flow in the simplest setting, on two-dimensional Riemannian manifolds. A number of results will be established for the Ricci flow on a two-dimensional manifold after first introducing the mathematical formulation of the Ricci flow as well as some relevant ideas from differential geometry which will be needed. The subject of equivalence of metrics is important here. Isometries provide a notion of equivalence between two Reimannian manifolds. An isometry between two Riemannian manifolds is defined to be a diffeomorphism $\varphi : M \to M'$ that preserves the metric structure. A weaker notion between Riemannian manifolds is that of conformal equivalence. Two metrics g and g' on a manifold are called conformal to each other if there exists a function $f : M \to \mathbb{R}^+$ such that $g = f \cdot f'$, so two Riemannian manifolds (M^n, g) and $(M^{n'}, g')$ are conformally equivalent if there is a diffeomorphism $\varphi : M \to M'$ such that g is conformal to $\varphi^* g'$. This leads to a discussion of the fact that a solution of the normalized Ricci flow exists for all time and converges to a constant curvature metric conformal to the initial metric. The existence of a constant curvature metric in each conformal class is a fact which is equivalent to the Uniformization Theorem. In this sense, the Ricci flow may be regarded as a natural homotopy between a given Riemannian metric and the canonical metric in its conformal class whose existence is guaranteed by the Uniformization Theorem [**14-15**].

In dimension 2, the situation is a bit special since a geometry is always isotropic, hence has a constant curvature, and there are only these possible models: $\mathbb{S}^2$, $\mathbb{E}^2$ and $\mathbb{H}^2$ with corresponding constant sectional curvatures $K = 1, 0, -1$.

Theorem (Riemann Uniformization Theorem.) Any compact surface admits one and only one of these three geometries. $\square$

The fact that a surface belongs to a unique geometric type follows from the Gauss-Bonnet formula. Similar results cannot hold in dimension 3. The product manifold $S^1 \times S^2$ cannot carry a Riemannian metric with constant curvature since its universal covering $S^2 \times \mathbb{R}$ has two ends while S^3, $\mathbb{E}^3$ and $\mathbb{H}^3$ have only one end.

In the 70's, Thurston observed up to equivalence there are only eight 3-dimensional geometries which are maximal in the sense that there is no $Isom(X)$-invariant Riemannian metric on X whose isometry group is strictly larger than $Isom(X)$.

Theorem (Classification of 3-Dimensional Geometries.) Up to equivalence there are exactly eight maximal geometries X in dimension 3 (i) Three isotropic geometries of constant curvature $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{H}^3$ (ii) Four anisotropic geometries with isotropy subgroup $SO(2)$, $\mathbb{S}^2 \times E^1$, Nil and $\tilde{S}L(2,\mathbb{R})$. Each of these geometries has a natural fibration by geodesic lines or circles over $\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$. (iii) The geometry Sol with trivial isotropy subgroup, based on the only simply-connected 3-dimensional Lie group which is solvable but not nilpotent.

1.1 Ricci Flow on Surfaces.

Let us begin by introducing the Ricci flow on Riemannian manifolds and then specialize to Ricci flow on surfaces. To understand the definition, $g(t)$ is considered to be a one parameter family of metrics on a manifold M^n. The parameter t which appears is an evolution parameter, or time. The time derivative of the metric as used in the differential equation will be defined first. A generic, arbitrary point on a manifold is denoted as x here.

Definition 1.1. Assume that $T(t)$ is a smooth one-parameter family on (m,n) tensor fields. The time derivative is a tensor field of type (m,n) such that

$$(\frac{\partial T}{\partial t})(X_1,\ldots,X_m,\omega_1,\ldots,\omega_n) = \frac{\partial}{\partial t}(T(t)(X_1,\ldots,X_m,\omega_1,\ldots,\omega_n)). \tag{1.1}$$

for all vector fields X_i and one-forms ω_j that are independent of t. In local coordinates, this is

$$\frac{\partial T}{\partial t} = (\frac{\partial}{\partial t}T^{i_1,\cdots,i_n}_{j_1,\cdots,j_m})\, dx^{j_1}\otimes\cdots\otimes dx^{j_m}\otimes\partial_{i_1}\otimes\cdots\otimes\partial_{i_n}. \tag{1.2}$$

Definition 1.2. Ricci flow is the evolution of the metric $g(t)$ on a manifold by means of the following differential equation

$$\frac{\partial g(t)}{\partial t} = -2\mathrm{Ric}[g(t)], \qquad g(0) = g_0. \tag{1.3}$$

Using (1.1), this can be interpreted as a differential equation, and it is often written as

$$\frac{\partial g(t)}{\partial t} = h[g(t)] \tag{1.4}$$

to permit generalization to other cases, such as normalized Ricci flow. Thus, in local coordinates, a solution of the Ricci flow (1.3) must obey the following differential equation

$$\frac{\partial g_{ij}(t)}{\partial t} = -2\mathrm{Ric}_{ij}[g(t)], \qquad g_{ij}(0) = (g_0)_{ij}.$$

The time dependence of the metric also means that the Levi-Civita connection $\nabla = \nabla_{g(t)}$ becomes time dependent. Although the operator $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is not a tensor field, the time derivative can be defined in a similar way

$$(\frac{\partial}{\partial t}\nabla)_X Y = \frac{\partial}{\partial t}(\nabla_X Y). \tag{1.5}$$

Thus, $(\partial/\partial t)\nabla$ defines a $(2,1)$ tensor field on M, in coordinates,

$$\frac{\partial}{\partial t}\nabla = (\frac{\partial \Gamma_{ij}^k}{\partial t})\, dx^i \otimes dx^j \otimes \partial_k. \tag{1.6}$$

To discuss the normalized Ricci flow, the manifold must admit a volume form, and so this form and its time derivative must be introduced. In any positively oriented coordinate system, the volume element depends on the metric $g(t)$ and is given by

$$dv_n = dv_n[g(t)] = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n.$$

If dv_n is integrated over M^n, the volume $\mathrm{Vol}(M)$ of the manifold is obtained. When it is desired to indicate that a particular quantity, such as the Laplacian, depends on the metric, g will appear as a subscript. Using the Jacobi formula for the derivative of a determinant and (1.4)

$$\frac{\partial}{\partial t}\sqrt{\det(g_{ij})} = \frac{1}{2}\frac{1}{\sqrt{\det(g_{ij})}} \det(g_{ij})\mathrm{tr}(g^{-1}h) = \frac{1}{2}(\mathrm{tr}h)\sqrt{\det(g_{ij})}.$$

Substituting this in the coordinate expression for dv_n, the following theorem results.

Theorem 1.1

$$\frac{\partial}{\partial t}\, dv_n = \frac{1}{2}(\mathrm{tr}_{g,12}\, h)\, dv_n. \tag{1.7}$$

This is an important result which will be fundamental in what follows. When g evolves according to the Ricci flow (1.3), $\mathrm{tr}\, h = -2\mathrm{Ric}$, and (1.7) implies that if the manifold M is orientable, then for n even as with a surface, its volume evolves as

$$\frac{\partial}{\partial t}\mathrm{Vol}(M) = \int_M \frac{\partial}{\partial t}\, dv_n = -\int_M (\mathrm{tr}\,\mathrm{Ric})\, dv_n = -\int_M R\, dv_n = -4\pi\chi(M). \tag{1.8}$$

The last equality is a consequence of the Gauss-Bonnet Theorem. Consequently, if the Euler characteristic $\chi(M)$ is not zero, the manifold either collapses to a point in finite time or expands infinitely. As the flow does not change the underlying manifold, by collapsing to a point means that all points are distance zero from the perspective of the Riemann distance. It is very useful to define a related modified flow and has the property that Vol(M) remains constant during the evolution.

Definition 1.3. (Normalized Ricci Flow.) On an orientable manifold, the normalized Ricci flow is the evolution of the metric $g(t)$ under the differential equation

$$\frac{\partial g(t)}{\partial t} = \frac{2r}{n} g(t) - 2\mathrm{Ric}[g(t)], \qquad g(0) = g_0. \tag{1.9}$$

In (1.9), r is defined to be the average scalar curvature

$$r = \frac{\int_M R\, dv_n}{\int_M dv_n} = \frac{4\pi\chi(M)}{\mathrm{Vol}(M)}. \tag{1.10}$$

Under the normalized Ricci flow with $h = (2r/n)g - 2\mathrm{Ric}$ in (1.4), the variation of the volume is now given by (1.7)

$$\frac{\partial}{\partial t}\mathrm{Vol}(M) = \int_M \frac{1}{n}(\mathrm{tr}_{1,2g} g) r\, dv_n - \int_M (\mathrm{tr}_{12,g}\mathrm{Ric})\, dv_n = r\mathrm{Vol}(M) - \int_M R\, dv_n = 0.$$

In the final step, (1.10) is needed. This is an important result because it implies that r, the average scalar curvature, remains constant during the normalized Ricci flow evolution.

For the case of a 2-manifold or surface, these equations can be written in terms of the Ricci curvature. Since the Gaussian curvature satisfies $R = 2K$, it is the case that

$$\mathrm{Ric} = K\, g = \frac{1}{2} R\, g, \tag{1.11}$$

and the equations for the normalized Ricci flow take the form,

$$\frac{\partial g(t)}{\partial t} = (r - R(t))\, g(t), \qquad g(0) = g_0. \tag{1.12}$$

This implies the following important result: A Riemannian metric on a surface is a fixed point of the normalized Ricci flow if and only if it has constant curvature. It is possible to determine an evolution equation for the evolution of the scalar curvature R which will be used in the following.

Lemma 1.1. If $g(t)$ is a smooth 1-parameter family of metrics on a Riemannian surface M^2 such that $\partial g/\partial t = h = f \cdot g$ for a scalar function f, then

$$\frac{\partial R}{\partial t} = -\Delta f - R f. \tag{1.13}$$

Proof: Let h be a fixed metric on M^2 and let g be conformally related to h by $g = e^u h$. By the Cartan structure equations **[16]**, the scalar curvature of g and h are related by

$$R_g = e^{-u}(-\Delta_h u + R_h). \tag{1.14}$$

When $\partial g/\partial t = f \cdot g$ then $u_t = f$ and so differentiating (1.14) with respect to t gives

$$\frac{\partial R_g}{\partial t} = -\frac{\partial u}{\partial t} e^{-u}\big(-\Delta_h u + R_h\big) - e^{-u}\Delta_h \frac{\partial u}{\partial t} = -fR_g - \Delta_g f.$$

Since f is just a function, Δ_h can be replaced by Δ_g.

□

In fact, suppose that $g(t) = v(t)\, g_0$ is a solution to the normalized Ricci flow on surfaces, then in (1.12),

$$\frac{\partial}{\partial t} g(t) = \frac{\partial v}{\partial t} g_0 = (r - R(t))\, g(t) = (\Delta_{g_0} \log v + rv - R_{g_0})g_0.$$

where $R(t) = \frac{1}{v}(R_{g_0} - \Delta_{g_0} \log v)$ is substituted. This implies that for $g(t) = v(t)g_0$ to be a solution of the normalized Ricci flow, v must evolve according to the equation

$$\frac{\partial v}{\partial t} = \Delta_{g_0} \log v + rv - R_{g_0}. \tag{1.15}$$

1.2 The Maximum Principle.

An enormous amount can be said with regard to the behavior of solutions of partial differential equations which resemble the heat equation by dropping the Laplacian term. A theorem referred to as the Maximum Principle is required and it permits the estimation of solutions of certain classes of partial differential equation.

Definition 1.4. (*Supersolution* and *Subsolution*) Consider the differential equation

$$\frac{\partial v}{\partial t} = \Delta v + \Phi(v)$$

with $\Phi : \mathbb{R} \to \mathbb{R}$ some function. A *supersolution* u to this differential equation is a solution which satisfies

$$\frac{\partial u}{\partial t} \geq \Delta u + \Phi(u) \tag{1.16}$$

everywhere on the domain of u. The concept of *subsoluion* is defined similarly, but reverses the inequality in (1.16).

Proposition 1.1. (*The Maximum Principle*)

Let M be a compact manifold and assume $g(t)$ is a smooth one-parameter family of metrics on the interval $[0,T)$. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function. Assume that $u : M \times [0,T) \to \mathbb{R}$ is a C^2 supersolution such that

$$\frac{\partial u(x,t)}{\partial t} \geq \Delta_{g(t)} u(x,t) + \Phi(u(x,t)) \tag{1.17}$$

holds for all $(x,t) \in M \times [0,T)$. Let $\varphi : [0,T) \to \mathbb{R}$ be a differentiable function such that

$$\frac{\partial \varphi(t)}{\partial t} = \Phi(\varphi(t)). \tag{1.18}$$

If at $t = 0$, $u(x,0) \geq \varphi(0)$ for all $x \in M$, then $u(x,t) \geq \varphi(t)$ for all $(x,t) \in M \times [0,T)$. A similar result holds for subsolutions and this result is called the minimum principle, but will be referred to generally as the maximum principle as well.

Corollary 1.1. Under the normalized Ricci flow on a surface,

$$\frac{\partial R}{\partial t} = \Delta R + R(R - r) \tag{1.19}$$

where the average scalar curvature r is constant with respect to t.

Proof: This is an immediate consequence of (1.13) upon using $f = -(R - r)$. $\square$

Thus an evolution equation for $R(t)$ has been found, and it will prove crucial. The Laplacian term in (1.19) can be thought of as a diffusion of R, the second term tends to concentrate R. If the last term were dropped, R would obey the heat equation. The Laplacian term can be ignored in (1.19) to compare its solutions with those of

$$\frac{d\varphi}{dt} = \varphi(\varphi - r), \qquad \varphi(0) = \varphi_0, \tag{1.20}$$

where φ plays the role of R. When $r \neq 0$ and $\varphi_0 \neq 0$, the solution is

$$\varphi(t) = \frac{r}{1 - (1 - \frac{r}{\varphi_0})e^{rt}}, \tag{1.21}$$

and when $r = 0$ and $\varphi_0 \neq 0$, the solution is

$$\varphi(t) = \frac{\varphi_0}{1 - \varphi_0 t}, \tag{1.22}$$

and $\varphi_0 = 0$, the solution is just $\varphi(t) \equiv 0$.

The maximum principle does not give a good upper bound for the curvature under the normalized Ricci flow, but useful lower bounds can be obtained. This is due to the fact that whenever $\varphi_0 < \max\{r, 0\}$, there is a $T < \infty$ given by

$$T = \begin{cases} -\frac{1}{r}\log(1 - \frac{r}{\varphi_0}) > 0, & r \neq 0, \\ \frac{1}{\varphi_0} > 0, & r = 0, \end{cases} \tag{1.23}$$

such that

$$\lim_{t \to T} \varphi(t) = \infty. \tag{1.24}$$

The differential equation behaves much better when $\varphi_0 < \min\{r, 0\}$, in which case we have

$$\varphi(t) - r \geq \varphi_0 - r. \tag{1.25}$$

If we define $R_m(t) = \inf_{x \in M^2} R(x,t)$, then Proposition 1.1 provides the following estimates.

Proposition 1.2. Let $g(t)$ be a complete solution with bounded curvature of the normalized Ricci flow on a compact manifold.

(i) If $r < 0$, then

$$R - r \geq \frac{r}{1 - (1 - \frac{r}{R_m(0)})e^{rt}} - r \geq (R_m(0) - r)e^{rt}, \tag{1.26}$$

(ii) If $r = 0$, then

$$R \geq \frac{R_m(0)}{1 - R_m(0)t} > -\frac{1}{t}. \tag{1.27}$$

(iii) If $r > 0$ and $R_m(0) < 0$, then

$$R \geq \frac{r}{1 - (1 - \frac{r}{R_m(0)})e^{rt}} \geq R_m(0)e^{-rt}. \tag{1.28}$$

The right-hand side tends to zero as $t \to \infty$ in all cases. □

This implies uniform lower bounds for the curvature under the normalized Ricci flow exist. If r is positive and $R_m(t)$ ever becomes nonnegative, it remains so for all t. If r is positive but $R_m(t)$ is negative, then $R_m(t)$ approaches zero exponentially fast. It will be a challenge here to obtain upper bounds as well.

2 Convergence of the Normalized Ricci Flow.

It turns out to be an interesting fact that metrics of constant curvature act as attractors of the Ricci flow. A theorem which discusses the convergence of the normalized Ricci flow would be very significant.

Proposition 2.1. Let (M, g_0) be a two-dimensional Riemannian manifold that is compact and orientable. Then a unique solution $g(t)$ of the normalized Ricci flow (1.9) exists on $[0, \infty)$. Furthermore, this solution converges to a smooth metric g_∞ which is conformal to g_0 and has constant curvature. □

This theorem actually implies the Uniformization Theorem because the limiting metric has all the required properties. In two dimensions, the Ricci flow is so well behaved, such a Proposition holds, whereas in higher dimensions, it does not hold in general. To prove a short time existence theorem, we will make use of the following consequence from the theory of partial differential equations.

Proposition 2.2. Let (M^n, g) be a compact Riemannian manifold. Consider the differential equation for a scalar function v,

$$\frac{\partial v(x,t)}{\partial t} = a^{ij}\big(x, v(x,t)\big)\nabla^2_{i,j} v + f\big(x, v(x,t), \nabla v(x,t)\big), \qquad v(x,0) = \phi(x). \tag{2.1}$$

In (2.1), a^{ij} are the components of a smooth symmetric $(0, 2)$ tensor field, which are smooth functions of their arguments and f is a smooth scalar function of its arguments with $\phi \in C^\infty(M)$. If the tensor field $a^{ij}(x, \phi(x))$ is positive definite everywhere, then a smooth and unique solution to the differential equation exists on an interval $[0, \epsilon)$ for some $\epsilon > 0$. This can be used to prove the following theorem.

Theorem 2.1. Let (M^2, g_0) be a compact Riemannian surface. Then a constant $\epsilon > 0$ and a function $v : M \times [0, \epsilon) \to \mathbb{R}$ exist such that $g(x,t) = v(x,t)g_0(x)$ is a smooth solution to the normalized Ricci flow with $g(0) = g_0$.

Proof: It has been shown that $g(t) = v(t)g_0$ is a solution to the normalized Ricci flow if and only if v satisfies (1.15). For every $f > 0$, it is the case that

$$\text{grad}(\log f) = (\frac{1}{f})\text{grad}(f),$$

and for every $f \in C^\infty(M)$ and vector field X,

$$\text{div}(fX) = X(f) + f\text{div}\, X.$$

Combining this collection of results we have,

$$\Delta_{g_0}(\log\, v) = -\frac{1}{v^2}(\text{grad}_{g_0} v)(v) + \frac{1}{v}\Delta_{g_0} v = -\frac{1}{v^2}||\text{grad}_{g_0} v||^2_{g_0} + \frac{1}{v}\Delta_{g_0} v. \tag{2.2}$$

since $(\text{grad}_{g_0} v)(u) = \langle \text{grad}_{g_0} v, \text{grad}_{g_0} v\rangle_{g_0} = ||\text{grad}_{g_0} v||^2$. Substituting (2.2) into (1.15) gives

$$\frac{\partial v}{\partial t} = \frac{1}{v}\Delta_{g_0} v - \frac{1}{v^2}||\text{grad}_{g_0} v||^2_{g_0} + rv - R_{g_0}. \tag{2.3}$$

Equation (2.3) has the form (2.1) in Proposition 2.2 provided we identify

$$a^{ij}(x,v) = \frac{1}{v}g_0^{ij}, \qquad f(x,v,\nabla v) = -\frac{1}{v^2}||\text{grad}_{g_0} v||^2_{g_0} + rv - R_{g_0}. \tag{2.4}$$

Now g_0 is symmetric and positive definite, so a^{ij} is a symmetric tensor field and positive definite everywhere. Applying Proposition 2.2, a smooth solution $v(t)$ to (2.2) exists on an interval of the form $[0, \epsilon)$ for some $\epsilon > 0$. It may be concluded that $g(x,t) = v(x,t)g_0$ defines a smooth solution to the normalized Ricci flow on surfaces over the interval $[0, \epsilon)$. $\square$

From this can be deduced the following short time existence theorem.

Theorem 2.2. Let (M^2, g_0) be a compact surface or Riemannian two-manifold. Then there exists a unique, smooth solution to the normalized Ricci flow with $g(0) = g_0$ which is defined on the interval $[0, \epsilon)$ for some $\epsilon > 0$. Furthermore, the solution $g(t)$ is conformal to g for all $t \in [0, \epsilon)$.

Proof: By Theorem 2.1 and uniqueness of solution, $g(t)$ exists on an interval $[0, \epsilon)$ for some $\epsilon > 0$ to the normalized Ricci flow on surfaces. By reducing ϵ, it is known by 2.1 that on this interval, a smooth and conformal solution $\tilde{g}(x,t) = v(x,t)g_0(x)$ exists to the normalized Ricci flow, and by uniqueness, these solutions coincide on $[0, \epsilon)$. Hence, a smooth and conformal solution to the flow exists on $[0, \epsilon)$ and the solution is unique. $\square$

To estimate the behavior of a solution obtained by applying the maximum principle, φ will be used to denote the function and the approach of the following Lemma is typically useful in applying the maximum principle to reach a conclusion concerning the behavior of solutions.

Lemma 2.1. Let $\alpha \geq 1$ with $r < 0$ and $A > 0$, and assume $\varphi(t)$ is a solution of the ordinary differential equation

$$\frac{d\varphi}{dt} = \alpha r \varphi + A e^{\frac{r}{2}t}.$$

Then a constant $C > 0$ exists such that $\varphi \leq C\, e^{\frac{r}{2}t}$.

Proof: The solution of this equation is given by

$$\varphi(t) = e^{\alpha r t}[\varphi(0) + \xi(e^{(\frac{1}{2}-\alpha)rt} - 1)],$$

where $\xi = A/r(\frac{1}{2} - \alpha)$. Clearly,

$$\varphi(t)e^{-\frac{r}{2}t} = e^{(\alpha-\frac{1}{2})rt}\,\varphi(0) + \xi - \xi e^{(\alpha-\frac{1}{2})rt},$$

converges to ξ as $t \to \infty$ since $r < 0$ and $\alpha - 1/2 > 0$. This implies that there exists a constant $C > 0$ such that $\varphi(t)e^{-\frac{r}{2}t} \leq C$. $\square$

3 Long Time Existence

The objective here is to ask: if appropriate bounds on the scalar curvature can be established, does this imply that a solution to the normalized Ricci flow exists for all time. The previous result establishes the short time existence and uniqueness of the normalized Ricci flow. This means a maximal interval $[0, T)$ for some $T \leq \infty$ can be defined on which the conformal solution $g(t)$ exists: maximal in the sense that if $T < \infty$, then no solution to the normalized flow exists on an interval of the form $[0, T + \epsilon)$ for any $\epsilon > 0$. Suppose M is a compact, orientable two-dimensional manifold and $g(t)$ evolves under the normalized Ricci flow. It may be conjectured that if $R(t)$ is bounded on every $[0, \tau) \subset [0, T)$, then we must have $T = \infty$, and the solution to the Ricci flow exists for all time. Let us formulate some general results concerning the convergence of the flow which will lead to a formulation and proof of this conjecture. For two symmetric $(0,2)$ tensors, A and B, by $A \leq B$ is meant that $B - A$ is a non-negative definite quadratic form, so for all $V \in TM$, $(B-A)(V,V) \geq 0$.

Theorem 3.1. Suppose the solution of the normalized Ricci flow exists on $[0,T)$ for some $0 < T \leq \infty$. The solution $g(t)$ can be expressed as $g(x,t) = e^{u(x,t)} g(x,0)$. Suppose that

$$\int_0^T |\frac{\partial}{\partial t} u(x,t)| \, dt \leq C, \qquad \forall x \in M \tag{3.1}$$

for some constant $C > 0$, then $g(t)$ converges to a continuous metric $g(T)$ which is conformal to g_0. There exist the uniform bounds

$$e^{-C} g(0) \leq g(t) \leq e^{C} g(0). \tag{3.2}$$

Proof: Integrating the derivative of $u(x,t)$ with respect to t, $u(x,T)$ can be written as

$$u(x,T) = u(x,0) + \int_0^T \frac{\partial u}{\partial \tau}(x,\tau) \, d\tau = \int_0^T \frac{\partial}{\partial \tau} u(x,\tau) \, d\tau,$$

with $u(x,0) = 0$. This is well defined under assumption (3.1), and based on it, we have

$$|u(x,T) - u(x,t)| = |\int_t^T \frac{\partial}{\partial \tau} u(x,\tau) \, d\tau| \leq \int_t^T |\frac{\partial}{\partial \tau} u(x,\tau)| \, d\tau.$$

The right-hand side approaches zero as $t \to T$. Since M is compact, the convergence is uniform, so $u(x,t)$ converges uniformly to $u(x,t)$. Hence $u(x,T)$ defines a continuous function on M, and moreover,

$$|u(x,t)| \leq \int_0^t |\frac{\partial u}{\partial \tau}(x,\tau)| \, d\tau \leq C, \quad t \leq T,$$

so $-C \leq u(x,t) \leq C$. Exponentiating yields $e^{-C} \leq e^{u(x,t)} \leq e^{C}$, and the uniform bounds (3.2) follow immediately. The limiting metric $g(x,T)$ is defined as $g(x,T) = e^{u(x,T)} g(x,0)$, and this defines a continuous metric on M conformal to g_0. □

From (1.12), recall that $\partial v / \partial t = r - R(x,t)$, so the hypothesis in Theorem 3.1 can be stated as

$$\int_0^T |r - R(x,t)| \, dt \leq C, \quad \forall x \in M. \tag{3.3}$$

For $k \geq 0$ an integer, for any real function $\varphi \in C^k(M^n)$, $\nabla^k \varphi$ denotes any kth covariant derivative of φ and we define

$$|\nabla^k \varphi|^2 = \nabla^{\alpha_1} \nabla^{\alpha_2} \cdots \nabla^{\alpha_k} \varphi \nabla_{\alpha_1} \nabla_{\alpha_2} \cdots \nabla_{\alpha_k} \varphi.$$

It is worth stating a generalization of the previous theorem which will be useful, however the long proof is omitted.

Theorem 3.2. Suppose that the solution to the normalized Ricci flow exists on $[0, T)$ for some $0 < T \leq \infty$. In addition, it may be assumed that for all $x \in M$ and every $k \geq 1$,

$$\int_0^T ||\nabla^k R(x,t)|| \, dt < \infty. \tag{3.4}$$

Then $g(t)$ converges to a smooth limiting metric $g(T)$ with respect to the C^p norm for every $p \geq 0$. □

It is desired to establish bounds on $R(x,t)$ which implies bounds on the derivative of $R(x,t)$ exist. The evolution equation for $||\nabla^k R||^2$ will be studied. This equation will contain several terms of lower order derivatives $\nabla^m R$ with $m < k$. It is convenient to introduce a way of writing a particular combination of these lower order terms as follows

$$\mathcal{R}^k = \sum_{j=1}^{\lfloor k/2 \rfloor} (\nabla^j R) * (\nabla^{k-j} R). \tag{3.5}$$

This states $\mathcal{R}^k$ is a linear combination of terms $\nabla^i R * \nabla^j R$ such that $i + j = k$ and $0 < i, j < k$. The following Lemma will be used frequently.

Lemma 3.1. (a) If $g(t)$ evolves on a Riemannian manifold by the normalized Ricci flow then

$$\frac{\partial}{\partial t}(\nabla^k R) = \nabla^k(\frac{\partial}{\partial t} R) + \mathcal{R}^k.$$

(b) For a time-dependent (k,m)-tensor field T, the norm $||T||^2$ evolves under this flow on surfaces as

$$\frac{\partial}{\partial t} ||T||^2_{g(t)} = (m-k)(r-R)||T||^2_{g(t)} + 2\langle T, \frac{\partial}{\partial t} T \rangle_{g(t)}.$$

(c) Under the flow on surfaces $||\nabla^k R||^2$ evolves as

$$\frac{\partial}{\partial t} ||\nabla^k R||^2 = \Delta ||\nabla^k R||^2 - 2||\nabla^{k+1} R||^2 + (k+2) r ||\nabla^k R||^2 + 4R||\nabla^k R||^2 + \nabla^k R * \mathcal{R}^k.$$

Proof: (a) Proceed by induction on k. Since $\partial(\nabla R)/\partial t = \nabla(\partial R/\partial t)$ as R is a function on M so $k = 1$ holds. Suppose it holds for $1 \le j \le k-1$, we get

$$\frac{\partial}{\partial t}(\nabla^k R) = \nabla(\frac{\partial}{\partial t}\nabla^{k-1}R) + (\frac{\partial \Gamma}{\partial t}) * \nabla^{k-1}R = \nabla(\nabla^{k-1}(\frac{\partial R}{\partial t}) + \mathcal{R}^{k-1}) + (\frac{\partial}{\partial t}\Gamma) * \nabla^{k-1}R.$$

Using the expression for $\partial\Gamma/\partial t$, it is seen that $(\partial\Gamma/\partial t) * \nabla^{k-1}R = (\nabla R) * \nabla^{k-1}R = \mathcal{R}^k$. Putting this in the previous equation gives the result.

(b) Using the fact that $\partial g^{ij}/\partial t = -(r-R)g^{ij}$, from (1.12),

$$\frac{\partial}{\partial t}||T||^2_{g(t)} = \frac{\partial}{\partial t}(g^{j_1 s_1}\cdots g^{j_k s_k} g_{i_1 r_1}\cdots g_{i_m r_m} T^{i_1\cdots i_m}_{j_1\cdots j_k} T^{r_1\cdots r_m}_{s_1\cdots s_k}) = (m-k)(r-R)||T||^2_{g(t)} + 2\langle T, \frac{\partial T}{\partial t}\rangle_{g(t)}.$$

(c) Using the commutator $\Delta\nabla\, R - \nabla\Delta R = \frac{1}{2}R\nabla R$ and the identities above as well as (1.19)

$$\frac{\partial}{\partial t}(\nabla^k R) = \nabla^k(\frac{\partial}{\partial t}R) + \mathcal{R}^k = \nabla^k(\Delta R) + \nabla^k(R^2) - r\nabla^k R + \mathcal{R}^k$$

$$= \Delta\nabla^k R + (2 - \frac{k}{2})R\nabla^k R - r\nabla^k R + \mathcal{R}^k.$$

Using (b), we have finally,

$$\frac{\partial}{\partial t}||\nabla^k R||^2 = k(R-r)||\nabla^k R||^2 + 2\langle\nabla^k R, \frac{\partial}{\partial t}\nabla^k R\rangle$$

$$= -(k+2)r||\nabla^k R||^2 + 4R||\nabla^k R||^2 + \langle\nabla^k R, \nabla^k R\rangle_{g(t)} + \nabla^k R * \mathcal{R}^k$$

$$= \Delta||\nabla^k R||^2 - 2||\nabla^{k+1}R||^2 - (k+2)r||\nabla^k R||^2 + 4R||\nabla^k R||^2 + \nabla^k R * \mathcal{R}^k.$$

□

Lemma 3.2. Suppose $T < \infty$ and there is a constant $C > 0$ such that $\sup_{x\in M}|R(x,t)| \le C$ for all $t \in [0,T)$. Then for every $k \ge 1$, there exists a constant $C_k > 0$ such that

$$\sup_{x\in M}||\nabla^k R(x,t)|| \le C_k, \tag{3.6}$$

for all $t \in [0,T)$.

Proof: Induction on k can be used. The hypothesis indicates that the result is true for $k = 0$, so suppose it is true for $0 \le j \le k-1$. To obtain the bound, the evolution equation for $||\nabla^k R||^2$ will be obtained. In terms of $\mathcal{R}^k$ in (3.5), Lemma 3.1 implies

$$\frac{\partial}{\partial t}||\nabla^k R||^2 = \Delta||\nabla^k R||^2 - 2||\nabla^{k+1}R||^2 - (k+2)r||\nabla^k R||^2 + 4R||\nabla^k R||^2 + \nabla^k R * \mathcal{R}^k. \tag{3.7}$$

From the induction hypothesis, a constant $b > 0$ must exist such that $||\mathcal{R}^k|| \leq b$, hence

$$\frac{\partial}{\partial t}||\nabla^k R||^2 \leq \Delta||\nabla^k R||^2 + [4R - (k+2)r] \cdot ||\nabla^k R||^2 + b||\nabla^k R||. \tag{3.8}$$

Since the polynomial $||\nabla^k R||^2 - b||\nabla^k R|| + b^2$ has a positive minimum, for some suitable $a > 0$,

$$\frac{\partial}{\partial t}||\nabla^k R||^2 \leq \Delta||\nabla^k R||^2 + a||\nabla^k R||^2 + b||\nabla^k R|| \leq \Delta||\nabla^k R||^2 + (a+1)||\nabla^k R||^2 + b.$$

Compactness of M^2 implies that $||\nabla^k R||^2$ is bounded, and so the maximum principle can be invoked to compare $||\nabla^k R||^2$ with the solution of the differential equation

$$\frac{\partial \varphi}{\partial t} = (a+1)\varphi + b.$$

This means there exists a $c > 0$ such that on $[0, T)$,

$$||\nabla^k R||^2 \leq \frac{b+c}{a+1} e^{(a+1)t} - \frac{b}{a+1}. \tag{3.9}$$

The right-hand side of this inequality is bounded on $[0, T)$, which implies that $||\nabla^k R||^2$ is bounded on $[0, T)$. □

The long time existence theorem can now be stated and proved.

Theorem 3.3. Let $[0, T)$ be the maximal time interval of existence for the normalized Ricci flow. Assume that $R(x, t)$ is bounded on every finite subinterval $[0, \tau) \subset [0, T)$, so this means that for every $0 < \tau < \infty$ with $\tau \leq T$, a constant $C_\tau > 0$ exists such that

$$\sup_{x \in M} |R(x, t)| \leq C_\tau, \qquad \forall t \in [0, \tau). \tag{3.10}$$

If this holds, it is the case that $T = \infty$, so the solution to the flow exists for all time.

Proof: Suppose that T is not infinite, $T < \infty$. By hypothesis, there exists $C > 0$ such that $\sup_{x \in M} |R(x, t)| \leq C$ for all $t \in [0, T)$. By Lemma 2.1, it follows that $\sup_{x \in M} ||\nabla^k R(x, t)|| \leq C_k$ for all $k \geq 1$ and appropriate constants $C_k > 0$. Consequently, using (3.3)

$$\int_0^T |\frac{\partial u(x,t)}{\partial t}| \, dt = \int_0^T |r - R(x, t)| \, dt \leq C \cdot T < \infty, \quad \forall x \in M,$$

and for every $k \geq 1$,

$$\int_0^t ||\nabla^k (\frac{\partial u}{\partial t}(x, t))|| \, dt = \int_0^T ||\nabla^k R|| \, dt \leq C_k \cdot T < \infty, \quad \forall x \in M.$$

Using Theorems 3.1 and 3.2, it may be concluded that $g(t)$ converges to a smooth metric $g(T)$. At this point, the short time existence theorem implies the solution to the normalized Ricci flow can be extended to a larger interval $[T, T+\epsilon)$. However, this contradicts the definition of the bound T, so it may be concluded that the assumption $T < \infty$ is not true, consequently $T = \infty$. $\square$

Some work has been done establishing lower bounds on the scalar curvature function already. This effort is continued by proving there exist upper and lower bounds on R on the domain of definition of the solution of the normalized Ricci flow, that is, to estimate bounds for the difference $R(x,t) - r$. A concept which arises in discussing this and in particular finding these bounds is the idea of the curvature potential function which is defined now.

Definition 3.1. (*Curvature Potential*) The curvature potential function is defined to be a smooth one-parameter family $\vartheta(x,t) \in C^\infty(M)$ such that each of the following equations hold for all $t \in [0, T)$,

$$\Delta_{g(t)}\vartheta(x,t) = R(x,t) - r, \qquad \frac{\partial \vartheta}{\partial t} = \Delta\vartheta + r\vartheta. \tag{3.11}$$

The following result will be used in due course.

Proposition 3.1. Let M^n be a compact manifold, let J be an interval and $p \in M^n$. Assume that $g(t)$ is a smooth family of metrics on M^n for $t \in J$. Let $\varphi(t) \in C^\infty(M^n)$ be a smooth one-parameter family of smooth functions on M^n for $t \in J$ such that

$$\int_{M^n} \varphi(t)\, dv_n = 0, \qquad \forall t \in J.$$

Then there exists a one-parameter family $\vartheta(t) \in C^\infty(M^n)$, $t \in J$ of smooth functions on M^n such that $\Delta_{g(t)}\,\vartheta(t) = \varphi(t)$ and $\vartheta(p,t) = 0$ for all $t \in J$. This one-parameter family is smooth on $M^n \times J$.

Lemma 3.3. The curvature potential function in Definition 3.1 exists.

Proof: Existence of a smooth one-parameter family $t \to \theta(t) \in C^\infty(M^2)$ is proved first with $\Delta_{g(t)}\theta = R(x,t) - r$, but with a different boundary condition. Choose $m \in M^2$ and $\varphi(t) = R(x,t) - r$, then Proposition 3.1 implies a smooth one-parameter family $[0,T) : t \to \theta \in C^\infty(M^2)$ exists such that $\Delta_{g(t)}\theta(t) = R(x,t) - r$ and $\theta(m,t) = 0$. A function $c : [0,T) \to \mathbb{R}$ is determined which depends only on t such that if we set $\vartheta(t) = \theta(t) + c(t)$, it is the case that

$$\frac{\partial \vartheta}{\partial t} = \Delta \vartheta + r\vartheta. \tag{3.12}$$

Differentiate the identity $\Delta\,\theta = R - r$ with respect to t to obtain,

$$\frac{\partial}{\partial t}(\Delta\,\theta) = \frac{\partial R}{\partial t}. \tag{3.13}$$

Since $(\partial/\partial t)\Delta_{g(t)} = (R - r)\Delta_{g(t)}$, it follows that

$$\frac{\partial R}{\partial t} = (R - r)(R - r) + \Delta_{g(t)}(\frac{\partial \theta}{\partial t}). \tag{3.14}$$

Using the evolution equation for the scalar curvature (1.18), in terms of θ and applying (3.11)

$$\frac{\partial R}{\partial t} = \Delta_{g(t)}R + R(R - r) = \Delta_{g(t)}\Delta_{g(t)}\,\theta + R\Delta_{g(t)}\,\theta. \tag{3.15}$$

Consequently,

$$\Delta_{g(t)}(\frac{\partial}{\partial t}\theta) = \Delta_{g(t)}\Delta_{g(t)}\theta + r\Delta_{g(t)}\theta. \tag{3.16}$$

Collecting terms, (3.16) implies

$$\Delta_{g(t)}(\frac{\partial \theta}{\partial t} - \Delta_{g(t)}\theta - r\theta) = 0. \tag{3.17}$$

Harmonic functions on compact manifolds are constant functions, so it may be concluded there exists a function $\gamma(t)$ depending only on t such that

$$\frac{\partial \theta}{\partial t} = \Delta_{g(t)}\,\theta + r\theta + \gamma(t). \tag{3.18}$$

Thus the function $\gamma(t)$ is smooth and uniquely determined by θ. Define $c(t)$ to be the solution of the differential equation

$$c_t(t) = rc(t) - \gamma(t), \qquad c(t) = -e^{-rt}\int_0^t e^{r\tau}\gamma(\tau)\,d\tau. \tag{3.19}$$

Clearly, the function $\vartheta + c$ satisfies the first equation of (3.11) since

$$\Delta_{g(t)}\vartheta(t) = \Delta_{g(t)}\theta(t) + \Delta_{g(t)}c(t) = \Delta_{g(t)}\theta(t) = R - r. \tag{3.20}$$

Differentiating $\vartheta(t)$ with respect to t and using the equation for $c_t(t)$, it is found that $\vartheta(t)$ satisfies

$$\frac{\partial \vartheta}{\partial t} = \frac{\partial \theta}{\partial t} + \frac{\partial c}{\partial t} = \Delta\theta + r\theta + \gamma + rc - \gamma = R - r + r(\vartheta - c) + \gamma + rc - \gamma = \Delta_{g(t)}\vartheta + r\vartheta. \quad (3.21)$$

Hence, the curvature potential function exists. □

Theorem 3.4. Let $[0, T)$ for $T \leq \infty$ be the maximal interval on which the solution to the normalized Ricci flow exists. Then a constant $C > 0$ exists which depends only on the initial metric such that

$$|R(x,t) - r| \leq Ce^{rt}, \quad \forall t \in [0, T). \quad (3.22)$$

Proof: The lower bound of (3.22) is not hard to get. Define the function $\Phi = R - r$ and obtain its evolution under (1.12)

$$\frac{\partial \Phi}{\partial t} = \frac{\partial R}{\partial t} = \Delta R + R(R - r) = \Delta(R - r) + (R - r)^2 + r(R - r) \geq \Delta\,\Phi + r\Phi. \quad (3.23)$$

Let $\varphi : [0, T) \to \mathbb{R}$ be a differentiable function of t such that $\varphi_t = r\varphi$ and which has solution $\varphi(t) = C_1 e^{rt}$. If at $t = 0$, $\Phi(x, 0) \geq \varphi(0)$ for all $x \in M$, then $\Phi(x, t) \geq \varphi(t)$ for all $(x, t) \in M \times [0, T)$, by the maximum principle. Since M is compact, there exists a constant C_1 such that $\Phi(0) = R(0) - r \geq -C_1$. The maximum principle implies that $\Phi(t) \geq -C_1 e^{rt}$ for all $t \in [0, T)$.

The upper bound is not as easy. However, the curvature potential can be made to play a crucial role. Define the function

$$Q = R - r + ||\nabla\vartheta||^2.$$

It is shown that Q satisfies an evolution equation of the form,

$$\frac{\partial Q}{\partial t} = \Delta Q - 2||F||^2 + rQ \quad (3.24)$$

where

$$F = \nabla^2\vartheta - \frac{1}{2}(\Delta\vartheta)g,$$

is the trace-free Hessian of ϑ. The maximum principle can be applied to (3.24). Let us first work out the derivative

$$\frac{\partial}{\partial t}(R-r) = \Delta R + R(R-r) = \Delta(R-r) + (R-r)^2 + r(R-r) = \Delta(R-r) + (\Delta\vartheta)^2 + r(R-r). \quad (3.25)$$

Using Lemma 3.1, the fact that $\partial/\partial t$ commutes with ∇ on functions and the identity $\Delta\nabla = \nabla\Delta + \frac{1}{2}R\nabla$, we obtain

$$\frac{\partial}{\partial t}||\nabla\vartheta||_g^2 = (R-r)||\nabla\vartheta||_g^2 + 2\langle\nabla\vartheta, \frac{\partial}{\partial t}\nabla\vartheta\rangle_{g(t)}$$

$$= (R-r)||\nabla\vartheta||_{g(t)}^2 + 2\langle\nabla\vartheta, \nabla\frac{\partial\vartheta}{\partial t}\rangle_g = (R-r)||\nabla\vartheta||_{g(t)}^2 + 2\langle\nabla\vartheta, \nabla(\Delta\vartheta + r\vartheta)\rangle_g$$

$$= (R-r)||\nabla\vartheta||_{g(t)}^2 + 2\langle\nabla\vartheta, \nabla(\Delta\vartheta + r\vartheta)\rangle_g = (R-r)||\nabla\vartheta||_g^2 + 2\langle\nabla\vartheta, \nabla\Delta\vartheta\rangle + 2\langle\nabla\vartheta, \nabla(r\vartheta)\rangle$$

$$= (R+r)||\nabla\vartheta||_g^2 + 2\langle\nabla\vartheta, \Delta\nabla\vartheta - \frac{1}{2}R\nabla\vartheta\rangle = (R+r)||\nabla\vartheta||_g^2 + 2\langle\nabla\vartheta, \Delta\nabla\vartheta\rangle - R||\nabla\vartheta||_g^2$$

$$= \Delta||\nabla\vartheta||_g^2 - 2||\nabla^2\vartheta||_g^2 + r||\nabla\vartheta||_g^2. \tag{3.26}$$

To complete the last line, the following identity is used

$$\langle\nabla\vartheta, \Delta\nabla\vartheta\rangle_g = \frac{1}{2}\Delta||\nabla\vartheta||_g^2 - ||\nabla^2\vartheta||_g^2. \tag{3.27}$$

Combining the two derivatives (3.25) and (3.26), the following evolution equation for Q results

$$\frac{\partial Q}{\partial t} = \frac{\partial}{\partial t}(R-r) + \frac{\partial}{\partial t}||\nabla\vartheta||_g^2 = \Delta(R-r) + (\Delta_g\vartheta)^2 + r(R-r) + \Delta_g||\nabla\vartheta||_g^2 - 2||\nabla^2\vartheta||_g^2 + r||\nabla\vartheta||_g^2$$

$$= \Delta_g Q + rQ + (\Delta_g\vartheta)^2 - 2||\nabla^2\vartheta||_g^2. \tag{3.28}$$

In order to make a comparison with the statement of the theorem, the following form of $||F||^2$ is required,

$$||F||^2 = \langle F, F\rangle_g = \langle\nabla^2\vartheta - \frac{1}{2}(\Delta\vartheta)g, \nabla^2\vartheta - \frac{1}{2}(\Delta\vartheta)g\rangle$$

$$= \langle\nabla^2\vartheta, \nabla^2\vartheta\rangle - \frac{1}{2}\Delta\vartheta\,\langle g, \nabla^2\vartheta\rangle - \frac{1}{2}\Delta\vartheta\langle\nabla^2\vartheta, g\rangle + \frac{1}{4}(\Delta\vartheta)^2\langle g, g\rangle$$

$$= ||\nabla^2\vartheta||_g^2 - (\Delta\vartheta)^2 + \frac{1}{2}(\Delta\vartheta)^2 = ||\nabla^2\vartheta||_g^2 - \frac{1}{2}(\Delta\vartheta)^2.$$

Therefore (3.24) is satisfied and since $||F||^2 \geq 0$ holds, it follows that

$$\frac{\partial Q}{\partial t} = \Delta Q + rQ - 2||F||_g^2 \leq \Delta Q + rQ. \tag{3.29}$$

Since M is compact, there exists a constant $C_2 > 0$ such that $Q(0) \leq C_2$. Once again, the maximum principle lets us conclude that

$$R(x,t) - r \leq R(x,t) - r + ||\nabla\vartheta||_g^2 = Q(t) \leq C_2 e^{rt}, \quad \forall t \in [0, T). \tag{3.30}$$

Combining (3.30) with the lower bound, it may be concluded that for an appropriate constant $C > 0$ (3.22) holds. □

Corollary 3.1. On a solution of the normalized Ricci flow on a compact surface, there exists a constant C depending on the initial metric such that

$$R - r \leq Q(t) \leq Ce^{rt}.$$

The proof is to note that this is just (3.30). The long-time existence theorem can now be stated.

Theorem 3.5. On a compact and orientable two-dimensional Riemannian manifold, the normalized Ricci flow (1.12) has a unique solution $g(t)$ which exists for all times $t \in [0, \infty)$.

Proof: The short time existence theorem indicates that for every smooth initial metric, a unique solution to the normalized flow exists on an interval $[0, \epsilon)$ for some $\epsilon > 0$. This implies that for some $T \leq \infty$, a maximal interval $[0, T)$ can be defined on which a unique solution exists. Using the bounds (3.22), it may be shown that the hypotheses of Theorem 3.2 are satisfied. To see this, suppose that $\tau < \infty$, and define the constant C_τ to be

$$C_\tau = |r| + \sup_{t \in [0,\tau]} Ce^{rt}. \tag{3.31}$$

The bound (3.22) implies $\sup_{x \in M} |R(x,t)| \leq C_\tau$ for all $t \in [0, \tau)$ and the long time existence Theorem 3.2 then gives $T = \infty$. Thus the solution to the normalized Ricci flow exists for all times $t \in [0, \infty)$. □

The result of Theorem 3.5 allows us to assume that all solutions of the normalized flow are defined on the whole of $[0, \infty)$ without further mentioning it. The final theorem of the section is related to equivalence of metrics.

Theorem 3.6. Let $(M^2, g(t))$ be the solution of the normalized Ricci flow on a surface with $r \leq 0$. Then the metrics $g(t)$ are equivalent, that is, there exists a $C \geq 1$ independent of t such that

$$\frac{1}{C} g(0) \leq g(t) \leq C\, g(0) \tag{3.32}$$

for all $t \in [0, \infty)$.

Proof: It is known that ϑ satisfies the second evolution equation of (3.11). Since M is a compact manifold, there exists an $A > 0$ which depends only on g_0 such that $|\vartheta(0)| \leq A$. The maximum principle implies that $|\vartheta(t)| \leq Ae^{rt}$. Substituting $g(x,t) = v(x,t)g_0(x)$ into the differential equation for the normalized flow, we find that

$$\frac{\partial v}{\partial t}(x,t) = (r - R(x,t))v(x,t) = -\Delta\vartheta \cdot v(x,t) = (r\vartheta - \frac{\partial \vartheta}{\partial t})v(x,t). \tag{3.33}$$

In this form, (3.33) is separable, so integrating it we obtain

$$\log\left(\frac{v(x,t)}{v(x,0)}\right) = r\int_0^t \vartheta(x,\tau)\,d\tau - \vartheta(x,t) + \vartheta(x,0). \tag{3.34}$$

Using the maximum principle bound, (3.34) implies that since $e^{rt} \leq 1$ for $t \geq 0$ when $r < 0$,

$$|\log\frac{v(x,t)}{v(x,0)}| = |r\int_0^t \vartheta(x,\tau)\,d\tau - \vartheta(x,t) + \vartheta(x,0)| \leq |r|B_1\int_0^t e^{r\tau}\,d\tau + B_1e^{rt} + B_1 \leq B.$$

Solving this for $v(x,t)$ gives the two bounds

$$e^{-B}v(x,0) \leq v(x,t) \leq e^{B}v(x,0),$$

for all $t \in [0,\infty)$. Defining $C = e^B$, the form of bounds given in the statement (3.32) follows. □

Combining Corollary 3.1 with Theorem 1.2, we can estimate R from both sides.

Theorem 3.7. For any solution $(M^2, g(t))$ of the normalized Ricci flow on a compact surface, there exists a constant $C > 0$ which depends only on the initial metric such that

(i) If $r < 0$ then

$$r - Ce^{rt} \leq R \leq r + Ce^{rt}. \tag{3.35}$$

(ii) If $r = 0$ then

$$-\frac{C}{1+Ct} \leq R \leq C. \tag{3.36}$$

(iii) If $r > 0$, then

$$-Ce^{-rt} \leq R \leq r + Ce^{rt}. \tag{3.37}$$

□

To summarize, time-dependent upper and lower bounds for the scalar curvature have been realized and are valid as long as a solution exists. The long-time existence of solutions is a consequence of these estimates.

4 Convergence Properties for $r < 0$ and $r = 0$.

Let us first consider the case in which $\chi(M) < 0$, so it follows from (1.10) that $r < 0$. Thus bounds (3.35) imply that the scalar curvature converges uniformly to its average. It remains to show that $g(t)$ converges to a smooth metric.

Theorem 4.1. On any solution $(M^2, g(t))$ of the normalized Ricci flow, $||\nabla R||^2$ evolves as

$$\frac{\partial}{\partial t}||\nabla R||^2 = \Delta||\nabla R||^2 - 2||\nabla\nabla R||^2 + (4R - 3r)||\nabla R||^2. \tag{4.1}$$

Proof: By (1.18) and the Ricci identity, $\nabla\Delta = \Delta\nabla - \frac{1}{2}R\nabla$, since R is a scalar function, we get

$$\frac{\partial}{\partial t}\nabla R = \nabla(\Delta R + R(R-r)) = \nabla\Delta R + \nabla R(R-r) = \Delta\nabla R - \frac{1}{2}R\nabla R + \nabla R^2 - r\nabla R$$

$$= \Delta\nabla R + \frac{3}{2}R\nabla R - r\nabla R.$$

It follows from this that, using Lemma 3.1,

$$\frac{\partial}{\partial t}||\nabla R||^2 = \frac{\partial}{\partial t}(g^{ij}\nabla_i R\nabla_j R) = \frac{\partial g^{ij}}{\partial t}\nabla_i R\nabla_j R + 2g^{ij}\nabla_i R\nabla_j\frac{\partial R}{\partial t}$$

$$= (R-r)||\nabla R||^2 + 2\langle\nabla R, \Delta\nabla R + \frac{3}{2}R\nabla R - r\nabla R\rangle$$

$$= (R-r)||\nabla R||^2 + 2\langle\nabla R, \Delta\nabla R\rangle + 3R\langle\nabla R, \nabla R\rangle - 2r\langle\nabla R, \nabla R\rangle$$

$$= \Delta||\nabla R||^2 - 2||\nabla\nabla R||^2 + ((R-r) + 3R - 2r)||\nabla R||^2 = \Delta||\nabla R||^2 - 2||\nabla\nabla R||^2 + (4R-3r)||\nabla R||^2,$$

since $\Delta||\nabla R||^2 = 2\langle\Delta\nabla R, \nabla R\rangle + 2||\nabla\nabla R||^2$, and we have (4.1). $\square$

Corollary 4.1. If $(M^2, g(t))$ is a solution of the normalized Ricci flow such that $r < 0$, then there exists a constant $C > 0$ such that

$$||\nabla R||^2 \leq Ce^{\frac{r}{2}t}. \tag{4.2}$$

Proof: By (3.35) and Theorem 4.1, it follows that

$$\frac{\partial}{\partial t}||\nabla R||^2 \leq \Delta||\nabla R||^2 - 2||\nabla\nabla R||^2 + (r + 4ce^{rt})||\nabla R||^2.$$

For all $t > 0$ sufficiently large, we can ensure that $4ce^{rt} < -\frac{r}{2}$, therefore

$$\frac{\partial}{\partial t}||\nabla R||^2 \leq \Delta||\nabla R||^2 + \frac{r}{2}||\nabla R||^2.$$

To apply the maximum principle, let $\varphi : [0, T) \to \mathbb{R}$ be a differentiable function such that $\varphi_t = \frac{r}{2}\varphi(t)$. If at $t = 0$, $||\nabla R(x, 0)||^2 \leq \varphi(0)$ for all $x \in M$, then $R(x, t) \leq \varphi(t)$ for all $(x, t) \in M \times [0, T)$ and so $||\nabla R(x, 0)|| \leq C$ implies (4.2). □

Theorem 4.2. On any solution $(M^2, g(t))$ of the normalized Ricci flow, $||\nabla\nabla R||^2$ evolves according to

$$\frac{\partial}{\partial t}||\nabla\nabla R||^2 = \Delta||\nabla\nabla R||^2 - 2||\nabla\nabla\nabla R||^2 + (2R - 4r)||\nabla\nabla R||^2 + 2R(\Delta R)^2 + 2\langle\nabla R, \nabla||\nabla R||^2\rangle. \tag{4.3}$$

Proof: In two dimensions where $n = 2$, the standard variational formula takes the form

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = -\frac{1}{2}(\nabla_i R\, \delta^k_j + \nabla_j R\, \delta^k_i - \nabla^k R\, g_{ij}). \tag{4.4}$$

In general, commuting derivatives moves the Laplacian to the front so

$$\nabla_i\nabla_j\Delta R = \Delta\nabla_i\nabla_j R + (-\nabla_j R_{il} + \nabla_l R_{ij} - \nabla_i R_{jl})\nabla_l R - 2R_{ikjl}\nabla_l\nabla_k R - R_{il}\nabla_j\nabla_l R - R_{jl}\nabla_i\nabla_l R.$$

For $n = 2$, the Riemann tensor is given by

$$R_{ijkl} = \frac{1}{2}R(g_{il}g_{jk} - g_{ik}g_{jl}). \tag{4.5}$$

Substituting (1.11) and (4.5) into $\nabla_i\nabla_j\Delta R$, we get

$$\nabla_i\nabla_j\Delta R = \Delta\nabla_i\nabla_j R + (\frac{1}{2}||\nabla R||^2 + R\Delta R)g_{ij} - 2R\nabla_i\nabla_j R - \nabla_i R\nabla_j R. \tag{4.6}$$

Consequently, differentiating $\nabla_i\nabla_j R$ with respect to t, we obtain

$$\frac{\partial}{\partial t}(\nabla_i\nabla_j R) = \nabla_i\nabla_j(\frac{\partial}{\partial t}R) - (\frac{\partial}{\partial t}\Gamma^k_{ij})\nabla_k R$$

$$= \nabla_i\nabla_j\big(\Delta R + R(R - r)\big) + \frac{1}{2}(\nabla_i R\, \delta^k_j + \nabla_j R\, \delta^k_i - \nabla^k R\, g_{ij})\nabla_k R$$

$$= \nabla_i\nabla_j\Delta R + \nabla_i\nabla_j R(R - r) + \nabla_i R\nabla_j R - \frac{1}{2}||\nabla R||^2 g_{ij}$$

$$= \nabla_i \nabla_j \Delta R + 2\nabla_i R \nabla_j R + 2R\nabla_i \nabla_j R - r\nabla_i \nabla_j R + \nabla_i R \nabla_j R - \frac{1}{2}||\nabla R||^2 g_{ij}. \tag{4.7}$$

The Laplacian in (4.7) can be moved to the front using (4.4) to obtain,

$$\frac{\partial}{\partial t}(\nabla_i \nabla_j R) = \Delta \nabla_i \nabla_j R + (\frac{1}{2}||\nabla R||^2 + R\Delta R) g_{ij} - 2R\nabla_i \nabla_j R$$

$$-\nabla_i R\nabla_j R + 3\nabla_i R \nabla_j R + 2R\nabla_i \nabla_j R - r\nabla_i \nabla_j R - \frac{1}{2}||\nabla R||^2 g_{ij}$$

$$= \Delta \nabla_i \nabla_j R + R\Delta R g_{ij} + 2\nabla_i R \nabla_j R - r\nabla_i \nabla_j R. \tag{4.8}$$

Finally, the required derivative can be evaluated

$$\frac{\partial}{\partial t}||\nabla\nabla R||^2 = \frac{\partial}{\partial t}(g^{ij} g^{kl} \nabla_i \nabla_k R \nabla_j \nabla_l R)$$

$$= 2(R-r) g^{ij} g^{kl} \nabla_i \nabla_k R \nabla_j \nabla_l R + 2g^{ij} g^{kl} (\frac{\partial}{\partial t} \nabla_i \nabla_k R)(\nabla_j \nabla_l R)$$

$$= 2(R-r)||\nabla\nabla R||^2 + 2g^{ij} g^{kl} (\Delta \nabla_i \nabla_k R + R\Delta R g_{ik} + 2\nabla_i R \nabla_k R - r\nabla_i \nabla_k R)(\nabla_j \nabla_l R) \tag{4.9}$$

Substituting

$$2g^{ij} g^{mn} \nabla_i R \nabla_j \nabla_m R \nabla_n R = g^{ij} \nabla_i R \nabla_j ||\nabla R||^2 = \langle \nabla R, \nabla ||\nabla R||^2 \rangle, \tag{4.10}$$

into (4.9), it is seen that (4.3) is the result. □

Corollary 4.2. If $(M, g(t))$ is a solution of the normalized Ricci flow with $r < 0$, then there exists a constant $C_2 > 0$ such that

$$||\nabla\nabla R||^2 \leq C_2 e^{\frac{r}{2}t}. \tag{4.11}$$

Proof: By (3.35), there exists $C > 0$ such that $R \leq r + Ce^{\frac{r}{2}t}$. Hence there exists a $t_0 \geq 0$ such that $R \leq 0$ for all $t \geq t_0$. Corollary 4.1 states there is a $C_1 > 0$ such that when $t \geq t_0$ we have $||\nabla R||^2 \leq C_1 e^{\frac{r}{2}t}$. It is then the case that

$$\frac{\partial}{\partial t}||\nabla\nabla R||^2 \leq \Delta ||\nabla\nabla R||^2 - 4r||\nabla\nabla R||^2 + C_1 e^{\frac{r}{2}t}||\nabla\nabla R||.$$

Introducing the new variable $\sigma = ||\nabla\nabla R||^2 - 3r||\nabla R||^2 \geq ||\nabla\nabla R||^2$ when $r < 0$ and noting that $4R - 3r = r + 4Ce^{\frac{r}{2}t} \leq (3/4)r$ which holds for $t_1 \geq t_0$ large enough so that $r + 16Ce^{rt/2} < 0$, we have

$$\frac{\partial}{\partial t}\sigma \leq 2||\nabla\nabla R|| \frac{\partial}{\partial t}||\nabla\nabla R|| - 6r||\nabla R|| \frac{\partial}{\partial t}||\nabla R||$$

$$\le \Delta\sigma + 2r||\nabla\nabla R||^2 + (4R-3r)(-3r)||\nabla R||^2) + C_1 e^{\frac{r}{2}t}||\nabla\nabla R||$$

$$\le \Delta\sigma + \frac{3}{4}r(||\nabla\nabla R||^2 - 3r||\nabla R||^2) + C_1' e^{\frac{r}{2}t}||\nabla\nabla R||$$

$$= \Delta\sigma + \frac{3}{4}r\sigma + C_1' e^{\frac{r}{2}t}\sqrt{\sigma} \le \Delta\sigma + \frac{2}{3}r\sigma + C_2' e^{rt}, \tag{4.12}$$

since, if $y = \sqrt{\sigma}$, there are the following bounds

$$0 \le -\frac{1}{12}ry^2 - C_1' e^{\frac{r}{2}t}y + C_2' e^{rt} \le (y + \frac{6}{r})C_1' e^{\frac{r}{2}t})^2 - \frac{12}{r}(\frac{3}{r}C_1'^2 + C_2')e^{rt}$$

provided that $C_2' \ge (3/|r|)C_1'^2$. Hence, by the maximum principle, there exists a $C_2'' > 0$ such that $\sigma \le C_2'' e^{rt/2}$, and this implies (4.11). □

Lemma 4.1. Let $(M, g(t))$ be a solution of the normalized Ricci flow on a closed surface with $r < 0$. Then for each positive integer k, there exists a constant $C_k < \infty$ such that for all $t \in [0, \infty)$

$$\sup_{x\in M} ||\nabla^k R(x,t)||^2 \le C_k e^{\frac{r}{2}t}. \tag{4.13}$$

Proof: This result can be proved by induction on k. The variation of $||\nabla^k R(x,t)||^2$ under the normalized flow is given by Lemma 3.1. The result has been shown for $k = 1, 2$, so assume the statement is true for $1 \le j \le k-1$. For $t \to \infty$, $R \to r$ and $r < 0$, so this implies that there exists $t_1 > 0$ such that if $t \ge t_1$, then $R(x,t) < 0$. Applying the induction hypothesis, there exists $A > 0$ such that $||\mathcal{R}^k|| \le A e^{\frac{r}{2}t}$. Since $R(x,t) < 0$, dropping negative terms for $t \ge t_1$, it follows that

$$\frac{\partial}{\partial t}||\nabla^k R(x,t)||^2 \le \Delta||\nabla^k R||^2 - (k+2)r||\nabla^k R||^2 + A e^{\frac{r}{2}t}||\nabla^k R||.$$

The induction hypothesis also permits a bound for the derivative of $||\nabla^{k-1} R(x,t)||^2$ to be established,

$$\frac{\partial}{\partial t}||\nabla^{k-1}R||^2 = \Delta||\nabla^{k-1}R||^2 - 2||\nabla^k R||^2 - (k+2)r||\nabla^{k-1}R||^2 + 4R||\nabla^{k-1}R||^2 + \nabla^{k-1}R * \mathcal{R}^{k-1}$$

$$\le \Delta||\nabla^{k-1} R||^2 - 2||\nabla^k R||^2 + C_1 e^{\frac{r}{2}t} + C_2 e^{\frac{r}{2}t} + C_3 e^{\frac{r}{2}t}||\mathcal{R}^{k-1}||$$

$$\le \Delta||\nabla^{k-1} R||^2 - 2||\nabla^k R||^2 + A_1 e^{\frac{r}{2}t}.$$

Now define the function Φ in terms of R as follows

$$\Phi = ||\nabla^k R||^2 - (k+1)r||\nabla^{k-1} R||^2, \tag{4.14}$$

and $\Phi \geq ||\nabla^k R||$ since $-(k+1)r$ is positive. Replacing $||\nabla^k R||$ in the equality $u^2 - u + 1 \geq 0$, the following upper bound holds,

$$\frac{\partial}{\partial t}\Phi \leq \Delta||\nabla^k R||^2 - (k+2)r||\nabla^k R||^2 + Ae^{\frac{r}{2}t}||\nabla^k R|| - (k+1)r\Delta||\nabla^k R||^2 + 2(k+1)r||\nabla^k R||^2 + A'e^{\frac{r}{2}t}$$

$$\leq \Delta\Phi + kr||\nabla^k R||^2 + Ae^{\frac{r}{2}t}||\nabla^k R|| + A'e^{\frac{r}{2}t} \leq \Delta\Phi + (kr + Ae^{\frac{r}{2}t})||\nabla^k R||^2 + (A + A')e^{\frac{r}{2}t}.$$

$$\leq \Delta\Phi + (kr + Ae^{\frac{r}{2}t})\Phi + Ce^{\frac{r}{2}t}.$$

To get this, use the inequality $x \leq 1 + x^2$ for $x \geq 0$ in the second last step and also replace $A + A'$ by C. The function $Ae^{\frac{r}{2}t} \to 0$ so there exists a $t_2 \geq t_1$ such that $t \geq t_2$ implies $Ae^{\frac{r}{2}t} \leq -\frac{r}{2}k$. For $t \geq t_2$, the following bound applies

$$\frac{\partial}{\partial t}\Phi \leq \Delta\Phi + k\frac{r}{2}\Phi + ce^{\frac{r}{2}t}. \tag{4.15}$$

Since M is compact, there is a constant $B > 0$ such that $\Phi(0) \leq B$, and the maximum principle allows us to compare Φ with the solution of the ordinary equation $\varphi_t = k(\frac{r}{2})\varphi + e^{rt/2}$ with $\varphi(0) = B$. The solution to the ordinary equation is

$$\varphi(t) = (\frac{2c}{r}e^{\frac{r}{2}t} + C_1)e^{\frac{r}{2}kt} \leq C_k\, e^{\frac{r}{2}t}. \tag{4.16}$$

So there is a positive constant C_k such that $\Phi(t) \leq \varphi(t) \leq C_k e^{\frac{r}{2}t}$, and hence $||\nabla^k\, R(x,t)||^2 \leq \Phi(t) \leq C_k\, e^{\frac{r}{2}t}$.

□

This immediately allows us to conclude with the following Proposition.

Proposition 4.1. Let $(M^2, g(t))$ be the solution to the normalized Ricci flow on a compact, orientable surface M with $r < 0$. Then a unique smooth metric g_∞ exists on M such that $g(t)$ converges to g_∞. This convergence is with respect to the C^k norm for any $k \in \mathbb{N}$. Furthermore, this limiting metric is conformal to g_0 and has constant scalar curvature.

Proof: Using the bounds (3.35), it may be concluded that

$$\int_0^\infty |r - R|\, dt \leq C \int_0^\infty e^{rt}\, dt < \infty.$$

Using Lemma 4.1, we have that for any $k \geq 1$,

$$\int_0^\infty ||\nabla^k R||^2\, dt \leq C_k \int_0^\infty e^{\frac{r}{2}t}\, dt < \infty.$$

By Theorems 3.1 and 3.2, it is found that $g(t)$ converges in any C_k norm to a smooth limiting metric g_∞, which is conformal to g_0. From the bounds estimate (3.35), we immediately see that g_∞ must have constant scalar curvature.

□

The next case to consider is $\chi(M) = 0$ so $r = 0$ and the bounds on scalar curvature (3.35) lead to the bound $|R(x,t)| \leq C$, which implies $R(x,t)$ remains bounded for all t. We would like to end up with a result like Proposition 4.1, so we must first prove R converges to $r = 0$. The next Theorem establishes that R is bounded by a function which converges to zero.

Lemma 4.3. The curvature potential function satisfies the bounds

$$||\nabla\vartheta(t)||^2 \leq \frac{A}{1+t}, \tag{4.17}$$

for some constant $A > 0$.

Proof: The evolution equation for $||\nabla\vartheta||^2$ is (3.26) with $r = 0$, and since $||\nabla\vartheta|| \geq 0$, (3.26) allows us to state that $||\nabla\vartheta||_t^2 \leq \Delta||\nabla\vartheta||^2$. Now M is compact, so $||\nabla\vartheta(0)||^2$ is bounded, thus using this with the maximum principle, it follows that $||\nabla\vartheta(t)||^2 \leq A_1$ for some constant $A_1 > 0$.

Since the curvature potential function must satisfy the heat equation $\vartheta_t = \Delta\,\vartheta$ when $r = 0$, it is clear that

$$\frac{\partial}{\partial t}\vartheta^2 = 2\vartheta\Delta\,\vartheta = \Delta\,\vartheta^2 - 2||\nabla\,\vartheta||^2.$$

Define the new potential function $\Phi = t||\nabla\vartheta||^2 + \vartheta^2$ and calculate the derivative of Φ with respect to t using the t-evolution equation for ϑ^2 and $||\nabla\vartheta||^2$ in the process

$$\frac{\partial}{\partial t}\Phi = ||\nabla\vartheta||^2 + t\frac{\partial}{\partial t}||\nabla\vartheta||^2 + \frac{\partial}{\partial t}\vartheta^2 = ||\nabla\vartheta||^2 + t(\Delta||\nabla\vartheta||^2 - 2||\nabla\vartheta||) + \Delta\vartheta^2 - 2||\nabla\vartheta||^2$$

$$= \Delta(t||\nabla\vartheta||^2) + \Delta\vartheta^2 - 2t||\nabla^2\vartheta|| - ||\nabla\vartheta||^2 = \Delta\Phi - 2t||\nabla^2\vartheta|| - ||\nabla\vartheta||^2 \leq \Delta\Phi,$$

for $t > 0$. Now M is compact so $\Phi(0)$ is bounded and the maximum principle implies that $\Phi(t) \leq A_2$, for some constant $A_2 > 0$. Since $\vartheta^2 \geq 0$, it follows that $||\nabla\vartheta(t)||^2 \leq A_2/t$. The estimate $||\nabla \vartheta(t)||^2 \leq A_1$ and this estimate can be unified into a single estimate which holds for $t \geq 0$,

$$||\nabla\vartheta(t)||^2 \leq \frac{A}{1+t},$$

for some A sufficiently large, which is (4.17). $\square$

Theorem 4.3. Let $(M^2, g(t))$ be the solution to the normalized Ricci flow on a compact and orientable surface M^2 such that $\chi(M) = 0$. The following upper bound for $|R(x,t)|$ holds,

$$|R(x,t)| \leq \frac{C}{1+t}, \tag{4.18}$$

for some constant $C > 0$ sufficiently large.

Proof: Under the normalized Ricci flow, the evolution equation for R is (1.18) with $r = 0$, that is, $R_t = \Delta R + R^2$. Since M is compact, there exists a $C_1 > 0$ such that $-C_1 \leq R(0)$. The maximum principle allows the comparison of $R(x,t)$ to the solution of $\varphi_t = \varphi^2$, $\varphi(0) = -C_1$, and this yields a lower bound

$$-\frac{C_1}{1+C_1 t} \leq R(x,t). \tag{4.19}$$

An upper bound is harder and the curvature potential ϑ is of use. In fact, there is the upper bound from Lemma 4.3.

Define the function

$$\Psi = R + 2||\nabla\vartheta||^2. \tag{4.20}$$

It has already been shown that $0 \leq ||F||^2 = ||\nabla^2\vartheta||^2 - \frac{1}{2}(\Delta\vartheta)^2$, and this implies that $2||\nabla^2\vartheta||^2 \geq (\Delta\vartheta)^2 = R^2$. This inequality leads to the bound

$$\frac{\partial}{\partial t}\Psi = \frac{\partial}{\partial t}R + 2\frac{\partial}{\partial t}||\nabla\vartheta||^2 = \Delta R + R^2 + 2\Delta||\nabla\vartheta||^2 - 4||\nabla^2\vartheta||^2 \leq \Delta\Psi + R^2 - 2R^2 = \Delta\Psi - R^2.$$

This implies that $\Psi_t \leq \Delta\Psi$, and given that $\Psi(0)$ is bounded above, the maximum principle implies that $\Psi(t) \leq C_2$ for some constant C_2 large enough. However, this upper bound can be improved by considering the related function $t\Psi$ and then differentiating it with respect to t,

$$\frac{\partial}{\partial t}(t\Psi) = \Psi + t\frac{\partial\Psi}{\partial t} = \Psi + t(\frac{\partial R}{\partial t} + 2\frac{\partial}{\partial t}||\nabla\vartheta||^2) = \Psi + t(\Delta R + R^2) + 2t(\Delta||\nabla\vartheta||^2 - 2||\nabla^2\vartheta||^2)$$

$$= t\Delta(R + 2||\nabla\vartheta||^2) + \Psi + tR^2 - 4t||\nabla^2\vartheta||^2$$

$$= t\Delta\Psi + \Psi + tR^2 - 4t||\nabla^2\vartheta||^2 \leq t\Delta\Psi + \Psi + tR^2 - 2tR^2 = t\Delta\Psi + \Psi - tR^2.$$

In fact the term $\Psi - tR^2$ on the right side can be expressed as

$$\Psi - tR^2 = -\frac{1}{2}tR^2 + R + 2||\nabla\vartheta||^2 - \frac{1}{2}tR^2 - 2tR||\nabla\vartheta||^2 - 2t||\nabla\vartheta||^4 + 2tR||\nabla\vartheta||^2 + 2t||\nabla\vartheta||^4$$

$$= -\frac{1}{2}tR^2 + R + 2||\nabla\vartheta||^2 - \frac{1}{2}t(R + 2||\nabla\vartheta||^2)^2 + 2t||\nabla\vartheta||^2(R + ||\nabla\vartheta||^2).$$

Consequently, an upper bound for the derivative of $t\Psi$ can be obtained,

$$\frac{\partial}{\partial t}(t\Psi) \leq \Delta(t\Psi)+\Psi-tR^2 = \Delta(t\Psi)-\frac{1}{2}tR^2+R+2||\nabla\vartheta||^2-\frac{1}{2}t(R+2||\nabla\vartheta||^2)^2+2t||\nabla\vartheta||^2(R+||\nabla\vartheta||^2).$$

An estimate is to be made at any point and time at which $R \geq 0$. From Lemma 4.3, we have $t||\nabla\vartheta||^2 \leq A$, so define $B = 2A > 0$. Then if it is the case that $t\Psi = t(R + 2||\nabla\vartheta||^2) \geq B$, it follows that $R \geq 0$. Now at any point and time where $t\Psi \geq B$, upon regrouping terms we have

$$R + 2||\nabla\vartheta||^2 + 2tR\,||\nabla\vartheta||^2 + 2t||\nabla\vartheta||^4 \leq R + 2||\nabla\vartheta||^2 + 2AR + 2A||\nabla\vartheta||^2$$

$$= R(1 + 2A) + 2(1 + A)||\nabla\vartheta||^2.$$

Then the following upper bound for $(t\Psi)_t$ can be determined,

$$\frac{\partial}{\partial t}(t\Psi) \leq \Delta(t\Psi) - \frac{1}{2}tR^2 + R + 2||\nabla\vartheta||^2 - \frac{1}{2}t(R + 2||\nabla\vartheta||^2)^2 + 2t||\nabla\vartheta||^2(R + ||\nabla\vartheta||^2)$$

$$\leq \Delta(t\Psi) - \frac{1}{2}tR^2 - \frac{1}{2}t(R + 2||\nabla\vartheta||^2)^2 + (1 + 2A)R + 2(A + 1)||\nabla\vartheta||^2$$

$$\leq \Delta(t\Psi) - \frac{1}{2t}[t(R + 2||\vartheta||^2)]^2 - [\sqrt{\frac{t}{2}}R - \sqrt{\frac{1}{2t}}(1 + 2A)]^2 + \frac{A_1}{t}$$

$$\leq \Delta(t\Psi) - \frac{1}{2t}[t(R + 2||\nabla\vartheta||^2)]^2 + \frac{A_1}{t}.$$

The constant $A_1 > 0$ is chosen large enough so that we have

$$2(1 + A)t||\nabla\vartheta||^2 \leq -\frac{1}{2}(1 + 2A)^2 + A_1.$$

Now if need be a larger constant $D > B$ can be found so that provided $t(R + 2||\nabla\vartheta||^2) \geq D$, the following upper bound holds,

$$\frac{\partial}{\partial t}(t\Psi) \leq \Delta(t\Psi) + \frac{1}{t}[A_1 - \frac{1}{2}[t(R + 2||\nabla\vartheta||^2)]^2] \leq \Delta(t\Psi).$$

The maximum principle can now be invoked to conclude there exists a constant $C_2 > 0$ sufficiently large such that

$$t\Psi \leq C_2. \tag{4.21}$$

Since it has just been shown that $\Psi(t)$ is bounded as well for all $t > 0$, a combination of these two estimates yields the following bound for $\Psi(t)$

$$\Psi(t) \leq \frac{C_2}{1+t}, \tag{4.22}$$

for some constant $C_2 > 0$ sufficiently large. □

Theorem 4.4. Let $(M^2, g(t))$ be the solution to the normalized Ricci flow on a compact and orientable surface M^2 with $\chi(M) = 0$. Then for every $k \geq 1$, a constant $C_k > 0$ exists so that,

$$||\nabla^k R(t)||^2 \leq \frac{C_k}{(1+t)^{k+2}}. \tag{4.23}$$

□

The last result is the following Theorem.

Theorem 4.5. Let $(M^2, g(t))$ be the solution to the normalized Ricci flow on a compact and orientable surface M^2 such that $\chi(M^2) = 0$. Then a unique smooth metric g_∞ exists on M^2 such that $g(t)$ converges to g_∞. The convergence is with respect to the C^k-norm for any $k \in \mathbb{N}$. Furthermore, this limiting metric is conformal to g_0 and has constant scalar curvature.

Proof: Apply the Sobolev inequality which says there exists a $C = C(M^2, g_0, 3)$ such that

$$\sup_{x \in M} |R(x,t)| \leq C||R||_{g_0, H_1^3} = C(||R||_{L^3} + ||\nabla R||_{g_0, L^3}).$$

The choice $p = 3$ is arbitrary since any $p > n$ will do. Now $r = 0$ and so there is a $C' = C'(M^2, g_0, 3)$ such that

$$||R||_{L^3} \leq C'||\nabla R||_{g_0, L^{6/5}}.$$

Combining these estimates gives

$$\sup_{x \in M} |R(x,t)| \leq C(||\nabla R(t)||_{g_0, L^3} + ||\nabla R(t)||_{g_0, L^{6/5}}).$$

Now the equivalence of metrics $g(t)$ and g_0 is used to estimate $||\nabla R(t)||_{g_0} \leq A||\nabla R(t)||_{g(t)}$ and there is also the upper bound from Theorem 4.4, so taking the square root

$$||\nabla R(t)||_{g(t)} \leq \frac{B}{(1+t)^{3/2}}.$$

This implies that for any $p \geq 1$, $||\nabla R(t)||_{g_0,L^p}$ can be bounded above,

$$||\nabla R||_{g_0,L^p} = (\int_M ||\nabla R||_{g_0}^p \, dv)^{1/p} \leq C(1+t)^{-3/2}(\int_M dv)^{1/p} \leq \frac{C\mathrm{Vol}(M)^{1/p}}{(1+t)^{3/2}}.$$

This implies that

$$\sup_{x \in M} |R(x,t)| \leq \frac{C'}{(1+t)^{3/2}}.$$

These estimates allow us to calculate the bounds

$$\int_0^\infty |R(x,\tau)| \, d\tau \leq \alpha \int_0^\infty \frac{d\tau}{(1+\tau)^{3/2}} < \infty, \quad \forall x \in M,$$

and for $k \geq 1$, we have that

$$\int_0^\infty ||\nabla^k R(x,\tau)|| \, d\tau \leq \beta \int_0^\infty \frac{d\tau}{(1+\tau)^{k/2+1}} < \infty, \quad \forall x \in M.$$

The previous results allow us to conclude that $g(t)$ converges in any C^k norm to a smooth limiting metric g_∞, which is conformal to g_0. The bounds on R imply this metric must have constant scalar curvature.

□

5 Positive Scalar Curvature

The cases in which $r \leq 0$ have been established so far. The case in which $r > 0$ is however far more difficult. It will only be proved that the normalized Ricci flow on a surface of positive Euler characteristic converges exponentially to a metric of constant positive curvature. In the special case in which the scalar curvature is initially nonnegative, by the strong maximum principle, one has $R_{min}(t) > 0$ for any $t > 0$, unless $R \equiv 0$ everywhere. Hence, it may be assumed that $R(x,t) > 0$.

The trace-free part of the Hessian of the potential function ϑ of the curvature was introduced earlier

$$F = \nabla\nabla\vartheta - \frac{1}{2}\Delta\vartheta \cdot g$$

where $\Delta\vartheta = R - r$. It might be possible to show that $g(t)$ converges to a metric of constant positive curvature by proving that F decays sufficiently rapidly. To this end, let us compute the evolution equation of F.

Theorem 5.1. On a solution $(M^2, g(t))$ of the normalized Ricci flow, the tensor F evolves by

$$\frac{\partial F}{\partial t} = \Delta\, F + (r - 2R)F. \tag{5.1}$$

Proof: Using (4.4) which gives that evolution of the Levi-Civita connection of g,

$$\frac{\partial}{\partial t}F_{ij} = \frac{\partial}{\partial t}(\nabla_i\nabla_j\vartheta - \frac{1}{2}(R-r)g_{ij}) = \nabla_i\nabla_j(\frac{\partial\vartheta}{\partial t}) - (\frac{\partial}{\partial t}\Gamma^k_{ij})\nabla_k\vartheta - \frac{1}{2}(\frac{\partial}{\partial t}R)g_{ij} - \frac{1}{2}(R-r)\frac{\partial g_{ij}}{\partial t}$$

$$= \nabla_i\nabla_j(\Delta\vartheta + r\vartheta) + \frac{1}{2}\Big(\nabla_i R\delta^k_j + \nabla_j R\delta^k_i - \nabla^k R g_{ij}\Big)\nabla_k\vartheta - \frac{1}{2}(\Delta R + R(R-r))g_{ij} + \frac{1}{2}(R-r)^2 g_{ij}$$

$$= \nabla_i\nabla_j\Delta\vartheta + \frac{1}{2}\Big(\nabla_i R\nabla_j\vartheta + \nabla_j R\nabla_i\vartheta - \langle\nabla^k R, \nabla_k\vartheta\rangle g_{ij}\Big) - \frac{1}{2}(\Delta R)g_{ij} + rF_{ij}.$$

Using (4.5) for a surface,

$$\nabla_i\nabla_j\Delta\vartheta = \nabla_i\nabla_j\nabla_k\nabla^k\vartheta = \nabla_i\nabla_k\nabla_j\nabla^k\vartheta - \nabla_i(R_{jl}\nabla^l\vartheta)$$

$$= \Delta\nabla_i\nabla_j\vartheta - \nabla^k(R^m_{ikj}\nabla_m\vartheta) - R^m_{ikj}\nabla_m\nabla^k\vartheta - R_{im}\nabla_j\nabla^m\,\vartheta - R_{jm}\nabla_i\nabla^m\vartheta - \nabla_i R_{jm}\nabla^m\vartheta$$

$$= \Delta\nabla_i\nabla_j\vartheta - \frac{1}{2}(\nabla_i R\nabla_j\vartheta + \nabla_i\vartheta\nabla_j R - \langle\nabla R, \nabla\vartheta\rangle g_{ij}) - 2R(\nabla_i\nabla_j\vartheta - \frac{1}{2}\Delta g_{ij}).$$

Combining these results, (5.1) results. □

Corollary 5.1. On a solution $(M^2, g(t))$ of the normalized Ricci flow, the norm squared of the tensor F evolves by

$$\frac{\partial}{\partial t}||F||^2 = \Delta||F||^2 - 2||\nabla F||^2 - 2R||F||^2. \tag{5.2}$$

Proof:

$$\frac{\partial}{\partial t}||F||^2 = \frac{\partial}{\partial t}(g^{ik}g^{jl}F_{ij}F_{kl}) = 2\langle F, \Delta F + (r-2R)F\rangle + 2(R-r)||F||^2 = \Delta||F||^2 - 2||\nabla F||^2 - 2R||F||^2.$$

□

This result motivates the following strategy. If it can be shown that $R \geq c$ for some constant $c > 0$ independent of t, an estimate of the type

$$|F| \leq Ce^{-ct} \tag{5.3}$$

is obtained. Then we can consider the modified Ricci flow equation

$$\frac{\partial g}{\partial t} = 2F = 2\nabla\nabla\vartheta - (R - r)g = (r - R)g + \mathcal{L}_{\nabla\vartheta} g. \tag{5.4}$$

The solution to (5.4) differs from the solution to the normalized Ricci flow (1.9) only by the one-parameter family of diffeomorphisms φ_t generated by the time-dependent vector fields $\nabla\vartheta(t)$. Since M^2 is compact, these diffeomorphisms exist by this result as long as the potential function does.

Theorem 5.2. If $\{X_t : 0 \leq t < T \leq \infty\}$ is a continuous time-dependent family of vector fields on a compact manifold M^n, then there exists a one-parameter family of diffeomorphisms $\{\phi_t : M^n \to M^n, 0 \leq t < T \leq \infty\}$ defined on some time interval such that

$$\frac{\partial \phi_t}{\partial t}(x) = X_t(\phi_t(x)), \qquad \phi_0(x) = x,$$

for all $x \in M^n$ and $t \in [0, T)$. □

Since $|F|^2$ is invariant under diffeomorphism, the estimate $|F|^2 \leq Ce^{-ct}$ will hold on the solution to the modified Ricci flow (5.4). Moreover one can obtain estimates for all derivatives of F, which prove that the solution $g(t)$ to the modified flow converges exponentially fast in all C^k to a metric g_∞ such that M_∞ vanishes identically. A solution $g(t)$ of the normalized Ricci flow on a surface M^2 is called a self-similar solution of the normalized Ricci flow if there exists a one-parameter family of conformal diffeomorphisms $\varphi(t)$ such that $g(t) = \varphi(t) * g(0)$. Differentiation of this with respect to time implies that

$$\frac{\partial}{\partial t} g(t) = \mathcal{L}_X g(t),$$

where $X(t)$ is a one-parameter family of vector fields generated by $\varphi(t)$ and $\mathcal{L}$ denotes the Lie derivative. By definition of Ricci flow, this is equivalent to

$$(r - R)g_{ij} = \nabla_i X_j + \nabla_j X_i,$$

and in this case, $g(t)$ is called a Ricci soliton. If $X = -\nabla\vartheta$ for some function $\vartheta(x,t)$, we have that $g(t)$ is a gradient soliton and it satisfies

$$(R - r)g_{ij} = 2\nabla_i\nabla_j\vartheta. \tag{5.5}$$

Thus (5.5) implies that g_∞ is a gradient soliton. The following Proposition implies that g_∞ is a metric of constant positive curvature.

Proposition 5.1. If $(M^2, g(t))$ is a self-similar solution of the normalized Ricci flow on a Riemannian surface, then $g(t) = g(0)$ is a metric of constant curvature.

Proof: Since any expanding or steady self-similar solution of the Ricci flow on a compact n-dimensional manifold is Einstein, we may assume $r > 0$. By passing to a cover space if necessary, we may assume that M^2 is diffeomorphic to S^2. Contracting the Ricci soliton equation $(r-R)g_{ij} = \nabla_i X_j + \nabla_j X_i$ by Rg^{-1} yields

$$2R(r - R) = 2R \operatorname{div} X,$$

and hence,

$$-\int_{S^2} (R - r)^2 \, dv_2 = \int_{S^2} R(r - R) \, dv_2 = \int_{S^2} R \operatorname{div} X \, dv_2.$$

Since X is a conformal Killing vector field, integrating by parts and using the Kazden-Warner identity implies that

$$\int_{S^2} (R - r)^2 \, dv_2 = \int_{S^2} \langle \nabla, X \rangle \, dv_2 = 0.$$

This simply means that $R = r$. □

It will then follow that there exist positive constants c_k, C_k for each $k \in \mathbb{N}$ such that the solution $g(t)$ of the modified flow satisfies

$$||\nabla^k R|| \leq C_k e^{-c_k t}. \tag{5.6}$$

By diffeomorphism invariance, the same estimate must hold for the solution of the unmodified flow. In this way, it may be concluded that the normalized Ricci flow starting at a metric of strictly positive scalar curvature converges exponentially fast to a constant curvature metric.

6 References.

[1] B. Chow and D. knopf, The Ricci flow: an introduction, volume 110, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, (2004).

[2] B. Chow, P. Lu and L. Ni, Hamilton's Ricci Flow, Graduate Studies in Mathematics, vol. 77, American Mathematical Society, Providence, RI, 2006.

[3] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.

[4] J. M. Lee, Riemannian Manifolds, Graduate Texts in Mathematics, vol 176, Springer-Verlag, NY, 1997.

[5] P. Topping, Lectures on the Ricci Flow, Cambridge University Press, Cambridge, 2006.

[6] C. Sinestrasi, Singularities of Three-Dimensional Ricci Flows, 71-104, in Ricci Flow and Geometric Applications, ed. M. Boileau, G. Besson, Springer, 2016.

[7] M. Anderson, Geometrization of 3-manifolds via the Ricci flow, Notices Amer. Math. Society, **51**, 184-193 (2004).

[8] R. S. Hamilton, Three-manifolds with positive Ricci curvature, J. Differential Geometry, **17**, 255-306, (1982).

[9] B. Chow, The Ricci flow on the 2 sphere, J. Differential Geometry, **33**, 325-334 (1991).

[10] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv: math.DG/0211159.

[11] G. Perelman, Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, arXiv:math.DG/0307245.

[12] B. G. Konopelchenko and I. Taimanov, Constant mean curvature surfaces by means of an integrable deformation, J. Phys. **A 29**, 1261-1265 (1996).

[13] P. Bracken, The Generalized Weierstrass System Inducing Surfaces in Euclidean Three Space and Higher Dimensional Spaces, Partial Differential Equations, ed. C. L. Jang, Nova Sciences Publishers, (2011).

[14] P. B. Gilkey, Invariance theory, the heat equation and the Atiyah-Singer index theorem, Sec-

ond edition, Studies in Advanced Mathematics, CRC Press, Boca Raton, Florida, 1995.

[15] J. Nash, The imbedding problem for Riemannian manifolds, Ann. of Math. (2), **63**, 20-63 (1956).

[16] S. S. Chern, W. H. Chen and K. S. Lam, Lectures on Differential Geometry, World Scientific, Singapore, 1999.