Shape Equations for Two-Dimensional Manifolds through a Moving Frame Variational Approach

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Abstract

A variational approach is considered which can be applied to functionals of a general form to determine a corresponding Euler-Lagrange or shape equation. It is the intention to formulate the theory in detail based on a moving frame approach. It is then applied to a functional of a general form which depends on both the mean and Gaussian curvatures as well as the area and volume elements of the manifold. Only the case of a two-dimensional closed manifold is considered. The first variation of the functional is calculated in terms of the variations of the basic variables of the manifold. The results of the first variation allow for the second variation of the functional to be evaluated.

Keywords: metric, geometry, manifold, curvature, shape equations, variational, moving frame

Mathematics Subject Classification: 49Q10, 53A05, 53Z05

1 Introduction

A membrane with or without a freely exposed edge can be thought of as a surface or differentiable manifold of dimension two in three-dimensional Euclidean space. The case in which the surface is closed so the boundary is the empty set can be formulated as a variational problem [1]. The result of carrying this out is a mathematical description of a surface in the form of differential equations. Equations derived in this manner are usually referred to as shape equations. Exterior differential forms are introduced to describe the surface in terms of a set of structure equations $[2-4]$. This mathematical approach has been used to discuss the relationship between non-linear partial differential equations and the theory of surfaces in terms of their applications [5].

The functiomal which is introduced and studied has a very general structure of the form

$$I = \int_M E\big(2H(\mathbf{r}), K(\mathbf{r})\big)\, dA + p \int_V dV. \tag{1.1}$$

In (1.1), the functions H and K are the mean curvature and Gaussian curvature at point $\mathbf{r}$ of M and p is a constant, M refers to integration on the surface and V is the volume enclosed within the surface. The mean curvature is the average of the principal curvatures, and the Gaussian curvature is the product of these two curvatures. The function E in the integrand of (1.1) should be differentiable with respect to both slots. A specific form for (1.1) which has been studied already is the case in which E is proportional to the square of the mean curvature. This functional gives the Willmore energy or the classical binding energy which comes up in many applications in many areas of science as diverse as biology [6] and string theory [7]. Associated with this functional is the Willmore conjecture. This is one of the more prominent results in the study of Willmore energies and has been proved by Marques and Neves [8] by applying min-max theory of minimal surfaces. The energy expresses how much the surface has deviated from a round sphere. The minimizers of the Willmore functional are referred to as Willmore surfaces $[9, 10]$. These correspond to the solutions of the resulting Euler-Lagrange equation corresponding to (1.1) in this Willmore case [11]. This Willmore energy is important in the context of conformal geometry, since it is known to be invariant under the conformal transformations of E^3.

To put some of this in a more physical context, liquid crystal structures in physics can be regarded as a smooth surface as well, with or without boundary depending on the nature of the system under consideration [12]. Examples of these appear in very distinct areas such as condensed matter physics and biology [13]. Both lipid bilayers and cell membranes are referred to collectively as bio-membranes. The geometric formalism that is proposed here is very well suited to investigate the shapes and stability of such membranes. These objects might be described as closed surfaces, or bilayers with exposed edges, which in a variational framework can be treated as a boundary term. Physical quantities such as the total free energy can be assigned to these structures. Theoretical study of such structures has the objectives of producing differential equations called shape equations by varying a given functional such as (1.1). The shape equations can be given in terms of a few parameters which can be used to model these kinds of physical systems.

The intention here is to first introduce the necessary concept of the moving frame which appears in from differential geometry and necessary information pertaining to variational theory of two-dimensional manifolds. The variations of the all the differential forms have to be defined. The work is primarily devoted to the case of closed manifolds. The general functional (1.1) shall undergo a variational treatment to formulate an Euler-Lagrange equation in the basic surface variables. This allows the discussion of the equilibrium shapes and mechanical stabilities of closed surfaces and admit the type of application already mentioned. The first and second variations of functional (1.1) is obtained first. The first variation produces the Euler-Lagrange equation corresponding to the associated functional. These results can be used later to produce shape equations of various kinds which are relevant to the study of many physical systems [14 − 16].

2 Surfaces in Three-Dimensional Euclidean Space

Let M be a smooth orientable manifold in three-dimensional Euclidean space $\mathbb{E}^3$. At every point P of M an orthonormal system $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ can be constructed such that $\mathbf{e}_3$ is normal to the surface so they satisfy $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. The set $\{P, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is called a moving frame for the manifold.

The difference between two frames at points P and P' very close to P is given as

$$d\mathbf{r} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2, \tag{2.1}$$

$$d\mathbf{e}_i = \omega_{ij}\mathbf{e}_j, \qquad i = 1, 2, 3, \tag{2.2}$$

where ω_1, ω_2 and ω_{ij} $(i, j = 1, 2, 3)$ are 1-forms.

From $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ it follows that $\omega_{ij} = -\omega_{ji}$, and from $dd\mathbf{r} = 0$, the structure equations of the surface satisfied by these differential forms follow,

$$d\omega_1 = \omega_{12} \wedge \omega_2, \tag{2.3}$$

$$d\omega_2 = \omega_{21} \wedge \omega_1, \tag{2.4}$$

$$\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} = 0, \tag{2.5}$$

$$d\,\omega_{ij} = \omega_{ik} \wedge \omega_{kj}, \qquad i, j = 1, 2, 3. \tag{2.6}$$

By Cartan's lemma, (2.5) implies that there exist functions on M such that ω_{13} and ω_{23} can be written

$$\omega_{13} = a\omega_1 + b\omega_2, \qquad \omega_{23} = b\omega_1 + c\omega_2. \tag{2.7}$$

The area form on M which appears in (1.1) is given by

$$d\,A = \omega_1 \wedge \omega_2, \tag{2.8}$$

Using these basis differential forms, the three fundamental forms for the surface can be expressed as

$$I = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2, \quad II = a\omega_1 \otimes \omega_1 + 2b\omega_1 \otimes \omega_2 + c\omega_2 \otimes \omega_2, \quad III = \omega_{31} \otimes \omega_{31} + \omega_{32} \otimes \omega_{32}. \tag{2.9}$$

In terms of the functions a, b and c, the mean curvature H and Gaussian curvature K of M are defined to be

$$H = \frac{1}{2}(a + c), \qquad K = a\,c - b^2. \tag{2.10}$$

The operator $*$ will appear frequently. It is called the Hodge operator and it is defined on M so that if f and g are ordinary functions, then

$$*f = f\,\omega_1 \wedge \omega_2, \qquad *\omega_1 = \omega_2, \qquad *\omega_2 = -\omega_1. \tag{2.11}$$

The operator $*$ can be used to define differential operators on M such as

$$d * df = \nabla^2 f \, \omega_1 \wedge \omega_2, \tag{2.12}$$

where ∇^2 is the Laplace-Beltrami operator.

With respect to a basis of one-forms ω_1, ω_2, the differential df is expressed as

$$df = f_1 \omega_1 + f_2 \omega_2, \tag{2.13}$$

which serves to define the derivatives f_i. The Gauss map $\mathcal{G} : M \to S^2$ is defined as $\mathcal{G}(\mathbf{r}) = \mathbf{e}_3(\mathbf{r})$, where S^2 is the unit sphere. It induces a linear map denoted $\mathcal{G}^* : \Lambda^1 \to \Lambda^1$ of forms so that using (2.13),

$$\mathcal{G}^* \omega_1 = \omega_{13}, \qquad \mathcal{G}^* \omega_2 = \omega_{23} \qquad \mathcal{G}^* \, df = f_1 \mathcal{G}^* \omega_1 + f_2 \mathcal{G}^* \omega_2. \tag{2.14}$$

Consequently, a new differential operator denoted $\tilde{d} = \mathcal{G}^* \, d$ can be defined with respect to (2.13) as

$$\tilde{d} f = f_1 \omega_{13} + f_2 \omega_{23}. \tag{2.15}$$

To represent this more effectively, an operator $\tilde{*}$ is defined such that $\tilde{*} \omega_{13} = \omega_{23}$ and $\tilde{*} \omega_{23} = -\omega_{13}$. Since $d \tilde{*} \tilde{d} f$ is a two form, there is a corresponding operator $\nabla \cdot \tilde{\nabla}$ defined such that

$$d \tilde{*} \tilde{d} f = \nabla \cdot \tilde{\nabla} f \, \omega_1 \wedge \omega_2. \tag{2.16}$$

The variational theory of surfaces which is required will be introduced. If we let M undergo an infinitesimal deformation such that every point $\mathbf{r}$ of M has a displacement $d\,\mathbf{r}$, a new surface $M' = \{\mathbf{r}' | \mathbf{r}' = \mathbf{r} + \delta \mathbf{r}\}$is obtained. The quantity $\delta \mathbf{r}$ is called the variation of the surface M and may be expressed as

$$\delta \mathbf{r} = \delta_1 \mathbf{r} + \delta_2 \mathbf{r} + \delta_3 \mathbf{r}, \qquad \delta_i \mathbf{r} = \Omega_{ij} \mathbf{e}_i, \tag{2.17}$$

and the repeated index is not summed in (2.17).

The following definition will be used. If f is a generalized function of $\mathbf{r}$, that is a scalar, vector function, an even r-form dependent on a point $\mathbf{r}$, define $\delta_i^{(p)}$ to have the action

$$\delta_i^{(p)} f = p! \, \mathcal{L}^{(p)} [f(\mathbf{r} + \delta_i \mathbf{r}) - f(\mathbf{r})] \tag{2.18}$$

and $p = 1, 2, 3, \ldots$, with the p-order variation of f as

$$\delta^{(p)} f = p!\, \mathcal{L}^{(p)}[f(\mathbf{r} + \delta\mathbf{r}) - f(\mathbf{r})], \tag{2.19}$$

where $\mathcal{L}^{(q)}[\cdots]$ represents the terms of $\Omega_1^{q_1}\Omega_2^{q_2}\Omega_3^{q_3}$ in a Taylor series of what is in the square bracket, with $p = p_1 + p_2 + p_3$ and p_1, p_2, p_3 non-negative integers.

Some consequences of these definitions which are useful to know are that $\delta_i^{(p)}$ and $\delta^{(q)}$, $p, q = 1, 2, 3, \ldots$ are linear operators; the operators $\delta_1^{(1)}, \delta_2^{(1)}, \delta_3^{(1)}$ and $\delta^{(1)}$ all commute; finally

$$\delta_i f(g(\mathbf{r})) = \big(\frac{\partial f}{\partial g}\big)\, \delta_i g, \tag{2.20}$$

and $\delta^p = (\delta_1 + \delta_2 + \delta_3)^p$.

Due to the deformation of M, the vectors $\mathbf{e}_i$ are also altered. The changes are denoted

$$\delta_l\, \mathbf{e}_i = \Omega_{lij}\, \mathbf{e}_j. \tag{2.21}$$

Clearly the fact that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ implies $\Omega_{lij} = -\Omega_{lji}$. As the operators δ_i are a set of linear mappings from M to M', they commute with the exterior differential operator d; that is, $d\delta_l\mathbf{r} = \delta_l d\mathbf{r}$ and $d\delta_l\mathbf{e}_j = \delta_l d\mathbf{e}_j$, and f a function, $\delta_i\, \tilde{d} f = \tilde{d}\delta_i f$. Finally, it remains to define how the ω_i, ω_{ij} are to be varied under the operation of the δ_i. The following set of relations will be used frequently in many of the calculations that follow.

$$\delta_1\omega_1 = d\Omega_1 - \omega_2\Omega_{121}, \tag{2.22}$$

$$\delta_1\omega_2 = \Omega_1\omega_{12} - \omega_1\Omega_{112}, \tag{2.23}$$

$$\Omega_{113} = a\Omega_1, \qquad \Omega_{123} = b\Omega_1, \tag{2.24}$$

$$\delta_2\omega_1 = \Omega_2\omega_{21} - \omega_2\Omega_{221}, \tag{2.25}$$

$$\delta_2\omega_2 = d\Omega_2 - \omega_1\Omega_{212}, \tag{2.26}$$

$$\Omega_{213} = b\Omega_2, \qquad \Omega_{223} = c\Omega_2, \tag{2.27}$$

$$\delta_3\omega_1 = \Omega_3\omega_{31} - \omega_2\Omega_{321}, \tag{2.28}$$

$$\delta_3\omega_2 = \Omega_3\omega_{32} - \omega_1\Omega_{312}, \tag{2.29}$$

$$d\Omega_3 = \Omega_{313}\omega_1 + \Omega_{323}\omega_2; \tag{2.30}$$

$$\delta_l \omega_{ij} = d\Omega_{lij} + \Omega_{lik}\omega_{kj} - \omega_{ik}\Omega_{lkj}. \tag{2.31}$$

These equations will be used in all the variational calculations that follow.

3 The First Variation of I

The first order variation of the functional (1.1) is calculated first. The general variational theory of surfaces requires the computation of

$$\delta\, I = \delta_1\, I + \delta_2 I + \delta_3\, I. \tag{3.1}$$

Each of the variations on the right have to be calculated independently. Since I in (1.1) is a sum of two integrals, the first term is referred to as I_e and the volume term as I_v. To calculate (3.1), we begin with the variations

$$\delta_i\, I_e = \int_M \big[\frac{\partial E}{\partial(2H)}\, \delta_i(2H)\, \omega_1 \wedge \omega_2 + \frac{\partial E}{\partial K}\, \delta_i K\, \omega_1 \wedge \omega_2 + E\, \delta_i(\omega_1 \wedge \omega_2)\big]. \tag{3.2}$$

To obtain $\delta_i I_v$, it suffices to calculate $\delta_i(dV)$ for $i = 1, 2, 3$. To carry out the first step of working out (3.2), the $\delta_i(2H)$, $\delta_i K$, as well as $\delta_i(\omega_1 \wedge \omega_2)$ must be determined. Finally the variations of dV are obtained.

3.1 The Variation δ_1

Theorem 3.1:

$$\delta_1(\omega_1 \wedge \omega_2) = d(\Omega_1 \omega_2), \qquad \delta_2(\omega_1 \wedge \omega_2) = -d(\Omega_2 \omega_1). \tag{3.3}$$

Proof: Using the rules from (2.22) to (2.27) it follows that

$$\delta_1(\omega_1 \wedge \omega_2) = \delta_1\omega_1 \wedge \omega_2 + \omega_1 \wedge \delta_1\omega_2 = (d\Omega_1 - \omega_2\Omega_{121}) \wedge \omega_2 + \omega_1 \wedge (\Omega_1\omega_{12} - \omega_1\Omega_{112})$$

$$= d\Omega_1 \wedge \omega_2 + \Omega_1\omega_1 \wedge \omega_{12} = d\Omega_1 \wedge \omega_2 + \Omega_1 d\omega_2 = d(\Omega_1\omega_2).$$

$$\delta_2(\omega_1 \wedge \omega_2) = (\Omega_2\omega_{21} - \omega_2\Omega_{221}) \wedge \omega_2 + \omega_1 \wedge (d\Omega_2 - \omega_1\Omega_{212}) = -d(\Omega_2\omega_1).$$

$\square$

The next large step is to calculate $\delta_i(2H)$ and $\delta_i K$ for the cases $i = 1, 2$. Begin with ω_{13} so using (2.7), we obtain

$$\delta_1 \omega_{13} = (\delta_1 a)\omega_1 + a(\delta_1 \omega_1) + (\delta_1 b)\omega_2 + b(\delta_i \omega_2)$$

$$= (\delta_1 a)\omega_1 + a(d\Omega_1 + \omega_2 \Omega_{112}) + (\delta_1 b)\omega_2 + b(\Omega_1 \omega_{12} - \omega_1 \Omega_{112}). \tag{3.4}$$

Another expression for $\delta_1 \omega_{13}$ can be found by using (2.31),

$$\delta_1 \omega_{13} = d(a\Omega_1) + \Omega_{112}\omega_{23} - \omega_{12}(b\,\Omega_1) = (da)\Omega_1 + ad\Omega_1 + \Omega_{112}(b\omega_1 + c\omega_2) - b\Omega_1 \omega_{12}. \tag{3.5}$$

Equating (3.4) and (3.5), a single equation is found

$$(\delta_1 a)\omega_1 + (\delta_1 b)\omega_2 + a\Omega_{112}\omega_2 + 2b\Omega_1\omega_{12} - 2b\Omega_{112}\omega_1 = (da)\Omega_1 + c\Omega_{112}\omega_2. \tag{3.6}$$

Based on (3.6), the wedge product on the right with ω_1 and ω_2 is calculated giving the following two equations

$$-(\delta_1 b)\omega_1 \wedge \omega_2 - (a-c)\,\Omega_{112}\omega_1 \wedge \omega_2 + 2b\Omega_1 \omega_{12} \wedge \omega_1 = \Omega_1\, da \wedge \omega_1, \tag{3.7}$$

$$(\delta_1 a)\omega_1 \wedge \omega_2 + 2b\Omega_1\omega_{12} \wedge \omega_2 - 2b\Omega_{112}\omega_1 \wedge \omega_2 = \Omega_1\, da \wedge \omega_2. \tag{3.8}$$

Beginning with ω_{23} in (2.7) and proceeding in exactly the same way, we calculate the pair

$$\delta_1 \omega_{23} = (\delta_1 b)\omega_1 + b(d\Omega_1 - \omega_2 \Omega_{121}) + (\delta_1 c)\omega_2 + c(\Omega_1 \omega_{12} - \omega_1 \Omega_{112}), \tag{3.9}$$

$$\delta_1 \omega_{23} = d\Omega_{123} + \Omega_{12k}\omega_{k3} - \Omega_{1k3}\omega_{2k} = d\,(b\,\Omega_1) + \Omega_{121}\omega_{13} - a\Omega_1\omega_{21}. \tag{3.10}$$

Equating (3.9) and (3.10) a single equation results

$$(\delta_1 b)\,\omega_1 + b\Omega_{112}\omega_2 + (\delta_1 c)\omega_2 + (c-a)\Omega_1\omega_{12} - c\Omega_{112}\omega_1 = (db)\Omega_1 - \Omega_{112}\omega_{13}. \tag{3.11}$$

The wedge product of ω_1 and then ω_2 can be calculated as was done with ω_{13} to give two more equations. Let us summarize equations (3.7) and (3.8) along with these two new equations using (2.3) and (2.4):

$$-(\delta_1 b)\omega_1 \wedge \omega_2 - (a-c)\Omega_{112}\omega_1 \wedge \omega_2 - 2b\Omega_1 d\omega_2 = \Omega_1 (da) \wedge \omega_1, \tag{3.12}$$

$$(\delta_1 a)\omega_1 \wedge \omega_2 + 2b\Omega_1 d\omega_1 - 2b\Omega_{112}\,\omega_1 \wedge \omega_2 = \Omega_1 (da) \wedge \omega_2, \tag{3.13}$$

$$-(2b)\Omega_{112}\omega_1 \wedge \omega_2 - (\delta_1 c)\omega_1 \wedge \omega_2 - (c-a)\Omega_1\, d\omega_2 = \Omega_1 (db) \wedge \omega_1, \tag{3.14}$$

$$(\delta_1 b)\omega_1 \wedge \omega_2 + (c-a)\Omega_1\, d\omega_1 + (a-c)\Omega_{112}\,\omega_1 \wedge \omega_2 = \Omega_1 (\delta b) \wedge \omega_2. \tag{3.15}$$

It is now a straightforward procedure to apply these equations to obtain the variations we need.

Lemma 3.1: The following constraint equations hold

$$(db) \wedge \omega_1 + 2b\, d\omega_1 = (a-c)\, d\omega_2 - (dc) \wedge \omega_2, \quad (db) \wedge \omega_2 + 2b\, d\omega_2 = (c-a)\, d\omega_1 - (da) \wedge \omega_1. \tag{3.16}$$

Proof: Add (3.12) and (3.15) to produce the second equation in (3.16) immediately. The first equation appears as a consequence of structure equation (2.6) by equating $d\omega_{23} = db \wedge \omega_1 + b\, d\omega_1 + dc \wedge \omega_2 + c d\omega_2$ and $\omega_{21} \wedge \omega_{23} = a\omega_{21} \wedge \omega_1 + b\omega_{21} \wedge \omega_2$.

□

Theorem 3.2: The variations $\delta_1(2H)$ and $\delta_1 K$ are given by

$$\delta_1(2H)\omega_1 \wedge \omega_2 = d(2H) \wedge \omega_2\, \Omega_1, \qquad (\delta_1 K)\, \omega_1 \wedge \omega_2 = dK \wedge \omega_2\, \Omega_1. \tag{3.17}$$

Proof: Take the linear combination of (3.13) and (3.14) and use the definition of mean curvature to get

$$\delta_1(2H)\omega_1 \wedge \omega_2 + 2b\Omega_1 d\omega_1 + (c-a)\Omega_1\, d\omega_2 = \Omega_1\, da \wedge \omega_2 - \Omega_1\, db \wedge \omega_1. \tag{3.18}$$

Using the results from Lemma 3.1, substitute $2b\, d\omega_1$ in (3.18) to obtain

$$\frac{1}{\Omega_1}\delta_1(2H)\, \omega_1 \wedge \omega_2 - db \wedge \omega_1 - dc \wedge \omega_2 = da \wedge \omega_2 - db \wedge \omega_1. \tag{3.19}$$

Equation (3.19) simplifies to the first equation in (3.17).

From the definition of the Gausian curvature, it follows that $\delta_1 K = c(\delta_1 a) + a(\delta_1 c) - 2b(\delta_1 b)$, write the linear combination c (3.13) - a (3.14)+ b (3.12) -b (3.15) as follows

$$c(\delta_1 a)\omega_1 \wedge \omega_2 + 2bc\Omega_1 d\omega_1 - 2bc\Omega_{112}\,\omega_1 \wedge \omega_2 + 2ab\Omega_{112}\omega_1 \wedge \omega_2 + a(\delta_1 c)\omega_1 \wedge \omega_2 + a(c-a)\Omega_1\, d\omega_2$$

$$-b(\delta_1 b)\,\omega_1 \wedge \omega_2 - b(a-c)\Omega_{112}\omega_1 \wedge \omega_2 - 2b^2\Omega_1\, d\omega_2 - b(\delta_1 b)\,\omega_1 \wedge \omega_2 - b(c-a)\Omega_1 d\omega_1 - b(a-c)\Omega_{112}\,\omega_1 \wedge \omega_2$$

$$= \Omega_1(c\,da \wedge \omega_2 - adb \wedge \omega_1 + bda \wedge \omega_1 - bdb \wedge \omega_2). \tag{3.20}$$

Collect like terms to find that those containing Ω_{112} cancel out. Using the equations from Lemma 3.1, we get

$$\frac{1}{\Omega_1}(\delta_1 K)\,\omega_1 \wedge \omega_2 + (c+a)\,b\,d\omega_1 + (a(c-a) - 2b^2)\,d\omega_2$$

$$= c\,da \wedge \omega_2 - a(-2b\,d\omega_1 + (a-c)\,d\omega_2 - dc \wedge \omega_2) + b((c-a)\,d\omega_1 - db \wedge \omega_2 - 2b\,d\omega_2) - b(db) \wedge \omega_2.$$

Upon simplifiying this the second result in (3.17) appears. □

Theorem 3.3:

$$\delta_1 I_e = \int_M d(E\omega_2\,\Omega_1). \tag{3.21}$$

Proof: Using the result in (3.2) and Theorem 3.2, we obtain that

$$\delta_1 I_e = \int_M \big[\frac{\partial E}{\partial(2H)}\,\delta_1(2H)\,\omega_1 \wedge \omega_2 + \frac{\partial E}{\partial K}\delta_1 K\,\omega_1 \wedge \omega_2 + Ed(\Omega_1\omega_2)\big]$$

$$= \int_M \big[\frac{\partial E}{\partial(2H)}\,d(2H) \wedge \omega_2\Omega_1 + \frac{\partial E}{\partial K}\,dK \wedge \omega_2\Omega_1 + Ed(\Omega_1\omega_2)\big) = \int_M \big[dE \wedge \omega_2\Omega_1 + E\,d(\omega_2\Omega_1)\big]$$

$$= \int_M d(E\omega_2\Omega_1).$$

□

As a consequence of Theorem 3.3, it follows that since M is a closed manifold,

$$\delta_1\, I_e = 0. \tag{3.22}$$

3.2 The Variation δ_2

In exactly the same manner, acting on ω_{13} and ω_{23} with δ_2 and projecting out with respect to ω_1, ω_2, the following four equations emerge

$$\frac{1}{\Omega_2}(\delta_2 b)\omega_1 \wedge \omega_2 - \frac{\Omega_{221}}{\Omega_2}(a-c)\omega_1 \wedge \omega_2 - (a-c)\,d\omega_2 = -(db) \wedge \omega_1, \tag{3.23}$$

$$\frac{1}{\Omega_2}(\delta_2 a)\omega_1 \wedge \omega_2 + \frac{\Omega_{221}}{\Omega_2}2b\,\omega_1 \wedge \omega_2 - (a-c)\,d\omega_1 = (db) \wedge \omega_2, \tag{3.24}$$

$$\frac{1}{\Omega_2}(\delta_2 c)\omega_1 \wedge \omega_2 - (2b)\frac{\Omega_{221}}{\Omega_2}\omega_1 \wedge \omega_2 - 2b\,d\omega_2 = -(dc) \wedge \omega_1, \tag{3.25}$$

$$\frac{1}{\Omega_2}(\delta_2 b)\omega_1 \wedge \omega_2 - \frac{\Omega_{221}}{\Omega_2}(a-c)\omega_1 \wedge \omega_2 - 2b\,d\omega_1 = (dc) \wedge \omega_2. \tag{3.26}$$

If the linear combination (3.26) minus (3.23) is calculated, the first equation of (3.16) appears.

Theorem 3.4:

$$\delta_2(2H)\,\omega_1\wedge\omega_2 = -d(2H)\wedge\omega_1\Omega_2, \qquad \delta_2(K)\,\omega_1\wedge\omega_2 = -dK\wedge\omega_1\Omega_2. \tag{3.27}$$

Proof: To obtain the first member of (3.27), add (3.24) and (3.25) together

$$(db)\wedge\omega_2 - (dc)\wedge\omega_1 + (a-c)\,d\omega_1 + 2b\,d\omega_2 = \frac{1}{\Omega_2}\delta_2(2H)\omega_1\wedge\omega_2.$$

Eliminate $db\wedge\omega_2$ using constraint equation (3.16) and then simplify.

Solve (3.23)-(3.26) for the δ_2 variations and then form the combination $\delta_2 K = c(\delta_2 a) + a(\delta_2 c) - 2b(\delta_2 b)$. After simplifying we have

$$\frac{1}{\Omega_2}\delta_2 K = (c(a-c)-2b^2)\,d\omega_1 + (2ab - b(a-c))\,d\omega_2 + c db\wedge\omega_2 - a\,dc\wedge\omega_1 + b(db)\wedge\omega_1 - b(dc)\wedge\omega_2.$$

Using (3.16) once more, this turns into

$$\frac{1}{\Omega_2}\delta_2 K = (c(a-c)-2b^2)\,d\omega_1 + (2ab-b(a-c))\,d\omega_2 - (2bc)d\omega_2 + c(c-a)\,d\omega_1$$

$$-c(da)\wedge\omega_1 - a(dc)\wedge\omega_1 + b\,(db)\wedge\omega_1 - b(a-c)\,d\omega_2 + 2b^2 d\omega_1 + b(db)\wedge\omega_1$$

$$= -(c\,da + a\,dc - b\,db - b\,db)\wedge\omega_1 = -dK\wedge\omega_1.$$

This is exactly the second equation in (3.27). □

Theorem 3.5:

$$\delta_2 I_e = \int_M d(E\omega_1\,\Omega_2). \tag{3.28}$$

The proof of (3.28) proceeds in exactly the same way as that of (3.21). Since M is a closed manifold, this theorem implies that

$$\delta_2 I_e = 0. \tag{3.29}$$

3.3 The Variation δ_3

Since it has been shown that $\delta_i I_e = 0$ when $i = 1, 2$, the variation δI_e is given entirely by means of the operator δ_3. To determine this variation, proceed as in the previous cases, but use the formulas for δ_3 and (2.28)-(2.30) and (2.31).

Lemma 3.2:

$$\delta_3(\omega_1 \wedge \omega_2) = -2H\,\Omega_3\,\omega_1 \wedge \omega_2. \tag{3.30}$$

Proof:

$$\delta_3\,(\omega_1 \wedge \omega_2) = \delta_3\omega_1 \wedge \omega_2 + \omega_1 \wedge \delta_3\omega_2 = (\Omega_3\omega_{31} - \omega_2\Omega_{321}) \wedge \omega_2 + \omega_1 \wedge (\Omega_3\omega_{32} - \omega_1\Omega_{312})$$

$$= \Omega_3(\omega_{31} \wedge \omega_1 + \omega_1 \wedge \omega_{32}).$$

Substituting (2.7), we see that $\omega_{13} \wedge \omega_2 + \omega_1 \wedge \omega_{32} = -(2H)\,\omega_1 \wedge \omega_2$, (3.30) follows. □

Lemma 3.3:

$$d(*d\Omega_3) = d\Omega_{313} \wedge \omega_2 + \Omega_{313}\,d\omega_2 - d\Omega_{313} \wedge \omega_1 - \Omega_{313}\,d\omega_1, \tag{3.31}$$

$$d * \tilde{d}\Omega_3 = d\Omega_{313} \wedge (b\omega_1 + c\omega_2) + \Omega_{313}a\,d\omega_2 - \Omega_{313}b\,d\omega_1 - d\Omega_{323} \wedge (a\omega_1 + b\omega_2) + \Omega_{323}\,b\,d\omega_2 - \Omega_{323}c\,d\omega_1. \tag{3.32}$$

Proof: Since $d\Omega_3 = \Omega_{313}\omega_1 + \Omega_{323}\omega_2$, we have $*d\Omega_3 = \Omega_{313}\omega_2 - \Omega_{323}\omega_1$. Differentiating with respect to d gives (3.31).

Since $\tilde{d}\Omega_3 = \Omega_{313}\omega_{13} + \Omega_{323}\omega_{23}$, then $*\tilde{d}\Omega_3 = \Omega_{313}\omega_{23} - \Omega_{323}\omega_{13}$. Differentiating once again with respect to d gives

$$d * \tilde{d}\Omega_3 = d\Omega_{313} \wedge \omega_{23} + \Omega_{313}d\omega_{23} - d\Omega_{323} \wedge \omega_{13} - \Omega_{323}\,d\omega_{13}.$$

Substituting ω_{13} and ω_{23} from (2.7) and using (2.3)-(2.4), equation (3.32) results. □

A system of equations can be obtained in a similar way to those obtained for δ_1 and δ_2. Calculating $\delta_3\,\omega_{13} = (\delta_3 a)\omega_1 + a(\delta_3\omega_1) + (\delta_3 b)\omega_2 + b(\delta_3\omega_2)$ and equating it to the expression obtained from (2.31), we obtain

$$(\delta_3 a)\omega_1 + a(-\Omega_3(a\omega_1 + b\omega_2) - \omega_2\Omega_{321}) + (\delta_3 b)\omega_2 + b(-\Omega_3(b\omega_1 + c\omega_2) - \omega_1\Omega_{312})$$

$$= d\Omega_{313} - \Omega_{321}(b\omega_1 + c\omega_2) - \omega_{12}\Omega_{323}. \tag{3.33}$$

Similarly, using ω_{23}, we deduce that

$$(\delta_3 b)\omega_1 + b(-\Omega_3(a\omega_1 + b\omega_2) - \omega_2\Omega_{321}) + (\delta_3 c)\omega_2 + c(-\Omega_3(b\omega_1 + c\omega_2) - \omega_1\Omega_{312})$$

$$= d\Omega_{323} + \Omega_{321}(a\omega_1 + b\omega_2) - \Omega_{313}\omega_{21}. \tag{3.34}$$

Taking (3.33) and (3.34) and projecting on ω_1 and ω_2 using the wedge product, each of these equations produces two equations and these are summarized here

$$\Omega_3(ab + \frac{\Omega_{321}}{\Omega_3}(a-c) - \frac{\delta_3 b}{\Omega_3} + bc)\omega_1 \wedge \omega_2 = d\Omega_{313} \wedge \omega_1 + \Omega_{323} d\omega_2, \tag{3.35}$$

$$\Omega_3(\frac{\delta_3 a}{\Omega_3} - a^2 - b^2 - 2b\frac{\Omega_{312}}{\Omega_3})\omega_1 \wedge \omega_2 = d\Omega_{313} \wedge \omega_2 - \Omega_{323} d\omega_1, \tag{3.36}$$

$$\Omega_3(b^2 + 2b\frac{\Omega_{321}}{\Omega_3} - \frac{\delta_3 c}{\Omega_3} + c^2)\omega_1 \wedge \omega_2 = d\Omega_{323} \wedge \omega_1 - \Omega_{313}\, d\omega_2, \tag{3.37}$$

$$\Omega_3(\frac{\delta_3 b}{\Omega_3} - ab - bc - (c-a)\frac{\Omega_{312}}{\Omega_3})\omega_1 \wedge \omega_2 = d\Omega_{323} \wedge \omega_2 + \Omega_{313}\, d\omega_1. \tag{3.38}$$

Theorem 3.6:

$$\delta_3(2H)\,\omega_1 \wedge \omega_2 = (4H^2 - 2K)\,\Omega_3\omega_1 \wedge \omega_2 + d * d\Omega_3. \tag{3.39}$$

Proof: Calculate the linear combination (3.36) minus (3.37) to obtain

$$\Omega_3\big(\frac{\delta_3 a}{\Omega_3} - a^2 - b^2 + 2b\frac{\Omega_{321}}{\Omega_3} - b^2 - 2b\frac{\Omega_{321}}{\Omega_3} + \frac{\delta_3 c}{\Omega_3} - c^2\big)\omega_1 \wedge \omega_2 = d\Omega_{313} \wedge \omega_2 + \Omega_{313}\, d\omega_2 - d\Omega_{323} \wedge \omega_1 - \Omega_{313}\, d\omega_1.$$

Using Lemma 3.3, this can be put in the form,

$$\Omega_3\big(\frac{\delta_3(2H)}{\Omega_3} - a^2 - 2ac - c^2 + 2ac - 2b^2\big)\,\omega_1 \wedge \omega_2 = d * d\Omega_3.$$

Since $4H^2 - 2K = a^2 + 2ac + c^2 - 2ac + 2b^2$, this is (3.39). $\square$

Theorem 3.7:

$$\delta_3 K\,\omega_1 \wedge \omega_2 = 2HK\,\omega_1 \wedge \omega_2\,\Omega_3 + d * \tilde{d}\Omega_3. \tag{3.40}$$

Proof: Form the linear combination of equations $-a$ (3.37) + c (3.36) + b (3.35) -b (3.38) and simplify to arrive at

$$\big(a\delta_3 c + c\delta_3 a - b\delta_3 b - b\delta_3 b + \Omega_3(ab^2 + b^2 c - 2ac^2)\big)\omega_1 \wedge \omega_2 = (\delta_3 K - 2HK\Omega_3)\,\omega_1 \wedge \omega_2.$$

Here the equation $(2H)K = (ac - b^2)(a + c) = 2ac^2 - ab^2 - b^2 c$ has been used. $\square$

Using Theorems 3.6 and 3.8 we arrive at the following important Theorem.

Theorem 3.8:

$$\delta_3 I_e = \int_M \big((4H^2 - 2K)\,\frac{\partial E}{\partial(2H)} + (2HK)\frac{\partial E}{\partial K} - 2H\,E\big)\Omega_3\,\omega_1 \wedge \omega_2$$
$$+ \int_M \big[\frac{\partial E}{\partial(2H)}\,d * d\Omega_3 + \frac{\partial E}{\partial K}\,d\tilde{*}\tilde{d}\Omega_3\big]. \tag{3.41}$$

Proof:

$$\delta_3 I_e = \int_M \delta_3 E\omega_1 \wedge \omega_2 + \int_M E\,\delta_3(\omega_1 \wedge \omega_2)$$
$$= \int_M \frac{\partial E}{\partial(2H)}\delta_3(2H)\omega_1 \wedge \omega_2 + \frac{\partial E}{\partial K}(\delta_3 K)\omega_1 \wedge \omega_2 + E\delta_3(\omega_1 \wedge \omega_2)$$
$$= \int_M \big((4H^2 - 2K)\,\frac{\partial E}{\partial(2H)} + 2HK\frac{\partial E}{\partial K} - 2HE\big)\Omega_3\omega_1 \wedge \omega_2$$
$$+ \int_M \big[\frac{\partial E}{\partial(2H)}\,d * d\Omega_3 + \frac{\partial E}{\partial K}\,d\tilde{*}\tilde{d}\Omega_3\big].$$

□

Corollary 3.1: For a closed surface M,

$$\delta_3 I_e = \int_M \big[(\nabla^2 + 4H^2 - 2K)\frac{\partial E}{\partial(2H)} + (\nabla \cdot \tilde{\nabla} + 2HK)\,\frac{\partial E}{\partial K} - 2HE\big]\Omega_3\,\omega_1 \wedge \omega_2. \tag{3.42}$$

Proof: Use (2.12) and (2.16) in the last term of (3.41), then applying Green's identity and Stokes Theorem. Namelt, it follows that if M is a closed manifold, then for functions f and h,

$$\int_M f\,d * dh = \int_M h\,d * df, \qquad \int_M f\tilde{*}\tilde{d}h = \int_M h\,d\tilde{*}\tilde{d}f.$$

Result (3.42) follows immediately. □

3.4 Variation of I_p

To finish, it remains to calculate the variation of I_p. This amounts to calculating the variations of element dV, which is not trivial.

Lemma 3.4:

$$\delta_1 \int_V dV = \frac{1}{3}\int_M d(\mathbf{r} \cdot \mathbf{e}_3\Omega_1\,\omega_2), \qquad \delta_2 \int_V dV = -\frac{1}{3}\int_M d(\mathbf{r} \cdot \mathbf{e}_3\Omega_2\,\omega_1). \tag{3.43}$$

Proof: Since M is closed, Stokes's theorem implies that, with $M = \partial V$,

$$\int_V 3\,dV = \int_V \nabla\cdot\mathbf{r}\,dV = \int_M \mathbf{r}\cdot\mathbf{n}\,\omega_1\wedge\omega_2. \tag{3.44}$$

Dividing by 3 and applying δ_1 to both sides yields

$$\delta_1\int_V dV = \frac{1}{3}\int_M \big(\delta_1\mathbf{r}\cdot\mathbf{e}_3\,\omega_1\wedge\omega_2 + \mathbf{r}\cdot(\delta_1\mathbf{e}_3)\,\omega_1\wedge\omega_2 + (\mathbf{r}\cdot\mathbf{e}_3)\delta_1\omega_1\wedge\omega_2\big).$$

The first and third terms are $\delta_1\mathbf{r}\cdot\mathbf{e}_3(\omega_1\wedge\omega_2) = \Omega_1\mathbf{e}_1\cdot\mathbf{e}_3\omega_1\wedge\omega_2 = 0$ and $\mathbf{r}\cdot\mathbf{e}_3\delta_1(\omega_1\wedge\omega_2) = \mathbf{r}\cdot\mathbf{e}_3\,d(\Omega_1\omega_2)$. To obtain the second term, note that $d\mathbf{r} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2$ so $\mathbf{e}_3\cdot d\mathbf{r} = 0$. Moreover, $d\mathbf{e}_3 = \omega_{3j}\,\mathbf{e}_j = \omega_{31}\mathbf{e}_1 + \omega_{32}\omega_2$ and so $\delta_1\mathbf{e}_3 = \Omega_{13j}\mathbf{e}_j = -\Omega_1(a\mathbf{e}_1 + b\mathbf{e}_2)$. Therefore, using (2.7)

$$\mathbf{r}\cdot\delta_1\mathbf{e}_3\,\omega_1\wedge\omega_2 = -\Omega_1(\mathbf{r}\cdot\mathbf{e}_1 a\omega_1 + \mathbf{r}\cdot\mathbf{e}_2\,b\omega_1)\wedge\omega_2 = \Omega_1(\mathbf{r}\cdot\mathbf{e}_1\omega_{13} + \mathbf{r}\cdot\mathbf{e}_2\omega_{23})\wedge\omega_2 = \Omega_1\mathbf{r}\cdot d\mathbf{e}_3\wedge\omega_2.$$

Substituting these into the variation formula above,

$$\delta_1\int_V dV = \frac{1}{3}\int_M \big(\Omega_1\mathbf{r}\cdot d\mathbf{e}_3\wedge\omega_2 + (\mathbf{r}\cdot\mathbf{e}_3)\,d(\Omega_1\omega_2)\big) = \frac{1}{3}\int_M d(\Omega_1\mathbf{r}\cdot\mathbf{e}_3\omega_2).$$

□

Since the variations in (3.43) have opposite signs, their contribution cancels out. The only contribution arises from variation δ_3.

Lemma 3.5:

$$\delta_3\int_V dV = \int_M \Omega_3\,\omega_1\wedge\omega_2. \tag{3.45}$$

Proof: Begin with (3.44) and apply δ_3

$$\delta_3\int_V dV = \frac{1}{3}\int_M \big[\delta_3\mathbf{r}\cdot\mathbf{e}_3\omega_1\wedge\omega_2 + \mathbf{r}\cdot(\delta_3\mathbf{e}_3)\omega_1\wedge\omega_2 + (\mathbf{r}\cdot\mathbf{e}_3)\delta_3(\omega_1\wedge\omega_2)\big]. \tag{3.46}$$

Using the facts that $\delta_3\mathbf{r} = \Omega_3\mathbf{e}_3$ and $\delta_3\mathbf{e}_3 = \Omega_{33j}\mathbf{e}_j$, the first and last terms are $(\delta_3\mathbf{r})\cdot\mathbf{e}_3\omega_1\wedge\omega_2 = \Omega_3\mathbf{e}_3\cdot\mathbf{e}_3\,\omega_1\wedge\omega_2 = \Omega_3\omega_1\wedge\omega_2$ and $\mathbf{r}\cdot\mathbf{e}_3\delta_3(\omega_1\wedge\omega_2) = -(2H)\mathbf{r}\cdot\mathbf{e}_3\,\omega_1\wedge\omega_2$, respectively. Now $d\Omega_3 = -(\Omega_{331}\omega_1 + \Omega_{332}\omega_2)$, so the second term in (3.46) is

$$\mathbf{r}\cdot(\delta_3\mathbf{e}_3)\omega_1\wedge\omega_2 = (\Omega_{331}\mathbf{r}\cdot\mathbf{e}_1 + \Omega_{332}\mathbf{r}\cdot\mathbf{e}_2)\,\omega_1\wedge\omega_2. \tag{3.47}$$

From the equations $d\Omega_3\wedge\omega_1 = \Omega_{332}\omega_1\wedge\omega_2$ and $d\Omega_3\wedge\omega_2 = -\Omega_{331}\,\omega_1\wedge\omega_2$, we obtain $-d\Omega_3\wedge(-\mathbf{r}\cdot\mathbf{e}_2\omega_1 + \mathbf{r}\cdot\mathbf{e}_1\omega_2) = (\mathbf{r}\cdot\mathbf{e}_2\Omega_{332} + \mathbf{r}\cdot\mathbf{e}_1\Omega_{331})\,\omega_1\wedge\omega_2$. Comparing this with (3.18), we conclude that

$$\mathbf{r}\cdot\delta_3\mathbf{e}_3\omega_1\wedge\omega_2 = -d\Omega_3\wedge(-\mathbf{r}\cdot\mathbf{e}_2\omega_1 + \mathbf{r}\cdot\mathbf{e}_1\omega_2)$$

Putting this in the integral then applying integration by parts and Stokes's Theorem, we arrive at

$$\int_M d\Omega_3 \wedge (-\mathbf{r}\cdot\mathbf{e}_2\omega_1 + \mathbf{r}\cdot\mathbf{e}_1\Omega_2) = \int_M \Omega_3 d(-\mathbf{r}\cdot\mathbf{e}_2\omega_1 + \mathbf{r}\cdot\mathbf{e}_2\omega_2). \tag{3.48}$$

Differentiating with d and using the structure equations, we get

$$d(-(\mathbf{r}\cdot\mathbf{e}_2)\omega_1 + (\mathbf{r}\cdot\mathbf{e}_1)\omega_2)$$

$$= -(d\mathbf{r}\cdot\mathbf{e}_2)\wedge\omega_1 - (\mathbf{r}\cdot d\mathbf{e}_2)\wedge\omega_1 - (\mathbf{r}\cdot\mathbf{e}_2)d\omega_1 + (d\mathbf{r}\cdot\mathbf{e}_1)\wedge\omega_2 + (\mathbf{r}\cdot d\mathbf{e}_1)\wedge\omega_2 + (\mathbf{r}\cdot\mathbf{e}_1)\,d\omega_2$$

$$= -\omega_2\wedge\omega_1 - (\mathbf{r}\cdot\mathbf{e}_j)\omega_{2j}\wedge\omega_1 - (\mathbf{r}\cdot\mathbf{e}_2)\omega_{12}\wedge\omega_2 + \omega_1\wedge\omega_2 + (\mathbf{r}\cdot\mathbf{e}_j)\omega_{1j}\wedge\omega_2 + (\mathbf{r}\cdot\mathbf{e}_1)\omega_{21}\wedge\omega_1$$

$$= 2\,\omega_1\wedge\omega_2 - c\,(\mathbf{r}\cdot\mathbf{e}_3)\omega_2\wedge\omega_1 + a\,(\mathbf{r}\cdot\mathbf{e}_3)\omega_1\wedge\omega_2$$

$$= 2\omega_1\wedge\omega_2 + (2H)(\mathbf{r}\cdot\mathbf{e}_3)\,\omega_1\wedge\omega_2.$$

Therefore, we have found that

$$-\int_M d\Omega_3 \wedge (-\mathbf{r}\cdot\mathbf{e}_2\omega_1 + \mathbf{r}\cdot\mathbf{e}_1\omega_2) = \int_M \Omega_3[2 + 2H(\mathbf{r}\cdot\mathbf{e}_3)]\,\omega_1\wedge\omega_2 \tag{3.49}$$

So the required variation is given by

$$\delta_3 \int_M dV = \frac{1}{3}\int_M [\Omega_3\omega_1\wedge\omega_2 - 2H(\mathbf{r}\cdot\mathbf{e}_3)\Omega_3\omega_1\wedge\omega_2 + \Omega_3[2 + 2H(\mathbf{r}\cdot\mathbf{e}_3)\omega_1\wedge\omega_2]$$

$$= \int_M \Omega_3\,\omega_1\wedge\omega_2.$$

□

Putting Theorem 3.8 and Lemmas 3.3-3.5 together, the variation $\delta\, I$ is given as

$$\delta I = \int_M [(\nabla^2 + 4H^2 - 2K)\,\frac{\partial E}{\partial(2H)} + (\nabla\cdot\tilde{\nabla} + 2HK)\,\frac{\partial E}{\partial K} - 2H\,E + p]\,\Omega_3\omega_1\wedge\omega_2. \tag{3.50}$$

This implies the following shape equation corresponding to (1.1).

Theorem 3.9: The Euler-Lagrange equation corresponding to functional I in (1.1) is given by

$$[(\nabla^2 + 4H^2 - 2K)\frac{\partial}{\partial(2H)} + (\nabla\cdot\tilde{\nabla} + 2HK)\frac{\partial}{\partial K} - 2H]\,E(2H,K) + p = 0. \tag{3.51}$$

□

4 Second Order Variation of I

The second order variation of (1.1) is determined. The calculations will be complete but concisely presented. It has already been shown that $\delta_1 I = \delta_2 I = 0$ for a closed M, so it follows that $\delta\delta_1 I = \delta\delta_2 I = 0$ and it suffices to calculate $\delta^2 I = \delta_3^2 I$. Notice that Ω_3 can come into and out of the expressions acted on by the operator δ_3 freely. To get this explicitly, the following two Lemmas are required.

Lemma 4.1: For every function f, the variation $\delta_3(d * df)$ is given by

$$\delta_3(d * df) = d * d\delta_3 f + d(2H\,\Omega_3 * df) - 2d(\Omega_3 \tilde{*}\tilde{d}f). \tag{4.1}$$

Proof: Since $df = f_1\omega_1 + f_2\omega_2$ and $*df = f_1\omega_2 - f_2\omega_1$, we find $\tilde{d}f = f_1\omega_{13} + f_2\omega_{23}$ and $\tilde{*}\tilde{d}f = f_1\omega_{23} + f_1\,\omega_{13}$. First of all we calculate $\delta_3 * df$,

$$\delta_3 * df = \delta_3 f_1\,\omega_2 + f_1(\delta_3\omega_2) - (\delta_3 f_2)\omega_1 - f_2(\delta_3\omega_1)$$

$$= (\delta_3 f_1)\omega_2 - (\delta_3 f_2)\omega_1 - \Omega_{312}(f_1\omega_1 + f_2\omega_2) + \Omega_3\,(f_1\omega_{32} - f_2\omega_{31})$$

$$= (\delta_2 f_1)\omega_2 - (\delta_3 f_2)\omega_1 - \Omega_{312}df + \Omega_3[f_2(a\omega_1 + b\omega_2) - f_1(b\omega_1 + c\omega_2)] \tag{4.2}$$

and $\delta_3\, df$ is given by,

$$\delta_3\, df = (\delta_3 f_1)\omega_1 + f_1(\delta_3\omega_1) + (\delta_3 f_2)\omega_2 + f_2(\delta_3\omega_2)$$

$$= (\delta_3 f_1)\omega_1 + (\delta_3 f_2)\omega_2 + f_1(\Omega_3\omega_{31} - \omega_2\Omega_{321}) + f_2(\Omega_3\omega_{32} - \omega_1\Omega_{312})$$

$$= (\delta_3 f_1)\omega_1 + (\delta_3 f_2)\omega_2 + \Omega_{312}[f_1\omega_2 - f_2\omega_1] + \Omega_3[f_1(-a\omega_1 - b\omega_2) + f_2(-b\omega_1 - c\omega_2)]. \tag{4.3}$$

Therefore, since the Hodge operator satisfies $*^2 = -1$, we have

$$*\delta_3\, df = (\delta_3 f_1)\omega_2 - (\delta_3 f_2)\omega_1 - \Omega_{312}\, df + \Omega_3[f_1(b\omega_1 - a\omega_2) + f_2(c\omega_1 - b\omega_2)] \tag{4.4}$$

The difference of (4.2) and (4.4) follows at once,

$$\delta_3 *\, df - *\delta_3\, df = \Omega_3[-2bf_1\omega_1 + 2bf_2\omega_2 - c\,f_1\omega_2 - cf_2\omega_1 + a\,f_1\omega_2 + a\,f_2\omega_1]. \tag{4.5}$$

The following equation has a very important application. Using the definition of mean curvature, we have $(2H) * df = (a+c)(f_1\omega_2 - f_2\omega_1)$, so

$$2H\, * df - 2\tilde{*}\tilde{d}f = af_1\omega_2 + cf_1\omega_2 - af_2\omega_1 - cf_2\omega_1 - 2f(b\omega_1 + c\omega_2) + 2f_2(a\omega_1 + b\omega_2)$$

$$= -2bf_1\omega_1 + 2bf_2\omega_2 - cf_1\omega_2 - cf_2\omega_1 + af_1\omega_2 + af_2\omega_1. \tag{4.6}$$

Using (4.6) in (4.5) we obtain

$$\delta_3 * df - *\delta_3\, df = (2H) * df\, \Omega_3 - 2\tilde{*}\tilde{d}f\, \Omega_3. \tag{4.7}$$

Differentiating both sides of (4.7) with respect to d and using the fact that δ_3 and d commute, (4.1) follows. □

Lemma 4.2: For every function f then

$$\delta_3\, d\tilde{*}\tilde{d}f = d\big[\delta_3(2H) * df + 2H\, \delta_3 * df + 2K\Omega_3 * df - 2H\Omega_3 * \tilde{d}f - *\tilde{d}\delta_3 f\big]. \tag{4.8}$$

Proof: Beginning with $*\tilde{d}f = (af_1 + bf_2)\omega_2 - (bf_1 + cf_2)\omega_1$ and (2.31),

$$\delta_3 * \tilde{d}f = \delta_3(af_1 + bf_2)\omega_2 - \delta_3(bf_1 + cf_2)\omega_1 + (af_1 + b\omega_2)\delta_3\omega_2 - (bf_1 + cf_2)\delta_3\omega_1$$

$$= \delta_3(af_1+bf_2)\omega_2 - \delta_3(bf_1+cf_2)\omega_1 - \Omega_{312}[f_1(a\omega_1+b\omega_2)+f_2(b\omega_1+c\omega_2)] + \Omega_3[(af_1+bf_2)\omega_{32} - (bf_1+cf_2)\omega_{31}]. \tag{4.9}$$

The last term in (4.9) is given by

$$(af_1 + bf_2)\omega_{32} - (bf_1 + cf_2)\omega_{31} = -(af_1 + bf_2)(b\omega_1 + c\omega_2) + (bf_1 + cf_2)(a\omega_1 + b\omega_2)$$

$$= (b^2 - ac)(-f_2\omega_1 + f_1\omega_2) = -K\, * df. \tag{4.10}$$

Using (4.10) in (4.9), it has been shown that

$$\delta_3\, \tilde{d}f = \delta_3\, (af_1 + bf_2)\omega_2 - \delta_3(bf_1 + cf_2)\omega_1 - \Omega_{312}\tilde{d}f - \Omega_3\, K * df. \tag{4.11}$$

Since $\tilde{d}\, f = f_1\omega_{13} + f_2\omega_{23}$, we find that

$$\delta_3\tilde{d}f = \delta_3(af_1 + bf_2)\omega_1 + \delta_3(bf_1 + cf_2)\omega_2 - \Omega_3[(af_1 + bf_2)\omega_{13} + (bf_1 + cf_2)\omega_{23}]$$

$$+\Omega_{312}[(af_1+bf_2)\omega_2-(bf_1+cf_2)\omega_1]$$

The following result will be required below

$$-(a+c)(f_1*\omega_{13}+f_2*\omega_{23})+(ac-b^2)(f_1\omega_2-f_2\omega_1)$$

$$=-(a^2f_1+abf_2+b^2f_1+bc\,f_1)\omega_2+(ab\,f_1-b^2f_2+bc\,f_1+c^2\,f_2)\,\omega_1.$$

Now apply $*$ to both sides of this to obtain

$$*\delta_3\,\tilde{d}f=\delta_3(af_1+bf_2)\omega_2-\delta_3(bf_1+cf_2)\omega_1-\Omega_{312}(f_1\omega_{13}+f_2\omega_{23})-\Omega_3[(a^2f_1+abf_2+b^2f_1+bcf_2)\omega_2$$

$$-(abf_1+b^2f_2+bcf_1+c^2f_2)\omega_1]$$

$$=\delta_3(af_1+bf_2)\omega_2-\delta_3(bf_1+cf_2)\omega_1-\Omega_{312}\,(f_1\omega_{13}+f_2\omega_{23})-(a+c)(f_1*\omega_{13}+f_2*\omega_{23})+(ac-b^2)(f_1\omega_2-f_2\omega_1)$$

$$=\delta_3(af_1+bf_2)\omega_2-\delta_3(bf_1+cf_2)\omega_1-(2H)\,\Omega_3*\tilde{d}f+K\Omega_3*\,df-\Omega_{312}\tilde{d}f. \tag{4.12}$$

The difference of (4.12) and (4.11) is given as

$$*\delta_3\tilde{d}f-\delta_3*\tilde{d}f=2K\Omega_3\,*df-(2H)\Omega_3\,*\tilde{d}f. \tag{4.13}$$

It is the case that

$$*\tilde{d}f+\tilde{*}\,\tilde{d}f=f_1*\omega_{13}+f_2*\omega_{23}+f_1\omega_{23}-f_2\omega_{13}=f_1(a+c)\omega_1-f_2(a+c)\omega_1=2H\,*df. \tag{4.14}$$

Consequently, $*\tilde{d}f+\tilde{*}\tilde{d}f=(2H)*\,df$ and so

$$d\delta_3\,*\tilde{d}f+d\delta_3\tilde{*}\tilde{d}f=d\,\delta_3(2H\,*df). \tag{4.15}$$

Differentiating (4.13) with the operator d, it is found that

$$d*\delta_3\tilde{d}\,f-d\delta_3\,*\tilde{d}f=d(2K\Omega_3\,*df)-d(2H\Omega_3\,*\tilde{d}f). \tag{4.16}$$

Adding (4.15) and (4.16), we find that

$$d\delta_3\tilde{*}\tilde{d}\,f=d[\delta_3(2H*df)+2K\Omega_3*df-2H\Omega_3\,*\tilde{d}f-*\delta_3\tilde{d}f]$$

$$=d[\delta_3(2H)\,*df+(2H)\,\delta_3*df+2K\Omega_3*df-(2H)\Omega_3*\tilde{d}f-*\tilde{d}\delta_3 f].$$

$\square$

Acting on $\delta_3 I_e$ with δ_3 again allows us to write

$$\delta_3^2 I_e = \int_M \delta_3[(4H^2-2K)\,\frac{\partial E}{\partial(2H)} + 2HK\,\frac{\partial E}{\partial K} - 2HK]\Omega_3\,\omega_1\wedge\omega_2$$
$$+\int_M [(2H^2-2K)\,\frac{\partial E}{\partial(2H)} + 2HK\,\frac{\partial E}{\partial K} - 2HK]\,\Omega_3\delta_3(\omega_1\wedge\omega_2)$$
$$+\int_M \delta_3[\frac{\partial E}{\partial(2H)}]d*d\Omega_3 + \int_M \frac{\partial E}{\partial(2H)}\delta_3(d*d\Omega_3) + \int_M \delta_3[\frac{\partial E}{\partial K}]\,d\tilde{*}\tilde{d}\Omega_3$$
$$+\int_M \frac{\partial E}{\partial K}\,\delta_3(d\tilde{*}\tilde{d}\Omega_3). \tag{4.17}$$

Lemmas 4.1-4.2 are required for the simplification of the last line of (4.17) so replacing function f by Ω_3 in Lemmas 4.1 and 4.2 and simplifying we obtain,

$$\delta_3\,d*d\Omega_3 = [\nabla(2H\Omega_3)\cdot\nabla\Omega_3 + (2H)\Omega_3\nabla^2\Omega_3 - 2\nabla\Omega_3\tilde{\nabla}\Omega_3 - 2\Omega_3\nabla\cdot\tilde{\nabla}\Omega_3]\omega_1\wedge\omega_2. \tag{4.18}$$

Recalling that $\delta_3(2H) = (4H^2-2K)\Omega_3 + \nabla^2\Omega_3$, using Lemmas 4.1, 4.2 and $\delta_3 * d\Omega_3 = (2H)\Omega_3 * d\Omega_3 - 2\Omega_3\tilde{*}\tilde{d}\Omega_3 + *\delta_3\,d\Omega_3$ it follows that

$$\delta_3(2H)*d\Omega_3 + 2H(2H\Omega_3*d\Omega_3 - 2\Omega_3\tilde{*}\tilde{d}\Omega_3) + 2K\Omega_3*d\Omega_3 - 2H\Omega_3*\tilde{d}\Omega_3$$
$$= [\delta_3(2H) + 4H^2\Omega_3 + 2K\Omega_3]\ *d\Omega_3 - 4H\Omega_3\tilde{*}\tilde{d}\Omega_3 - 2H\Omega_3*\tilde{d}\Omega_3 \tag{4.19}$$
$$= [8H^2\Omega_3 + \nabla^2\Omega_3]*d\Omega_3 - 4H\Omega_3*\tilde{d}\Omega_3 - 2H\Omega_3*\tilde{d}\Omega_3.$$

Operating on this result with the operator d gives an alternate expression for $d\delta_3\tilde{*}\tilde{d}\Omega_3$ in Lemma 4.2,

$$d\delta_3\,\tilde{*}\tilde{d}\Omega_3 = [\nabla(8H^2\Omega_3+\nabla^2\Omega_3)\nabla\Omega_3 + (8H^2\Omega_3+\nabla^2\Omega_3)\nabla^2\Omega_3 - \nabla*4H\Omega_3)\cdot\tilde{\nabla}\Omega_3$$
$$-(4H)\Omega_3\nabla\cdot\tilde{\nabla}\Omega_3 - \nabla(2H\Omega_3)\cdot\tilde{\nabla}\Omega_3 - (2H)\Omega_3\nabla\cdot\tilde{\nabla}\Omega_3]\omega_1\wedge\omega_2. \tag{4.20}$$

Substituting the known results for the variations $\delta_3(2H)$ and $\delta_3 K$ given in Theorems 3.6-3.7 into $\delta_3^2\,I_e$ then using the results of the Lemmas 4.1, 4.2, we obtain, after collecting terms, $\delta_3^2\,I_e$,

$$\int_M [(4H^2-2K)^2\frac{\partial^2 E}{\partial(2H)^2} + (4H(4H^2-2K) - 4HK - 2H(4H^2-2K) - 2H(4H^2-2K))\frac{\partial E}{\partial(2H)}$$

$$+(K(4H^2-2K)+4H^2K-4H^2K-4H^2K)\frac{\partial E}{\partial K}+(2HK(4H^2-2K)+2HK(4H^2-2K))\frac{\partial^2 E}{\partial(2H)\partial K}$$

$$+(2HK)^2\frac{\partial^2 E}{\partial K^2}+(-(4H^2-2K)+4H^2)E]\,\Omega_3^2\,\omega_1\wedge\omega_2+\int_M[4H\,\frac{\partial E}{\partial(2H)}+2HK\frac{\partial^2 E}{\partial(2H)\partial K} \quad (4.21)$$

$$+8H^2\frac{\partial E}{\partial K}-E+K\frac{\partial E}{\partial K}+(4H^2-2K)\frac{\partial^2 E}{\partial(2H)^2}+2HK\frac{\partial^2 E}{\partial K\partial(2H)}$$

$$+(4H^2-2K)\frac{\partial^2 E}{\partial(2H)^2}]\Omega_3\,\nabla^2\Omega_3\,\omega_1\wedge\omega_2+[-2\frac{\partial E}{\partial(2H)}+(4H^2-2K)\frac{\partial^2 E}{\partial K\partial(2H)}$$

$$+2H\frac{\partial E}{\partial H}+2HK\frac{\partial^2 E}{\partial K^2}-2H\frac{\partial E}{\partial K}-2\frac{\partial E}{\partial(2H)}+(4H^2-2K)\frac{\partial^2 E}{\partial K\partial(2H)}+2HK\frac{\partial^2 E}{\partial K^2}-4H\frac{\partial E}{\partial K}]\Omega_3\nabla\cdot\tilde{\nabla}\Omega_3\omega_1\wedge\omega_2$$

$$+\int_M[\frac{\partial^2 E}{\partial(2H)^2}+\frac{\partial E}{\partial K}](\nabla^2\Omega_3)^2\,\omega_1\wedge\omega_2+\int_M[\frac{\partial^2 E}{\partial(2H)\partial K}\nabla^2\Omega_3\cdot\nabla\cdot\tilde{\nabla}\Omega_3+\frac{\partial E}{\partial(2H)}\nabla(2H\Omega_3)\cdot\nabla\Omega_3]\,\omega_1\wedge\omega_2$$

$$+\int_M[\frac{\partial^2 E}{\partial K^2}(\nabla\cdot\tilde{\nabla}\Omega_3)^2-2\frac{\partial E}{\partial(2H)}(\nabla\Omega_3)\cdot(\tilde{\nabla}\Omega_3)]\,\omega_1\wedge\omega_2$$

$$+\int_M\frac{\partial E}{\partial K}[\nabla(8H^2\Omega_3+\nabla^2\Omega_3)\nabla\Omega_3-\nabla(4H\Omega_3)\cdot\tilde{\nabla}\Omega_3-4H\Omega_3\nabla\cdot\tilde{\nabla}\Omega_3-\nabla(2H\Omega_3)\cdot\tilde{\nabla}\Omega_3-2H\nabla\cdot\tilde{\nabla}\Omega_3]\,\omega_1\wedge\omega_2.$$

Collecting like terms and simplifying the coefficients in this we arrive at the following Theorem pertaining to the second order variation of I_e.

Theorem 4.3: The second order variation of the functional I_e in (1.1) is given as

$$\delta^2 I=\int_M\big[(4H^2-2K)^2\frac{\partial^2 E}{\partial(2H)^2}-4HK\,\frac{\partial E}{\partial(2H)}-2K^2\frac{\partial E}{\partial K}$$

$$+4HK(4H^2-2K)\frac{\partial^2 E}{\partial(2H)\partial K}+4(KH)^2\frac{\partial^2 E}{\partial K^2}+2K\,E-2H\,p]\,\omega_1\wedge\omega_2$$

$$+\int_M\Omega_3\nabla^2\Omega_3[4H\frac{\partial E}{\partial(2H)}+4(2H^2-E)\frac{\partial^2 E}{\partial(2H)^2}+K\frac{\partial E}{\partial K}+4HK\frac{\partial^2 E}{\partial K\partial(2H)}-E+8H^2\frac{\partial E}{\partial K}]\,\omega_1\wedge\omega_2$$

$$+\int_M[2(4H^2-2K)\frac{\partial^2 E}{\partial(2H)\partial K}-4\frac{\partial E}{\partial(2H)}+4HK\frac{\partial^2 E}{\partial K^2}-4H\frac{\partial E}{\partial K}]\Omega_3\nabla\cdot\tilde{\nabla}\Omega_3\,\omega_1\wedge\omega_2$$

$$+\int_M[\frac{\partial^2 E}{\partial(2H)^2}+\frac{\partial E}{\partial K}](\nabla^2\Omega_3)^2\omega_1\wedge\omega_2+\int_M[\frac{\partial^2 E}{\partial(2H)\partial K}\nabla^2\Omega_3\cdot\nabla\tilde{\nabla}\Omega_3+\frac{\partial E}{\partial(2H)}\nabla(2H\Omega_3)\cdot\nabla\Omega_3]\,\omega_1\wedge\omega_2$$

$$+\int_M[\frac{\partial^2 E}{\partial K^2}(\nabla\cdot\tilde{\nabla}\Omega_3)^2-2\frac{\partial E}{\partial(2H)}(\nabla\Omega_3)\cdot(\tilde{\nabla}\Omega_3)]\omega_1\wedge\omega_2 \quad (4.22)$$

$$+\int_M\frac{\partial E}{\partial K}\,[\nabla(8H^2\Omega_3+\nabla^2\Omega_3)\cdot\nabla\Omega_3-\nabla(4H\Omega_3)\cdot\tilde{\nabla}\Omega_3-4H\Omega_3\nabla\cdot\tilde{\nabla}\Omega_3$$

$$-\nabla(2H\Omega_3)\cdot\tilde{\nabla}\Omega_3-2H\Omega_3\nabla\cdot\tilde{\nabla}\Omega_3]\,\omega_1\wedge\omega_2.$$

□

5 Conclusions

A variational approach to obtaining the Euler-Lagrange or shape equation from a given functional under deformations of the underlying manifold has been developed here. This leads to many other things to be done and many directions this could take. It is some work to take the differential operators that were produced in this process and give tensor forms to them. This will result in differential equations which can also be studied and solutions found as well. The closed manifolds tested here can be perturbed so that boundaries appear which leads to a new variational problem, that of including boundary terms in the functional defining the problem.

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