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The global existence of small self-interacting scalar field propagating in the contracting universe

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Abstract

We present a condition on the self-interaction term that guaranties the existence of the global in time solution of the Cauchy problem for the semilinear Klein-Gordon equation in the Friedmann-Lamaître-Robertson-Walker model of the contracting universe. For the Klein-Gordon equation with the Higgs potential we give a lower estimate for the lifespan of solution.
Keywords: FLRW space-time, Klein-Gordon equation; semilinear equation; global solution; Higgs potential
MSC (2010) 35L71, 35Q40, 35Q75, 81E20

0 Introduction and statement of results

In the present paper we prove the global in time existence of the solutions of the Cauchy problem for the semilinear Klein-Gordon equation in the FLRW (Friedmann-Lamaître-Robertson-Walker) space-time of the contracting universe for the self-interacting scalar field.

The metric g in the FLRW space-time of the contracting universe in the Lamaître-Robertson coordinates (see, e.g., [16]) is as follows, $g_{00} = g^{00} = -1$, $g_{0j} = g^{0j} = 0$, $g_{ij}(x, t) = e^{-2t}\sigma_{ij}(x)$, $i, j = 1, 2, \dots, n$, where $\sum_{j=1}^n \sigma^{ij}(x)\sigma_{jk}(x) = \delta_{ik}$, and δ_{ij} is Kronecker's delta. The metric $\sigma^{ij}(x)$ describes the time slices. The covariant Klein-Gordon equation in that space-time in the coordinates is

$$\psi_{tt} - \frac{e^{2t}}{\sqrt{|\det \sigma(x)|}} \sum_{i,j=1}^n \frac{\partial}{\partial x^i} \left(\sqrt{|\det \sigma(x)|} \sigma^{ij}(x) \frac{\partial}{\partial x^j} \psi \right) - n\psi_t + m^2\psi = F(\psi). \quad (0.1)$$

It is obvious that the properties of this equation and of its solutions are not time invertible. In fact, the equation obtained from (0.1) by the time reversal $t \rightarrow -t$ is the semilinear Klein-Gordon equation in the de Sitter spacetime. The Cauchy problem for the later equation is well investigated and the conditions for the existence of small data global in time solutions for some important metrics are discovered [1, 10, 11, 12, 13, 17, 28]. The equation (0.1) is a special case of the equation

$$\psi_{tt} - e^{2t} A(x, \partial_x) \psi - n\psi_t + m^2\psi = F(x, \psi), \quad (0.2)$$

where $A(x, \partial_x) = \sum_{|\alpha| \leq 2} a_\alpha(x) \partial_x^\alpha$ is a second order elliptic partial differential operator. We also assume that the mass m can be a complex number, $m^2 \in \mathbb{C}$.

To formulate the main theorem of this paper we need some description of the nonlinear term F . Furthermore, we are only concerned with the behavior of F at the origin in ψ space.

Condition (L). The smooth in x function $F = F(x, \psi)$ is said to be Lipschitz continuous with exponent $\alpha \geq 0$ in the space $H_{(s)}(\mathbb{R}^n)$ if there is a constant $C \geq 0$ such that

$$\|F(x, \psi_1) - F(x, \psi_2)\|_{H_{(s)}(\mathbb{R}^n)} \leq C \|\psi_1 - \psi_2\|_{H_{(s)}(\mathbb{R}^n)} \left(\|\psi_1\|_{H_{(s)}(\mathbb{R}^n)}^\alpha + \|\psi_2\|_{H_{(s)}(\mathbb{R}^n)}^\alpha \right) \quad (0.3)$$

for all $\psi_1, \psi_2 \in H_{(s)}(\mathbb{R}^n)$.

The polynomial in ψ functions and functions $F(\psi) = \pm|\psi|^{\alpha+1}$, $F(\psi) = \pm|\psi|^\alpha\psi$ are important examples of the Lipschitz continuous with exponent $\alpha > 0$ in the Sobolev space $H_{(s)}(\mathbb{R}^n)$, $s > n/2$, functions.

In what follows we use the metric space

$$X(R, H_{(s)}, \gamma) := \left\{ \psi \in C([0, \infty); H_{(s)}) \mid \|\psi\|_X := \sup_{t \in [0, \infty)} e^{\gamma t} \|\psi(x, t)\|_{H_{(s)}} \leq R \right\},$$

where $\gamma \in \mathbb{R}$, $R > 0$, with the metric

$$d(\psi_1, \psi_2) := \sup_{t \in [0, \infty)} e^{\gamma t} \|\psi_1(x, t) - \psi_2(x, t)\|_{H_{(s)}}.$$

We define solution of the Cauchy problem via corresponding integral equation that contains the following resolving operator

$$G := \mathcal{K} \circ \mathcal{E}\mathcal{E},$$

where $\mathcal{E}\mathcal{E}$ stands for the resolving operator of the evolution equation. More precisely, for the function $f(x, t)$ we define

$$v(x, t; b) := \mathcal{E}\mathcal{E}[f](x, t; b),$$

where the function $v(x, t; b)$ is a solution to the Cauchy problem

$$\partial_t^2 v - A(x, \partial_x)v = 0, \quad x \in \mathbb{R}^n, \quad t \geq 0, \quad (0.4)$$

$$v(x, 0; b) = f(x, b), \quad v_t(x, 0; b) = 0, \quad x \in \mathbb{R}^n, \quad (0.5)$$

while the integral transform \mathcal{K} is given by

$$\mathcal{K}[v](x, t) := 2e^{\frac{\gamma}{2}t} \int_0^t db \int_0^{e^t - e^b} dr e^{-\frac{\gamma}{2}b} v(x, r; b) E(r, t; 0, b; M).$$

Here the principal square root $M = (n^2/4 - m^2)^{\frac{1}{2}}$ is the main parameter that controls estimates and solvability of the integral equation (0.6) that will be written below. The kernel $E(r, t; 0, b; M)$ was introduced in [26] and [24] (see also (1.2)). Hence,

$$G[f](x, t) = 2e^{\frac{\gamma}{2}t} \int_0^t db \int_0^{e^t - e^b} dr e^{-\frac{\gamma}{2}b} \mathcal{E}\mathcal{E}[f](x, r; b) E(r, t; 0, b; M).$$

Thus the integral equation that corresponds to the Cauchy problem for (0.2) is

$$\Phi(x, t) = \Phi_0(x, t) + G[e^{-\Gamma \cdot} F(\cdot, \Phi)](x, t) \quad (0.6)$$

with $\Gamma = 0$, where the function $\Phi_0(x, t)$ is generated by the initial data. The main result of this paper is the following theorem that states the existence of global in time solution for small initial data in Sobolev spaces.

Theorem 0.1 *Let $F(x, \Phi)$ be Lipschitz continuous in the space $H_{(s)}(\mathbb{R}^n)$ with $s > n/2 \geq 1$, $F(x, 0) \equiv 0$, and $\alpha > 0$. Assume that the real part of M is positive $\Re M > 0$ and that, $M = \Re M$ if $\Re M = 1/2$. If one of the following three conditions is fulfilled:*

- (i) $\frac{n}{2} + \Re M + \gamma(\alpha + 1) + \Gamma > 0$, $\frac{n}{2} + \max\{\frac{1}{2}, \Re M\} + \gamma \leq 0$,
- (ii) $\frac{n}{2} + \Re M + \gamma(\alpha + 1) + \Gamma = 0$, $\frac{n}{2} + \max\{\frac{1}{2}, \Re M\} + \gamma < 0$,
- (iii) $\frac{n}{2} + \Re M + \gamma(\alpha + 1) + \Gamma < 0$, $\frac{n}{2} + \max\{\frac{1}{2}, \Re M\} + \gamma \leq 0$, $\gamma\alpha + \Gamma \geq 0$,

then, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$ and every given functions $\varphi_0, \varphi_1 \in H_{(s)}(\mathbb{R}^n)$, satisfying the estimate

$$\|\varphi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^n)} \leq \varepsilon,$$

there exists a solution $\Phi \in C([0, \infty); H_{(s)}(\mathbb{R}^n))$ of the Cauchy problem

$$\Phi_{tt} - n\Phi_t - e^{2t}\Delta\Phi + m^2\Phi = e^{-\Gamma t}F(x, \Phi), \quad (0.7)$$

$$\Phi(x, 0) = \varphi_0(x), \quad \Phi_t(x, 0) = \varphi_1(x). \quad (0.8)$$

The solution $\Phi(x, t)$ belongs to the space $X(2\varepsilon, H_{(s)}(\mathbb{R}^n), \gamma)$, that is,

$$\sup_{t \in [0, \infty)} e^{\gamma t} \|\Phi(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq 2\varepsilon.$$

The theorem also includes the equations with polynomial in Φ function $F(x, \Phi) = F(\Phi)$, with $F(0) = 0$ and the functions $F(\Phi) = \pm|\Phi|^{\alpha+1}$ and $F(\Phi) = \pm|\Phi|^\alpha\Phi$. The sharpness of the conditions on α , γ , and Γ is an interesting open problem that will not be discussed here.

For the given real numbers Γ , n , γ , and the complex number M , define the function

$$I(t) := e^{t(\frac{n}{2} + \max\{\frac{1}{2}, \Re M\} + \gamma)} \int_0^t e^{-(\frac{n}{2} + \max\{\frac{1}{2}, \Re M\} + \gamma(\alpha+1) + \Gamma)b} db.$$

Theorem 0.2 *If $\Re M > 0$, $\gamma \leq -(\frac{n}{2} + \max\{\frac{1}{2}, \Re M\})$, and none of the conditions (i)-(iii) is fulfilled, then the lifespan T_{ls} of the solution of (0.7)-(0.8) can be estimated from below as follows*

$$T_{ls} \geq \mathcal{I} \left(C(M, n, \alpha, \gamma, \Gamma) \left(\|\varphi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^n)} \right)^{-\alpha} \right)$$

with some constant $C(M, n, \alpha, \gamma, \Gamma)$ for sufficiently small $\|\varphi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^n)}$. Here \mathcal{I} is the function inverse to $I = I(t)$.

In particular, Theorem 0.2 states an estimate for the lifespan T_{ls} of the solution of the Higgs boson equation with the Higgs potential in the contracting universe; which reads

$$\psi_{tt} - e^{2t}\Delta\psi - n\psi_t = \mu^2\psi - \lambda\psi^3.$$

Here $\lambda > 0$ and $\mu > 0$. For instance, if $n = 3$, $\alpha = 2$, $\Gamma = 0$, $\gamma = -3/2$, $\mu^2 = 7/4$, that is $m^2 = -7/4$ and $M = 2$, then

$$T_{ls} \geq -\frac{2}{3} \ln \left(\|\varphi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^n)} \right) + C.$$

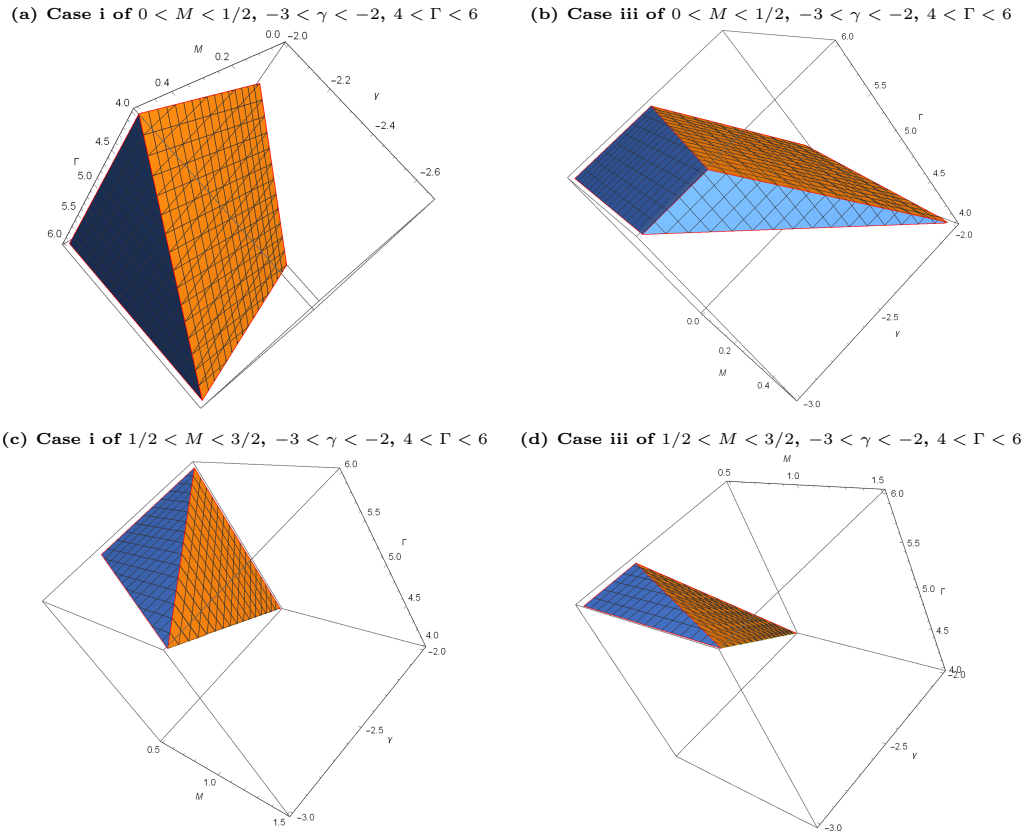
Example. To discuss the conditions of Theorem 0.1 we consider equation relevant to the physically interesting case of $n = 3$ and $\alpha = 2$. Then for the real M the conditions (i)-(iii) read as follows. If $M \leq 1/2$, then

$$\begin{aligned} (i_{M \leq 1/2}) \quad & \frac{3}{2} + M + 3\gamma + \Gamma > 0, \quad 2 + \gamma \leq 0, \\ (ii_{M \leq 1/2}) \quad & \frac{3}{2} + M + 3\gamma + \Gamma = 0, \quad 2 + \gamma < 0, \\ (iii_{M \leq 1/2}) \quad & \frac{3}{2} + M + 3\gamma + \Gamma < 0, \quad 2 + \gamma \leq 0, \quad 2\gamma + \Gamma \geq 0. \end{aligned}$$

If $M > 1/2$, then

$$\begin{aligned} (i_{M > 1/2}) \quad & \frac{3}{2} + M + 3\gamma + \Gamma > 0, \quad \frac{3}{2} + M + \gamma \leq 0, \\ (ii_{M > 1/2}) \quad & \frac{3}{2} + M + 3\gamma + \Gamma = 0, \quad \frac{3}{2} + M + \gamma < 0, \\ (iii_{M > 1/2}) \quad & \frac{3}{2} + M + 3\gamma + \Gamma < 0, \quad \frac{3}{2} + M + \gamma \leq 0, \quad 2\gamma + \Gamma \geq 0. \end{aligned}$$

Figure 1: Feasible domain for M, γ, Γ if $n = 3$ and $\alpha = 2$



In order to prove Theorem 0.1 and Theorem 0.2 we establish in Section 3 estimates in the Besov spaces $B_p^{s,q}$ for the linear equation obtained from (0.2) although estimates in the Sobolev spaces are sufficient for our goal. In fact, the proofs of the estimates for these two cases are identical. For

the linear equation without a source term, these estimates for the large time t imply the limitation for the rate of growth as follows:

$$\|\Phi(x, t)\|_{X'} \leq C e^{(\frac{n}{2} + a + \max\{\frac{1}{2}, \Re M\})t} \|\varphi_0\|_X + C e^{(\frac{n}{2} + a + \Re M)t} \|\varphi_1\|_X,$$

where if $X = B_p^{s,q}$, then $X' = B_{p'}^{s',q}$, $a := s - s' - 2n(1/p - 1/2)$, $1/p + 1/p' = 1$, while $X' = L^{p'}$ if $X = L^p$. In the case of Sobolev spaces $X = X' = H_{(s)}(\mathbb{R}^n)$ and $p = 2$. The integral transform \mathcal{K} allows us to avoid consideration in the phase space and to apply immediately the well-known decay estimates for the solution of the wave equation (operator $\mathcal{E}\mathcal{E}$) (see, e.g., [2, 3, 18]).

Ebert and Nunes do Nascimento [6] have studied the long time behavior of the energy of solutions for a class of linear equations with time-dependent mass and speed of propagation. They introduced a classification of the potential term, which clarifies whether the solution behaves like the solution to the wave equation or the Klein-Gordon equation. For the equation

$$u_{tt} - e^{2t}\Delta u + m^2 u = |u|^p,$$

with $n \leq 4$, $m > 0$, $2 \leq p \leq \frac{n}{[n-2]_+}$ they established the existence of energy class solution for small data. Their proof is based on splitting the phase space into pseudo-differential and hyperbolic zones. That method of zones was invented for the hyperbolic operators with multiple characteristics (see [23] and references therein) and then modified and successfully used in studying the large time behavior of the solutions (see [5, 8, 14, 19, 20, 21], and references therein).

Hirosawa and Nunes do Nascimento [15] have proved some energy estimates for Klein-Gordon equations with time-dependent potential:

$$u_{tt} - \Delta u + m(t)u = 0,$$

where

$$m(t) = \frac{\mu^2}{(1+t)^2} + \delta(t), \quad 0 < \mu < \frac{1}{2}, \quad |\delta(t)| \lesssim (1+t)^{-2\beta}, \quad \beta < 1,$$

under some conditions on the possible oscillations of $\delta(t)$. Here, $m(t)$ is not necessarily positive function.

The present paper is organized as follows. In Section 1 we describe the integral transform from [27] and the representations generated by that transform for the solutions of the Cauchy problem for the linear equation. In Section 2 some estimates for the kernels of those integral transformers are derived. Then, in Section 3 we quote the $B_p^{s,q} - B_{p'}^{s',q}$ -estimates from [2] for the second order hyperbolic operator and using integral transform from Section 1 we demonstrate how these estimates can be pushed forward to the source free equation with the exponentially increasing in time coefficient. In Section 4 we obtain similar estimates for the equation with the source term. Section 5 is devoted to the solvability of the associated integral equation while Section 6 completes the proof of Theorems 0.1-0.2.

1 Resolving operators for the linear equation

The following partial Liouville transform (change of unknown function)

$$u = e^{-\frac{n}{2}t}\Phi, \quad \Phi = e^{\frac{n}{2}t}u,$$

eliminates the term with time derivative of the equation (0.7). We obtain

$$u_{tt} - e^{2t}A(x, \partial_x)u - M^2 u = e^{-(\Gamma + \frac{n}{2})t}F(e^{\frac{n}{2}t}u), \quad (1.1)$$

where

$$M = \left(\frac{n^2}{4} - m^2 \right)^{\frac{1}{2}}.$$

In this section we consider the linear part of the equation (1.1) with $M \in \mathbb{C}$. The equation (1.1) covers two important cases. The first one is the Higgs boson equation, which has $F(\phi) = -\lambda\phi^3$ and $M^2 = \frac{n^2}{4} + \mu^2$ with $\lambda > 0$, $\mu > 0$, and $n = 3$. This includes also equation of *tachyonic scalar fields* living on the de Sitter universe. (See, e.g. [4, 7].) The second case is the case of the small physical mass (the light scalar field), that is $0 \leq m \leq \frac{n}{2}$.

We introduce the kernel functions $E(x, t; x_0, t_0; M)$, $K_0(z, t; M)$, and $K_1(z, t; M)$ (see also [26] and [24]). First, for $M \in \mathbb{C}$ we define the function

$$\begin{aligned} E(x, t; x_0, t_0; M) &= 4^{-M} e^{-M(t_0+t)} \left((e^t + e^{t_0})^2 - (x - x_0)^2 \right)^{-\frac{1}{2}+M} \\ &\quad \times F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^t - e^{t_0})^2 - (x - x_0)^2}{(e^t + e^{t_0})^2 - (x - x_0)^2}\right). \end{aligned} \quad (1.2)$$

Next we define also the kernels $K_0(z, t; M)$ and $K_1(z, t; M)$ by

$$\begin{aligned} K_0(z, t; M) &:= - \left[\frac{\partial}{\partial b} E(z, t; 0, b; M) \right]_{b=0} \\ &= -4^{-M} e^{-tM} ((1 + e^t)^2 - z^2)^M \frac{1}{[(e^t - 1)^2 - z^2] \sqrt{(e^t + 1)^2 - z^2}} \\ &\quad \times \left[(e^t - 1 + M(e^{2t} - 1 - z^2)) F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^t - 1)^2 - z^2}{(e^t + 1)^2 - z^2}\right) \right. \\ &\quad \left. + (1 - e^{2t} + z^2) \left(\frac{1}{2} + M\right) F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^t - 1)^2 - z^2}{(e^t + 1)^2 - z^2}\right) \right] \end{aligned}$$

and $K_1(z, t; M) := E(z, t; 0, 0; M)$, that is,

$$\begin{aligned} K_1(z, t; M) &= 4^{-M} e^{-Mt} ((e^t + 1)^2 - z^2)^{-\frac{1}{2}+M} \\ &\quad \times F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^t - 1)^2 - z^2}{(e^t + 1)^2 - z^2}\right), \quad 0 \leq z \leq e^t - 1, \end{aligned}$$

respectively. For the solution Φ of the the Cauchy problem

$$\Phi_{tt} - n\Phi_t - e^{2t} A(x, \partial_x) \Phi + m^2 \Phi = f, \quad \Phi(x, 0) = \varphi_0(x), \quad \Phi_t(x, 0) = \varphi_1(x),$$

according to [27] we obtain

$$\begin{aligned} &\Phi(x, t) \\ &= 2e^{\frac{n}{2}t} \int_0^t db \int_0^{e^t - e^b} dr e^{-\frac{n}{2}b} v_f(x, r; b) E(r, t; 0, b; M) \\ &\quad + e^{\frac{n-1}{2}t} v_{\varphi_0}(x, \phi(t)) + e^{\frac{n}{2}t} \int_0^1 v_{\varphi_0}(x, \phi(t)s) (2K_0(\phi(t)s, t; M) - nK_1(\phi(t)s, t; M)) \phi(t) ds \\ &\quad + 2e^{\frac{n}{2}t} \int_0^1 v_{\varphi_1}(x, \phi(t)s) K_1(\phi(t)s, t; M) \phi(t) ds, \quad x \in \mathbb{R}^n, \quad t > 0, \end{aligned} \quad (1.3)$$

where the function $v_f(x, t; b)$ is a solution to the Cauchy problem (0.4)-(0.5), while $\phi(t) := e^t - 1$. Here, for $\varphi \in C_0^\infty(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$, the function $v_\varphi(x, \phi(t)s)$ coincides with the value $v(x, \phi(t)s)$ of the solution $v(x, t)$ of the Cauchy problem for the equation (0.4) with the first initial datum $\varphi(x)$, while the second datum is zero.

2 Some estimates for the kernel functions K_0 and K_1

Lemma 2.1 *Let $a > -1$, $\Re M > 0$, and $\phi(t) = e^t - 1$. Then*

$$\int_0^1 \phi(t)^a s^a |K_1(\phi(t)s, t; M)| \phi(t) ds \leq C_M e^{-\Re M t} (e^t - 1)^{a+1} (e^t + 1)^{2\Re M - 1} \quad \text{for all } t > 0.$$

In particular,

$$\begin{aligned} \int_0^1 \phi(t)^a s^a |K_1(\phi(t)s, t; M)| \phi(t) ds &\leq C_{M,a} e^{(\Re M + a)t} \quad \text{for all } t \in [1, \infty), \\ \int_0^1 \phi(t)^a s^a |K_1(\phi(t)s, t; M)| \phi(t) ds &\leq C_{M,a} t^{a+1} \quad \text{for all } t \in (0, 1). \end{aligned}$$

Proof. By the definition of the kernel K_1 , we obtain

$$\begin{aligned} \int_0^1 \phi(t)^a s^a |K_1(\phi(t)s, t; M)| \phi(t) ds &= \int_0^{e^t - 1} r^a |K_1(r, t; M)| dr \\ &\leq 4^{-\Re M} e^{-\Re M t} \int_0^{e^t - 1} r^a ((e^t + 1)^2 - r^2)^{-\frac{1}{2} + \Re M} \left| F \left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^t - 1)^2 - z^2}{(e^t + 1)^2 - z^2} \right) \right| dr. \end{aligned}$$

On the other hand, for $\Re M > 0$ we have (see pages 8,9 [28])

$$\int_0^1 \phi(t)^a s^a |K_1(\phi(t)s, t; M)| \phi(t) ds \leq C_M e^{-\Re M t} \int_0^{e^t - 1} y^a ((e^t + 1)^2 - y^2)^{-\frac{1}{2} + \Re M} dy.$$

and again due to pages 8,9 [28] for $M > 0$ we have

$$\begin{aligned} &\int_0^{z-1} y^a ((z+1)^2 - y^2)^{-\frac{1}{2} + M} dy \tag{2.1} \\ &= \frac{1}{1+a} (z-1)^{1+a} (z+1)^{2M-1} F \left(\frac{1+a}{2}, \frac{1}{2} - M; \frac{3+a}{2}; \frac{(z-1)^2}{(z+1)^2} \right), \end{aligned}$$

where $a > -1$ and $z \geq 1$. Hence, for $\Re M > 0$ we have

$$\int_0^1 \phi(t)^a s^a |K_1(\phi(t)s, t; M)| \phi(t) ds \leq C_M e^{-\Re M t} (e^t - 1)^{a+1} (e^t + 1)^{2\Re M - 1} \quad \text{for all } t > 0.$$

Thus the lemma is proved. □

Lemma 2.2 *Let $a > -1$, $\Re M > 0$, and $\phi(t) = e^t - 1$. Then*

$$\begin{aligned} &\int_0^1 \phi(t)^a s^a |K_0(\phi(t)s, t; M)| \phi(t) ds \\ &\leq C_{M,a} (e^t - 1)^{a+1} (e^t + 1)^{\Re M - 1} \begin{cases} 1 & \text{if } \Re M > 1/2 \\ t^{\text{sgn}|\Im M|} + e^{(\frac{1}{2} - \Re M)t} & \text{if } \Re M \leq 1/2 \end{cases} \end{aligned}$$

for all $t > 0$. In particular, for all $t \in [1, \infty)$

$$\int_0^1 \phi(t)^a s^a |K_0(\phi(t)s, t; M)| \phi(t) ds \leq C_{M,a} e^{t(a + \Re M)} \begin{cases} 1 & \text{if } \Re M > 1/2, \\ t^{\text{sgn}|\Im M|} + e^{(\frac{1}{2} - \Re M)t} & \text{if } \Re M \leq 1/2, \end{cases}$$

while

$$\int_0^1 \phi(t)^a s^a |K_0(\phi(t)s, t; M)| \phi(t) ds \leq C_{M,a} t^{a+1} \quad \text{for all } t \in (0, 1).$$

Proof. It is evident that

$$\int_0^1 \phi(t)^a s^a |K_0(\phi(t)s, t; M)| \phi(t) ds = \int_0^{e^t-1} y^a |K_0(y, t; M)| dy.$$

By definition of the kernel K_0 we obtain with $z = e^t$

$$\begin{aligned} & \int_0^1 \phi(t)^a s^a |K_0(\phi(t)s, t; M)| \phi(t) ds \\ & \leq 4^{-\Re M} z^{-\Re M} \int_0^{z-1} y^a ((1+z)^2 - y^2)^{\Re M} \frac{1}{[(z-1)^2 - y^2] \sqrt{(z+1)^2 - y^2}} \\ & \quad \times \left| \left[(z-1 + M(z^2 - 1 - y^2)) F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right. \right. \\ & \quad \left. \left. + (1 - z^2 + y^2) \left(\frac{1}{2} + M\right) F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right] \right| dy. \end{aligned} \quad (2.2)$$

Then we follow arguments used in the proof of Proposition 1.6 [28]. For the large number N if $z \in (1, N)$, then

$$\frac{(z-1)^2 - y^2}{(z+1)^2 - y^2} \leq \frac{(z-1)^2}{(z+1)^2} \leq \frac{(N-1)^2}{(N+1)^2} < 1 \quad \text{for all } y \in (0, z-1).$$

Here and henceforth, if A and B are two non-negative quantities, we use $A \lesssim B$ to denote the statement that $A \leq CB$ for some absolute constant $C > 0$. For these values of z and $\Re M \geq 0$ we consider the function

$$\begin{aligned} & \frac{1}{[(z-1)^2 - y^2]} \left| \left[(z-1 + M(z^2 - 1 - y^2)) F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right. \right. \\ & \quad \left. \left. + (1 - z^2 + y^2) \left(\frac{1}{2} + M\right) F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right] \right| \end{aligned}$$

that is bounded since $F\left(\frac{1}{2}, \frac{1}{2}; 1; 0\right) = F\left(-\frac{1}{2}, \frac{1}{2}; 1; 0\right) = 1$ and, consequently, from (2.2) by (2.1) we derive

$$\begin{aligned} & \int_0^1 \phi(t)^a s^a |K_0(\phi(t)s, t; M)| \phi(t) ds \\ & \leq 4^{-\Re M} z^{-\Re M} \int_0^{z-1} y^a ((1+z)^2 - y^2)^{\Re M - \frac{1}{2}} dy \\ & \leq 4^{-\Re M} z^{-\Re M} \frac{1}{1+a} (z-1)^{1+a} (z+1)^{2\Re M - 1} F\left(\frac{1+a}{2}, \frac{1}{2} - \Re M; \frac{3+a}{2}; \frac{(z-1)^2}{(z+1)^2}\right) \\ & \lesssim z^{-\Re M} (z-1)^{1+a} (z+1)^{2\Re M - 1}, \quad 1 < z \leq N. \end{aligned}$$

Finally

$$\int_0^1 \phi(t)^a s^a |K_0(\phi(t)s, t; M)| \phi(t) ds \lesssim z^{-\Re M} (z-1)^{1+a} (z+1)^{2\Re M - 1}, \quad 1 < z \leq N.$$

Thus, we can restrict ourselves to the case of large $z \geq N$. We fix $\varepsilon \in (0, 1)$ and divide the domain of integration into two zones,

$$\begin{aligned} Z_1(\varepsilon, z) &:= \left\{ (z, y) \mid \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2} \leq \varepsilon, 0 \leq y \leq z-1 \right\}, \\ Z_2(\varepsilon, z) &:= \left\{ (z, y) \mid \varepsilon \leq \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}, 0 \leq y \leq z-1 \right\}. \end{aligned}$$

Furthermore, we split the integral into two parts:

$$\int_0^{z-1} y^a |K_0(y, t; M)| dy = \int_{(z,r) \in Z_1(\varepsilon, z)} y^a |K_0(y, t; M)| dy + \int_{(z,r) \in Z_2(\varepsilon, z)} y^a |K_0(y, t; M)| dy.$$

In the first zone we have (7.11) [24]:

$$\begin{aligned} F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \varepsilon\right) &= 1 + \left(\frac{1}{4} - M\right)^2 \varepsilon + O(\varepsilon^2), \\ F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \varepsilon\right) &= 1 + \left(M^2 - \frac{1}{4}\right) \varepsilon + O(\varepsilon^2). \end{aligned}$$

We use the last formulas to estimate the terms of (2.2) containing the hypergeometric functions:

$$\begin{aligned} &\left| \left[(z-1 + M(z^2 - 1 - y^2)) F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right. \right. \\ &\quad \left. \left. + (1 - z^2 + y^2) \left(\frac{1}{2} + M\right) F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right] \right| \\ &= \left| \left[(z-1 + M(z^2 - 1 - y^2)) \left[1 + (M^2 - M + \frac{1}{4}) \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2} + O\left(\left(\frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right)^2\right) \right] \right. \right. \\ &\quad \left. \left. + (1 - z^2 + y^2) \left(\frac{1}{2} + M\right) \left(1 + (M^2 - \frac{1}{4}) \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2} + O\left(\left(\frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right)^2\right) \right) \right] \right| \\ &\leq \frac{1}{2} ((z-1)^2 - y^2) \\ &\quad + \frac{1}{8} \left| (2M-1)(6M(y^2 - z^2) + 4Mz + 2M + y^2 - z^2 - 2z + 3) \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2} \right. \\ &\quad \left. + (z-1 + M(z^2 - 1 - y^2)) O\left(\left(\frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right)^2\right) \right. \\ &\quad \left. + (1 - z^2 + y^2) \left(\frac{1}{2} + M\right) O\left(\left(\frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right)^2\right) \right|. \end{aligned}$$

Hence, we have to consider the following three integrals, which can be easily evaluated and estimated (see, also, (2.1))

$$\begin{aligned} A_1 &:= \int_0^{z-1} y^a ((1+z)^2 - y^2)^{\Re M} \frac{1}{[(z-1)^2 - y^2] \sqrt{(z+1)^2 - y^2}} ((z-1)^2 - y^2) dy \\ &\lesssim \frac{1}{a+1} (z-1)^{a+1} (z+1)^{2\Re M-1}, \end{aligned}$$

$$A_2 := \int_0^{z-1} y^a ((1+z)^2 - y^2)^{\Re M} \frac{1}{[(z-1)^2 - y^2] \sqrt{(z+1)^2 - y^2}} \\ \times \left| (2M-1)(6M(y^2 - z^2) + 4Mz + 2M + y^2 - z^2 - 2z + 3) \right| \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2} dy.$$

Consider the real and imaginary parts of the function $Z = Z(y)$:

$$\Re Z(y) = \frac{(6\Re M(y^2 - z^2) + 4\Re Mz + 2\Re M + y^2 - (z+1)^2 + 4)}{(z+1)^2 - y^2}, \\ \Im Z(y) = \frac{(6\Im M(y^2 - z^2) + 4\Im Mz + 2\Im M)}{(z+1)^2 - y^2}.$$

The function $Z = Z(y)$ takes the following values at two end points

$$Z(z-1) = -\frac{(2M+1)(z-1)}{z}, \quad Z(0) = -\frac{(z-1)(M(6z+2) + z+3)}{(z+1)^2}.$$

Then with $N := \Im M$, $\widetilde{M} := \Re M$ and $b = y^2$ we have

$$\frac{d}{db} [(\Re Z(y))^2 + (\Im Z(y))^2] \\ = \frac{8}{((z+1)^2 - y^2)^3} \left[(12b\widetilde{M}^2 + 8b\widetilde{M} + 12bN^2 + b + 4\widetilde{M}^2 + 8\widetilde{M} + 4N^2 + 3) \right. \\ \left. + z(24b\widetilde{M}^2 + 4b\widetilde{M} + 24bN^2 + 16\widetilde{M}^2 + 12\widetilde{M} + 16N^2 - 2) \right. \\ \left. + z^2(4\widetilde{M}^2 - 16\widetilde{M} + 4N^2 - 1) + z^3(-24\widetilde{M}^2 - 4\widetilde{M} - 24N^2) \right] \\ < 0 \quad \text{for large } z \quad \text{and all } y \in [0, z-1].$$

Indeed,

$$\frac{d}{db} [(\Re Z(y))^2 + (\Im Z(y))^2] \Big|_{b=0} = -(z+1)^{-6} \left[8(z-1)(z^2(24\widetilde{M}^2 + 4\widetilde{M} + 24N^2) + z(20\widetilde{M}^2 \right. \\ \left. + 20\widetilde{M} + 20N^2 + 1) + (4\widetilde{M}^2 + 8\widetilde{M} + 4N^2 + 3)) \right]$$

and

$$\frac{d}{db} [(\Re Z(y))^2 + (\Im Z(y))^2] \Big|_{b=(z-1)^2} \\ = -2^{-1}z^{-3} \left[(z-1)(\widetilde{M}^2(8z+4) + 4\widetilde{M}(z+1) + N^2(8z+4) + 1) \right].$$

Hence,

$$A_2 \lesssim (z-1)^{a+1} (z+1)^{2\Re M-1}.$$

In the first zone we have

$$A_3 := \int_0^{z^{-1}} y^a ((1+z)^2 - y^2)^{\Re M} \frac{|z-1 + M(z^2 - 1 - y^2)|}{[(z-1)^2 - y^2] \sqrt{(z+1)^2 - y^2}} \left| \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2} \right|^2 dy \\ + \int_0^{z^{-1}} y^a ((1+z)^2 - y^2)^{\Re M} \frac{(z^2 - 1 - y^2)^{\frac{1}{2}} + M}{[(z-1)^2 - y^2] \sqrt{(z+1)^2 - y^2}} \left| \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2} \right|^2 dy.$$

Then

$$A_3 \lesssim \int_0^{z^{-1}} y^a ((1+z)^2 - y^2)^{\Re M} \frac{|z-1| + |z^2 - 1 - y^2|}{\sqrt{(z+1)^2 - y^2}} \frac{[(z-1)^2 - y^2]}{((z+1)^2 - y^2)^2} dy \\ \lesssim z^2 \int_0^{z^{-1}} y^a ((1+z)^2 - y^2)^{\Re M - \frac{3}{2}} dy \\ = z^2 \frac{1}{a+1} (z-1)^{a+1} (z+1)^{2\Re M - 3} F\left(\frac{a+1}{2}, \frac{3}{2} - M; \frac{a+3}{2}; \frac{(z-1)^2}{(z+1)^2}\right) \\ \lesssim (z-1)^{a+1} (z+1)^{2\Re M - 1} \times \begin{cases} 1 & \text{if } \Re M > 1/2 \\ \ln z & \text{if } \Re M = 1/2 \\ z^{\frac{1}{2} - \Re M} & \text{if } 1/2 > \Re M \end{cases}.$$

Finally in the first zone we obtain

$$\int_{(z,r) \in Z_1(\varepsilon, z)} y^a |K_0(y, t; M)| dy \lesssim (z-1)^{a+1} (z+1)^{\Re M - 1} \times \begin{cases} 1 & \text{if } \Re M > 1/2 \\ \ln z & \text{if } \Re M = 1/2 \\ z^{\frac{1}{2} - \Re M} & \text{if } 1/2 > \Re M \end{cases}.$$

In the second zone

$$\varepsilon \leq \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2} \leq 1 \quad \text{implies} \quad \frac{1}{(z-1)^2 - y^2} \leq \frac{1}{\varepsilon[(z+1)^2 - y^2]}.$$

Further, the hypergeometric functions for $\Re M > 0$ obey the estimates

$$\left| F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \zeta\right) \right| \leq C \quad \text{and} \quad \left| F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \zeta\right) \right| \leq C_M \quad \text{for all } \zeta \in [0, 1].$$

This allows us to estimate the integral over the second zone:

$$\int_{0, (z,r) \in Z_2(\varepsilon, z)}^{z^{-1}} y^a ((1+z)^2 - y^2)^{\Re M} \frac{1}{[(z-1)^2 - y^2] \sqrt{(z+1)^2 - y^2}} \\ \times \left[\left| (z-1 + M(z^2 - 1 - y^2)) F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right. \right. \\ \left. \left. + (1 - z^2 + y^2) \left(\frac{1}{2} + M\right) F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right| \right] dy \\ \lesssim z^2 \int_0^{z^{-1}} y^a ((1+z)^2 - y^2)^{\Re M - \frac{3}{2}} dy \\ = (a+1)^{-1} z^2 (z-1)^{a+1} (z+1)^{2\Re M - 3} F\left(\frac{a+1}{2}, \frac{3}{2} - \Re M; \frac{a+3}{2}; \frac{(z-1)^2}{(z+1)^2}\right).$$

According to Lemma A1 [28] with $a > -1$ we have

$$\begin{aligned} \lim_{z \rightarrow \infty} F\left(\frac{a+1}{2}, \frac{3}{2} - M; \frac{a+3}{2}; \frac{(z-1)^2}{(z+1)^2}\right) &= \frac{\Gamma(\frac{a+3}{2})\Gamma(M - \frac{1}{2})}{\Gamma(\frac{a}{2} + M)} \quad \text{if } M > 1/2, \\ \lim_{z \rightarrow \infty} \frac{1}{\ln z} F\left(\frac{a+1}{2}, \frac{3}{2} - M; \frac{a+3}{2}; \frac{(z-1)^2}{(z+1)^2}\right) &= \frac{1+a}{2} \quad \text{if } M = 1/2, \\ \lim_{z \rightarrow \infty} z^{M-\frac{1}{2}} F\left(\frac{a+1}{2}, \frac{3}{2} - M; \frac{a+3}{2}; \frac{(z-1)^2}{(z+1)^2}\right) &= 2^{2M-1} \frac{1+a}{1-2M} \quad \text{if } 1/2 > M. \end{aligned}$$

Hence,

$$\begin{aligned} &\int_{(z,r) \in Z_2(\varepsilon, z)} y^\alpha |K_0(y, t; M)| dy \\ &\lesssim (z-1)^{a+1} (z+1)^{\Re M - 1} \times \begin{cases} 1 & \text{if } \Re M > 1/2 \\ \ln z & \text{if } \Re M = 1/2 \\ z^{\frac{1}{2} - \Re M} & \text{if } 1/2 > \Re M \end{cases}. \end{aligned}$$

Then we combine the estimates obtained in two zones

$$\begin{aligned} \int_0^{z^{-1}} y^\alpha |K_0(y, t; M)| dy &= \int_{(z,r) \in Z_1(\varepsilon, z)} y^\alpha |K_0(y, t; M)| dy + \int_{(z,r) \in Z_2(\varepsilon, z)} y^\alpha |K_0(y, t; M)| dy \\ &\lesssim (z-1)^{a+1} (z+1)^{\Re M - 1} \begin{cases} 1 & \text{if } \Re M > 1/2, \\ \ln z & \text{if } \Re M = 1/2, \\ z^{\frac{1}{2} - \Re M} & \text{if } 1/2 > \Re M. \end{cases} \end{aligned}$$

The lemma is proved. \square

3 Estimates for the equation without a source

Let $\varphi_j = \varphi(2^{-j}\xi)$, $j > 0$, and $\varphi_0 = 1 - \sum_{j=1}^{\infty} \varphi_j$, where $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi \geq 0$ and $\text{supp } \varphi \subseteq \{\xi \in \mathbb{R}^n; 1/2 < |\xi| < 2\}$, is that

$$\sum_{-\infty}^{\infty} \varphi(2^{-j}\xi) = 1, \quad \xi \neq 0.$$

The norm $\|v\|_{B_p^{s,q}}$ of the Besov space $B_p^{s,q}$ is defined as follows

$$\|v\|_{B_p^{s,q}} = \left(\sum_{j=0}^{\infty} (2^{js} \|\mathcal{F}^{-1}(\varphi_j \hat{v})\|_p)^q \right)^{1/q},$$

where \hat{v} is the Fourier transform of v . The following theorem by Brenner [2] is crucial for this and the next sections.

Theorem 3.1 (Brenner [2]) *Let $A = A(x, D)$ be a second order negative elliptic differential operator with real C^∞ -coefficients such that $A(x, D) = A(\infty, D)$ for $|x|$ large enough. Let $u(t) = G_0(t)g_0 + G_1(t)g_1$ be the solution of*

$$\partial_t^2 u - A(x, D)u = 0, \quad x \in \mathbb{R}^n, \quad t \geq 0, \quad (3.1)$$

$$u(x, 0) = g_0(x), \quad u_t(x, 0) = g_1(x), \quad x \in \mathbb{R}^n. \quad (3.2)$$

Then for each $T < \infty$ there is a constant $C = C(T)$ such that if $(n+1)\delta \leq \nu + s - s'$, $\nu = 0, 1$,

$$\|G_\nu(t)g\|_{B_{p'}^{s',q}} \leq C(T)t^{\nu+s-s'-2n\delta} \|g\|_{B_p^{s,q}}, \quad 0 < t \leq T. \quad (3.3)$$

Here $s, s' \geq 0$, $q \geq 1$, $1 \leq p \leq 2$, $1/p + 1/p' = 1$, and $\delta = 1/p - 1/2$.

The estimate (3.3) can be also written as

$$\|G_\nu(t)g\|_{B_{p'}^{s',q}} \leq C(T)t^{\nu+s-s'-n(1/p-1/p')} \|g\|_{B_p^{s,q}}, \quad 0 < t \leq T, \quad \nu = 0, 1.$$

Remark 3.2 In the case of $A(x, D) = \Delta$ the constant $C(T)$ can be chosen independent of T , that is,

$$\|G_\nu(t)g\|_{B_{p'}^{s',q}} \leq ct^{\nu+s-s'-n(1/p-1/p')} \|g\|_{B_p^{s,q}} \quad \text{for all } t \in (0, \infty), \quad \nu = 0, 1.$$

(See, also, e.g. [2, 18, 22].)

Theorem 3.3 Suppose that for the equation (3.1) the conditions of Theorem 3.1 are fulfilled. Assume that $s, s' \geq 0$, $q \geq 1$, $1 \leq p \leq 2$, $1/p + 1/p' = 1$, and $\delta = 1/p - 1/2$, $(n+1)\delta \leq s - s'$, $-1 < s - s' - 2n\delta$. Denote $a := s - s' - 2n\delta$.

Then for each $T < \infty$ there is a constant $C = C(T)$ such that the solution $\Phi = \Phi(x, t)$ of the Cauchy problem

$$\Phi_{tt} - n\Phi_t - e^{2t}A(x, D)\Phi + m^2\Phi = 0, \quad \Phi(x, 0) = \varphi_0(x), \quad \Phi_t(x, 0) = \varphi_1(x), \quad (3.4)$$

where $\Re M > 0$, satisfies the following estimate

$$\begin{aligned} \|\Phi(x, t)\|_{B_{p'}^{s',q}} &\leq C(T)\|\varphi_0\|_{B_p^{s,q}} e^{\frac{n}{2}t} (e^t - 1)^a \left(e^{-\frac{1}{2}t} + (e^t - 1)(e^t + 1)^{\Re M - 1} \left[e^{-\Re M t} (e^t + 1)^{\Re M} \right. \right. \\ &\quad \left. \left. + \begin{cases} 1 & \text{if } \Re M > 1/2 \\ t^{\text{sgn}|\Im M|} + e^{(\frac{1}{2} - \Re M)t} & \text{if } \Re M \leq 1/2 \end{cases} \right] \right) \\ &\quad + C(T)\|\varphi_1\|_{B_p^{s,q}} e^{\frac{n}{2}t} e^{-\Re M t} (e^t - 1)^{a+1} (e^t + 1)^{2\Re M - 1} \quad \text{for all } 0 < t < T. \end{aligned}$$

If $A(x, D) = \Delta$, then the constant $C(T)$ is independent of T , that is, the estimate holds with $T = \infty$.

Corollary 3.4 The solution $\Phi = \Phi(x, t)$ of the Cauchy problem (3.4) satisfies the following estimates

$$\begin{aligned} \|\Phi(x, t)\|_{B_{p'}^{s',q}} &\leq C(T)\|\varphi_0\|_{B_p^{s,q}} e^{(\frac{n}{2}+a+\Re M)t} \begin{cases} 1 & \text{if } \Re M > 1/2 \\ t^{\text{sgn}|\Im M|} + e^{(\frac{1}{2}-\Re M)t} & \text{if } \Re M \leq 1/2 \end{cases} \\ &\quad + C(T)\|\varphi_1\|_{B_p^{s,q}} e^{(\frac{n}{2}+a+\Re M)t}, \quad \text{for all } t \in (1, T), \\ \|\Phi(x, t)\|_{B_{p'}^{s',q}} &\leq C(T)t^a \|\varphi_0\|_{B_p^{s,q}} + t^{a+1} \|\varphi_1\|_{B_p^{s,q}} \quad \text{for all } t \in (0, 1). \end{aligned}$$

If $A(x, D) = \Delta$, then the constant $C(T)$ is independent of T , that is, the estimates hold with $T = \infty$.

Proof of Theorem 3.3. The proofs for the cases of the Laplace operator Δ and the operator $A(x, \partial_x)$ are identical and we set now $A(x, \partial_x) = \Delta$. First we consider the case of $\varphi_1 = 0$. Then according to (1.3)

$$\begin{aligned} &\Phi(x, t) \\ &= e^{\frac{n-1}{2}t} v_{\varphi_0}(x, \phi(t)) + e^{\frac{n}{2}t} \int_0^1 v_{\varphi_0}(x, \phi(t)s) (2K_0(\phi(t)s, t; M) - nK_1(\phi(t)s, t; M)) \phi(t) ds \end{aligned}$$

and, consequently,

$$\begin{aligned} \|\Phi(x, t)\|_{B_{p'}^{s', q}} &\leq e^{\frac{n-1}{2}t} \|v_{\varphi_0}(x, \phi(t))\|_{B_{p'}^{s', q}} \\ &\quad + e^{\frac{n}{2}t} \int_0^1 \|v_{\varphi_0}(x, \phi(t)s)\|_{B_{p'}^{s', q}} |2K_0(\phi(t)s, t; M) - nK_1(\phi(t)s, t; M)| \phi(t) ds. \end{aligned} \quad (3.5)$$

If $n \geq 2$, then according to Theorem 3.1, for the solution $v = v(x, t)$ of the Cauchy problem (3.1)-(3.2) with $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ one has the estimate (3.3) provided that $s, s' \geq 0$, $q \geq 1$, $1 \leq p \leq 2$, $1/p + 1/p' = 1$, and $\delta = 1/p - 1/2$, $(n+1)\delta \leq s - s'$. Hence,

$$\|v_{\varphi_0}(x, \phi(t))\|_{B_{p'}^{s', q}} \leq C \phi(t)^{s-s'-2n\delta} \|\varphi_0\|_{B_p^{s, q}} \quad \text{for all } t > 0,$$

where $\phi(t) = e^t - 1$. Consequently, for the first term of the right-hand side of (3.5) we have

$$e^{\frac{n-1}{2}t} \|v_{\varphi_0}(x, \phi(t))\|_{B_{p'}^{s', q}} \leq C e^{\frac{n-1}{2}t} (e^t - 1)^{s-s'-2n\delta} \|\varphi_0\|_{B_p^{s, q}} \quad \text{for all } t > 0.$$

For the second term of (3.5) we obtain

$$\begin{aligned} &e^{\frac{n}{2}t} \int_0^1 \|v_{\varphi_0}(x, \phi(t)s)\|_{B_{p'}^{s', q}} |2K_0(\phi(t)s, t; M) - nK_1(\phi(t)s, t; M)| \phi(t) ds \\ &\leq \|\varphi_0\|_{B_p^{s, q}} e^{\frac{n}{2}t} \int_0^1 \phi(t)^{s-s'-2n\delta} s^{s-s'-2n\delta} (|2K_0(\phi(t)s, t; M)| + n|K_1(\phi(t)s, t; M)|) \phi(t) ds. \end{aligned}$$

Denote $a := s - s' - 2n\delta$. We have to estimate the following two terms of the last inequality:

$$\int_0^1 \phi(t)^a s^a |K_i(\phi(t)s, t; M)| \phi(t) ds, \quad i = 0, 1,$$

where $\phi(t) = e^t - 1$ and $t > 0$. To complete the estimate of the second term of (3.5) we apply Lemma 2.1 and Lemma 2.2. Thus, if $\varphi_1 = 0$, then from (3.5) we derive

$$\begin{aligned} \|\Phi(x, t)\|_{B_{p'}^{s', q}} &\lesssim e^{\frac{n-1}{2}t} (e^t - 1)^a \|\varphi_0\|_{B_p^{s, q}} \\ &\quad + \|\varphi_0\|_{B_p^{s, q}} e^{\frac{n}{2}t} \left\{ \int_0^1 \phi(t)^a s^a (|K_0(\phi(t)s, t; M)| + |K_1(\phi(t)s, t; M)|) \phi(t) ds \right\} \\ &\lesssim \|\varphi_0\|_{B_p^{s, q}} \left(e^{\frac{n-1}{2}t} (e^t - 1)^a + e^{\frac{n}{2}t} \left[e^{-\Re M t} (e^t - 1)^{a+1} (e^t + 1)^{2\Re M - 1} \right. \right. \\ &\quad \left. \left. + (e^t - 1)^{a+1} (e^t + 1)^{\Re M - 1} \begin{cases} 1 & \text{if } \Re M > 1/2 \\ t^{\text{sgn}|\Im M|} + e^{(\frac{1}{2} - \Re M)t} & \text{if } \Re M \leq 1/2 \end{cases} \right] \right) \\ &\lesssim \|\varphi_0\|_{B_p^{s, q}} \left(e^{\frac{n-1}{2}t} (e^t - 1)^a + e^{\frac{n}{2}t} (e^t - 1)^{a+1} \left[e^{-\Re M t} (e^t + 1)^{2\Re M - 1} \right. \right. \\ &\quad \left. \left. + (e^t + 1)^{\Re M - 1} \begin{cases} 1 & \text{if } \Re M > 1/2 \\ t^{\text{sgn}|\Im M|} + e^{(\frac{1}{2} - \Re M)t} & \text{if } \Re M \leq 1/2 \end{cases} \right] \right). \end{aligned}$$

In the case of $\varphi_0 = 0$ from (1.3) we have

$$\begin{aligned} \|\Phi(x, t)\|_{B_{p'}^{s', q}} &\leq 2e^{\frac{n}{2}t} \int_0^1 \|v_{\varphi_1}(x, \phi(t)s)\|_{B_{p'}^{s', q}} |K_1(\phi(t)s, t; M)| \phi(t) ds \\ &\leq 2\|\varphi_1\|_{B_p^{s, q}} e^{\frac{n}{2}t} \int_0^1 \phi(t)^a s^a |K_1(\phi(t)s, t; M)| \phi(t) ds, \quad t > 0. \end{aligned}$$

Due to Lemma 2.1 we obtain the statement of the theorem. Theorem is proved. \square

Remark 3.5 *The estimates in the Lebesgue spaces for the case of large mass $m \geq n/2$ were discussed in [9].*

4 Estimates for the equation with a source

Theorem 4.1 *Assume that for the equation (3.1) the conditions of Theorem 3.1 are fulfilled. Let $\Phi = \Phi(x, t)$ be a solution of the Cauchy problem*

$$\Phi_{tt} - n\Phi_t - e^{2t}A(x, \partial_x)\Phi + m^2\Phi = f, \quad \Phi(x, 0) = 0, \quad \Phi_t(x, 0) = 0.$$

Then for each $T < \infty$ there is a constant $C = C(T)$ such that the solution $\Phi = \Phi(x, t)$ with $\Re M > 0$ satisfies the following estimate:

$$\|\Phi(x, t)\|_{B_{p'}^{s', q}} \leq C(T) e^{t(\frac{n}{2} + \Re M + s - s' - n(\frac{1}{p} - \frac{1}{p'}))} \int_0^t e^{-(\frac{n}{2} + \Re M)b} \|f(x, b)\|_{B_p^{s, q}} db.$$

for all $t \in (0, C(T))$, provided that $s, s' \geq 0$, $q \geq 1$, $1 \leq p \leq 2$, $1/p + 1/p' = 1$, and $\delta = 1/p - 1/2$, $(n+1)\delta \leq s - s'$, $-1 < s - s' - 2n\delta$. If $A(x, D) = \Delta$, then the constant $C(T)$ is independent of T , that is, the estimate holds with $T = \infty$.

Proof. From (1.3) we have

$$\Phi(x, t) = 2e^{\frac{n}{2}t} \int_0^t db \int_0^{e^t - e^b} dr e^{-\frac{n}{2}b} v(x, r; b) E(r, t; 0, b; M).$$

Denote $a := s - s' - n(\frac{1}{p} - \frac{1}{p'}) > -1$, then according to (3.3) we can write

$$\|v(x, r; b)\|_{B_{p'}^{s', q}} \leq Cr^a \|f(x, b)\|_{B_p^{s, q}} \quad \text{for all } r > 0.$$

Hence,

$$\begin{aligned} \|\Phi(x, t)\|_{B_{p'}^{s', q}} &\lesssim e^{\frac{n}{2}t} \int_0^t db \int_0^{e^t - e^b} dr e^{-\frac{n}{2}b} \|v(x, r; b)\|_{B_{p'}^{s', q}} |E(r, t; 0, b; M)| \\ &\lesssim e^{\frac{n}{2}t} \int_0^t e^{-\Re M(b+t)} e^{-\frac{n}{2}b} \|f(x, b)\|_{B_p^{s, q}} db \int_0^{e^t - e^b} r^a \\ &\quad \times \left((e^t + e^b)^2 - r^2 \right)^{-\frac{1}{2} + \Re M} \left| F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^t - e^b)^2 - r^2}{(e^t + e^b)^2 - r^2}\right) \right| dr. \end{aligned}$$

Following the outline of the proof of Lemma 1.5 [28] we set $r = ye^b$ and obtain

$$\begin{aligned} & \|\Phi(x, t)\|_{B_{p'}^{s', q}} \\ & \lesssim e^{\frac{n}{2}t} e^{-\Re M t} \int_0^t e^{-(\frac{n}{2} - \Re M)b} e^{ab} \|f(x, b)\|_{B_p^{s, q}} db \\ & \quad \times \int_0^{e^{t-b}-1} y^a \left((e^{t-b} + 1)^2 - y^2 \right)^{-\frac{1}{2} + \Re M} \left| F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{t-b} - 1)^2 - y^2}{(e^{t-b} + 1)^2 - y^2}\right) \right| dy. \end{aligned}$$

In order to estimate the integral

$$I_{a, M}(z) := \int_0^{z-1} y^a \left((z+1)^2 - y^2 \right)^{-\frac{1}{2} + \Re M} \left| F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right| dy,$$

where $z = e^{t-b} > 1$ and $a > -1$ we use

$$\left| F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right| \leq C_M$$

and estimate the integral

$$\begin{aligned} I_{a, M}(z) & \leq \int_0^{z-1} y^a \left((z+1)^2 - y^2 \right)^{-\frac{1}{2} + \Re M} dy \\ & = \frac{1}{a+1} (z-1)^{a+1} (z+1)^{2\Re M - 1} F\left(\frac{a+1}{2}, \frac{1}{2} - \Re M; \frac{a+3}{2}; \frac{(z-1)^2}{(z+1)^2}\right) \\ & \lesssim (z-1)^{a+1} (z+1)^{2\Re M - 1}. \end{aligned}$$

Hence,

$$\begin{aligned} & \|\Phi(x, t)\|_{B_{p'}^{s', q}} \\ & \lesssim e^{\frac{n}{2}t} e^{-\Re M t} \int_0^t e^{-(\frac{n}{2} - \Re M)b} e^{ab} (e^{t-b} - 1)^{1+a} (e^{t-b} + 1)^{(2\Re M - 1)} \|f(x, b)\|_{B_p^{s, q}} db \\ & \lesssim e^{\frac{n}{2}t} e^{-\Re M t} e^{t(1+a)} \int_0^t e^{-(\frac{n}{2} + 1 - \Re M)b} (e^{t-b} + 1)^{(2\Re M - 1)} \|f(x, b)\|_{B_p^{s, q}} db \\ & \lesssim e^{t(\frac{n}{2} + \Re M + a)} \int_0^t e^{-(\frac{n}{2} + \Re M)b} \|f(x, b)\|_{B_p^{s, q}} db. \end{aligned}$$

Theorem is proved. \square

5 Integral equation. Global existence in Sobolev spaces

We are going to apply the Banach fixed-point theorem. In order to estimate nonlinear term we use the Lipschitz condition (\mathcal{L}) . Evidently, the Condition (\mathcal{L}) imposes some restrictions on n, α, s . First for $A(x, \partial_x) = \Delta$ we consider the integral equation (0.6),

$$\Phi(x, t) = \Phi_0(x, t) + G[e^{-\Gamma \cdot} F(\cdot, \Phi)](x, t),$$

where the function $\Phi_0 \in C([0, \infty); H_{(s)}(\mathbb{R}^n))$ is given. Every solution to the Cauchy problem (0.7)-(0.8) solves also the integral equation (0.6) with some function $\Phi_0 = \Phi_0(x, t)$ which is a solution to the Cauchy problem for the linear equation (3.4).

The operator G and the structure of the nonlinear term determine the solvability of the integral equation (0.6). For the operator G generated by the linear part of the equation (0.2) with $m^2 < 0$ the global solvability of the integral equation (0.6) was studied in [25]. For the case of $m^2 < 0$ and the nonlinearity $F(\Phi) = c|\Phi|^{\alpha+1}$, $c \neq 0$, the results of [25] imply the nonexistence of the global solution even for arbitrary small function $\Phi_0(x, 0)$ under some conditions on n and α .

Theorem 5.1 *Assume that $F(x, \Phi)$ is Lipschitz continuous in the space $H_{(s)}(\mathbb{R}^n)$, $F(x, 0) \equiv 0$, and $\alpha > 0$. Suppose that $\Re M > 0$ and one of the following conditions is satisfied*

$$\begin{aligned} (i_0) \quad & \frac{n}{2} + \Re M + \gamma(\alpha + 1) + \Gamma > 0, \quad \frac{n}{2} + \Re M + \gamma \leq 0, \\ (ii_0) \quad & \frac{n}{2} + \Re M + \gamma(\alpha + 1) + \Gamma = 0, \quad \frac{n}{2} + \Re M + \gamma < 0, \\ (iii_0) \quad & \frac{n}{2} + \Re M + \gamma(\alpha + 1) + \Gamma < 0, \quad \gamma\alpha + \Gamma \geq 0. \end{aligned}$$

Then for every given function $\Phi_0(x, t) \in X(\varepsilon, H_{(s)}(\mathbb{R}^n), \gamma)$ such that

$$\sup_{t \in [0, \infty)} e^{\gamma t} \|\Phi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} < \varepsilon,$$

and for sufficiently small ε , the integral equation (0.6) has a unique solution $\Phi(x, t) \in X(2\varepsilon, H_{(s)}(\mathbb{R}^n), \gamma)$, that is

$$\sup_{t \in [0, \infty)} e^{\gamma t} \|\Phi(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} < 2\varepsilon.$$

Theorem 5.2 *Assume that $F(x, \Phi)$ is Lipschitz continuous in the space $H_{(s)}(\mathbb{R}^n)$, $F(x, 0) \equiv 0$, and $\alpha > 0$. Suppose that one of the following conditions is satisfied*

$$\begin{aligned} (iv_0) \quad & \frac{n}{2} + \Re M + \gamma(\alpha + 1) + \Gamma > 0, \quad \frac{n}{2} + \Re M + \gamma > 0, \\ (v_0) \quad & \frac{n}{2} + \Re M + \gamma(\alpha + 1) + \Gamma = 0, \quad \frac{n}{2} + \Re M + \gamma \geq 0, \\ (vi_0) \quad & \frac{n}{2} + \Re M + \gamma(\alpha + 1) + \Gamma < 0, \quad \gamma\alpha + \Gamma < 0. \end{aligned}$$

If $\Re M > 0$, then for the function $\Phi_0(x, t) \in X(\varepsilon, H_{(s)}(\mathbb{R}^n), \gamma)$ the unique solution $\Phi(x, t)$ of the integral equation (0.6) has the lifespan T_{ls} that can be estimated from below

$$T_{ls} \geq \mathcal{I} \left(C(M, n, \alpha, \gamma, \Gamma) \left(\max_{\tau \in [0, \infty)} e^{\gamma \tau} \|\Phi_0(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right)^{-\alpha} \right)$$

for sufficiently small $\max_{\tau \in [0, \infty)} e^{\gamma \tau} \|\Phi_0(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)}$ with some number $C(M, n, \alpha, \gamma, \Gamma) > 0$.

We need the following elementary lemma.

Lemma 5.3 *For $\mathcal{M} \in [0, \infty)$ and $\alpha > 0$ the inequality*

$$e^{t(\frac{n}{2} + \mathcal{M} + \gamma)} \int_0^t e^{-(\frac{n}{2} + \mathcal{M} + \gamma(\alpha + 1) + \Gamma)b} db \leq \text{const} \quad \text{for all } t \in [0, \infty) \quad (5.1)$$

holds true for all γ and Γ such that one of these conditions is satisfied

$$\begin{aligned} (i_{\mathcal{M}}) \quad & \frac{n}{2} + \mathcal{M} + \gamma(\alpha + 1) + \Gamma > 0, \quad \frac{n}{2} + \mathcal{M} + \gamma \leq 0, \\ (ii_{\mathcal{M}}) \quad & \frac{n}{2} + \mathcal{M} + \gamma(\alpha + 1) + \Gamma = 0, \quad \frac{n}{2} + \mathcal{M} + \gamma < 0, \\ (iii_{\mathcal{M}}) \quad & \frac{n}{2} + \mathcal{M} + \gamma(\alpha + 1) + \Gamma < 0, \quad \gamma\alpha + \Gamma \geq 0. \end{aligned}$$

If one of the following conditions

$$\begin{aligned} (iv_{\mathcal{M}}) \quad & \frac{n}{2} + \mathcal{M} + \gamma(\alpha + 1) + \Gamma > 0, \quad \frac{n}{2} + \mathcal{M} + \gamma > 0, \\ (v_{\mathcal{M}}) \quad & \frac{n}{2} + \mathcal{M} + \gamma(\alpha + 1) + \Gamma = 0, \quad \frac{n}{2} + \mathcal{M} + \gamma \geq 0, \\ (vi_{\mathcal{M}}) \quad & \frac{n}{2} + \mathcal{M} + \gamma(\alpha + 1) + \Gamma < 0, \quad \gamma\alpha + \Gamma < 0, \end{aligned}$$

is fulfilled, then the function

$$I_{\mathcal{M}}(t) := e^{t(\frac{n}{2} + \mathcal{M} + \gamma)} \int_0^t e^{-(\frac{n}{2} + \mathcal{M} + \gamma(\alpha + 1) + \Gamma)b} db$$

is monotonic and unbounded, $\lim_{t \rightarrow \infty} I_{\mathcal{M}}(t) = \infty$.

Proof of Theorem 5.1. Consider the mapping

$$S[\Phi](x, t) := \Phi_0(x, t) + G[e^{-\Gamma \cdot} F(\cdot, \Phi)](x, t).$$

We are going to prove that S maps $X(R, H_{(s)}(\mathbb{R}^n), \gamma)$ into itself and that S is a contraction, provided that ε and R are sufficiently small. Theorem 4.1 implies

$$\|S[\Phi](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq \|\Phi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M e^{t(\frac{n}{2} + \Re M)} \int_0^t e^{-(\frac{n}{2} + \Re M + \Gamma)b} \|\Phi(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)}^{\alpha+1} db.$$

Taking into account the Condition (\mathcal{L}) we arrive at

$$\begin{aligned} e^{\gamma t} \|S[\Phi](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} & \leq e^{\gamma t} \|\Phi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M \left(\sup_{\tau \in [0, \infty)} e^{\gamma \tau} \|\Phi(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right)^{\alpha+1} \\ & \quad \times e^{t(\frac{n}{2} + \Re M + \gamma)} \int_0^t e^{-(\frac{n}{2} + \Re M + \gamma(\alpha + 1) + \Gamma)b} db. \end{aligned}$$

Then, for given α , Γ , and $\gamma \in \mathbb{R}$ according to Lemma 5.3, we have

$$\sup_{t \in [0, \infty)} e^{\gamma t} \|S[\Phi](x, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq \sup_{t \in [0, \infty)} e^{\gamma t} \|\Phi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + C \left(\sup_{t \in [0, \infty)} e^{\gamma t} \|\Phi(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right)^{\alpha+1}.$$

Thus, the last inequality proves that the operator S maps $X(R, H_{(s)}(\mathbb{R}^n), \gamma)$ into itself if ε and R are sufficiently small, namely, if $\varepsilon + CR^{\alpha+1} < R$.

It remains to prove that S is a contraction mapping. As a matter of fact, we just use the estimate (0.3) in order to obtain the contraction property

$$e^{\gamma t} \|S[\Phi](\cdot, t) - S[\Psi](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq CR(t)^\alpha d(\Phi, \Psi), \quad (5.2)$$

where $R(t) := \max\left\{\sup_{0 \leq \tau \leq t} e^{\gamma\tau} \|\Phi(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)}, \sup_{0 \leq \tau \leq t} e^{\gamma\tau} \|\Psi(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)}\right\} \leq R$.

Indeed, the inequality (5.1) with $\mathcal{M} = \Re M$ holds true for α and γ satisfying conditions of the theorem. Together with

$$e^{\gamma t} \|S[\Phi](\cdot, t) - S[\Psi](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_M d(\Phi, \Psi) R(t)^\alpha e^{t(\frac{n}{2} + \Re M + \gamma)} \int_0^t e^{-(\frac{n}{2} + \Re M + \gamma(\alpha+1) + \Gamma)b} db$$

it leads to (5.2). Next we choose ε and R sufficiently small. Banach's fixed point theorem completes the proof. \square

Proof of Theorem 5.2. We have

$$e^{\gamma t} \|\Phi(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq e^{\gamma t} \|\Phi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M I(t) \left(\sup_{\tau \in [0, \infty)} e^{\gamma\tau} \|\Phi(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right)^{\alpha+1}.$$

Set

$$T_\varepsilon := \inf\{T : \max_{\tau \in [0, T]} e^{\gamma\tau} \|\Phi(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \geq 2\varepsilon\}, \quad \varepsilon := \max_{\tau \in [0, \infty)} e^{\gamma\tau} \|\Phi_0(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)}.$$

Then

$$2\varepsilon \leq \varepsilon + C_{M,n,\alpha} \varepsilon^{\alpha+1} I_M(T_\varepsilon)$$

implies

$$I_M(T_\varepsilon) \geq C_{M,n,\alpha}^{-1} \varepsilon^{-\alpha}.$$

If \mathcal{I} is a function inverse to $I = I(t)$, then

$$T_{I\varepsilon} \geq \mathcal{I}_M \left(C_{M,n,\alpha}^{-1} \left(\max_{\tau \in [0, \infty)} e^{\gamma\tau} \|\Phi_0(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right)^{-\alpha} \right).$$

for sufficiently small $\max_{\tau \in [0, \infty)} e^{\gamma\tau} \|\Phi_0(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)}$. Theorem is proved. \square

6 The Cauchy problem. Global existence of small data solutions

Proof of Theorem 0.1. For the function Φ_0 , that is, for the solution of the Cauchy problem (3.4) and for $n \geq 2$, according to Theorem 3.3 we have with $a = 0$ the estimate

$$\begin{aligned} & e^{\gamma t} \|\Phi_0(x, t)\|_{H_{(s)}} \\ & \lesssim \|\varphi_0\|_{H_{(s)}} e^{(\frac{n}{2} + \Re M + \gamma)t} \begin{cases} 1 & \text{if } \Re M > 1/2 \\ t^{\text{sgn}|\Im M|} + e^{(\frac{1}{2} - \Re M)t} & \text{if } \Re M \leq 1/2 \end{cases} \\ & \quad + \|\varphi_1\|_{H_{(s)}} e^{(\frac{n}{2} + \Re M + \gamma)t} \\ & \lesssim e^{(\frac{n}{2} + \Re M + \gamma)t} \left(\|\varphi_1\|_{H_{(s)}} + \|\varphi_0\|_{H_{(s)}} \begin{cases} 1 & \text{if } \Re M > 1/2 \\ t^{\text{sgn}|\Im M|} + e^{(\frac{1}{2} - \Re M)t} & \text{if } \Re M \leq 1/2 \end{cases} \right) \end{aligned}$$

for all $t \in (1, \infty)$. The factors of the norms of the initial data functions of the last inequality are bounded provided that

$$\frac{n}{2} + \max\left\{\frac{1}{2}, \Re M\right\} + \gamma \leq 0, \quad \Re M < 1/2 \quad \text{or} \quad \Re M > 1/2 \quad \text{or} \quad M = \Re M = 1/2.$$

Thus, we have $\Phi_0 \in X(R, H_{(s)}(\mathbb{R}^n), \gamma)$. Consequently, according to Theorem 5.1 the function Φ belongs to the space $X(R, H_{(s)}(\mathbb{R}^n), \gamma)$, where the operator S is a contraction. Theorem is proved. \square

Proof of Theorem 0.2. According to Theorem 4.1 we have the estimate

$$\begin{aligned} e^{\gamma t} \|S[\Phi](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} &\leq e^{\gamma t} \|\Phi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M \left(\sup_{\tau \in [0, \infty)} e^{\gamma \tau} \|\Phi(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right)^{\alpha+1} \\ &\quad \times e^{t(\frac{n}{2} + \Re M + \gamma)} \int_0^t e^{-(\frac{n}{2} + \Re M + \gamma(\alpha+1) + \Gamma)b} db. \end{aligned}$$

Due to Corollary 3.4 we have $\Phi_0 \in X(R, H_{(s)}(\mathbb{R}^n), \gamma)$. Thus,

$$\begin{aligned} &e^{\gamma t} \|S[\Phi](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \\ &\leq C e^{(\frac{n}{2} + \max\{\frac{1}{2}, \Re M\} + \gamma)t} \left(\|\varphi_0\|_{L^p} + \|\varphi_1\|_{H_{(s)}} \right) + C_M \left(\sup_{\tau \in [0, \infty)} e^{\gamma \tau} \|\Phi(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right)^{\alpha+1} I(t) \\ &\leq C \left(\|\varphi_0\|_{L^p} + \|\varphi_1\|_{H_{(s)}} \right) + C_M \left(\sup_{\tau \in [0, \infty)} e^{\gamma \tau} \|\Phi(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right)^{\alpha+1} I(t). \end{aligned}$$

Set

$$T_\varepsilon := \inf \{ T : \max_{\tau \in [0, T]} e^{\gamma \tau} \|\Phi(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \geq 2\varepsilon \}, \quad \varepsilon := C(\|\varphi_0\|_{H_{(s)}} + \|\varphi_1\|_{H_{(s)}}).$$

Then

$$2\varepsilon \leq \varepsilon + C_{M, n, \alpha} \varepsilon^{\alpha+1} I(T_\varepsilon)$$

implies the statement of the theorem. Thus, the theorem is proved. \square

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