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THE SPECTRAL PICTURE AND JOINT SPECTRAL RADIUS OF THE GENERALIZED SPHERICAL ALUTHGE TRANSFORM

CHAFIQ BENHIDA, RAÚL E. CURTO, SANG HOON LEE, AND JASANG YOON

ABSTRACT. For an arbitrary commuting d -tuple \mathbf{T} of Hilbert space operators, we fully determine the spectral picture of the generalized spherical Aluthge transform $\Delta_t(\mathbf{T})$ and we prove that the spectral radius of \mathbf{T} can be calculated from the norms of the iterates of $\Delta_t(\mathbf{T})$. Let $\mathbf{T} \equiv (T_1, \dots, T_d)$ be a commuting d -tuple of bounded operators acting on an infinite dimensional separable Hilbert space, let $P := \sqrt{T_1^*T_1 + \dots + T_d^*T_d}$, and let

$$\begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix} = \begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix} P$$

be the canonical polar decomposition, with (V_1, \dots, V_d) a (joint) partial isometry and

$$\bigcap_{i=1}^d \ker T_i = \bigcap_{i=1}^d \ker V_i = \ker P.$$

For $0 \leq t \leq 1$, we define the generalized spherical Aluthge transform of \mathbf{T} by

$$\Delta_t(\mathbf{T}) := (P^t V_1 P^{1-t}, \dots, P^t V_d P^{1-t}).$$

We also let $\|\mathbf{T}\|_2 := \|P\|$. We first determine the spectral picture of $\Delta_t(\mathbf{T})$ in terms of the spectral picture of \mathbf{T} ; in particular, we prove that, for any $0 \leq t \leq 1$, $\Delta_t(\mathbf{T})$ and \mathbf{T} have the same Taylor spectrum, the same Taylor essential spectrum, the same Fredholm index, and the same Harte spectrum. We then study the joint spectral radius $r(\mathbf{T})$, and prove that $r(\mathbf{T}) = \lim_n \|\Delta_t^{(n)}(\mathbf{T})\|_2$ ($0 < t < 1$), where $\Delta_t^{(n)}$ denotes the n -th iterate of Δ_t . For $d = t = 1$, we give an example where the above formula fails.

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1. INTRODUCTION

Let \mathcal{H} be a complex infinite dimensional Hilbert space, let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on \mathcal{H} , and let $T \in \mathcal{B}(\mathcal{H})$. For $T \equiv V|T|$ the canonical polar decomposition of T , we let $\Delta(T) \equiv \tilde{T} := |T|^{1/2}V|T|^{1/2}$ denote the Aluthge transform of T [1]. It is well known that T is invertible if and only if \tilde{T} is invertible; moreover, the spectra of T and \tilde{T} are equal. Over the last two decades, considerable attention has been given to the study of the Aluthge transform; cf. [2, 10, 22, 24, 27–29, 33, 45]. Moreover, the Aluthge transform has been generalized to the case of powers of $|T|$ different from $\frac{1}{2}$ [3, 9, 21, 40] and to the case of commuting d -tuples of operators [4, 5, 18–20, 23, 30, 31].

Recall that, for $t \in [0, 1]$, the *generalized Aluthge transform* of T is $\tilde{T}^t := |T|^tV|T|^{1-t}$; the special instance of $t = 1$ is known as the *Duggal transform*, defined as $\tilde{T}^D := |T|V$ (see [24, 31]).

In this paper we will focus on the generalized *spherical* Aluthge transform of commuting d -tuples, which we now describe. Let $\mathbf{T} \equiv (T_1, \dots, T_d)$ be a commuting d -tuple of operators on \mathcal{H} , and consider the canonical polar decomposition of the column operator

$$D_{\mathbf{T}} := \begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix}; \quad (1.1)$$

that is,

$$D_{\mathbf{T}} = D_{\mathbf{V}}P : \mathcal{H} \longrightarrow \mathcal{H} \oplus \dots \oplus \mathcal{H}, \quad (1.2)$$

where $P := |D_{\mathbf{T}}| \equiv \sqrt{T_1^*T_1 + \dots + T_d^*T_d}$ is a positive operator on \mathcal{H} and $D_{\mathbf{V}} := \begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix}$ is a (joint) partial isometry from \mathcal{H} to $\mathcal{H} \oplus \dots \oplus \mathcal{H}$. Then, $(D_{\mathbf{V}})^*D_{\mathbf{V}}$ is the (orthogonal) projection onto the initial space of the partial isometry $D_{\mathbf{V}}$, which in turn is

$$(\ker D_{\mathbf{T}})^\perp = (\ker T_1 \cap \dots \cap \ker T_d)^\perp = (\ker P)^\perp.$$

We define the *spherical* polar decomposition of \mathbf{T} by

$$\mathbf{T} \equiv (V_1P, \dots, V_dP) \quad (1.3)$$

(cf. [16, 18, 30, 31]). We will also define the 2-norm of \mathbf{T} by $\|\mathbf{T}\|_2 := \|P\|$.

For $0 \leq t \leq 1$, the *generalized spherical Aluthge transform* of \mathbf{T} is the d -tuple

$$\Delta_t(\mathbf{T}) := (P^tV_1P^{1-t}, \dots, P^tV_dP^{1-t}). \quad (1.4)$$

We remark that when $t = \frac{1}{2}$ (resp. $t = 1$), we get the spherical Aluthge (resp. Duggal) transform $\Delta_{\text{sph}}(\mathbf{T})$ (resp. $\widehat{\mathbf{T}}^D$) of \mathbf{T} ; that is,

$$\begin{aligned} \Delta_{\frac{1}{2}}(\mathbf{T}) &\equiv \Delta_{\text{sph}}(\mathbf{T}) := (P^{\frac{1}{2}}V_1P^{\frac{1}{2}}, \dots, P^{\frac{1}{2}}V_dP^{\frac{1}{2}}) \\ (\text{resp. } \Delta_1(\mathbf{T}) &\equiv \widehat{\mathbf{T}}^D := (PV_1, \dots, PV_d). \end{aligned} \tag{1.5}$$

In [18–20], it was shown that the spherical Aluthge transform ($t = \frac{1}{2}$) respects the commutativity of the pair; the key step was the identity $V_iPV_j = V_jPV_i$, for all $i, j = 1, \dots, d$. Using this, one can prove that $\Delta_t(\mathbf{T})$ is commutative whenever \mathbf{T} is, for all $0 \leq t \leq 1$ (cf. (2.1) below), a fact also observed in [21]. (However, the d -tuple \mathbf{V} is, in general, not commutative.)

Before we state our main results, we pause to recall the notation we will use for several spectral systems. We shall let σ_T denote the *Taylor spectrum*, σ_{T_e} the *Taylor essential spectrum*, σ_p the *point spectrum*, σ_ℓ the *left spectrum* (also known as the *approximate point spectrum*), σ_r the *right spectrum*, and σ_H the *Harte spectrum* ($\sigma_H := \sigma_\ell \cup \sigma_r$).

Theorem 1.1. *Let $\mathbf{T} = (T_1, \dots, T_d)$ be a commuting d -tuple of operators on \mathcal{H} , and let $0 < t \leq 1$. Then*

$$\sigma_T(\Delta_t(\mathbf{T})) = \sigma_T(\mathbf{T}).$$

As a consequence, we have

$$r(\Delta_t(\mathbf{T})) = r_T(\mathbf{T}),$$

where r_T denotes the joint spectral radius; i.e.,

$$r(\mathbf{T}) := \sup\{\|\boldsymbol{\lambda}\|_2 := \sqrt{|\lambda_1|^2 + \dots + |\lambda_d|^2} : \boldsymbol{\lambda} \equiv (\lambda_1, \dots, \lambda_d) \in \sigma_T(\mathbf{T})\}, \tag{1.6}$$

and similarly for $r_T(\Delta_t(\mathbf{T}))$.

Theorem 1.2. *Let $\mathbf{T} = (T_1, \dots, T_d)$ be a commuting d -tuple of operators on \mathcal{H} , and let $0 < t \leq 1$. Then*

$$\sigma_{T_e}(\Delta_t(\mathbf{T})) = \sigma_{T_e}(\mathbf{T}).$$

Moreover, the Fredholm index satisfies

$$\text{index}(\Delta_t(\mathbf{T}) - \boldsymbol{\lambda}) = \text{index}(\mathbf{T} - \boldsymbol{\lambda}),$$

for all $\boldsymbol{\lambda} \notin \sigma_{T_e}(\mathbf{T})$.

Theorem 1.3. *Let $\mathbf{T} = (T_1, \dots, T_d)$ be a commuting d -tuple of operators on \mathcal{H} . Then, for $0 \leq t \leq 1$, the following statements hold.*

- (i) $\sigma_p(\Delta_t(\mathbf{T})) = \sigma_p(\mathbf{T})$.
- (ii) $\sigma_\ell(\Delta_t(\mathbf{T})) = \sigma_\ell(\mathbf{T})$.
- (iii) $\sigma_H(\Delta_t(\mathbf{T})) = \sigma_H(\mathbf{T})$.
- (iv) Assume $t > 0$. Then $\sigma_r(\mathbf{T}) \subseteq \sigma_r(\Delta_t(\mathbf{T})) \subseteq \sigma_r(\mathbf{T}) \cup \{0\}$.
- (v) More generally, if $\sigma_{\pi,k}$ and $\sigma_{\delta,k}$ denote the Słodkowski's spectral systems [38], then we have $\sigma_{\pi,k}(\Delta_t(\mathbf{T})) = \sigma_{\pi,k}(\mathbf{T})$ ($k = 0, \dots, d$) and $\sigma_{\delta,k}(\mathbf{T}) \subseteq \sigma_{\delta,k}(\Delta_t(\mathbf{T})) \subseteq \sigma_{\delta,k}(\mathbf{T}) \cup \{0\}$ ($k = 0, \dots, d$).

The first inclusion in Theorem 1.1(iv) can be proper, as the following example shows.

Example 1.4. For $0 < t \leq 1$, consider the commuting pair $\mathbf{T} := (U_+^*, 0)$, where U_+^* denotes the adjoint of the (unweighted) unilateral weighted shift acting on $\ell^2(\mathbb{Z}_+)$, with canonical orthonormal basis $\{e_0, e_1, \dots\}$. The polar decomposition of U_+^* is $U_+^*(I - E_0)$, where E_0 denotes the orthogonal projection onto the 1-dimensional subspace generated by e_0 . It is clear that the pair \mathbf{T} has positive part $P = (I - E_0)$; moreover, $U_+^*(I - E_0) = U_+^*$. Therefore, $\Delta_t(\mathbf{T}) = ((I - E_0)^t U_+^*(I - E_0)^{1-t}, 0) = ((I - E_0)U_+^*, 0) = (U_+^* - e_0 \otimes e_1, 0)$, for all $t \in (0, 1]$. This pair is unitarily equivalent to $(0 \oplus U_+^*, 0)$, and it follows that $\sigma_r(\Delta_t(\mathbf{T})) = (\{0\} \cup \mathbb{T}) \times \{0\} = \mathbb{T} \times \{0\} \cup \{(0, 0)\}$, for all $t \in (0, 1]$. (Here \mathbb{T} denotes the closed unit circle.) On the other hand, $\sigma_r(\mathbf{T}) = \mathbb{T} \times \{0\}$. \square

As we will see in Section 3, the key ingredient in the Proof of Theorem 1.1 is a careful analysis of the role of P in establishing an isomorphism between the Koszul complexes of \mathbf{T} and $\Delta_t(\mathbf{T})$. As a consequence, we shall see in Section 3 that P also enters in the Proof of Theorem 1.3.

We now turn our attention to the spectral radius of \mathbf{T} . For $n = 1$, T. Yamazaki proved in [45] that the Aluthge transform of an operator T can be used to calculate the spectral radius of T , via iteration. Concretely, $r(T) = \lim_n \|\Delta^{(n)}(T)\|$, where $\Delta^{(n)}$ denotes the n -th iterate of Δ .

We first extend this result to the case of single operators and the generalized Aluthge transform.

Theorem 1.5. *For $0 \leq t \leq 1$, there exists $s_t \geq 0$ such that $r(T) \leq s_t \leq \|T\|$ and*

$$\lim_{n \rightarrow \infty} \left\| \Delta_t^{(n)}(T) \right\| = s_t.$$

Theorem 1.6. *For $0 \leq t < 1$, we have*

$$\lim_{n \rightarrow \infty} \left\| \Delta_t^{(n)}(T) \right\| = r(T). \quad (1.7)$$

We now extend the above results to the case of commuting d -tuples. For $t = \frac{1}{2}$, Theorem 1.8 was first proved in [23].

Theorem 1.7. *Let $\mathbf{T} = (T_1, \dots, T_d)$ be a commuting d -tuple of operators on \mathcal{H} . For $t \in [0, 1]$ and $n \geq 1$, the sequence $\left\{ \left\| \Delta_t^{(n)}(\mathbf{T}) \right\|_2 \right\}_{n=1}^{\infty}$ satisfies*

$$r_{\mathbf{T}}(\mathbf{T}) \leq \left\| \Delta_t^{(n+1)}(\mathbf{T}) \right\|_2 \leq \left\| \Delta_t^{(n)}(\mathbf{T}) \right\|_2 \leq \|\mathbf{T}\|_2.$$

Theorem 1.8. *Let $\mathbf{T} = (T_1, \dots, T_d)$ be a commuting d -tuple of operators on \mathcal{H} . For $0 < t < 1$, we have*

$$r_{\mathbf{T}}(\mathbf{T}) = \lim_{n \rightarrow \infty} \left\| \Delta_t^{(n)}(\mathbf{T}) \right\|_2. \quad (1.8)$$

The organization of the paper is as follows. In Section 2 we collect notation and terminology needed throughout the paper, and we also list some standard results needed for our proofs. We devote Section 3 to the spectral results, which allow us to compare the spectral pictures of \mathbf{T} and $\Delta_t(\mathbf{T})$, for all $t \in [0, 1]$. In Section 4 we discuss the multivariable version of the spectral radius formula for the generalized spherical Aluthge transform. We extend the recent result of K. Feki and T. Yamazaki [23] for $t = \frac{1}{2}$ to the general case of $0 < t < 1$; we also discuss what happens when $t = 1$. The proof of Theorem 1.8 is rather subtle, in that we need to study in great detail the behavior of the norms of \mathbf{T} , its generalized spherical Aluthge transform $\Delta_t(\mathbf{T})$, the successive iterates $\Delta_T^{(n)}(\mathbf{T})$, and the norms of the respective powers, of the form $\left\| (\Delta_t^{(n)}(\mathbf{T}))^k \right\|_2$. There is a rather fascinating interplay between the norms of the powers and the norms of the iterates.

2. NOTATION AND PRELIMINARIES

Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting pair of operators on \mathcal{H} , and consider the Koszul complex of \mathbf{T} on \mathcal{H} ; that is,

$$K(\mathbf{T}, \mathcal{H}) : 0 \longrightarrow \mathcal{H} \xrightarrow{\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}} \mathcal{H} \oplus \mathcal{H} \xrightarrow{\begin{pmatrix} -T_2 & T_1 \end{pmatrix}} \mathcal{H} \longrightarrow 0.$$

Recall that $P = \sqrt{T_1^* T_1 + T_2^* T_2}$ and that the generalized Aluthge transform is given by $\Delta_t(\mathbf{T}) = (P^t V_1 P^{1-t}, P^t V_2 P^{1-t})$, for $0 \leq t \leq 1$, where (V_1, V_2) is the joint partial isometry in the polar decomposition of $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$. We know that

$$(V_1 P V_2 - V_2 P V_1) P = T_1 T_2 - T_2 T_1 = 0, \quad (2.1)$$

and therefore the operator $C := V_1 P V_2 - V_2 P V_1$ vanishes on the range of P . Since C also vanishes on the kernel of P (because $\ker P = \ker V_1 \cap \ker V_2$), we easily conclude that $C = 0$. A direct consequence of this is the commutativity of $\Delta_t(\mathbf{T})$ for all $0 \leq t \leq 1$. In an entirely similar way one can prove that for a commuting d -tuple $\mathbf{T} \equiv (T_1, \dots, T_d)$ the identities

$$V_i P V_j = V_j P V_i \quad (2.2)$$

always hold ($1 \leq i, j \leq d$), and therefore the generalized spherical Aluthge transform $\Delta_t(\mathbf{T})$ is commutative for all $t \in [0, 1]$. It follows that we can form two Koszul complexes, one for \mathbf{T} and one for $\Delta_t(\mathbf{T})$, with boundary maps denoted by $D_{\mathbf{T}}^0, D_{\mathbf{T}}^1, \dots, D_{\mathbf{T}}^{d-1}$ and by $D_{\Delta_t(\mathbf{T})}^0, D_{\Delta_t(\mathbf{T})}^1, \dots, D_{\Delta_t(\mathbf{T})}^{d-1}$, respectively:

$$K(\mathbf{T}, \mathcal{H}) : 0 \longrightarrow \mathcal{H} \xrightarrow{D_{\mathbf{T}}^0} \mathcal{H} \oplus \dots \oplus \mathcal{H} \xrightarrow{D_{\mathbf{T}}^1} \dots \xrightarrow{D_{\mathbf{T}}^{d-2}} \mathcal{H} \oplus \dots \oplus \mathcal{H} \xrightarrow{D_{\mathbf{T}}^{d-1}} \mathcal{H} \longrightarrow 0$$

and

$$K(\Delta_t(\mathbf{T}), \mathcal{H}) : 0 \longrightarrow \mathcal{H} \xrightarrow{D_{\Delta_t(\mathbf{T})}^0} \mathcal{H} \oplus \dots \oplus \mathcal{H} \xrightarrow{D_{\Delta_t(\mathbf{T})}^1} \dots \xrightarrow{D_{\Delta_t(\mathbf{T})}^{d-2}} \mathcal{H} \oplus \dots \oplus \mathcal{H} \xrightarrow{D_{\Delta_t(\mathbf{T})}^{d-1}} \mathcal{H} \longrightarrow 0.$$

(For the precise definition of the Koszul complex and its boundary maps, the reader is referred to [14, 41, 43]; observe that $D_{\mathbf{T}}^0 = D_{\mathbf{T}}$, as defined in (1.1).)

From the definition of $\Delta_t(\mathbf{T})$ we readily obtain d key identities:

$$\begin{aligned}
(1) \quad & (P^t \oplus \cdots \oplus P^t) D_{\mathbf{T}}^0 = D_{\Delta_t(\mathbf{T})}^0 P^t; \\
(2) \quad & (P^t \oplus \cdots \oplus P^t) D_{\mathbf{T}}^1 = D_{\Delta_t(\mathbf{T})}^1 (P^t \oplus \cdots \oplus P^t); \\
& \vdots \qquad \qquad \qquad \vdots \quad \vdots \\
(d) \quad & P^t D_{\mathbf{T}}^{d-1} = D_{\Delta_t(\mathbf{T})}^{d-1} (P^t \oplus \cdots \oplus P^t).
\end{aligned}$$

These identities establish, for each $0 \leq t \leq 1$, a co-chain homomorphism $\Phi_t : K(\mathbf{T}, \mathcal{H}) \longrightarrow K(\Delta_t(\mathbf{T}), \mathcal{H})$. This homomorphism will allow us to compare, at each stage, the exactness of one complex with the exactness of the other. We will do this in Section 3.

Remark 2.1. Observe that, for $0 < t \leq 1$, the fixed points of the generalized spherical Aluthge transform are the *spherically quasinormal* d -tuples; these are the d -tuples for which P commutes with each T_i , or equivalently, with each V_i , for $i = 1, \dots, d$. For, if $0 < t \leq 1$ and $\Delta_t(\mathbf{T}) = \mathbf{T}$, we must have $(P^t V_i - V_i P^t) P^{1-t} = 0$ for $i = 1, \dots, d$. Let $C_i := P^t V_i - V_i P^t$ ($i = 1, \dots, d$). When $t < 1$, C_i vanishes on the range of P^{1-t} , and therefore it vanishes on the range of P (for all $i = 1, \dots, d$). Since it also vanishes on the kernel of P , we conclude that $C_i = 0$ ($i = 1, \dots, d$). It follows that V_i commutes with P , for all $i = 1, \dots, d$. When $t = 1$, the commutativity of V_i and P ($i = 1, \dots, d$) is obvious. \square

We now recall the definitions of Taylor and Taylor essential spectrum, and of Fredholm index.

Definition 2.2. A commuting d -tuple $\mathbf{T} \equiv (T_1, \dots, T_d)$ is said to be (Taylor) invertible if its associated Koszul complex $K(\mathbf{T}, \mathcal{H})$ is exact (at each stage). The Taylor spectrum of \mathbf{T} is

$$\sigma_T(\mathbf{T}) := \{\boldsymbol{\lambda} \equiv (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d : K(\mathbf{T} - \boldsymbol{\lambda}, \mathcal{H}) \text{ is not exact}\}.$$

\mathbf{T} is said to be Fredholm if all the homology quotients in $K(\mathbf{T}, \mathcal{H})$ are finite-dimensional; this implies, in particular, that the boundary maps have closed range. The Taylor essential spectrum is the set

$$\sigma_{Te}(\mathbf{T}) := \{\boldsymbol{\lambda} \in \mathbb{C}^d : \mathbf{T} - \boldsymbol{\lambda} \text{ is not Fredholm}\}.$$

If, given $k = 0, 1, \dots, d$, one only requires that the Koszul complex be exact at the last $k + 1$ stages $d - k, d - k + 1, \dots, d - 1, d$, we obtain Z. Słodkowski's spectral systems $\sigma_{\delta, k}$. These spectral systems lie between the right spectrum and the Taylor spectrum; that is, $\sigma_r \equiv \sigma_{\delta, 0} \subseteq \sigma_{\delta, 1} \subseteq \cdots \subseteq \sigma_{\delta, d-1} \subseteq \sigma_{\delta, d} \equiv \sigma_T$. Similarly, if, given $k = 0, 1, \dots, d$, one only requires that the Koszul complex be exact at the first $k + 1$ stages $0, 1, \dots, k - 1, k$, together with the closed range of $D_{\mathbf{T}}^k$, we obtain the spectral system $\sigma_{\pi, k}$. These spectral systems lie between the left spectrum and the Taylor spectrum; that is, $\sigma_\ell \equiv \sigma_{\pi, 0} \subseteq \sigma_{\pi, 1} \subseteq \cdots \subseteq \sigma_{\pi, d} \equiv \sigma_T$. If σ_Z denotes one of the above mentioned spectral systems, we will say that \mathbf{T} is Z -invertible whenever $\mathbf{0} \equiv (0, \dots, 0) \notin \sigma_Z(\mathbf{T})$; for instance, we will refer to \mathbf{T} as being Taylor invertible, or left invertible, or Harte invertible, and so on.

J.L. Taylor showed in [41] that, if $\mathcal{H} \neq \{0\}$, then $\sigma_T(\mathbf{T})$ is a nonempty, compact subset of the polydisc of multiradius $(r(T_1), \dots, r(T_d))$, where $r(T_i)$ is the spectral radius of T_i ($i = 1, \dots, d$). As a consequence, if $\boldsymbol{\lambda} \in \sigma_T(\mathbf{T})$, then $\|\boldsymbol{\lambda}\|_2 \leq \sqrt{d} \|P\|$; for, $|\lambda_i|^2 \leq r(T_i)^2 \leq \|T_i^* T_i\|$ for every $i = 1, \dots, d$, and therefore $\|\boldsymbol{\lambda}\|_2^2 \leq \sum_{i=1}^d \|T_i^* T_i\| \leq d \|P\|^2$. However, one can do better. For, consider the elementary operator $M_{\mathbf{T}} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ given as $M_{\mathbf{T}}(X) := \sum_{i=1}^d T_i^* X T_i$ ($X \in \mathcal{B}(\mathcal{H})$). We know from [15] that $\sigma(M_{\mathbf{T}}) = \{\bar{\lambda}_1 \lambda_1 + \dots + \bar{\lambda}_d \lambda_d : \boldsymbol{\lambda} \in \sigma_T(\mathbf{T})\}$. As a result, if $\boldsymbol{\lambda} \in \sigma_T(\mathbf{T})$, we have $\|\boldsymbol{\lambda}\|_2^2 \leq r(M_{\mathbf{T}}) \leq \|M_{\mathbf{T}}\| = \|P\|^2$. Thus, $r(\mathbf{T}) \leq \|P\| = \|\mathbf{T}\|_2$. (For additional facts about these joint spectra, the reader is referred to [12–15, 38, 39, 42].)

As shown in [13, 14], the Fredholmness of \mathbf{T} can be detected in the Calkin algebra $\mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. (Here \mathcal{K} denotes the closed two-sided ideal of compact operators on \mathcal{H} ; we also let $\pi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{Q}(\mathcal{H})$ denote the quotient map.) Concretely, \mathbf{T} is Fredholm on \mathcal{H} if and only if the d -tuple of left multiplication operators $L_{\pi(\mathbf{T})} := (L_{\pi(T_1)}, \dots, L_{\pi(T_d)})$ is Taylor invertible when acting on $\mathcal{Q}(\mathcal{H})$. In particular, \mathbf{T} is left Fredholm on \mathcal{H} if and only if $L_{\pi(\mathbf{T})}$ is left invertible on $\mathcal{Q}(\mathcal{H})$.

Given a commuting d -tuple \mathbf{T} , by the *spectral picture* of \mathbf{T} , denoted $SP(\mathbf{T})$, we will refer to the collection of the sets $\sigma_T(\mathbf{T})$, $\sigma_{T_e}(\mathbf{T})$, $\sigma_{\pi,k}(\mathbf{T})$, $\sigma_{\delta,k}(\mathbf{T})$ and $\sigma_H(\mathbf{T})$ (for all $k = 0, 1, \dots, d$), together with the index function defined on the Fredholm domain of \mathbf{T} (that is, the complement in \mathbb{C}^d of $\sigma_{T_e}(\mathbf{T})$).

Our strategy to describe the spectral picture of $\Delta_t(\mathbf{T})$ will partially rely on the theory of criss-cross commutativity for pairs of d -tuples of operators acting on \mathcal{H} . Given two d -tuples \mathbf{A} and \mathbf{B} , we say that \mathbf{A} and \mathbf{B} *criss-cross commute* if $A_i B_j A_k = A_k B_j A_i$ and $B_i A_j B_k = B_k A_j B_i$, for all $i, j, k = 1, \dots, d$. (Observe that in this definition we do not assume that \mathbf{A} or \mathbf{B} is commuting.) We now let

$$\mathbf{AB} := (A_1 B_1, \dots, A_d B_d) \quad \text{and} \quad \mathbf{BA} := (B_1 A_1, \dots, B_d A_d).$$

It is an easy exercise to prove that if \mathbf{A} and \mathbf{B} criss-cross commute, then each of \mathbf{AB} and \mathbf{BA} is commuting.

Now, fix a commuting d -tuple \mathbf{T} and a real number $t \in [0, 1]$, and let $\mathbf{A} \equiv (A_1, \dots, A_d) := (V_1 P^{1-t}, \dots, V_d P^{1-t})$ and $\mathbf{B} \equiv (B_1, \dots, B_d) := (P^t, \dots, P^t)$. Then $\mathbf{AB} = \mathbf{T}$ and $\mathbf{BA} = \Delta_t(\mathbf{T})$; moreover, the commutativity of \mathbf{T} implies that \mathbf{A} and \mathbf{B} criss-cross commute (using (2.2)). It follows that $\Delta_t(\mathbf{T})$ is also commuting.

We now briefly describe unilateral weighted shifts, since we use them to generate examples. For $\omega \equiv \{\omega_n\}_{n=0}^\infty$ a bounded sequence of positive real numbers (called *weights*), let $W_\omega \equiv \text{shift}(\omega_0, \omega_1, \dots) : \ell^2(\mathbb{Z}_+) \rightarrow \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by

$$W_\omega e_n := \omega_n e_{n+1} \quad (\text{all } n \geq 0),$$

where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. As we noted in Example 1.4, $U_+ := \text{shift}(1, 1, \dots)$ is the (unweighted) unilateral shift, and for $0 < a < 1$ we will let $S_a := \text{shift}(a, 1, 1, \dots)$.

We will also need 2-variable weighted shifts. Consider the nonnegative quadrant

$$\mathbb{Z}_+^2 := \{\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}^2 : k_1 \geq 0 \text{ and } k_2 \geq 0\},$$

and the Hilbert space $\ell^2(\mathbb{Z}_+^2)$ of square-summable complex sequences indexed by \mathbb{Z}_+^2 ; the canonical orthonormal basis for this space is $\{e_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}_+^2}$. Observe that $\ell^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to the Hilbert space tensor product $\ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$, via the map $e_{\mathbf{k}} \mapsto e_{k_1} \otimes e_{k_2}$ ($\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$).

Assume now that we are given two double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, where $\mathbf{k} \in \mathbb{Z}_+^2$. We can then define the *2-variable weighted shift* $W_{(\alpha, \beta)} \equiv (T_1, T_2)$ by

$$T_1 e_{(k_1, k_2)} := \alpha_{(k_1, k_2)} e_{(k_1+1, k_2)} \quad \text{and} \quad T_2 e_{(k_1, k_2)} := \beta_{(k_1, k_2)} e_{(k_1, k_2+1)}.$$

For all $\mathbf{k} \in \mathbb{Z}_+^2$, we have

$$T_1 T_2 = T_2 T_1 \iff \beta_{(k_1+1, k_2)} \alpha_{(k_1, k_2)} = \alpha_{(k_1, k_2+1)} \beta_{(k_1, k_2)}. \quad (2.3)$$

Given two unilateral weighed shifts W_ω and W_τ , a trivial way to build a 2-variable weighted shift is to let $\alpha_{(k_1, k_2)} := \omega_{k_1}$ and $\beta_{(k_1, k_2)} := \tau_{k_2}$ for all $\mathbf{k} \in \mathbb{Z}_+^2$. It is not hard to see that, in this case, $W_{(\alpha, \beta)} \cong (W_\omega \otimes I, I \otimes W_\tau)$. For additional facts about 2-variable weighted shifts, the reader is referred to [16–19].

For a given \mathbf{T} , the calculation of $\Delta_t(\mathbf{T})$ is not always straightforward, even for relatively simple forms of the components of \mathbf{T} . We now present an example, which helps to visualize the type of complications one may encounter.

Example 2.3. Let $\mathcal{H} := \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$, let $\mathbf{T} = (T_1, T_2) \cong (I \otimes U_+^*, U_+ \otimes I)$, and fix $0 < t \leq 1$. Then

$$\sigma_r(\mathbf{T}) = \sigma_r(\Delta_t(\mathbf{T})) = \mathbb{T} \times \overline{\mathbb{D}}.$$

Observe first that

$$P^2 = I \otimes (E_0 + 2E_0^\perp),$$

where E_0 is the projection onto $\langle e_0 \rangle$ and $E_0^\perp := I - E_0$. From the functional calculus for P , we get

$$P = I \otimes (E_0 + \sqrt{2}E_0^\perp), \quad P^t = I \otimes (E_0 + 2^{t/2}E_0^\perp), \quad \text{and} \quad P^{1-t} = I \otimes (E_0 + 2^{(1-t)/2}E_0^\perp).$$

Also, observe that $\ker P = 0$. Let $V_1 := I \otimes \frac{1}{\sqrt{2}}U_+^*$. Then,

$$\begin{aligned} V_1 P &= I \otimes \left[\frac{1}{\sqrt{2}}U_+^* (E_0 + \sqrt{2}E_0^\perp) \right] = I \otimes \left[\frac{1}{\sqrt{2}} (U_+^* E_0 + \sqrt{2}U_+^* E_0^\perp) \right] \\ &= I \otimes U_+^* E_0^\perp = I \otimes U_+^*. \end{aligned}$$

Let $V_2 := U_+ \otimes (E_0 + \frac{1}{\sqrt{2}}E_0^\perp)$. Then,

$$V_2 P = U_+ \otimes \left(E_0 + \frac{1}{\sqrt{2}}E_0^\perp \right) (E_0 + \sqrt{2}E_0^\perp) = U_+ \otimes I.$$

Thus,

$$\begin{aligned}
V_1^*V_1 + V_2^*V_2 &= I \otimes (2^{-\frac{1}{2}}U_+ \cdot 2^{-\frac{1}{2}}U_+^*) \\
&\quad + [U_+^* \otimes (E_0 + 2^{-\frac{1}{2}}E_0^\perp)][U_+ \otimes (E_0 + 2^{-\frac{1}{2}}E_0^\perp)] \\
&= I \otimes \frac{1}{2}U_+U_+^* + U_+^*U_+ \otimes (E_0 + \frac{1}{2}E_0^\perp) \\
&= I \otimes \frac{1}{2}E_0^\perp + I \otimes (E_0 + \frac{1}{2}E_0^\perp) \\
&= I \otimes I.
\end{aligned}$$

Therefore, (V_1, V_2) is a (noncommuting) joint isometry, and $\ker V_1 \cap \ker V_2 = 0 = \ker P$. It follows that (V_1P, V_2P) is the spherical polar decomposition of (T_1, T_2) . Let $0 < t \leq 1$. Then,

$$\begin{aligned}
P^tV_1P^{1-t} &= [I \otimes (E_0 + 2^{\frac{t}{2}}E_0^\perp)] \left(I \otimes [2^{-\frac{1}{2}}U_+^*] [I \otimes (E_0 + 2^{\frac{1-t}{2}}E_0^\perp)] \right) \\
&= I \otimes \left[(E_0 + 2^{\frac{t}{2}}E_0^\perp) 2^{-\frac{t}{2}}U_+^*E_0^\perp \right] = I \otimes (2^{-\frac{t}{2}}E_0U_+^* + E_0^\perp U_+^*).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
P^tV_2P^{1-t} &= [I \otimes (E_0 + 2^{\frac{t}{2}}E_0^\perp)][U_+ \otimes (E_0 + 2^{-\frac{1}{2}}E_0^\perp)][I \otimes (E_0 + 2^{\frac{1-t}{2}}E_0^\perp)] \\
&= U_+ \otimes I.
\end{aligned}$$

Thus,

$$\sigma_r(\Delta_t(\mathbf{T})) = \sigma_r(2^{-\frac{t}{2}}E_0U_+^* + E_0^\perp U_+^*) \times \overline{\mathbb{D}}.$$

We now need to study the right spectrum of $W(t) := 2^{-\frac{t}{2}}E_0U_+^* + E_0^\perp U_+^*$. A moment's thought reveals that $W(t)$ is the adjoint of $S_a \equiv \text{shift}(a, 1, 1, \dots)$, where $a := 2^{-\frac{t}{2}}$. Then,

$$\sigma_r(W(t)) = \sigma_r(S_a^*) = \overline{\sigma_\ell(S_a)} = \mathbb{T},$$

where the bar over σ_ℓ denotes complex conjugation. It follows that

$$\sigma_r(\Delta_t(\mathbf{T})) = \mathbb{T} \times \overline{\mathbb{D}}. \tag{2.4}$$

On the other hand,

$$\sigma_r(\mathbf{T}) = \sigma_r(I \otimes U_+^*, U_+ \otimes I) = \sigma_r(U_+^*) \times \sigma_r(U_+) = \mathbb{T} \times \overline{\mathbb{D}}. \tag{2.5}$$

Therefore, by (2.4) and (2.5) we have $\sigma_r(\Delta_t(\mathbf{T})) = \sigma_r(\mathbf{T})$. \square

3. THE SPECTRAL PICTURE OF THE GENERALIZED SPHERICAL ALUTHGE TRANSFORM

In [29], I.B. Jung, E. Ko and C. Pearcy proved that, for $T \in \mathcal{B}(\mathcal{H})$, $\sigma(\Delta_{\frac{1}{2}}(T)) = \sigma(T)$; subsequently, M. Chō, I. Jung, and W.Y. Lee also proved that $\sigma(\Delta_1(T)) = \sigma(T)$ [10]. It is also known that for $r > \epsilon \geq 0$ and $T \equiv V|T| \in \mathcal{B}(\mathcal{H})$, T and $|T|^\epsilon V|T|^{r-\epsilon}$ have the same spectrum ([26, Lemma 5]). If we put $r = 1$ and $\epsilon = t$, then T and $\Delta_t(T)$ have the same spectrum, for all $0 \leq t \leq 1$. As a consequence, we obtain $r(T) = r(\Delta_t(T))$ for all $0 \leq t \leq 1$, where $r(T)$ denotes the spectral radius.

In what follows, we will extend these results to the case of commuting d -tuples of operators on \mathcal{H} . We will first look at Taylor invertibility, which requires an analysis of the exactness of the Koszul complexes $K(\mathbf{T} - \boldsymbol{\lambda}, \mathcal{H})$ and $K(\Delta(\mathbf{T}) - \boldsymbol{\lambda}, \mathcal{H})$, for $\boldsymbol{\lambda} \in \mathbb{C}^d$. We will split the discussion into two cases: (i) $\boldsymbol{\lambda} = \mathbf{0}$, and (ii) $\boldsymbol{\lambda} \neq \mathbf{0}$. We will first consider the case $\boldsymbol{\lambda} = \mathbf{0}$.

3.1. Taylor invertibility when $\boldsymbol{\lambda} = \mathbf{0}$. Given a commuting d -tuple $\mathbf{T} \equiv (T_1, \dots, T_d)$, we begin with a key connection between the invertibility of P and the Koszul complex homomorphism Φ_t introduced on page 6, right before Definition 2.2. We recall that \mathbf{T} is left invertible if and only if P is invertible.

Theorem 3.1. *Let \mathbf{T} be a commuting d -tuple, let $K(\mathbf{T}, \mathcal{H})$ be its associated Koszul complex, and let P be the positive factor in the polar decomposition of $D_{\mathbf{T}}^0$, that is, $P = \sqrt{T_1^* T_1 + \dots + T_d^* T_d}$. Assume that $0 < t < 1$. The following statements are equivalent:*

- (i) \mathbf{T} is left invertible;
- (ii) P is invertible;
- (iii) $\Delta_t(\mathbf{T})$ is left invertible.

Proof. (i) \Leftrightarrow (ii): This is straightforward.

(ii) \Rightarrow (iii): Assume that P is invertible. Then the map Φ_t is an isomorphism of Koszul complexes. We also know that \mathbf{T} is left invertible; that is, the Koszul complex $K(\mathbf{T}, \mathcal{H})$ is exact at stage 0 and the range of $D_{\mathbf{T}}^0$ is closed. Since Φ_t is an isomorphism, we must therefore have that $K(\Delta_t(\mathbf{T}), \mathcal{H})$ is exact at stage 0 and the range of $D_{\Delta_t(\mathbf{T})}^0$ is closed. Thus, $\Delta_t(\mathbf{T})$ is left invertible.

(iii) \Rightarrow (ii): Observe that $(D_{\Delta_t(\mathbf{T})}^0)^* D_{\Delta_t(\mathbf{T})}^0 = P^{1-t}(V_1^* P V_1 + \dots + V_d^* P V_d) P^{1-t}$. By assumption, $D_{\Delta_t(\mathbf{T})}^0$ is left invertible, and therefore $P^{1-t}(V_1^* P V_1 + \dots + V_d^* P V_d) P^{1-t}$ is onto. It follows that P^{1-t} is onto, which implies that P is invertible. □

Remark 3.2. Careful study of the Proof of Theorem 3.1 reveals that left invertibility can be replaced by the invertibility associated with any of the spectral systems $\sigma_{\pi,k}$, for $0 \leq k \leq d$. For, if $0 \notin \sigma_{\pi,k}(\mathbf{T})$, then $0 \notin \sigma_{\pi,0}(\mathbf{T})$, which means that \mathbf{T} is left invertible, and therefore P is invertible, and the two Koszul complexes are isomorphic. We leave the details to the reader. □

We now deal with right invertibility (which will also include invertibility relative to the spectral systems $\sigma_{\delta,k}$).

Theorem 3.3. *Let \mathbf{T} be a commuting d -tuple, let $K(\mathbf{T}, \mathcal{H})$ be its associated Koszul complex, and let P be the positive factor in the polar decomposition of $D_{\mathbf{T}}^0$. Assume that $0 < t \leq 1$, and consider the following statements.*

- (i) \mathbf{T} is right invertible;
- (ii) P is invertible;
- (iii) $\Delta_t(\mathbf{T})$ is right invertible.

Then (iii) \Rightarrow (ii), and (iii) \Rightarrow (i). The implication (i) \Rightarrow (iii) is not true in general (cf. Example 1.4).

Proof. (iii) \Rightarrow (ii): Assume that $\Delta_t(\mathbf{T})$ is right invertible; that is, the boundary map $D_{\Delta_t(\mathbf{T})}^{d-1}$ is onto. Now recall that $D_{\Delta_t(\mathbf{T})}^{d-1}(D_{\Delta_t(\mathbf{T})}^{d-1})^* = P^t(\sum_{i=1}^d V_i P^{2(1-t)} V_i^*) P^t$. Since $t > 0$, it follows that P^t is onto, and therefore P is invertible.

(iii) \Rightarrow (i): Assume that $\Delta_t(\mathbf{T})$ is right invertible. By what we just proved, P is invertible. It follows that the two Koszul complexes are isomorphic, and therefore \mathbf{T} is right invertible. \square

Remark 3.4. Just as in the case of Remark 3.2, the Proof of Theorem 3.3 reveals that right invertibility can be replaced by the invertibility associated with any of the spectral systems $\sigma_{\delta,k}$, for $0 \leq k \leq d$. For, if $0 \notin \sigma_{\delta,k}(\Delta_t(\mathbf{T}))$, then $0 \notin \sigma_{\delta,0}(\Delta_t(\mathbf{T}))$, which means that $\Delta_t(\mathbf{T})$ is right invertible, and therefore P is invertible, and the two Koszul complexes are isomorphic. Again, we leave the details to the reader. \square

Corollary 3.5. *Let \mathbf{T} be a commuting d -tuple, let $0 < t \leq 1$, and let $\Delta_t(\mathbf{T})$ be the generalized spherical Aluthge transform of \mathbf{T} . The following statements are equivalent.*

- (i) \mathbf{T} is Taylor invertible.
- (ii) $\Delta_t(\mathbf{T})$ is Taylor invertible.

Proof. Assume first that \mathbf{T} is Taylor invertible. Then it is left invertible, and by Theorem 3.1, the operator P is invertible. As a result, the Koszul complexes $K(\mathbf{T}, \mathcal{H})$ and $K(\Delta_t(\mathbf{T}), \mathcal{H})$ are isomorphic. Since $K(\mathbf{T}, \mathcal{H})$ is exact, so is $K(\Delta_t(\mathbf{T}), \mathcal{H})$. Therefore, $\Delta_t(\mathbf{T})$ is Taylor invertible.

Conversely, assume that $\Delta_t(\mathbf{T})$ is Taylor invertible, and therefore right invertible. By Theorem 3.3, P is invertible, and it follows that the two Koszul complexes are isomorphic. Then \mathbf{T} is Taylor invertible, as desired. \square

Corollary 3.6. *Let \mathbf{T} , $0 < t \leq 1$ and $\Delta_t(\mathbf{T})$ be as above. The following statements are equivalent.*

- (i) \mathbf{T} is Harte invertible.
- (ii) $\Delta_t(\mathbf{T})$ is Harte invertible.

Proof. Assume first that \mathbf{T} is Harte invertible. Then it is left invertible, and by Theorem 3.1, the operator P is invertible. As a result, the Koszul complexes $K(\mathbf{T}, \mathcal{H})$ and $K(\Delta_t(\mathbf{T}), \mathcal{H})$ are isomorphic. It follows that $\Delta_t(\mathbf{T})$ is Harte invertible.

Conversely, assume that $\Delta_t(\mathbf{T})$ is Harte invertible, and therefore right invertible. By Theorem 3.3, P is invertible, and it follows that the two Koszul complexes are isomorphic. Then \mathbf{T} is Harte invertible, as desired. \square

The following corollary is a simple consequence of the preceding results. We omit the proof.

Corollary 3.7. *Let \mathbf{T} , $0 < t \leq 1$ and $\Delta_t(\mathbf{T})$ be as above, and fix $0 \leq k \leq d$. The following statements hold.*

- (i) $\mathbf{0} \notin \sigma_{\pi,k}(\Delta_t(\mathbf{T})) \Leftrightarrow \mathbf{0} \notin \sigma_{\pi,k}(\mathbf{T})$.
- (ii) If, in addition, $t < 1$ then $\mathbf{0} \notin \sigma_{\delta,k}(\Delta_t(\mathbf{T})) \Leftrightarrow \mathbf{0} \notin \sigma_{\delta,k}(\mathbf{T})$.

3.2. Fredholmness when $\lambda = \mathbf{0}$. Continuing with the case $\lambda = \mathbf{0}$, we will now study the Fredholmness of \mathbf{T} and $\Delta_t(\mathbf{T})$. By now, the analogues of Theorems 3.1 and 3.3 should be

natural. We will give an abridged proof of Theorem 3.8, and leave the proof of Theorem 3.9 to the interested reader.

Theorem 3.8. *Let \mathbf{T} be a commuting d -tuple, and let $K(\mathbf{T}, \mathcal{H})$ and P be as above. Assume that $0 < t < 1$. The following statements are equivalent:*

- (i) \mathbf{T} is left Fredholm;
- (ii) P is Fredholm;
- (iii) $\Delta_t(\mathbf{T})$ is left Fredholm.

Sketch of Proof. From page 6, recall the co-chain homomorphism

$$\Phi_t : K(\mathbf{T}, \mathcal{H}) \rightarrow K(\Delta_t(\mathbf{T}), \mathcal{H})$$

induced by P^t . Let π denote the Calkin map, and let $\pi(P^t)$ denote the corresponding element in $\mathcal{Q}(\mathcal{H})$. At the Calkin algebra level, Φ_t becomes φ_t , a co-chain homomorphism (induced by $L_{\pi(P^t)}$) between the Koszul complexes $K(L_{\pi(\mathbf{T})}, \mathcal{Q}(\mathcal{H}))$ and $K(L_{\pi(\Delta_t(\mathbf{T}))}, \mathcal{Q}(\mathcal{H}))$, where $\pi(\mathbf{T})$ and $\pi(\Delta_t(\mathbf{T}))$ have the obvious definitions, and L denotes the d -tuple of left multiplications by the coordinates of \bullet .

(ii) \Rightarrow (iii): Assume that P is Fredholm. Then P^t is Fredholm, and therefore $\pi(P^t)$ is an invertible element of $\mathcal{Q}(\mathcal{H})$. It follows that the associated map φ_t is an isomorphism. Since \mathbf{T} is left Fredholm, $L_{\pi(\mathbf{T})}$ is left invertible, and therefore $L_{\pi(\Delta_t(\mathbf{T}))}$ is left invertible, which implies that $\Delta_t(\mathbf{T})$ is left Fredholm. \square

Theorem 3.9. *Let \mathbf{T} be a commuting d -tuple, and let $K(\mathbf{T}, \mathcal{H})$ and P be as above. Assume that $0 < t \leq 1$, and consider the following statements.*

- (i) \mathbf{T} is right Fredholm;
- (ii) P is Fredholm;
- (iii) $\Delta_t(\mathbf{T})$ is right Fredholm.

Then (iii) \Rightarrow (ii), and (iii) \Rightarrow (i). The implication (i) \Rightarrow (iii) is not true in general.

Corollary 3.10. *Let \mathbf{T} , $0 < t \leq 1$ and $\Delta_t(\mathbf{T})$ be as above. The following statements are equivalent.*

- (i) \mathbf{T} is Fredholm.
- (ii) $\Delta_t(\mathbf{T})$ is Fredholm.

3.3. Taylor invertibility when $\lambda \neq \mathbf{0}$. We now consider the various forms of invertibility of $\mathbf{T} - \lambda$ when $\lambda \neq \mathbf{0}$. From page 7, recall that two (not necessarily commuting) d -tuples \mathbf{A} and \mathbf{B} criss-cross commute if $A_i B_j A_k = A_k B_j A_i$ and $B_i A_j B_k = B_k A_j B_i$ for all $i, j, k = 1, \dots, d$. Recall also that, under this condition, both \mathbf{AB} and \mathbf{BA} are commuting.

Lemma 3.11. *(cf. [6, 7]) Assume that \mathbf{A} and \mathbf{B} criss-cross commute. Then*

$$\sigma_T(\mathbf{BA}) \setminus \{\mathbf{0}\} = \sigma_T(\mathbf{AB}) \setminus \{\mathbf{0}\}.$$

3.4. Fredholmness when $\lambda \neq \mathbf{0}$. Here we need the analogue of Lemma 3.11 for the Taylor essential spectrum.

Lemma 3.12. *(cf. [6, 7, 34, 35]) Assume that \mathbf{A} and \mathbf{B} criss-cross commute. Then*

$$\sigma_{Te}(\mathbf{BA}) \setminus \{\mathbf{0}\} = \sigma_{Te}(\mathbf{AB}) \setminus \{\mathbf{0}\}.$$

Moreover, for $\mathbf{0} \neq \boldsymbol{\lambda} \notin \sigma_{Te}(\mathbf{T})$, one has

$$\text{index}(\Delta_t(\mathbf{T}) - \boldsymbol{\lambda}) = \text{index}(\mathbf{T} - \boldsymbol{\lambda}).$$

3.5. The Spectral Picture of \mathbf{T} and $\Delta_t(\mathbf{T})$. We are now ready to prove Theorems 1.1, 1.2 and 1.3, which we will restate as a single result, encompassing the various spectral systems.

Theorem 3.13. *Let $\mathbf{T} = (T_1, \dots, T_d)$ be a commuting d -tuple of operators on \mathcal{H} , and let $0 < t \leq 1$. Then*

(Taylor)

$$\sigma_T(\Delta_t(\mathbf{T})) = \sigma_T(\mathbf{T}).$$

As a consequence, we have

$$r(\Delta_t(\mathbf{T})) = r_T(\mathbf{T}),$$

where r_T denotes the joint spectral radius.

(Taylor ess.)

$$\sigma_{Te}(\Delta_t(\mathbf{T})) = \sigma_{Te}(\mathbf{T}).$$

(index) The Fredholm index satisfies

$$\text{index}(\Delta_t(\mathbf{T}) - \boldsymbol{\lambda}) = \text{index}(\mathbf{T} - \boldsymbol{\lambda}),$$

for all $\boldsymbol{\lambda} \notin \sigma_{Te}(\mathbf{T})$.

(Other σ 's): (i) $\sigma_p(\Delta_t(\mathbf{T})) = \sigma_p(\mathbf{T})$.

(ii) $\sigma_\ell(\Delta_t(\mathbf{T})) = \sigma_\ell(\mathbf{T})$.

(iii) $\sigma_H(\Delta_t(\mathbf{T})) = \sigma_H(\mathbf{T})$.

(iv) Assume $t > 0$. Then $\sigma_r(\mathbf{T}) \subseteq \sigma_r(\Delta_t(\mathbf{T})) \subseteq \sigma_r(\mathbf{T}) \cup \{0\}$.

(v) More generally, if $\sigma_{\pi,k}$ and $\sigma_{\delta,k}$ denote the Słodkowski's spectral systems, we have

$$\sigma_{\pi,k}(\Delta_t(\mathbf{T})) = \sigma_{\pi,k}(\mathbf{T}) \quad (k = 0, \dots, d)$$

and

$$\sigma_{\delta,k}(\mathbf{T}) \subseteq \sigma_{\delta,k}(\Delta_t(\mathbf{T})) \subseteq \sigma_{\delta,k}(\mathbf{T}) \cup \{0\} \quad (k = 0, \dots, d).$$

Proof. Let $\boldsymbol{\lambda} \in \sigma_T(\Delta_t(\mathbf{T}))$. If $\boldsymbol{\lambda} = \mathbf{0}$, we know that $\Delta_t(\mathbf{T})$ is not Taylor invertible. By Corollary 3.5, \mathbf{T} is not Taylor invertible, and therefore $\boldsymbol{\lambda} \in \sigma_T(\mathbf{T})$. Similarly, $\mathbf{0} \in \sigma_T(\mathbf{T}) \Rightarrow \mathbf{0} \in \sigma_T(\Delta_t(\mathbf{T}))$.

Assume now that $\boldsymbol{\lambda} \neq \mathbf{0}$, and recall that $\mathbf{T} = \mathbf{A}\mathbf{B}$ and $\Delta_t(\mathbf{T}) = \mathbf{B}\mathbf{A}$ for $\mathbf{A} := (V_1 P^{1-t}, \dots, V_d P^{1-t})$ and $\mathbf{B} := (P^t, \dots, P^t)$ (cf. page 7). By Lemma 3.11, the Taylor invertibility of $\Delta_t(\mathbf{T}) - \boldsymbol{\lambda}$ is equivalent to the Taylor invertibility of $\mathbf{T} - \boldsymbol{\lambda}$. This completes the proof of the equality of $\sigma_T(\Delta_t(\mathbf{T}))$ and $\sigma_T(\mathbf{T})$.

To deal with the Taylor essential spectra, we can adapt the above arguments, replacing Taylor invertibility by Fredholmness, Corollary 3.5 by Corollary 3.10, and Lemma 3.11 by Lemma 3.12, respectively. In terms of the Fredholm index, the results in [34] are based on an isomorphism between the homology spaces of $K(\Delta_t(\mathbf{T}), \mathcal{H})$ and $K(\mathbf{T}, \mathcal{H})$, which guarantees the equality of the Fredholm indices away from $\mathbf{0}$. However, the Fredholm

index is continuous on the complement in \mathbb{C}^d of $\sigma_T(\mathbf{T})$ [13, 14], and integer-valued, and therefore $\text{index}(\Delta_t(\mathbf{T}) - \boldsymbol{\lambda}) = \text{index}(\mathbf{T} - \boldsymbol{\lambda})$ must also hold whenever $\mathbf{0} \notin \sigma_T(\mathbf{T})$.

As for other spectral systems, we can use Remarks 3.2 and 3.4 together with the fact that the results in [6, 7, 34, 35] are based on isomorphisms of the Koszul complexes, or of the related homology spaces. However, for the case of the point spectrum σ_p and $\boldsymbol{\lambda} = \mathbf{0}$, we can simply invoke the condition $\ker T_1 \cap \cdots \cap \ker T_d = \ker P = \ker V_1 \cap \cdots \cap \ker V_d$. For instance, if $0 \neq x \in \mathcal{H}$ and $T_1 x = \cdots = T_d x = 0$, then $Px = 0$ and $V_1 x = \cdots = V_d x = 0$, which readily implies that $P^t V_1 P^{1-t} x = \cdots = P^t V_d P^{1-t} x = 0$. As a result, $\sigma_p(\mathbf{T}) \subseteq \sigma_p(\Delta_t(\mathbf{T}))$. Conversely, if $P^t V_1 P^{1-t} x = \cdots = P^t V_d P^{1-t} x = 0$ for some $0 \neq x \in \mathcal{H}$, then P must not be one-to-one (otherwise we get a contradiction) and therefore $T_1 x = \cdots = T_d x = 0$. \square

We conclude this Section with a simple application of Theorem 3.13.

Corollary 3.14. *Let $\mathbf{T} = (T_1, \dots, T_d)$ be a commuting d -tuple of operators on \mathcal{H} , and let $0 < t < 1$. Assume that $\mathbf{0} \in \sigma_r(\Delta_t(\mathbf{T})) \setminus \sigma_r(\mathbf{T})$. Then $\mathbf{0} \in \sigma_\ell(\mathbf{T}) \cap \sigma_\ell(\Delta_t(\mathbf{T}))$.*

Proof. First recall that the two Koszul complexes for \mathbf{T} and $\Delta_t(\mathbf{T})$ are isomorphic when P is invertible. By assumption, the Koszul complexes are not isomorphic, so P is not invertible. Let $\{x_n\}_{n=1}^\infty$ be a sequence of unit vectors in \mathcal{H} such that $Px_n \rightarrow 0$ as $n \rightarrow \infty$. Since $t < 1$, it follows that $P^t V_i P^{1-t} x_n \rightarrow 0$ as $n \rightarrow \infty$, for all $i = 1, \dots, d$. This means that $\Delta_t(\mathbf{T})$ is not bounded below; that is, $\mathbf{0} \in \sigma_\ell(\Delta_t(\mathbf{T}))$. But $\mathbf{0}$ is also in $\sigma_\ell(\mathbf{T})$, because P is not bounded below. It follows that $\mathbf{0} \in \sigma_\ell(\mathbf{T}) \cap \sigma_\ell(\Delta_t(\mathbf{T}))$, as desired. \square

4. JOINT SPECTRAL RADIUS OF THE GENERALIZED SPHERICAL ALUTHGE TRANSFORM

In this Section we will extend K. Feki and T. Yamazaki's proof of the spectral radius formula for the spherical Aluthge transform in d -variables ($d > 1$ and $t = \frac{1}{2}$) [23] to the case of the generalized spherical Aluthge transform ($d > 1$ and $0 < t < 1$). First, we present a simple example that shows that the spectral radius formula (1.8) cannot be extended to the case of $t = 1$, even for $d = 1$.

Example 4.1. Consider an operator $T \in \mathcal{B}(\mathcal{H})$ with polar decomposition $T \equiv VP$, where the partial isometry V satisfies the algebraic equation $V^2 = -I$. It is clear that V must be unitary, and $V^* = -V$. Recall that, in general, the polar decomposition of the adjoint of T is given by $T^* \equiv V^*Q$, where $Q := \sqrt{TT^*}$. Thus, in the case at hand, we must have $Q = -VPV$ and $T^* = -PV$. It follows that

$$\Delta_1(T) = PV = -T^*,$$

and therefore

$$\Delta_1(\Delta_1(T)) = \Delta_1(-T^*) = \Delta_1(-V^*Q) = -QV^* = -(-VPV)(-V) = -VPV^2 = VP = T.$$

We conclude that

$$\Delta_1^{(n)}(T) = \begin{cases} T, & n \text{ is even} \\ -T^*, & n \text{ is odd.} \end{cases}$$

As a consequence, $\left\| \Delta_1^{(n)}(T) \right\| = \|T\|$ for all $n \geq 1$, which implies that $\lim_{n \rightarrow \infty} \left\| \Delta_1^{(n)}(T) \right\| = \|T\|$. On the other hand, within this collection of operators, $r(T)$ may be smaller than

$\|T\|$, and therefore, $r(T) \neq \lim_{n \rightarrow \infty} \left\| \Delta_1^{(n)}(T) \right\|$. For a concrete example where $r(T) < \|T\|$, let $\mathcal{H} := \mathbb{C}^2$, let $1 < k \in \mathbb{R}$ and let

$$T_k := \begin{pmatrix} 1 & k \\ -k & -1 \end{pmatrix},$$

with polar decomposition

$$T_k \equiv VP = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} k & 1 \\ 1 & k \end{pmatrix}. \quad \square$$

As we mentioned in the Introduction, T. Yamazaki proved, for single operators, that $r(T) = \lim_n \left\| \Delta_{1/2}^{(n)}(T) \right\|$. Subsequently, T.Y. Tam [40], building on D. Wang's [44] simplified proof of Yamazaki's result, extended the spectral radius formula to the case of $0 < t < 1$, under the assumption that T is invertible. Very recently, K. Feki and T. Yamazaki have extended these results to the case of $d > 1$ and $t = \frac{1}{2}$ [23]. In what follows, we will extend the spectral radius formula to all commuting d -tuples and all $0 < t < 1$.

First, we need to record three operator inequalities that can be traced back to the work of E. Heinz [25] and A. McIntosh [36]; for two of them, the formulation below is taken from [32].

Lemma 4.2. [36] *Let \mathcal{H} be a complex Hilbert space and let $A, B, X \in \mathcal{B}(\mathcal{H})$. Then,*

$$\|A^*XB\| \leq \|AA^*X\|^{\frac{1}{2}} \|XBB^*\|^{\frac{1}{2}}. \quad (4.1)$$

Lemma 4.3. ([25], [32, Theorems 1 and 2]) *Let \mathcal{H} be a complex Hilbert space and let $A, B, X \in \mathcal{B}(\mathcal{H})$. Then, for $t \in [0, 1]$,*

$$\|A^tXB^t\| \leq \|AXB\|^t \|X\|^{1-t} \quad (4.2)$$

and

$$\|A^tXB^{1-t}\| \leq \|AX\|^t \|XB\|^{1-t} \quad (4.3)$$

We will also need the multivariable analogue of the classical spectral radius formula, established by V. Müller and A. Soltysiak in 1992. For this, first recall that the joint spectral radius of a commuting d -tuple is given by $r(\mathbf{T}) := \sup\{\|\boldsymbol{\lambda}\|_2 : \boldsymbol{\lambda} \in \sigma_T(\mathbf{T})\}$ (cf. [8, 11, 37]). It is straightforward to see that $r(\mathbf{T}) \leq \|\mathbf{T}\|$. Also, given a d -tuple \mathbf{T} and an integer $k \geq 1$, we will let \mathbf{T}^k denote the d^k -tuple defined inductively as follows:

$$\begin{aligned} \mathbf{T}^1 &:= \mathbf{T} \\ \mathbf{T}^2 &:= (T_1T_1, T_1T_2, \dots, T_1T_d, T_2T_1, T_2T_2, \dots, T_dT_1, T_dT_2, \dots, T_dT_d), \\ &\vdots \\ \mathbf{T}^{k+1} &:= \mathbf{T}\mathbf{T}^k. \end{aligned} \quad (4.4)$$

Lemma 4.4. [37] *Let \mathbf{T} be a commuting d -tuple of Hilbert space operators. Then*

$$r(\mathbf{T}) = \lim_{k \rightarrow \infty} \|\mathbf{T}^k\|_2^{\frac{1}{k}}.$$

As a way to initiate our discussion, recall that, for $T \in \mathcal{B}(\mathcal{H})$ and $0 < t \leq 1$, one has $\|\Delta_t(T)\| \leq \|T\|$ [21, 40]. (Briefly, $\|P^t V P^{1-t}\| \leq \|PV\|^t \|VP\|^{1-t}$ (by (4.3)), and it is always true that $\|PV\| = \|V^*P\| \leq \|P\| = \|T\| = \|VP\|$.) As a consequence, the sequence of iterates of $\Delta_t(T)$ has decreasing norms, and therefore $\lim \left\| \Delta_t^{(n)}(T) \right\|_2$ exists and is majorized by $\|T\|_2$.

The other ingredient that enters into the 1-variable proof is the comparison between the norms of the integer powers of two consecutive iterates, $\left\| (\Delta_t^{(n+1)}(T))^k \right\|_2$ and $\left\| (\Delta_t^{(n)}(T))^k \right\|_2$. Our goal is to provide the appropriate analogues in the case of d -variables.

For the reader's convenience, we now restate Theorem 1.8.

Theorem 4.5. *Let $\mathbf{T} = (T_1, \dots, T_d)$ be a commuting d -tuple of operators on \mathcal{H} . For $0 < t < 1$, we have*

$$r_{\mathbf{T}}(\mathbf{T}) = \lim_{n \rightarrow \infty} \left\| \Delta_t^{(n)}(\mathbf{T}) \right\|_2. \quad (4.5)$$

To formulate a strategy to prove this result, we remind ourselves that there are three elements that should enter into the proof: (i) an application of Müller and Soltysiak's joint spectral radius formula [37], which will require a good handle on the powers of \mathbf{T} , its generalized spherical Aluthge transform $\Delta_t(\mathbf{T})$ and its iterates; (ii) the equality of the Taylor spectra of \mathbf{T} and any iterate $\Delta_T^{(n)}(\mathbf{T})$; and (iii) the asymptotic behavior of the norms of the iterates. The first goal is to establish the inequality $r(\mathbf{T}) \leq \left\| \Delta_t^{(n)}(\mathbf{T}) \right\|_2$ for all $t \in (0, 1), n \geq 1, k \geq 1$. Once this is done, we will look carefully at the behavior of $\left\| (\Delta_t^{(n)}(\mathbf{T}))^k \right\|_2$ as a function of the parameters t, n and k . Our proof will be structured around nine key steps, each of independent interest; we will record them in a series of nine auxiliary lemmas.

Lemma 4.6. *(cf. [23, Lemma 4.5]) For $k \geq 1$,*

$$\left\| \mathbf{T}^{k+1} \right\|_2 \leq \|\mathbf{T}\|_2 \left\| \mathbf{T}^k \right\|_2.$$

Proof. By (4.4), we have

$$\begin{aligned} \left\| \mathbf{T}^{k+1} \right\|_2^2 &= \left\| \sum_{i_1, \dots, i_{k+1}=1}^d T_{i_1}^* \cdots T_{i_{k+1}}^* T_{i_{k+1}} \cdots T_{i_1} \right\| \\ &= \left\| \sum_{i_1=1}^d T_{i_1}^* \left(\sum_{i_2, \dots, i_{k+1}=1}^d T_{i_2}^* \cdots T_{i_{k+1}}^* T_{i_{k+1}} \cdots T_{i_2} \right) T_{i_1} \right\| \\ &\leq \left\| \sum_{i_1=1}^d T_{i_1}^* T_{i_1} \right\| \left\| \sum_{i_2, \dots, i_{k+1}=1}^d T_{i_2}^* \cdots T_{i_{k+1}}^* T_{i_{k+1}} \cdots T_{i_2} \right\| \\ &= \|\mathbf{T}\|_2^2 \left\| \mathbf{T}^k \right\|_2^2, \end{aligned}$$

where the inequality in the third line requires an application of the well-known inequality $\|X^*AX\| \leq \|X^*X\| \|A\|$, for $A \geq 0$ and X arbitrary; in the present situation, X is the column vector with operator entries which are products of T_1, \dots, T_d , and A is the diagonal operator matrix with constant diagonal entry $\sum_{i_2, \dots, i_{k+1}=1}^d T_{i_2}^* \cdots T_{i_{k+1}}^* T_{i_{k+1}} \cdots T_{i_2}$. \square

Lemma 4.7. For $0 < t \leq 1$,

$$\|\Delta_t(\mathbf{T})\|_2 \leq \|\mathbf{T}\|_2.$$

Proof. On the Hilbert space $\mathcal{H}^d := \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (d orthogonal summands), consider the operators $A := \text{diag}(P, \dots, P)$, $B := A$, and

$$X := \begin{pmatrix} V_1 & 0 & 0 & 0 \\ V_2 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ V_d & 0 & 0 & 0 \end{pmatrix}.$$

Clearly, $\Delta_t(\mathbf{T})$ is the first column of the operator matrix $A^t X B^{1-t}$, whose other columns are all zero. Now apply (4.3) to obtain

$$\|\Delta_t(\mathbf{T})\|_2 = \|A^t X B^{1-t}\| \leq \|AX\|^t \|XB\|^{1-t}.$$

Observe now that

$$\|AX\| = \|X^*A\| \leq \|A\| = \|P\| = \|\mathbf{T}\|_2,$$

and $\|XB\| = \|\mathbf{T}\|_2$. It follows that

$$\|\Delta_t(\mathbf{T})\|_2 \leq \|\mathbf{T}\|_2^t \|\mathbf{T}\|_2^{1-t} = \|\mathbf{T}\|_2,$$

as desired. \square

Lemma 4.8. (cf. [23, Lemma 4.3] for $t = \frac{1}{2}$) For $0 < t \leq 1$ and $k \geq 1$,

$$\|(\Delta_t(\mathbf{T}))^k\|_2 \leq \|\mathbf{T}^k\|_2.$$

Proof. We imitate the Proof of Lemma 4.7, this time letting A, B, X act on the space \mathcal{H}^{d^k} . Thus, $A := \text{diag}(P, \dots, P)$, $B := A$, and X is a d^k by d^k operator matrix, with the only nonzero entries appearing in the first column; the entries in that column are given by $V_{i_1} P \cdots P V_{i_k}$, for $i_1, \dots, i_k = 1, \dots, d$. It is easy to show that $\|(\Delta_t(\mathbf{T}))^k\|_2 = \|A^t X B^{1-t}\|$. We can once again appeal to (4.3) to obtain

$$\|(\Delta_t(\mathbf{T}))^k\|_2 = \|A^t X B^{1-t}\| \leq \|AX\|^t \|XB\|^{1-t}. \quad (4.6)$$

(In the special case of $t = \frac{1}{2}$, one can use instead Lemma 4.2, as done in [23].) Now,

$$\begin{aligned}
\|AX\|^2 &= \|X^*AX\| \\
&= \left\| \sum_{i_1, \dots, i_k}^d V_{i_1}^* P \dots P V_{i_k}^* P^2 V_{i_k} P \dots P V_{i_1} \right\| \\
&= \left\| \sum_{i_1, \dots, i_k}^d V_{i_1}^* P \dots P V_{i_k}^* P \left(\sum_{i_1, \dots, i_{k+1}}^d V_{i_{k+1}}^d V_{i_{k+1}} \right) P V_{i_k} P \dots P V_{i_1} \right\| \\
&\quad (\text{since } V_1^* V_1 + \dots + V_d^* V_d \text{ is the projection onto the closure of } \text{Ran } P) \\
&= \left\| \sum_{i_1, \dots, i_k}^d V_{i_1}^* \left(\sum_{i_2, \dots, i_{k+1}}^d T_{i_2}^* \dots T_{i_{k+1}}^* T_{i_{k+1}} \dots T_{i_2} \right) V_{i_1} \right\| \\
&\leq \left\| \sum_{i_2, \dots, i_{k+1}}^d T_{i_2}^* \dots T_{i_{k+1}}^* T_{i_{k+1}} \dots T_{i_2} \right\| \\
&= \|\mathbf{T}^k\|_2^2. \tag{4.7}
\end{aligned}$$

On the other hand, it is straightforward to verify that $\|XA\| = \|\mathbf{T}^k\|$. Inserting this and (4.7) in (4.6) yields the desired conclusion. \square

Corollary 4.9. (cf. [23, bottom of page 13]) *With the notation in Lemma 4.8, we also have:*

$$\|AXA\| = \|\mathbf{T}^{k+1}\|_2.$$

Proof.

$$\begin{aligned}
\|AXA\|^2 &= \|AX^*A^2XA\| \\
&= \left\| \sum_{i_1, \dots, i_k}^d P V_{i_1}^* P \dots P V_{i_k}^* P^2 V_{i_k} P \dots P V_{i_1} P \right\| \\
&= \left\| \sum_{i_1, \dots, i_k}^d P V_{i_1}^* P \dots P V_{i_k}^* P \left(\sum_{i_1, \dots, i_{k+1}}^d V_{i_{k+1}}^d V_{i_{k+1}} \right) P V_{i_k} P \dots P V_{i_1} P \right\| \\
&\leq \left\| \sum_{i_1, \dots, i_{k+1}}^d T_{i_1}^* \dots T_{i_{k+1}}^* T_{i_{k+1}} \dots T_{i_1} \right\| \\
&= \|\mathbf{T}^{k+1}\|_2^2.
\end{aligned}$$

\square

Lemma 4.10. *For $0 < t \leq 1$ and $k, n \geq 1$,*

$$\left\| (\Delta_t^{(n+1)}(\mathbf{T}))^k \right\|_2 \leq \left\| (\Delta_t^{(n)}(\mathbf{T}))^k \right\|_2.$$

Proof. Since $\Delta_t^{(n+1)}(\mathbf{T}) = \Delta_t(\Delta_t^{(n)}(\mathbf{T}))$, it suffices to prove the conclusion for $n = 0$. But this is the content of Lemma 4.8, when applied to $\Delta_t^{(n)}(\mathbf{T})$. \square

Corollary 4.11. *For $0 < t \leq 1$ and $k, n \geq 1$,*

$$\left\| (\Delta_t^{(n)}(\mathbf{T}))^k \right\|_2 \leq \cdots \leq \left\| (\Delta_t^{(2)}(\mathbf{T}))^k \right\|_2 \leq \left\| (\Delta_t(\mathbf{T}))^k \right\|_2 \leq \|\mathbf{T}^k\|_2.$$

Proof. Immediate from Lemmas 4.8 and 4.10. \square

Lemma 4.12. *For $0 < t < 1$ and $k, n \geq 1$,*

$$r(\mathbf{T}) \leq \left\| (\Delta_t^{(n)}(\mathbf{T}))^k \right\|_2^{\frac{1}{k}}.$$

Proof. By Theorem 1.1, we know that $\sigma_T(\Delta_t(\mathbf{T})) = \sigma_T(\mathbf{T})$. It follows that the same is true for any iterate of $\Delta_t(\mathbf{T})$; that is, $\sigma_T(\Delta_t^{(n)}(\mathbf{T})) = \sigma_T(\mathbf{T})$ for all $n \geq 1$. Also, let us recall that the Spectral Mapping Theorem holds for σ_T [14, 42, 43]. We claim that $r(\mathbf{T}^k) = (r(\mathbf{T}))^k$. For, using (4.4) we have

$$\begin{aligned} \sigma_T(\mathbf{T}^k) &= \sigma_T(T_1^k, T_1^{k-1}T_2, \dots, T_{d-1}T_d^{k-1}) \\ &= \{(\lambda_1^k, \lambda_1^{k-1}\lambda_2, \dots, \lambda_{d-1}\lambda_d^{k-1}, \lambda_d^k) : (\lambda_1, \dots, \lambda_d) \in \sigma_T(\mathbf{T})\}. \end{aligned}$$

Since

$$\left\| (\lambda_1^k, \lambda_1^{k-1}\lambda_2, \dots, \lambda_{d-1}\lambda_d^{k-1}, \lambda_d^k) \right\|_2^2 = (|\lambda_1|^2 + \cdots + |\lambda_d|^2)^k = \|\boldsymbol{\lambda}\|_2^{2k},$$

it follows that $r(\mathbf{T}^k) = (r(\mathbf{T}))^k$. Therefore,

$$\begin{aligned} (r(\mathbf{T}))^k &= (r(\Delta_t^{(n)}(\mathbf{T}))^k = r((\Delta_t^{(n)}(\mathbf{T}))^k) \\ &\quad \text{(by the above-mentioned Spectral Mapping Property applied to } \Delta_t^{(n)}(\mathbf{T})) \\ &\leq \left\| (\Delta_t^{(n)}(\mathbf{T}))^k \right\|_2. \end{aligned}$$

It follows that $r(\mathbf{T}) \leq \left\| (\Delta_t^{(n)}(\mathbf{T}))^k \right\|_2^{\frac{1}{k}}$, as desired. \square

Remark 4.13. Corollary 4.11 and Lemma 4.12 provide a proof of Theorem 1.7. \square

Lemma 4.14. *For $0 < t < 1$ and $k \geq 1$, $(\left\| (\Delta_t(\mathbf{T})^{(n)})^k \right\|_2)_{n=1}^\infty$ is a decreasing sequence. If we let $L_{t,k}$ denote its limit, then $r(\mathbf{T})^k \leq L_{t,k}$.*

Proof. This is a straightforward consequence of Corollary 4.11 and Lemma 4.12. \square

Lemma 4.15. *For $0 \leq t \leq \frac{1}{2}$ and $k \geq 1$, we have*

$$\left\| (\Delta_t(\mathbf{T}))^k \right\|_2 \leq \|\mathbf{T}^{k+1}\|_2^t \|\mathbf{T}^{k-1}\|_2^{1-t} \|\mathbf{T}\|_2^{1-2t}.$$

Proof. With the notation used in the Proof of Lemma 4.8, we have

$$\begin{aligned} \left\| (\Delta_t(\mathbf{T}))^k \right\|_2 &= \|A^t X A^{1-t}\| = \|(A^t X A^t) A^{1-2t}\| \\ &\leq \|A^t X A^t\| \|A^{1-2t}\| \\ &\leq \|A X A\|^t \|X\|^{1-t} \|\mathbf{T}^{1-2t}\|_2 \\ &\leq \|\mathbf{T}^{k+1}\|_2^t \|\mathbf{T}^{k-1}\|_2^{1-t} \|\mathbf{T}^{1-2t}\|_2, \end{aligned}$$

using Corollary 4.9 to get the first factor in the last line. \square

Lemma 4.16. For $\frac{1}{2} \leq t \leq 1$ and $k \geq 1$, we have

$$\|(\Delta_t(\mathbf{T}))^k\|_2 \leq \|\mathbf{T}^{k+1}\|_2^{1-t} \|\mathbf{T}^{k-1}\|_2^t \|\mathbf{T}\|_2^{2t-1}.$$

Proof. As in Lemma 4.15,

$$\begin{aligned} \|(\Delta_t(\mathbf{T}))^k\|_2 &= \|A^t X A^{1-t}\| = \|A^{2t-1} (A^{1-t} X A^{1-t})\| \\ &\leq \|A^{2t-1}\| \|A^{1-t} X A^{1-t}\| \\ &\leq \|A^{2t-1}\| \|A X A\|^{1-t} \|X\|^t \\ &\leq \|\mathbf{T}\|_2^{2t-1} \|\mathbf{T}^{k+1}\|_2 \|\mathbf{T}^{k-1}\|_2, \end{aligned}$$

as desired. \square

Lemma 4.17. Let \mathbf{T} be a commuting d -tuple of operators on Hilbert space, and assume that $r(\mathbf{T}) > 0$. Then, for $0 < t < 1$ and $k \geq 1$, we have

$$L_{t,k} = L_{t,1}^k.$$

Proof. We will split the proof into two cases, and use induction on k .

Case 1: $0 < t \leq \frac{1}{2}$. For $k = 1$, we use Lemma 4.14. Assume now that $L_{t,k} = L_{t,1}^k$ for all integers $k \leq m$; we will establish the identity for $k = m + 1$. By Lemma 4.15, with \mathbf{T} replaced by $\Delta_t^{(n)}(\mathbf{T})$, we have

$$\|(\Delta_t^{(n+1)}(\mathbf{T}))^m\|_2 \leq \|(\Delta_t^{(n)}(\mathbf{T}))^{m+1}\|_2^t \|(\Delta_t^{(n)}(\mathbf{T}))^{m-1}\|_2^{1-t} \|\Delta_t^{(n)}(\mathbf{T})\|_2^{1-2t} \quad (4.8)$$

$$\begin{aligned} &\leq \|(\Delta_t^{(n)}(\mathbf{T}))^m\|_2^t \|(\Delta_t^{(n)}(\mathbf{T}))\|_2^t \|(\Delta_t^{(n)}(\mathbf{T}))^{m-1}\|_2^{1-t} \|\Delta_t^{(n)}(\mathbf{T})\|_2^{1-2t} \\ &\quad (\text{by Lemma 4.6}) \end{aligned}$$

$$= \|(\Delta_t^{(n)}(\mathbf{T}))^m\|_2^t \|(\Delta_t^{(n)}(\mathbf{T}))^{m-1}\|_2^{1-t} \|\Delta_t^{(n)}(\mathbf{T})\|_2^{1-2t}. \quad (4.9)$$

We now take the limit as $n \rightarrow \infty$, in (4.8) and (4.9) to obtain

$$L_{t,m} \leq L_{t,m+1}^t L_{t,m-1}^{1-t} L_{t,1}^{1-2t} \leq L_{t,m}^t L_{t,m-1}^{1-t} L_{t,1}^{1-2t}.$$

In view of the inductive hypothesis, and letting $L := L_{t,1}$, we can write

$$L^m \leq L_{t,m+1}^t L^{(m-1)(1-t)} L^{1-2t} \leq L^{mt} L^{(m-1)(1-t)} L^{1-2t},$$

equivalently,

$$L^m \leq L_{t,m+1}^t L^{m-t(m+1)} \leq L^m.$$

It follows that $L_{t,m+1}^t L^{m-t(m+1)} = L^m$. By Lemma 4.14, $L \geq r(\mathbf{T}) > 0$, so we obtain $L_{t,m+1} = L^{m+1}$, as desired.

Case 2: $\frac{1}{2} \leq t < 1$. Here,

$$\left\| (\Delta_t^{(n+1)}(\mathbf{T}))^m \right\|_2 \leq \left\| (\Delta_t^{(n)}(\mathbf{T}))^{m+1} \right\|_2^{1-t} \left\| (\Delta_t^{(n)}(\mathbf{T}))^{m-1} \right\|_2^t \left\| \Delta_t^{(n)}(\mathbf{T}) \right\|_2^{2t-1} \quad (4.10)$$

$$\begin{aligned} &\leq \left\| (\Delta_t^{(n)}(\mathbf{T}))^m \right\|_2^{1-t} \left\| (\Delta_t^{(n)}(\mathbf{T})) \right\|_2^{1-t} \left\| (\Delta_t^{(n)}(\mathbf{T}))^{m-1} \right\|_2^t \left\| \Delta_t^{(n)}(\mathbf{T}) \right\|_2^{2t-1} \\ &\quad (\text{by Lemma 4.6}) \\ &= \left\| (\Delta_t^{(n)}(\mathbf{T}))^m \right\|_2^{1-t} \left\| (\Delta_t^{(n)}(\mathbf{T}))^{m-1} \right\|_2^t \left\| \Delta_t^{(n)}(\mathbf{T}) \right\|_2^t. \end{aligned} \quad (4.11)$$

As above, we then take the limit as $n \rightarrow \infty$, in (4.10) and (4.11) to obtain

$$L_{t,m} \leq L_{t,m+1}^{1-t} L_{t,m-1}^t L_{t,1}^{2t-1} \leq L_{t,m}^{1-t} L_{t,m-1}^t L_{t,1}^t.$$

In view of the inductive hypothesis, we then have

$$L^m \leq L_{t,m+1}^{1-t} L^{(m-1)t} L^{2t-1} \leq L^{m(1-t)} L^{(m-1)t} L^t;$$

equivalently,

$$L^m \leq L_{t,m+1}^{1-t} L^{(m+1)t-1} \leq L^m.$$

It follows that $L_{t,m+1}^t L^{m+(m+1)(t-1)} = L^m$. By Lemma 4.14, $L \geq r(\mathbf{T}) > 0$, so we obtain $L_{t,m+1} = L^{m+1}$, as desired. \square

We are now ready to put all the pieces together.

Proof of Theorem 4.5. We first consider the case when $r(\mathbf{T}) > 0$. By Corollary 4.11 and Lemma 4.17, for each $k \geq 1$ the sequence $\left\{ \left\| (\Delta_t^{(n)}(\mathbf{T}))^k \right\|_2^{\frac{1}{k}} \right\}_{n=1}^\infty$ is decreasing to $L := L_{t,1}$. Moreover, $L \geq r(\mathbf{T})$. We claim that $L = r(\mathbf{T})$. Suppose that $L > r(\mathbf{T})$. Fix $n_0 \geq 1$ and consider the sequence $\left\{ \left\| (\Delta_t^{(n_0)}(\mathbf{T}))^k \right\|_2^{\frac{1}{k}} \right\}_{k=1}^\infty$. By Lemmas 4.14 and 4.17, this sequence is bounded from below by L . It follows that $\liminf_{k \rightarrow \infty} \left\| (\Delta_t^{(n_0)}(\mathbf{T}))^k \right\|_2^{\frac{1}{k}} \geq L$. But we know that this limit is the spectral radius $r(\Delta_t^{(n_0)}(\mathbf{T}))$, so $r(\mathbf{T}) \geq L > r(\mathbf{T})$, a contradiction.

Next, we look at the case when $r(\mathbf{T}) = 0$. On the orthogonal direct sum $\mathcal{H} \oplus \mathcal{H}$, let $\mathbf{S} := \mathbf{T} \oplus (cI, 0, \dots, 0)$, where $c > 0$. Clearly, $\sigma_T(\mathbf{S}) = \sigma_T(\mathbf{T}) \cup \{(c, 0, \dots, 0)\} = \{(c, 0, \dots, 0)\}$, so $r(\mathbf{S}) = c > 0$. Moreover, $\Delta_t^{(n)}(\mathbf{S}) = \Delta_t^{(n)}(\mathbf{T}) \oplus (cI, 0, \dots, 0)$. It follows that $\left\| \Delta_t^{(n)}(\mathbf{T}) \right\|_2 \leq \left\| \Delta_t^{(n)}(\mathbf{S}) \right\|_2$, and therefore

$$\limsup \left\| \Delta_t^{(n)}(\mathbf{T}) \right\|_2 \leq \limsup \left\| \Delta_t^{(n)}(\mathbf{S}) \right\|_2 = r(\mathbf{S}) = c.$$

Since c can be taken arbitrarily small, we conclude that $\lim \left\| \Delta_t^{(n)}(\mathbf{T}) \right\|_2 = 0 = r(\mathbf{T})$, as desired. \square

Remark 4.18. Theorem 1.6 is the $d = 1$ instance of Theorem 4.5. Similarly, Theorem 1.5 follows from Lemma 4.14 when $d = k = 1$. \square

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