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On closed subsets of non-commutative association schemes of rank 6

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ON CLOSED SUBSETS OF NON-COMMUTATIVE ASSOCIATION SCHEMES
OF RANK 6

By

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A Thesis Presented to the Graduate Faculty of the
College of Science, Mathematics and Technology
in Partial Fulfillment of the Requirements for the Degree of
Master of Science
in the field of
Mathematics

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Abstract

The notion of an association scheme is a generalization of the concept of a group. In fact, the so-called thin association schemes correspond in a well-understood way to groups. In this thesis, we look at the structure of non-commutative association schemes of rank 6. We will show that a non-normal closed subset of a non-commutative association scheme of rank 6, must have rank 2. The so-called Coxeter schemes of rank 6 which we present in Section 4 provide examples of association schemes of rank 6 with non-normal closed subsets of rank 2.

It is shown that normal closed subsets of imprimitive non-commutative schemes of rank 6 must have rank 2 or 3, so we will also look at the structure of association schemes of rank 6, with symmetric normal closed subsets of rank 3, and the structure of association schemes of rank 6 when they have non-symmetric closed subsets of rank 3. We will additionally see how to construct an association scheme of rank 6 with non-symmetric normal closed subsets.

Contents

1. Introduction

One might ask "Why are we interested in schemes of rank 6?". Well, as one knows, each group of order 5 or less is commutative. It just so happens, a corresponding result holds for association schemes[1]. In fact, in 1975, Donald G. Higman was able to prove that each association scheme of rank 5 or less is commutative.

What we also observe is that we can obtain a thin scheme from a group, and, conversely, we can obtain a group from a thin scheme; cf. [7; Theorem 5.5.1, Theorem 5.5.2]. Now, the non-commutative group of order 6 provides us with a non-commutative scheme of rank 6. However, although any two groups of order 6 are isomorphic, there are infinitely many non-commutative association schemes of rank 6 which are pairwise not isomorphic.

Up to a certain point, all imprimitive non-commutative association schemes of rank 6 were known to belong to one of three classes.

One of these three classes is the class of the *Coxeter schemes of rank* 6. Coxeter schemes are schemes that are generated by constrained sets of involutions, which satisfy what is called the *exchange condition.* Much of the motivation for analyzing Coxeter schemes comes from the being able to interpret what are called buildings[2].

A second class of these schemes are the *semidirect products with kernel of rank* 3 *and complement of rank* 2. Semidirect products are generalizations of direct products.

And our third class is *thin schemes with normal closed subset of rank* 3.

As it turns out the intersection of any two of these three classes is the class of thin Coxeter schemes of order 6. Therefore we can view these three classes as generalizations of Coxeter groups of order 6. Much of the interest in the investigation of Coxeter sets, which generate Coxeter schemes, lies on the observation that closed subsets generated by Coxeter sets share many characteristics with Coxeter groups; cf. [8; Theorem 3.1.4, Theorem 3.1.5, Theorem 3.1.8, Section 3.3].

However, as was shown recently in [1], a fourth class of association schemes of rank 6 was constructed, which are not Coxeter schemes or semidirect products, and do not have a thin closed subset of rank 3. In fact, a scheme can be constructed for every Mersenne prime[3] p greater than 3, having order $p(p+2)$, and having a non-symmetric closed subset of rank 3.

We will now state the organization of our thesis.

In Section 2, we compile the definitions and the notation which is needed in our thesis. In Section 3, we show that non-commutative schemes of rank 6 do not have non-normal closed subsets of rank different from 2. We closely follow the arguments of [5]. In Section 4, we give the definition of Coxeter schemes of rank 6. They all come from projective planes. In Section 5, we compile results on symmetric normal closed subsets of rank 3 in schemes of rank 6. All results stem from [2].

We will mention that normal closed subsets of non-commutative schemes of rank 6 have rank 2 or 3. There is a proof for this in [4], in which the authors use

[1] We shall define an association scheme and all other notation and terminology after we state our investigations

[2] Buildings were first brought up as means of outlining the simple algebraic groups over an arbitrary field.

[3] We recall that Mersenne prime is a prime number M_p of the form $M_p = 2^p - 1$.

representation theory of association schemes. Looking into this case we have a two subcases.

The subcase where the rank of T is 2 is widely open since not much is known in this case. For the subcase where the rank of T is equal to 3, we have that either T is symmetric, in which there is some progress in [2], or T is non-symmetric, which we look at in [1]. For our symmetric case, we let X be a finite set, and S be a scheme on X, and assume that S is not commutative with rank 6. Let U be a symmetric normal closed subset of S, such that $|U| = 3$. Since S is assumed to be non-commutative, $S \setminus U$ contains exactly one symmetric element, by [7; Lemma 1.2.5]. We will denote this element by s, and we assume that $n_s = n_{s^U}$. Based on this hypothesis our goal is to prove the following:

Proposition 1 n_U *divides* $n_{S//U} - 2$.

Our final section is devoted to recent results obtained by Drabkin and French, towards the non-symmetric case. We will see a certain construction of a scheme [1], where we use multiplicative group of a finite field, to form a non-symmetric scheme (particularly we will be looking at the case where our scheme is of rank 3).

2. Definitions

For the rest of this paper, let X be a finite set, and let S be a partition of $X \times X$. We define the set 1_X to be the set of all pairs (x, x) such that $x \in X$. For every $s \in S$, we define s^* to be the set of all (y, z) such that $(z, y) \in s$. And for every $y \in X$ and $p \in S$, we define yp to be the set of all z such that $(y, z) \in p$. S is called an *association scheme* on X (or simply a *scheme* on X), if the following conditions hold:

(i) $1_X \in S$

(ii) $s^* \in S \; \forall s \in S$

(iii) $\forall p, q, r \in S$ there exists an integer a_{pqr} such that $\forall y \in X$ and $\forall z \in yr$, $|yp \cap zq^*| = a_{pqr}$.

If S is a scheme on a set X, we call the cardinality of X the *order* of S and the cardinality of S the *rank* of S. The integers a_{pqr} are called *structure constants* of S. An association scheme is called *commutative* if $a_{pqr} = a_{qpr}$ for any three elements $p, q, r \in S$. For the rest of this section S will be a scheme on a finite set X.

An element $s \in S$ is called *symmetric* if $s = s^*$. A subset of S is called *symmetric* if each of its elements is symmetric.

For simplicity we will write 1 instead of 1_X. For any $s \in S$, we define $n_s = a_{ss^*1}$. The integer n_s is called the *valency* of s. An association scheme is called *thin*, if $n_s = 1$ for every $s \in S$.

For every $p, q \in S$, we define the set pq to be the set of all elements $r \in S$ such that $a_{pqr} \neq 0$. We call the set pq the *complex product* of p and q.

Let $T \subseteq S$, then T is called *closed* if for any two elements $p, q \in T$, $pq \subseteq T$. It is easy to see that if T is a closed subset of S, then $1 \in T$ and $t^* \in T$ for every $t \in T$; cf.[7; Lemma 2.1.6].

We note that $\{1\}$ and S are closed subsets of S. A scheme S is called *primitive* if $\{1\}$ and S are the only closed subsets of S.

For any $T \subseteq S$ and $p \in S$, we define pT to be the union of the sets pt with $t \in T$. Now we define the set S/T to be the set of all cosets. We can see that S/T forms a partition of S, shown in [7; Lemma 2.1.4]. A closed subset T of S is called *normal* if for every $s \in S$, $sT = Ts$.

An element s in $S \setminus \{1\}$ is called an *involution* if $\{1, s\}$ is closed.

Let T be a closed subset of S. We define the set xT to be the union of all sets xt with $x \in X$ and $t \in T$. For each element $t \in T$, we set $t_{xT} := t \cap xT \times xT$. We define the set T_{xT} to be the set of all sets t_{xT} with $t \in T$. It is not very hard to see that T_{xT} is a scheme on the set xT, and this is shown in [7; Theorem 2.1.8(ii)]. T_{xT} is called the *subscheme of S defined by xT*.

Let T be a closed subset of S. We denote X/T to be the set of all sets xT with $x \in X$. From [7; Lemma 2.1.4] we see that X/T is a partition of X. For each element s in S, we define

$$s^T := \{(yT, zT) | z \in ys\}.$$

We define $S//T$ to be the set of all sets s^T with $s \in S$. Then, by [7; Theorem 4.1.3(i)], $S//T$ is a scheme on X/T. The scheme $S//T$ is called the *quotient (scheme) of S over T*.

Let X' be a finite set, with S' being a scheme on X'. A bijective map ϕ from X to X' is called an *isomorphism* from S to S' if there exists a bijective map σ from S to S' such that $\phi(xs) = \phi(x)\sigma(s)$ for any two elements $x \in X$ and $s \in S$. The map σ is called the *bijection associated with* ϕ. The schemes S and S' are called *isomorphic* if there exists an isomorphism from S to S'. An isomorphism from S to S is called an *automorphism* of S.

It is important to note that the set of all *automorphisms* of S is a group under composition. We can denote this group $\mathrm{Aut}(S)$.

An automorphism α of S is called $faithful$ if the bijection associated with α is the identity on S. We can easily see that the set of all faithful automorphisms of S is a subgroup of $\mathrm{Aut}(S)$.

A scheme S is called *schurian* if there exists an isomorphism from S to a quotient of a thin scheme.

Now we are done with the basic definitions and theoretic notation for association schemes. Much of this notation is adapted from group theory. We will define other information needed in later sections.

3. Non-Normal Closed Subsets of Schemes of Rank 6

In this section, we will see some results that help us declare the rank of non-normal closed subsets of schemes of rank 6. For the rest of this section S will be a scheme, and T will be a closed subset of S. For each subset R of S, we define R^* to be the set of all $s \in S$ such that $s^* \in R$.

Lemma 3.1 *Let p be an element in S with $pT \subseteq Tp$, and let q be an element in Tp such that q^*T=qT. Then $q \in pT$.*

Proof: Since $q \in Tp$, then q^* must be in p^*T. Therefore $p^* \in q^*T$. Since $q^*T = qT$, it follows that $p^* \in qT \subseteq TpT = Tp = Tq$. Thus $p^* \in Tq$, so $p \in q^*T = qT$. Thus $q \in pT$. □

Lemma 3.2 *Let s be an element in S such that sT=$\{s\}$ and s^*T=$\{s^*\}$. Then $Ts = sT$.*

Proof: From our assumption s^*T= $\{s^*\}$ we obtain $Ts = \{s\}$. Since $sT = \{s\}$, then $Ts = sT$. □

Lemma 3.3 *Let s be an element in S such that $(sT)^* \in S/T$. Then $Ts = sT$.*

Proof: We assume that $(sT)^* \in S/T$. Since $s^* \in (sT)^*$, $(sT)^* = s^*T$. Thus we obtain $Ts = sT$. □

Lemma 3.4 *Assume that $|S \setminus T| = |S/T|$ and that $S \setminus T$ possesses exactly $|S \setminus T| - 2$ symmetric elements. Then $Ts = sT$.*

Proof: Assuming that $|S \setminus T| = |S/T|$, we obtain that $S/T \setminus \{T\}$ possesses exactly one element of cardinality 2, and all other elements of $S/T \setminus \{T\}$ consist of a single element. Let p and q be the two elements in the uniquely determined element of $S/T \setminus \{T\}$ of cardinality 2. If $\{p, q\}^* = \{p, q\}$, then we are done by Lemma 3.3. Therefore we are done if we are able to show that $\{p, q\}^* \neq \{p, q\}$ leads to a contradiction. If $\{p, q\}^* \neq \{p, q\}$, then either $p^* \notin \{p, q\}$ or $q^* \notin \{p, q\}$. Without loss of generality, we may assume that $p^* \notin \{p, q\}$. Since $\{p, q\}$ is the only element in $S/T \setminus \{T\}$ that contains more than one element, $p^* \notin \{p, q\}$ forces $p^*T = \{p^*\}$. Thus, $p^*T \subseteq Tp^*$. We are assuming that $S \setminus T$ has exactly $|S \setminus T| - 2$ symmetric elements. Then as $p^* \neq p \neq q$, we must have $q^* = q$. Therefore $q^*T = qT$ and since $q \in pT$, $q \in Tp^*$. It follows from $p^*T \subseteq Tp^*$, $q \in Tp^*$, and $q^*T = qT$ that $q \in p^*T$ by Lemma 3.1. Thus, as $p^*T = \{p^*\}$, $q = p^*$, contradiction. □

Theorem 3.5 *Assume that $|S \setminus T| \leq 3$. Then T is a normal in S.*

Proof: If $|S/T| = 1$, $T = S$, and we are done. If $|S/T| = 2$, we have $Ts = S \setminus T = sT$ for each element $s \in S \setminus T$. Thus T is normal. Assume that $3 \leq |S/T|$. Then as $|S/T| \leq |S \setminus T| + 1 \leq 4$, we have $|S/T| = |S \setminus T| = 3$ or $|S/T| = |S \setminus T| + 1$. In our first case, we are done by Lemma 3.3 and Lemma 3.4. In our second case, we are done by Lemma 3.2. □

Corollary 3.6 *Assume that the rank of S is 6, and T is not normal in S. Then* $|T|$ *=2.*

This result is an immediate consequence of Theorem 3.5.

4. Coxeter Schemes of Rank 6

In this section we define a Coxeter scheme. We will also formulate a Coxeter scheme from a projective plane and analyze this scheme. Let's begin with some definitions. We recall S is a scheme on a finite set X.

Let R be a non-empty subset of S. We define $R^0 = \{1\}$, and inductively define $R^n := R^{n-1}R$ for $n \in \mathbb{N}$. For $s \in S$, we define $\ell_R(s)$ as the smallest non-negative integer n such that $s \in (R^* \cup R)^n$. We also define $\langle R \rangle$ to be the intersection of all closed subsets of S which contain R.

Let L be a set of fixed involutions in S. Instead of ℓ_L, we will write ℓ. For $q \in \langle L \rangle$, we define $S_1(q, L)$ as the set of all p such that there exists and element $r \in pq$, with $\ell(r) = \ell(p) + \ell(q)$. L is called *constrained* if for any two elements $q \in \langle L \rangle$ and $p \in S_1(q, L)$, $|pq| = 1$. L is said to satisfy the *exchange condition* if, for any three elements $h, k \in L$ and $s \in S_1(k, L)$, $h \in S_1(s, L)$ implies $hs \subseteq sk \cup S_1(k, L)$. L is called a *Coxeter set* if, L is constrained and satisfies the exchange condition. S is called a *Coxeter scheme with respect to L* if it is a scheme generated by a Coxeter set L.

Now, lets define a projective plane. Let Γ be the set of the triples (P, L, I) where $I \subseteq P \times L$. We call I an *incidence* relation between points $p \in P$ and lines $l \in L$. Then Γ is a *projective plane* if the following hold:

(i) For every $p_1, p_2 \in P$, there exists exactly one $l \in L$ such that $p_1 Il$ and $p_2 Il$.

(ii) For every $l_1, l_2 \in L$, there exists exactly one $p \in P$ such that pIl_1 and pIl_2.

(iii) For every $p \in P$, the number of lines l incident to p is constant and for every $l \in L$ the number of points $p \in P$ incident to l is also constant.

A *flag* of Γ is an ordered pair (p, l), where $p \in P$ and $l \in L$, such that pIl. Let X denote the set of all flags of Γ. We consider the following relations:

$$\begin{aligned} 1_X &= \{((p_1, l_1), (p_2, l_2)) \in X \times X : p_1 = p_2 \text{ and } l_1 = l_2\} \\ h &= \{((p_1, l_1), (p_2, l_2)) \in X \times X : p_1 = p_2 \text{ and } l_1 \neq l_2\} \\ k &= \{((p_1, l_1), (p_2, l_2)) \in X \times X : p_1 \neq p_2 \text{ and } l_1 = l_2\} \end{aligned}$$

Let $h \circ k$ denote the relation product of h and k. Let $P = \{1_X, h, k, h \circ k, k \circ h, h \circ k \circ h\}$. We note that $h \circ k \circ h = k \circ h \circ k$. Now, our claim is the following:

Theorem 4.1 *P is a Coxeter scheme on X of rank 6.*

Proof: Clearly P is of rank 6, and it is not very difficult to show P is a partition of $X \times X$. We also see that the first condition for a scheme holds for P.

Now, we know, $1_X, h^*, k^* \in P$. We can also see that $(h \circ k)^* = k^* \circ h^* = k \circ h$. Therefore $(h \circ k)^*, (k \circ h)^* \in P$. And $(h \circ k \circ h)^* = h^* \circ k^* \circ h^* = h \circ k \circ h$. Thus, the second condition for a scheme holds.

And by [3; Lemma 5.1] we see that the third condition for a scheme holds for P.

Due to [5; Theorem 2.4.2], P is a Coxeter scheme. □

Now from Theorem 4.1, we can see 1_X, h, and k are involutions, therefore $\{1_X, h\}$ and $\{1_X, k\}$ are closed subsets of P. Furthermore we can see that these closed subsets are non-normal.

Now we look at some special cases for schemes of rank 6, in which we are able to classify these schemes as Coxeter if some conditions hold. Let's assume that S is a scheme of rank 6 and let T be a non-normal closed subset of S. By Corollary 3.6 , $|T| = 2$. Therefore $T = \{1_X, t\}$ for some symmetric element $t \in S$, and S possesses elements s,m, and j with $s = s^*$, $j = j^*$ and $S = \{1_X, t, s, m, m^*, j\}$; cf.[5; Lemma 2.1.2]. We now obtain the following results.

Theorem 4.2 *If $a_{tss} = 0$, S is a semidirect product with complement T or a Coxeter scheme (with respect to $\{s, t\}$).*

Theorem 4.3 *Assume that $a_{tss} = 1$. Then S is a Coxeter scheme with respect to $\{t, j\}$ and has valency 21.*

These results are shown in [5,(2.4)], in which some representation theory is needed.

5. Symmetric Normal Closed Subsets of Rank 3

From [5; Theorem 3.1.4] we see that normal closed subsets of imprimitive non-commutative association schemes of rank 6 must have rank 2 or 3. In this section we will look at some results corresponding to symmetric normal closed subsets of rank 3. First lets look at some results for structure constants.

Let's recall that S is a scheme on finite set X. We let $s \in S$, and $R \subseteq S$, which is non-empty. We define a_{sRs} to be the sum of all integers a_{srs} for $r \in R$, and a_{Rss} as the sum of all integers a_{rss} for $r \in R$.

Lemma 5.1 *Let s be an element in S, and let T be a normal closed subset of S. Then the following hold*

(i) We have $a_{sTs} = a_{Tss}$
(ii) We have $a_{sTs} = a_{s^*Ts^*}$
(iii) We have $n_s = a_{sTs} n_{sT}$

Proof: (i) If we set one of the two closed subsets in [7; Lemma 2.3.3] equal to $\{1\}$ we obtain $a_{sTs} n_{sT} = n_s n_T$ as well as $a_{Tss} n_{Ts} = n_T n_s$. Since T is assumed to be normal in S, we have $Ts = sT$. It follows that $a_{sTs} = a_{Tss}$.

(ii) From [7; Lemma 1.1.1(ii)], $a_{Tss} = a_{s^*Ts^*}$. Therefore, our desired result is obtained from (i).

(iii) From [7; Theorem 4.1.3(iii)] we see that $n_{s^T} n_T = n_{TsT}$. Assuming that T is normal in S, we must have $TsT = sT$. Thus, $ns^T nT = n_{sT}$. And now from [7; Lemma 2.3.4(ii)], $a_{sTs} n_{sT} = n_s n_T$. Thus, $n_s = a_{sTs} n_{sT}$. □

Lemma 5.2 *Let $s \in S$, let T be a closed subset of S, and assume that $a_{sTs} = 1$. Let p and q be two different elements in T. Then the following hold:*

(i) The set $sp \cap sq$ is empty.
(ii) If T is normal in S, $ps \cap qs$ is empty.

Proof: (i) Let's assume that $sp \cap sq$ is not empty. Then by [7; Lemma 1.3.4], we obtain that $s^*s \cap pq^*$ is not empty. Let $t \in s^*s \cap pq^*$. Since $t \in s^*s$, we obtain that $s \in st$ by [7; Lemma 1.3.3(ii)]. It follows that, $1 \leq a_{sts}$. From $t \in pq^*$ and $p \neq q$, we therefore obtain $t \neq 1$ by [7; Lemma 1.3.2(i)]. Thus, we have $2 \leq a_{s1s} + a_{sts} \leq a_{sTs}$, contradiction.

(ii) Let's assume that T is normal in S. Then by Lemma 5.1(ii), $a_{sTs} = a_{s^*Ts^*}$. Therefore, by (i) $s^*p^* \cap s^*q^*$ is empty. By [7; Lemma 1.3.2(iii)], $s^*p^* = (ps)^*$ and $s^*q^* = (qs)^*$. Thus since $(ps)^* \cap (qs)^*$, $ps \cap qs$ is empty as well. □

Lemma 5.3 *Let T be a closed subset of S where $|S \setminus T| \leq |T|$, and let $s \in S \setminus T$ where $a_{sTs} = 1$. Then the following hold:*

(i) For each element $t \in T$, we have $|st| = 1$.

(ii) We have $S \setminus T = sT$.
(iii) We have $|S \setminus T| = |T|$.

Proof: From Lemma 5.2(i) we see that the $|T|$ different products st with $t \in T$ are pairwise disjoint. Thus, since sT is equal to the union of all sets st with $t \in T$, we obtain

$$|T| \leq \sum_{t \in T} |st| = |sT| \leq |S \setminus T| \leq |T|.$$

From this result we obtain all three results. □

Lemma 5.4 *Let T be a normal closed subset of S, and let $s \in S \setminus T$. Assume $a_{sTs} = 1$ and that $|st| = 1$ for each element $t \in T$. Then for each element $t \in T$, there exists exactly one element $t' \in T$ such that $ts = st'$.*

Proof: Let $t \in T$, and let $r \in ts$. Therefore $r \in Ts$. Thus, since T is assumed to be normal in S, $r \in sT$. Thus, T contains an element t' such that $r \in st'$. Therefore, since st' is assumed to contain only one element, we must have $st' = \{r\}$. It follows that $st' \subseteq ts$.

From Lemma 5.2(i) we see that $\{st|t \in T\}$ is a partition of sT and has cardinality $|T|$. From Lemma 5.2(ii) we see that $\{ts|t \in T\}$ is a partition of sT and has cardinality $|T|$ as well. Thus we must have $ts = st'$ for any two elements $t, t' \in T$ with $st' \subseteq ts$. □

Before going on to our next result we will define a unique map. Let T be a normal closed subset of S, let $s \in S \setminus T$, and assume that $a_{sTs} = 1$. Let y and z be elements in X, where $(y, z) \in s$. From $s \notin T$ we obtain that $yT \neq zT$. Since T is normal in S, we have $1 \leq |xs \cap zT|$ for each element $x \in yT$. Since $a_{sTs} = 1$, we have $|xs \cap zT| \leq 1$ for each element $x \in yT$. Therefore, $|xs \cap zT| = 1$ for each element $x \in yT$. We define $\phi_{yT,zT}$ to be the map from yT to zT, which sends each $x \in yT$ to the unique element in $xs \cap zT$.

Lemma 5.5 *Let T be a normal closed subset of S, let $s \in S \setminus T$, and assume that $a_{sTs} = 1$. Let y and z be elements in X, where $(y, z) \in s$. Then the following hold:*
(i) The map $\phi_{yT,zT}$ is a bijection from yT to zT.
(ii) Assume that $|st| = 1$ for each element $t \in T$. Then $\phi_{yT,zT}$ is an isomorphism from the scheme T_{yT} to the scheme T_{zT}.

Proof: Set $\phi := \phi_{yT,zT}$.
(i) From $a_{sTs} = 1$, and assuming that T is normal in S, $a_{Tss} = 1$ by Lemma 5.1(i). That means that ϕ is injective. Therefore, since $|yT|$ is finite and $|yT| = |zT|$, ϕ must be bijective.

(ii) From (i) we know that ϕ is a bijective map from yT to zT. Let v and w be elements of yT, and let $t \in T$ which satisfies $(v, w) \in t$. By t' we denote the uniquely determined element in T satisfying $ts = st'$, in Lemma 5.4. Then

$$\phi(w) \in ws \cap zT \subseteq vts \cap zT = vst' \cap zT.$$

Therefore, vs contains an element x such that $\phi(w) \in xt' \cap zT$. From $\phi(w) \in xt' \cap zT$ we obtain that $x \in zT$. Thus, since $x \in vs$, $x = \phi(v)$. Therefore, as $\phi(w) \in xt'$, $\phi(w) \in \phi(v)t'$, meaning $(\phi(v), \phi(w)) \in t'$. □

Now we will see some results on symmetric schemes of rank 3, and their associated automorphisms.

Lemma 5.6 *Let S be a symmetric scheme on a finite set X, let $|S| = 3$, and let α be a non-faithful automorphism of S. Then the following hold:*

(i) The length of a non-trivial cycle of α on X is a multiple of 4.

(ii) The automorphism α has exactly one fixed point.

Proof: (i) This follows immediately from the definition of a non-faithful automorphism, with the condition that the two non-identity elements of S are symmetric.

(ii) Assuming that α is not faithful, we obtain that the bijection associated with α interchanges the two non-identity elements of S. Therefore, the two non-identity elements of S must have the same valency, and this implies that $|X|$ is odd. Thus, by (i), α must have at least one fixed point. □

Lemma 5.7 *Let S be a symmetric scheme on a finite set X, let $|S| = 3$, and assume that S admits a non-faithful automorphism. Then the following hold:*

(i) We have $|X| \equiv 1$ mod 4.

(ii) The subgroup of Aut (S) consisting of all faithful automorphisms of S has index 2 in Aut(S).

Proof: (i) Since S admits a non-faithful automorphism, this follows from Lemma 5.6.

(ii) This result follows from the fact that products of two elements in Aut(S) are faithful. □

Now we move one to our main result for this section, Proposition 1. For the rest of this section, we will assume that S is not commutative with rank 6, and U will be a symmetric normal closed subset of S, such that $|U| = 3$. We will denote s as the unique symmetric element in $S \setminus U$, and we assume that $n_s = n_{s^U}$.

Lemma 5.8 *The following hold:*

(i) We have $a_{sUs} = 1$

(ii) The scheme $S//U$ has rank 2.

Proof: (i) From $n_s = n_{s^U}$, this result follows from Lemma 5.1(iii).

(ii) Considering (i) this follows from Lemma 5.3(ii). □

For the rest of this section, the two non-identity (symmetric) elements of U will be denoted by u_1 and u_2.

Corollary 5.9 *The following hold:*

(i) The sets $\{s\}$, su_1, *and* su_2 *are pairwise disjoint and all have cardinality 1.*

(ii) We have $su_1 = u_2 s$ *and* $su_2 = u_1 s$

Proof: (i) Considering Lemma 5.8(i), this follows from Lemma 5.2(i) and Lemma 5.3(i).

(ii) For this result we recall that s is the only symmetric element in $S \backslash U$. Therefore, we obtain from (i) that $(su_2)^* = su_1$. Thus, as s and u_2 are symmetric, $su_1 = u_2 s$, by [6; Lemma 1.3.2(iii)]. The equality $su_2 = u_1 s$ follows similarly.

It is important to observe that the map $\phi_{yU,zU}$ is defined for any two elements $y, z \in X$, with $yU \neq zU$, not only for elements $y, z \in X$ with $(y, z) \in s$. This is due to Lemma 5.8(ii). □

Lemma 5.10 *Let* $x, y \in X$ *with* $yU \neq zU$. *Then the following hold:*

(i) The map $\phi_{yU,zU}$ *is an isomorphism from* U_{yU} *to* U_{zU}.

(ii) Set $\phi := \phi_{yU,zU}$. *Then* $(\phi(v), \phi(w)) \in u_2$ *for any two elements v and w in yU with* $(v, w) \in u_1$ *and* $(\phi(v), \phi(w)) \in u_1$ *for any two elements v and w in yU with* $(v, w) \in u_2$.

(iii) The maps $\phi_{yU,zU}$ *and* $\phi_{zU,yU}$ *are inverses of each other.*

Proof: (i) From Lemma 5.1(i) we know that $a_{sUs} = 1$. From Corollary 5.9(i), we know that both su_1 and su_2 have cardinality 1. Therefore, by Lemma 5.5(ii), ϕ is an isomorphism from U_{yU} to U_{zU}.

(ii) Assume that $(v, w) \in u_1$. Thus, by Corollary 5.9(ii), $(\phi(v), \phi(w)) \in u_2$. Similarly, we obtain $(\phi(v), \phi(w)) \in u_1$ if $(v, w) \in u_2$.

This result follows from the fact that s is symmetric. □

For the rest of this section we set $l := n_{S//U}$, and assume that $3 \leq l$. Furthermore, we will fix an element $x \in X$ and representatives $x_1, x_2, ..., x_{l-1}$ of the cosets of U in X which are different from xU.

For any two different elements i and j in $\{1, ..., l-1\}$, we define

$$\psi_{ij} := \phi_{xU,x_iU}\phi_{x_iU,x_jU}\phi_{x_jU,xU}$$

Our next result is a direct consequence of Lemma 4.10.

Corollary 5.11 *Let i and j be two different elements in* $\{1, ..., l-1\}$. *Then the following hold:*

(i) The map ψ_{ij}*is a non-faithful automorphism of* U_{xU}.

(ii) The maps ψ_{ij} *and* ψ_{ji} *are inverses of each other.*

Lemma 5.12 *We have* $a_{sss}n_U = l - 2$

Proof: We set $z := x_{l-1}$ and also define τ to be the set of all triples (x', y', z') of elements in X such that

$$x' \in y's \cap xU \text{ and } z' \in x's \cap y's \cap zU.$$

We compute the cardinality of τ in two different ways.

We note first that any two elements $x' \in X$ and $z' \in x's$, $|\tau| = a_{sss}n_U$, recalling that s is symmetric.

On the other hand, we also note that for any three elements $x', y', z' \in X$ and $(x', y', z') \in \tau$, there exists an element i in $\{1, ..., l-2\}$ with $y' \in x_iU$.

Let $i \in \{1, ..., l-2\}$, let $y' \in x_iU$, and let x' and z' be elements in X such that $(x', y', z') \in \tau$. Then x' is a fixed point of $\psi_{i,l-1}$. Conversely, if x' is a fixed point of $\psi_{i,l-1}$, then there exists elements $y' \in x_iU$ and $z' \in X$ such that $(x', y', z') \in \tau$.

Now if we recall from Corollary 5.11(i) and Lemma 5.6(ii) that $\psi_{i,l-1}$ has exactly one fixed point, then τ contains exactly one triple (x', y', z') such that $y' \in x_iU$. Since i was chosen arbitrarily in $\{1, ..., l-2\}$, we have shown that $|\tau| = l - 2$. $\square$

By Lemma 5.12, we have proved Proposition 1.

6. Non-Symmetric Normal Closed Subsets of Rank 3

In the previous section we saw some results on symmetric closed subsets of rank 3, in this section we look at the case where our closed subset of rank 3 is non-symmetric. We will look at some results needed for our investigation, and we will then show a certain construction of a scheme of rank 6, with non-symmetric normal closed subset of rank 3. First let's look at what are called adjacency matrices of an association scheme.

Recall S is a scheme on a finite set X, and let $s \in S$. The *adjacency matrix of* s, denoted by σ_s, is the matrix with rows and columns that are indexed by X and defined by

$$(\sigma_s)_{x,y} = \begin{cases} 1 : (x,y) \in s \\ 0 : (x,y) \notin s \end{cases}$$

for $x, y \in X$.

Remark 6.1 By definition, we can see that the matrix σ_1 is the identity matrix indexed by X, since $(x,y) \in 1$ if and only if $x = y$. Also, for each $s \in S$, we have σ_{s^*} is equal to the transpose $(\sigma_s)^T$ of σ_s, since $(\sigma_s^*)_{x,y} = 1$ if and only if $(x,y) \in s^*$, which holds if and only if $(y,x) \in s$, and $(\sigma_s)^T_{x,y} = 1$ if and only if $(\sigma_s)_{y,x} = 1$, which holds if and only if $(y,x) \in s$.

Remark 6.2 If S is a scheme, then for any $p, q \in S$, we have

$$\sigma_p \sigma_q = \sum_{r \in S} a_{pqr} \sigma_r.$$

We see this result in [7; Lemma 9.1.1(i)], or we can also use the argument obtained in the following result Lemma 6.3.

Let U be a subset of a scheme S on a finite set X, we define

$$\Phi_U = \{\sum_{u \in U} a_u \sigma_u : a_u \in \mathbb{Z} \ \forall u \in U\}$$

Lemma 6.3 *Suppose X is a set and S is a partition of $X \times X$, with S satisfying the first two conditions for an association scheme, and Φ_S is closed under matrix multiplication. Then S is an association scheme. If U is a nonempty subset of S such that $u^* \in U$, for every $u \in U$, and Φ_U is closed under matrix multiplication, then U is a closed subset of S. If Φ_S is closed under matrix multiplication and is commutative, then S is a commutative association scheme.*

Proof: Given three elements $p, q, r \in S$, we then have

$$\sigma_p \sigma_q = \sum_{s \in S} b_{pqs} \sigma_s.$$

where $b_{pqs} \in \mathbb{Z}$ for each $s \in S$. Now, if we have $(x,y) \in r$, then

$$(\sigma_p\sigma_q)_{x,y} = b_{pqs}.$$

We also obtain

$$\begin{aligned}(\sigma_p\sigma_q)_{x,y} &= \sum_{z\in X}(\sigma_s)_{x,z}(\sigma_s)_{z,y}\\ &= \sum_{z\in X:(x,y)\in p,(z,y)\in q} 1 = |\{z\in X:(x,z)\in p,(z,y)\in q\}|\end{aligned}$$

Therefore b_{pqr} is the number of elements $z \in X$ such that $(x, z) \in p$ and $(z, y) \in q$. From this we can see that S satisfies the last condition of an association scheme, as well as the first two.

Now assume that U is a nonempty subset of S, such that $u^* \in U$ whenever $u \in U$ and assume that Φ_U is closed under matrix multiplication. Given a pair of elements $p, q \in U$, we can use a similar argument to the previous one to show that $b_{pqr} = 0$ unless $r \in U$. Since $b_{pqr} = a_{pqr}$, we can conclude that pq consists only of elements which are contained in U. Similarly, since $q \in U$ implies that $q^* \in U$, we have that pq^* consists only of elements contained in U.

And now assume that Φ_S is closed under matrix multiplication and is commutative. Then for any pair of elements $p, q \in S$, we obtain $\sigma_p\sigma_q = \sigma_q\sigma_p$. It follows that $b_{pqr} = b_{qpr}$ for any $r \in S$. Since $b_{pqr} = a_{pqr}$ and $b_{qpr} = a_{qpr}$, S is commutative. □

In the following results, we will assume that p is a prime such that $p \equiv 3 \bmod 4$, and we let $\mathbb{F}_p$ denote the field of order p. Let $\mathbb{F}_p^\times$ denote the multiplicative group of units in $\mathbb{F}_p$, and we fix a chosen generator γ for $\mathbb{F}_p^\times$. Given that k is a divisor of $p-1$, let Q_k denote the subgroup of $\mathbb{F}_p^\times$ with index k. Therefore Q_k is generated by γ^k and has order $\frac{p-1}{k}$. Then the *cyclotomic scheme* of rank $k+1$, denoted $Cyc(p, k)$, is the scheme on $\mathbb{F}_p$ with relations $\{1_{\mathbb{F}_p}, u_1, ..., u_k\}$ defined by

$$\begin{aligned}1_{\mathbb{F}_p} &= \{(x,y)\in\mathbb{F}_p\times\mathbb{F}_p : y-x=0\}\\ u_i &= \{(x,y)\in\mathbb{F}_p\times\mathbb{F}_p : y-x\in\gamma^{i-1}Q_k\},\ 1\le i\le k\end{aligned}$$

Particularly, we will be looking at the scheme $U := Cyc(p, 2)$ of rank 3. Therefore, let $Q = Q_2$. Then, since $\frac{p-1}{2} \equiv 1 \bmod 2$ and $\gamma^{\frac{p-1}{2}} = -1$, we obtain $\gamma Q = \gamma^{\frac{p-1}{k}} Q = -Q$, so the elements of U

$$\begin{aligned}1_{\mathbb{F}_p} &= \{(\alpha,\beta)\in\mathbb{F}_p\times\mathbb{F}_p : \beta-\alpha=0\}\\ u &= \{(\alpha,\beta)\in\mathbb{F}_p\times\mathbb{F}_p : \beta-\alpha\in Q\}\\ u^* &= \{(\alpha,\beta)\in\mathbb{F}_p\times\mathbb{F}_p : \beta-\alpha\in -Q\}\end{aligned}$$

Proposition 6.4 *We have*

$$\sigma_u\sigma_{u^*} = \tfrac{p-1}{2}\sigma_{1_{\mathbb{F}_p}} + \tfrac{p-3}{4}\sigma_u + \tfrac{p-3}{4}\sigma_{u^*}.$$

Proof: For any $\alpha, \beta \in \mathbb{F}_p$, $(\sigma_u\sigma_{u^*})_{\alpha,\beta}$ is equal to the number of elements $\gamma \in \mathbb{F}_p$, with $\alpha - \gamma \in Q$ and $\gamma - \beta \in -Q$. This is equal to one fourth of the total number of pairs $r, s \in \mathbb{F}_p^\times$ such that $r^2 - s^2 = \alpha - \beta$. Indeed, given any γ such that $\alpha - \gamma \in Q$

and $\gamma - \beta \in -Q$, we can find two elements r such that $\alpha - \gamma = r^2$ and two elements s such that $\beta - \gamma = s^2$, and conversely, given a pair of elements $r, s \in \mathbb{F}_p^\times$ such that $\alpha - \beta = r^2 - s^2$, we can define $\gamma := \alpha - r^2 = \beta - s^2$.

For $t \in \mathbb{F}_p$, let E_t denote the set of pairs $r, s \in \mathbb{F}_p^\times$ such that $r^2 - s^2 = t$. We can easily see that $|E_0| = 2(p-1)$. On the other hand, we also obtain that for an $t \in \mathbb{F}_p^\times$, there is a bijection between E_t and E_{-t} taking (r, s) to (s, r). For any $q \in Q$, there is also a bijection between E_t and E_{qt} taking (r, s) to (mr, ms), where is an element in $\mathbb{F}_p^\times$, with $m^2 = q$. Therefore, $|E_t|$ is the same for all $t \in \mathbb{F}_p^\times$. Since we have $(p-1)^2$ pairs of elements $r, s \in \mathbb{F}_p^\times$, we obtain

$$(p-1)^2 = \sum_{t \in \mathbb{F}_p} |E_t| = |E_0| + (p-1)|E_1| = 2(p-1) + (p-1)|E_1|.$$

Therefore, $|E_t| = p - 3$ for each $t \in \mathbb{F}_p$. Thus,

$$(\sigma_u \sigma_{u^*})_{\alpha,\beta} = \begin{cases} \frac{p-1}{2} & : \alpha = \beta \\ \frac{p-3}{4} & : \alpha \neq \beta \end{cases}$$

and with this our result follows. □

Corollary 6.5 *We have*

$$\begin{aligned} \sigma_{u^*}\sigma_u &= \tfrac{p-1}{2}\sigma_{1_{\mathbb{F}_p}} + \tfrac{p-3}{4}\sigma_u + \tfrac{p-3}{4}\sigma_{u^*} \\ \sigma_u\sigma_u &= \tfrac{p-3}{4}\sigma_u + \tfrac{p+1}{4}\sigma_{u^*} \\ \sigma_{u^*}\sigma_{u^*} &= \tfrac{p+1}{4}\sigma_u + \tfrac{p-3}{4}\sigma_{u^*} \end{aligned}$$

Proof: Let J be the matrix indexed on $\mathbb{F}_p$ with all entries equal to 1. Therefore $\sigma_{1_{\mathbb{F}_p}} + \sigma_u + \sigma_{u^*} = J$. Clearly

$$J\sigma_u = \sigma_u J = \tfrac{p-1}{2} J = J\sigma_{u^*} = \sigma_{u^*} J.$$

Therefore, σ_u and σ_{u^*} commute, so our first equation follows Proposition 6.4. Our second equation is now obtained by rewriting the expression on the left as $\sigma_u(J - \sigma_{1_{\mathbb{F}_p}} - \sigma_{u^*})$ and using proposition 6.4 as well. And our third equation is most easily obtained by taking the transpose of both sides on the second equation. □

It is interesting to investigate the automorphism group of the scheme U. The following is useful result.

Proposition 6.6 *A map $\phi : \mathbb{F}_p \to \mathbb{F}_p$ is an automorphism of the scheme U on $\mathbb{F}_p$ if and only if there exists an elements $a \in \mathbb{F}_p^\times$ and $b \in \mathbb{F}_p$ such that for each $x \in \mathbb{F}_p$, $\phi(x) = ax + b$. The map ϕ is strict if and only if $a \in Q$.*

Proof: Assume there exist two elements $a \in \mathbb{F}_p^\times$ and $b \in \mathbb{F}_p$, such that for each $x \in \mathbb{F}_p$, $\phi(x) = ax + b$. We can clearly see that ϕ is a bijection. For any two elements $(\alpha, \beta) \in \mathbb{F}_p \times \mathbb{F}_p$, we obtain

$$\phi(\beta) - \phi(\alpha) = a(\beta - \alpha).$$

If $a \in Q$, the elements $a(\beta - \alpha)$ and $(\beta - \alpha)$ belong to the same subset $\{0\}$, Q, or $-Q$. Therefore, (ϕ, id_U) is an automorphism of U, so ϕ is strict automorphism.

If we have $a \in -Q$, then $a(\beta - \alpha) \in -Q$ if $\beta - \alpha \in -Q$ and $a(\beta - \alpha) \in -Q$ if $\beta - \alpha \in Q$. And clearly, $a(\beta - \alpha) = 0$ if and only if $\beta - \alpha = 0$. So we have

$$(\alpha, \beta) \in \begin{cases} 1_{\mathbb{F}_p} \\ u \\ u^* \end{cases} \text{if and only if } (\phi(\alpha), \phi(\beta)) \in \begin{cases} 1_{\mathbb{F}_p} \\ u \\ u^* \end{cases}$$

Therefore, by this ϕ is an automorphism of U which is not strict.

Now, let's assume that ϕ is a strict automorphism. So whenever we have $\alpha - \beta \in Q$, we obtain $\phi(\alpha) - \phi(\beta) \in Q$. Since ϕ is a permutation of $\mathbb{F}_p$ and Q is a non-empty proper subset of $\mathbb{F}_p^\times$, it follows from [6; Proposition 1] that there exist $a, b \in \mathbb{F}_p$ such that $\phi(x) = ax + b$ for all $x \in \mathbb{F}_p$. Now, since we have $(0, 1) \in u$ and ϕ is a strict automorphism, we must have $(\phi(0), \phi(1)) \in u$, in other words

$$a = (a + b) - (b) = \phi(1) - \phi(0) \in Q$$

From our first paragraph, we can see that the map $\psi : \mathbb{F}_p \to \mathbb{F}_p$, which is given by $\psi(x) = -x$, is an automorphism which is not strict. If ϕ is an automorphism which is not strict, then the associated map $\omega : U \to U$ must take u to u^* and u^* to u. Therefore the composition $\phi \circ \omega$ is a strict automorphism. From the previous paragraph we can see that there are elements $a \in Q$ and $b \in \mathbb{F}_p$ such that for each $x \in \mathbb{F}_p$, we obtain

$$\phi(-x) = (\phi \circ \omega)(x) = ax + b$$

And from this, it follows that $\phi(x) = -ax+b$, for every $x \in \mathbb{F}_p$. From $-a \in -Q \subseteq \mathbb{F}_p$ and $b \in \mathbb{F}_p$, we obtain our result. □

Corollary 6.7 *Every automorphism of U is either the identity, has exactly one fixed point, or has no fixed points. An automorphism which has no fixed points has the form $x \to x + b$ for some $b \in \mathbb{F}_p$ such that $b \neq 0$. Particularly, every automorphism that is not strict, has exactly one fixed point.*

Proof: Let ψ be an automorphism of U. Then by Proposition 6.6, there exist elements $a \in \mathbb{F}_p^\times$ and $b \in \mathbb{F}_p$ such that $\psi(x) = ax + b$ for all $x \in \mathbb{F}$. We have $ax + b = x$ if and only if $(a - 1)x = -b$. Therefore, if $a \neq 1$, ψ has exactly one fixed point, namely $\frac{-b}{a-1}$. If $a = 1$ and $b = 0$, then ψ is the identity. If $a = 1$ and $b \neq 0$, then $\psi(x) = x + b$ has no fixed points.

If ψ is not strict, then $a \in -Q$, meaning $a = 1$, so the last result follows. □

Now, the notion of a complete p-array will be introduced. We will still assume that p is a prime.

A *complete p-array* is a $(p + 2) \times (p + 2)$ matrix M, with its rows and columns indexed by the set $\{0, 1, 2, ..., p+1\}$, and entries come from the set $\mathbb{F}_p \cup \{*\}$ satisfying the following three conditions:

(i) For any $i, j \in \{0, 1, 2, ..., p + 1\}$, we have $M_{i,j} = *$ if and only if $i = j$.

(ii) For any $i, j \in \{0, 1, 2, ..., p+1\}$, we have $M_{i,j} = M_{j,i}$.

(iii) For any $i, j \in \{0, 1, 2, ..., p+1\}$ with $i \neq j$, and any $t \in \mathbb{F}_p$, there is some $k \in \{0, 1, 2, ..., p+1\} \setminus \{i, j\}$ such that $M_{i,k} - M_{j,k} = t$.

The following is an example of complete 3-array:

$$\begin{pmatrix} * & 2 & 1 & 0 & 0 \\ 2 & * & 0 & 1 & 0 \\ 1 & 0 & * & 2 & 0 \\ 0 & 1 & 2 & * & 0 \\ 0 & 0 & 0 & 0 & * \end{pmatrix}$$

Let $\mathcal{F}_p$ denote the group of functions from $\{0, 1, 2, ..., p+1\}$ to $\mathbb{F}_p$, under pointwise addition. Given $\gamma \in \mathbb{F}_p$, let $l_\gamma : \mathbb{F}_p \to \mathbb{F}_p$, defined as $l_\gamma(\alpha) = \gamma\alpha$ for $\alpha \in \mathbb{F}_p$. Let ϕ denote the homomorphism from $\mathbb{F}_p^\times \times \sum_{\{0,1,2,\ldots,p+1\}}$ to $\mathrm{Aut}(\mathcal{F})$ given by:

$$\phi(\gamma, \sigma) = l_\gamma \circ f \circ \sigma^{-1}$$

Let $\mathcal{G}_p$ denote the semidirect product $\mathcal{F}_p \times_\phi (\mathbb{F}_p^\times \times \sum_{\{0,1,2,\ldots,p+1\}})$. Therefore in $\mathcal{G}_p$, we have

$$(f_1, \gamma_1, \sigma_1)(f_2, \gamma_2, \sigma_2) = (f_1 + l_{\gamma_1} \circ f_2 \circ \sigma_1^{-1}, \gamma_1\gamma_2, \sigma_1 \circ \sigma_2)$$

Remark 6.8 The semidirect product $\mathcal{G}_p$ is a group.

Proof: For any two elements in $\mathcal{G}_p$, we can easily see that the product is back in $\mathcal{G}_p$.

For any three elements in $\mathcal{G}_p$, we obtain

$$\begin{aligned} & ((f_1, \gamma_1, \sigma_1)(f_2, \gamma_2, \sigma_2))(f_3, \gamma_3, \sigma_3) \\ = \; & (f_1 + l_{\gamma_1} \circ f_2 \circ \sigma_1^{-1}, \gamma_1\gamma_2, \sigma_1 \circ \sigma_2)(f_2, \gamma_2, \sigma_2) \\ = \; & ((f_1 + l_{\gamma_1} \circ f_2 \circ \sigma_1^{-1}) + l_{\gamma_1\gamma_2} \circ f_3 \circ (\sigma_1 \circ \sigma_2)^{-1}, (\gamma_1\gamma_2)\gamma_3, (\sigma_1 \circ \sigma_2) \circ \sigma_3) \\ = \; & (f_1 + l_{\gamma_1} \circ f_2 \circ \sigma_1^{-1} + l_{\gamma_1\gamma_2} \circ f_3 \circ \sigma_2^{-1} \circ \sigma_1^{-1}, \gamma_1\gamma_2\gamma_3, \sigma_1 \circ \sigma_2 \circ \sigma_3). \end{aligned}$$

and

$$\begin{aligned} & (f_1, \gamma_1, \sigma_1)((f_2, \gamma_2, \sigma_2)(f_3, \gamma_3, \sigma_3)) \\ = \; & (f_1, \gamma_1, \sigma_1)(f_2 + l_{\gamma_2} \circ f_3 \circ \sigma_2^{-1}, \gamma_2\gamma_3, \sigma_2 \circ \sigma_3) \\ = \; & (f_1 + l_{\gamma_1} \circ (f_2 + l_{\gamma_2} \circ f_3 \circ \sigma_2^{-1}) \circ \sigma_1^{-1}, \gamma_1(\gamma_2\gamma_3), \sigma_1(\sigma_2 \circ \sigma_3)) \\ = \; & (f_1 + l_{\gamma_1} \circ f_2 \circ \sigma_1^{-1} + l_{\gamma_1\gamma_2} \circ f_3 \circ \sigma_2^{-1} \circ \sigma_1^{-1}, \gamma_1\gamma_2\gamma_3, \sigma_1 \circ \sigma_2 \circ \sigma_3). \end{aligned}$$

Since $\mathcal{F}_p$ is the group of functions under pointwise addition, the multiplication in $\mathbb{F}_p$ is associative, and map composition is associative, we can see that associativity holds in $\mathcal{G}_p$.

We can easily see that the identity element in $\mathcal{G}_p$ is $(f_0, 1, id)$, where f_0 denotes the function that maps all elements in $\{0, 1, 2, ..., p+1\}$ to 0, and $id \in \sum_{\{0,1,2,\ldots,p+1\}}$ is the identity permutation.

Now, for any element $(f_1, \gamma_1, \sigma_1)$, we obtain its inverse as the element $(l_{\gamma_1}^{-1} \circ -f_1 \circ \sigma_1, \gamma_1^{-1}, \sigma_1^{-1})$, since we know that $l_{\gamma_1}^{-1} \circ -f_1 \circ \sigma_1 \in \mathcal{F}_p$, and for any $\gamma_1 \in \mathbb{F}_p$ and $\sigma_1 \in \sum_{\{0,1,2,\ldots,p+1\}}$ we have a corresponding inverse. Thus, $\mathcal{G}_p$ is a group. $\square$

We now obtain a nice result for complete p-arrays in relation to the semidirect product $\mathcal{G}_p$.

Lemma 6.9 *Suppose M is a complete p-array, and $(f, \gamma, \sigma) \in \mathcal{G}_p$. Then the matrix M' defined by*

$$M'_{i,j} = \begin{cases} \gamma M_{\sigma^{-1}(i),\sigma^{-1}(j)} + f(i) + f(j) : i \neq j \\ * : i = j \end{cases}$$

is a complete p-array.

Proof: Assume $i, j \in \{0, 1, 2, ..., p + 1\}$. If $i \neq j$, then $\sigma(i) \neq \sigma(j)$, meaning $M_{\sigma(i),\sigma(j)} \neq *$. Therefore the empty diagonal condition holds (condition (i)).

If $i \neq j$, we obtain $M'_{j,i} = \gamma M_{\sigma^{-1}(j),\sigma^{-1}(i)} + f(j) + f(i)$. Due to the symmetry of M and commutativity of addition, $M'_{j,i}$ is also equal to $\gamma M_{\sigma^{-1}(i),\sigma^{-1}(j)} + f(i) + f(j)$. Thus, the symmetry condition (condition (ii)) holds as well for M'.

Now, if $i \neq j$, then given $t \in \mathbb{F}_p$, let $i' = \sigma^{-1}(i)$, $j' = \sigma^{-1}(j)$, and $t' = \gamma^{-1}(t + f(i) - f(i))$. Then by row difference property (condition (iii)), there is a $k' \in \{0, 1, 2, ..., p + 1\} \setminus \{i, j\}$ such that $M_{i',k'} - M_{j',k'} = t'$. Let $k = \sigma(k')$. Then

$$M'_{i,k} - M'_{j,k} = \gamma(M_{i',k'} - M_{j',k'}) + f(i) - f(j) = \gamma t' + f(i) + f(k) - f(j) - f(k) = t$$

Therefore the row difference property holds for M'. $\square$

For the rest of this section, assume p is a prime number with $p \equiv 3 \bmod 4$.

Now, we will construct a scheme S_M from a complete p-array. For the remainder of this section, we assume that M is a complete p-array, and we let X denote the set $\{0, 1, 2, ...p + 1\} \times \mathbb{F}_p$. We let Q denote the non-zero quadratic residues in $\mathbb{F}_p^\times$. Clearly, Q is a subgroup of $\mathbb{F}_p^\times$ with index 2. Since $p \equiv 3 \bmod 4$, $-1 \notin Q$, so the coset of Q which does not contain the identity is the coset containing -1, which we can denote as $-Q$.

We will consider the following six relations on the set $X \times X$:

$$\begin{aligned}
1_X &= \{((a, \alpha), (b, \beta)) \in X \times X : a = b \ \text{ and } \ \alpha - \beta = 0\} \\
u &= \{((a, \alpha), (b, \beta)) \in X \times X : a = b \ \text{ and } \ \beta - \alpha \in Q\} \\
u' &= \{((a, \alpha), (b, \beta)) \in X \times X : a = b \ \text{ and } \ \beta - \alpha \in -Q\} \\
s &= \{((a, \alpha), (b, \beta)) \in X \times X : a \neq b \ \text{ and } \ \alpha + \beta - M_{a,b} = 0\} \\
m_1 &= \{((a, \alpha), (b, \beta)) \in X \times X : a \neq b \ \text{ and } \ \alpha + \beta - M_{a,b} \in Q\} \\
m_2 &= \{((a, \alpha), (b, \beta)) \in X \times X : a \neq b \ \text{ and } \ \alpha + \beta - M_{a,b} \in -Q\}
\end{aligned}$$

Let $S_M = \{1_x, u, u', s, m_1, m_2\}$ and let $U_M = \{1_X, u, u'\}$.

Lemma 6.10 *The subsets 1_X, u, u', s, m_1, and m_2 form a partition of $X \times X$.*

Proof: Since $\{0\}$, Q, and $-Q$ form a partition of $\mathbb{F}_p$, it is easy to see that the six given elements are mutually disjoint and the union of the sets is $X \times X$. □

Now, let's recall that for a subset s of $X \times X$, we have

$$s^* = \{(x, y) \in X \times X : (y, x) \in s\}$$

Lemma 6.11 *We have $u^* = u'$, $1_X^* = 1_X$, $s^* = s$, $m_1^* = m_1$, and $m_2^* = m_2$.*

Proof: If $((b, \beta), (a, \alpha)) \in u^*$, then $((a, \alpha), (b, \beta)) \in u$, which implies $a = b$ and $\beta - \alpha \in Q$. Then $b = a$ and $\alpha - \beta \in -Q$, which implies $((b, \beta), (a, \alpha)) \in u'$. With this, we show that $u^* \subseteq u'$. Since both sets have cardinality $\frac{p-1}{2}$,we have $u^* = u'$.
Now, assume $((b, \beta), (a, \alpha)) \in s^*$. Then $((a, \alpha), (b, \beta)) \in s$, which means $a \neq b$ and $\alpha + \beta = M_{a,b}$. Due to the symmetry condition of M, this implies that $b \neq a$ and $\beta - \alpha = M_{b,a}$, so $((b, \beta), (a, \alpha)) \in s$. Therefore, $s^* \subseteq s$, and since both sets have cardinality $p + 1$, we obtain $s^* = s$. We can apply similar justification to the other cases. □

We have observed that S_M is a partition of $X \times X$ and it satisfies the first two conditions of an association scheme. All that's left to show is our third condition towards the structure constants. We will use Lemma 6.3 to complete the proof that S_M is an association scheme on X.

Before going on to the main theorem for this section, we will need some results on the adjacency matrices for elements of S_M.

Corollary 6.12 *We have $\sigma_s\sigma_u = \sigma_{m_1}$ and $\sigma_s\sigma_{u^*} = \sigma_{m_2}$.*

Corollary 6.13 *We have $\sigma_u\sigma_s = \sigma_{m_2}$ and $\sigma_{u^*}\sigma_s = \sigma_{m_1}$.*

These results stem from [1; Section 4]. Now let's look at the main theorem for this section.

Theorem 6.14 *If M is a complete p-array, then the following hold:*

(i) S_M of subsets of $X \times X$ is a scheme.

(ii) S_M has rank 6.

(iii) S_M is non-commutative.

(iv) U_M is a non-symmetric normal closed subset of S_M.

Proof: (i) We know that S_M contains 1_X, and we see that S_M is a scheme due to Lemma 6.10, Lemma 6.11, Lemma 6.3, and [1, Corollary 4.13].

(ii) Undoubtedly, we can see S_M has rank 6.

(iii) Due to Remark 6.2 and Corollary 6.12, we obtain $a_{sum_1} = 1$, and $a_{usm_1} = 0$.

(iv) By Lemma 6.3 and [1; Corollary 4.12], U_M is a closed subset of S_M. Since $u \neq u^*$, U_M is non-symmetric. Since U_M is closed, we have $vU_M = U_M = U_M v$, for any $v \in U_M$. By Remark 6.2, Corollary 6.12, and Corollary 6.13, we have $su = u^*s = \{m_1\}$, and $su^* = us = \{m_2\}$. Therefore, for each $v \in S_M \setminus U_M$, we have $vU_M = \{s, m_1, m_2\} = U_m v$, for example, $m_1 U_M = suU_M = sU_M = \{s, m_1, m_2\}$ and $U_M m_1 = U_M u^* s = U_M s = \{s, m_1, m_2\}$. Therefore, U_M is normal. □

An interesting result is shown as well in [1; Theorem 4.16, Theorem 4.17], given two complete p-arrays M and M' in the same $\mathcal{G}_p$ orbit, it follows that S_M and $S_{M'}$ are isomorphic schemes and vice-versa.

7. References

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