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Existence of positive solutions to semilinear elliptic systems with supercritical growth

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Abstract

We establish the existence of positive entire solutions to cooperative systems of semilinear elliptic equations involving nonlinearities with critical and supercritical growth. Consequently, we obtain existence results to several well-known model examples such as systems of the Hénon, Lane-Emden and stationary Schrödinger types. The main technique for generating our results relies on a topological approach for the shooting method combined with non-existence results to closely related boundary value problems.

Keywords: Degree theory; entire solutions; existence theory; Hénon system; Lane-Emden system; Schrödinger system; shooting method.
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1 Introduction

Throughout the paper, assume that L and $N \geq 3$ are positive integers, $\Omega \subseteq \mathbb{R}^N$ is an open domain containing the origin, $\mathbb{R}^L_+$ denotes the convex cone $\{\mathbf{u} \in \mathbb{R}^L \,|\, u_1, u_2, \ldots, u_L \geq 0\}$, and

$$\mathbf{F} = (f_1, f_2, \ldots, f_L) : \Omega\backslash\{0\} \times \mathbb{R}^L_+ \longrightarrow \mathbb{R}^L \tag{1.1}$$

is a non-trivial, continuous vector-valued map with $\mathbf{F}(|x|, 0) \equiv 0$ in $\Omega\backslash\{0\}$. Consider the system of semilinear elliptic equations involving the Laplace operator: $-\Delta u_i = f_i(|x|, \mathbf{u})$ in Ω, for $i = 1, 2, \ldots, L$, which we write more concisely as

$$-\Delta \mathbf{u} = \mathbf{F}(|x|, \mathbf{u}) \text{ in } \Omega. \tag{1.2}$$

Our goal is to establish the existence of positive solutions in $\Omega = \mathbb{R}^N$ to cooperative systems of the form (1.2) where the nonlinearity $\mathbf{F}$ has supercritical (or critical) growth. Here we say $\mathbf{F}$ has *supercritical growth* if

$$\lambda^{-(N+2)/(N-2)}\mathbf{F}(\lambda^{-2/(N-2)}x, \lambda\mathbf{u})$$

is non-decreasing in $\lambda \geq 1$ for all $(x, \mathbf{u}) \in \Omega\backslash\{0\} \times \mathbb{R}_+^L$. We say that system (1.2) is *cooperative* if for $i = 1, 2, \ldots, L$, $(x, \mathbf{u})$, $(x, \mathbf{v}) \in \Omega\backslash\{0\} \times \mathbb{R}_+^L$ with $\mathbf{u} \leq \mathbf{v}$ and $u_i = v_i$ implies that $f_i(|x|, \mathbf{u}) \leq f_i(|x|, \mathbf{v})$. When referring to a positive solution $\mathbf{u} = \mathbf{u}(x)$ of system (1.2), it should always be understood to belong to $C^2(\Omega\backslash\{0\}) \cap C(\bar{\Omega})$ if Ω is bounded or $C^2(\Omega\backslash\{0\}) \cap C(\Omega)$ if $\Omega = \mathbb{R}^N$, and it satisfies the equations pointwise in $\Omega\backslash\{0\}$. The typical model here is perhaps the weighted elliptic system

$$\begin{cases} -\Delta u = |x|^{\sigma_1}(\lambda_1 u^{p_1} v^{p_2} + \lambda_2 v^{p_3}) \text{ in } \Omega, \\ -\Delta v = |x|^{\sigma_2}(\lambda_3 v^{q_1} u^{q_2} + \lambda_4 u^{q_3}) \text{ in } \Omega, \end{cases} \tag{1.3}$$

where $\sigma_1, \sigma_2 \in (-2, \infty)$, $\lambda_j \geq 0$, $p_i, q_i > 0$, $i = 1, 2, 3$ and $j = 1, 2, 3, 4$. Included in this model case are several familiar examples from conformal geometry and mathematical physics. For instance, the particular system

$$\begin{cases} -\Delta u = u^p v^q \text{ in } \Omega, \\ -\Delta v = v^p u^q \text{ in } \Omega, \end{cases} \tag{1.4}$$

is closely related to the Schrödinger systems for the Bose-Einstein condensate [15, 18, 19]. System (1.3) can also be reduced to the system

$$\begin{cases} -\Delta u = v^q \text{ in } \Omega, \\ -\Delta v = u^p \text{ in } \Omega, \end{cases} \tag{1.5}$$

which is often called the Lane-Emden system. Now, system (1.5) and its scalar version, i.e., when $p = q$ and $u \equiv v$, arise in a number of mathematical problems such as the Yamabe problem, on finding the best constant in the Sobolev inequality in $\mathbb{R}^N$ and other geometric problems [5, 6, 13], and they are also connected with the sharp Hardy-Littlewood-Sobolev inequality [17]. Another special case of (1.3) that has received recent attention is the Hénon-Lane-Emden system [11, 25, 33],

$$\begin{cases} -\Delta u = |x|^{\sigma_1} v^q \text{ in } \Omega, \\ -\Delta v = |x|^{\sigma_2} u^p \text{ in } \Omega. \end{cases} \tag{1.6}$$

These well-known models, especially the Lane-Emden system, have been examined extensively in the past several decades. Of particular interest to many are the necessary and sufficient conditions for the non-existence of positive entire solutions. One obvious motivation for this is the fact that such a Liouville type theorem is an important ingredient in establishing a priori bounds on positive solutions to second-order elliptic problems with Dirichlet data (cf. [10] and [12]). Specifically, the following question remains an open problem although an affirmative answer is known for dimension $N \leq 4$ [3, 29, 31] and for radial solutions in any dimension $N \geq 3$ [22]. Namely, it is conjectured that system (1.5) has no positive solution in $\Omega = \mathbb{R}^N$ if and only if (p, q) satisfies the subcritical condition

$$\frac{N}{1+q} + \frac{N}{1+p} > N - 2.$$

An analogous conjecture with similar partial results holds for the Hénon-Lane-Emden system [11, 25]. Interestingly enough, our method covers the Lane-Emden and Hénon-Lane-Emden systems thus verifying this subcritical condition is indeed necessary for the nonexistence of positive solutions in any dimension $N \geq 3$. However, this particular existence result was already known, at least for the Lane-Emden system [30]. Our method also applies to higher-order poly-harmonic systems. For example, given an integer $m \in [1, N/2)$, $p, q > 0$, $\sigma_1, \sigma_2 > -2$, and if

$$\frac{N+\sigma_1}{1+q} + \frac{N+\sigma_2}{1+p} \leq N - 2m,$$

then the Hardy-Littlewood-Sobolev type system,

$$\begin{cases} (-\Delta)^m u = |x|^{\sigma_1} v^q \ \text{ in } \Omega, \\ (-\Delta)^m v = |x|^{\sigma_2} u^p \ \text{ in } \Omega, \end{cases}$$

admits a positive solution in $\Omega = \mathbb{R}^N$ [20, 32].

Nevertheless, in this paper we obtain existence results to a general class of systems with power nonlinearities. Namely, if we take $\mathbf{p}, \mathbf{q}, \mathbf{u} \in \mathbb{R}^L_+$ and write

$$|\mathbf{p}| = p_1 + p_2 + \cdots + p_L \ \text{ and } \ \mathbf{u}^{\mathbf{p}} = u_1^{p_1} u_2^{p_2} \cdots u_L^{p_L},$$

then we may consider nonlinearities of the form

$$\begin{cases} f_i(|x|, \mathbf{u}) = |x|^\sigma (\mathbf{u}^{\mathbf{p^i}} + u_{i+1}^{q_i}), \ i = 1, 2, \ldots, L-1, \\ f_L(|x|, \mathbf{u}) = |x|^\sigma (\mathbf{u}^{\mathbf{p^L}} + u_1^{q_L}) \end{cases} \tag{1.7}$$

or

$$f_i(|x|, \mathbf{u}) = |x|^\sigma (u_1^{p_1^i} + u_2^{p_2^i} + \ldots + u_L^{p_L^i}), \ i = 1, 2, \ldots, L. \tag{1.8}$$

Theorem 1. *Let $\Omega = \mathbb{R}^N$ and $\sigma \in (-2, \infty)$. Then there hold the following.*

(a) Let $\mathbf{F}$ be defined as in (1.7), and suppose $\mathbf{p^i} \geq (1, 1, \ldots, 1)$ and

$$|\mathbf{p^i}|, q_i \ \geq \frac{N+2+2\sigma}{N-2}, \ \text{ for } i = 1, 2, \ldots, L.$$

Then system (1.2) admits a positive entire solution.

(b) Let $\mathbf{F}$ *be defined as in* (1.8) *and suppose*

$$p_j^i \ \geq \frac{N+2+2\sigma}{N-2} \ \text{ for } \ i,j = 1,2,\ldots,L.$$

Then system (1.2) *admits a positive entire solution.*

In fact, we establish an even more general existence result than Theorem 1 assuming only that $\mathbf{F}$ is non-negative, cooperative, supercritical and satisfies certain non-degeneracy conditions (to be specified shortly below).

1.1 Preliminaries and the Main Result

We start with some notation that will be used throughout the rest of the paper. Let $B_R(x) \subset \mathbb{R}^N$ denote the usual open ball with radius $R > 0$, center x, and boundary $\partial B_R(x)$. For simplicity, we sometimes use B_R in place of $B_R(0)$. We denote by $\partial\mathbb{R}^L_+$ the boundary of the convex cone $\mathbb{R}^L_+$, and we write $\mathbf{u} > 0$ ($= 0$, respectively) in $\mathbb{R}^L_+$ to mean $u_1, u_2, \ldots, u_L > 0$ ($= 0$, respectively).

Now, for any fixed non-zero $\alpha \in \partial\mathbb{R}^L_+$, we may assign a permutation, which we denote by $\Sigma(\alpha) = \{i_1, i_2, \ldots, i_j, \ldots, i_L\}$, of the set $\{1, 2, \ldots, j, \ldots, L\}$ such that

$$\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_j} = 0 \ \text{ and } \ \alpha_{i_{j+1}}, \alpha_{i_{j+2}}, \ldots, \alpha_L > 0.$$

In addition to the previous assumptions we placed on our systems, e.g., supercritical growth, Lipschitz continuous in $\mathbf{u}$, etc., we will further impose the following non-degeneracy conditions.

Condition A: The map $\mathbf{F}$ satisfy the following criteria:

(a) For a given positive constant c_1 there exist constants $c_2 > 0$ and $\sigma \in (-2, \infty)$ such that for $|\mathbf{u}| \leq c_1$,

$$|\mathbf{F}(x, \mathbf{u})| \leq c_2 |x|^\sigma \ \text{ in } \ \Omega\backslash\{0\}.$$

(b) For any non-zero $\overline{\alpha} \in \partial\mathbb{R}^L_+$, there exist $C(\overline{\alpha}) > 0$ and $\delta = \delta(\overline{\alpha}) > 0$ such that

$$\sum_{k=j+1}^{L} f_{i_k}(|x|, \mathbf{u}) \leq \frac{C(\overline{\alpha})}{L} \sum_{k=1}^{j} f_{i_k}(|x|, \mathbf{u}), \quad x \in \Omega\backslash\{0\}, \tag{1.9}$$

whenever $|\overline{\alpha} - \mathbf{u}| < \delta$ (note that the summation above is taken with respect to some $\Sigma(\overline{\alpha})$ as defined earlier).

Our main result is the following.

Theorem 2. *Let $\Omega = \mathbb{R}^N$ and suppose $\mathbf{F}$ is non-negative, locally Lipschitz in $\mathbf{u}$, has supercritical growth, and satisfies Condition $\mathbf{A}$. Then system* (1.2) *admits a positive entire solution $\mathbf{u}(x)$.*

Remark 1. *Recall a map $\mathbf{F}$ defined on $\Omega_1 \times \mathbb{R}^L_+$ is said to be locally Lipschitz continuous in $\mathbf{u}$ if for $\mathbf{v} \in \mathbb{R}^L_+$ and bounded $\Omega_2 \subset \Omega_1$, there exists an open neighborhood $\mathcal{N} \subset \mathbb{R}^L_+$ of $\mathbf{v}$ such that $\mathbf{F}$ is Lipschitz continuous in $\mathbf{u}$ on $\Omega_2 \times \mathcal{N}$.*

2 Proof of Theorem 2

To better communicate our ideas and for the reader's convenience, let us attempt to clarify the precise roles the conditions on $\mathbf{F}$ will play in the proof of Theorem 2. Our proof will invoke a degree theoretic approach for the shooting method and a non-existence result for the Dirichlet problem to (1.2) where $\Omega = B_R$ is any ball domain. Our method can be outlined as follows.

STEP 1. Reformulate the elliptic system in radial coordinates and set up an initial value problem with the assumption that no global solution exists for any positive initial condition.

STEP 2. Construct the target map, which aims the shooting method, i.e., it maps initial values (initial shooting positions) to target values related to the solution of the initial value problem.

STEP 3. Show that Condition $\mathbf{A}$ ensures the target map is continuous so that we may apply a standard topological degree argument to obtain a non-trivial zero of the target map.

STEP 4. The zeros of the target map correspond to positive solutions to a closely related Dirichlet problem, but then we arrive at a contradiction with a known non-existence result for that boundary value problem.

In this final section, we first define the target map then go over several lemmas we require in our proof of the main result.

2.1 The Target Map

For any initial value $\alpha > 0$, set $r = |x|$ and consider the initial value problem

$$\begin{cases} -\left(\mathbf{u}''(r) + \dfrac{N-1}{r}\mathbf{u}'(r)\right) = \mathbf{F}(r, \mathbf{u}(r)), \\ \mathbf{u}'(0) = 0, \ \mathbf{u}(0) = \alpha, \end{cases} \tag{2.1}$$

where $'$ denotes d/dr. Indeed, positive solutions to this problem, which we denote by $\mathbf{u}(r, \alpha)$, exhibit an important monotonicity property. The result is elementary so we omit its proof.

Lemma 1. *Let $\mathbf{u}(r, \alpha) : [0, r_{max}) \subseteq \mathbb{R}_+ \longrightarrow \mathbb{R}_+^L$ be the unique positive solution of (2.1) where $[0, r_{\max})$ is the maximal interval of existence. Then there holds*

$$\mathbf{u}'(r, \alpha) \leq 0 \ \textit{in} \ (0, r_{max}).$$

Obviously, if we can find an initial condition $\alpha > 0$ for which the associated positive solution $\mathbf{u}(r, \alpha)$ is global with $r_{max} = \infty$, then this would produce a desired positive solution to system (1.2) thereby completing the proof of Theorem 2. Otherwise, assuming that (2.1) admits no global positive solution for any $\alpha > 0$, we may then define the following mapping.

Definition 1. *Define the target map $\psi : \mathbb{R}_+^L \longrightarrow \partial\mathbb{R}_+^L$ as follows.*

(a) For $\alpha > 0$, set $\psi(\alpha) = \mathbf{u}(r_0, \alpha)$ where $r_0 = r_0(\alpha)$ is the smallest finite value of r for which

$$u_{i_0}(r_0, \alpha) = 0 \ \textit{for some} \ 1 \leq i_0 \leq L.$$

(b) Set $\psi \equiv Identity$ on $\partial\mathbb{R}_+^L$.

Remark 2. *As illustrated in Figure 1, this definition makes sense because of the positivity of solutions, the monotonicity property of Lemma 1, and our assumption that no global solution exists for any initial condition, i.e., the trajectory must eventually touch the wall at some finite time. In particular, we shall refer to (a) in the definition as the case when the solution touches the wall.*

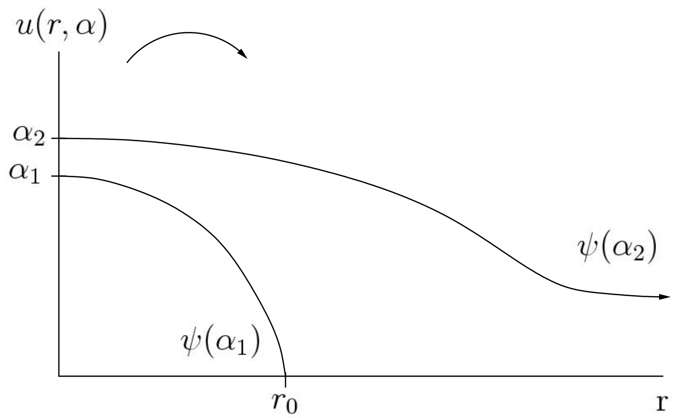

Figure 1: The target map $\psi : \mathbb{R}^L_+ \longrightarrow \partial\mathbb{R}^L_+$.

2.2 Preparations

Prior to our proof of Theorem 2, we prepare several results required in its proof. First, we recall a basic property from Brouwer degree theory, and the reader is referred to [1] and [24] for more details.

Lemma 2. *Let $U \subset \mathbb{R}^L$ be a bounded open set and let $\mathbf{G}, \mathbf{H} : \bar{U} \mapsto \mathbb{R}^L$ be continuous maps. Suppose that $\mathbf{G} \equiv \mathbf{H}$ on ∂U and $a \notin \mathbf{G}(\partial U) = \mathbf{H}(\partial U)$. Then*

$$degree(\mathbf{G}, U, a) = degree(\mathbf{H}, U, a).$$

Lemma 3. *The target map $\psi : \mathbb{R}^L_+ \longrightarrow \partial\mathbb{R}^L_+$ is continuous.*

We defer the proof of this until the end, however, we state and prove here a simple result that we will use in our proof of Lemma 3.

Lemma 4. *For each given initial value $\alpha > 0$, the unique positive solution $\mathbf{u}(r,\alpha)$ of (2.1) satisfies the integral representation*

$$\alpha - \mathbf{u}(r,\alpha) = \int_0^r \frac{1}{\tau^{n-1}} \int_0^\tau s^{n-1}\mathbf{F}(s, \mathbf{u}(s,\alpha))\, ds\, d\tau =: \mathbf{G}(r,\alpha), \qquad (2.2)$$

for $0 \le r \le r_0(\alpha)$.

Proof. Indeed, from the system of ordinary differential equations in (2.1), it follows that

$$-(r^{n-1}\mathbf{u}'(r,\alpha))' = r^{n-1}\mathbf{F}(r, \mathbf{u}(r,\alpha)).$$

By using the given initial conditions and carefully noting that part (a) of Condition **A** justifies the integration of the above twice, we can easily arrive at identity (2.2). $\square$

Lemma 5. *For every $a > 0$, there exists an $\alpha_a > 0$ in A_a where*

$$A_a := \left\{ \alpha \in int(\mathbb{R}^L_+) \mid \sum_{i=1}^{L} \alpha_i = a \right\}$$

such that $\psi(\alpha_a) = 0$.

Proof. Define B_a to be the set

$$B_a := \left\{ \alpha \in \partial\mathbb{R}^L_+ \mid \sum_{i=1}^{L} \alpha_i \leq a \right\}.$$

It is clear that ψ maps A_a into B_a.
Now define the homeomorphism $\varphi : B_a \longrightarrow A_a$ by

$$\varphi(\alpha) = \alpha + \frac{1}{L}\left(a - \sum_{i=1}^{L} \alpha_i \right)(1, 1, \cdots, 1)$$

with continuous inverse $\varphi^{-1} : A_a \longrightarrow B_a$ defined

$$\varphi^{-1}(\alpha) = \alpha - \left(\min_{i=1,\cdots,L} \alpha_i \right)(1, 1, \cdots, 1).$$

Set $\eta = \varphi \circ \psi : A_a \longrightarrow A_a$. Then η is continuous on A_a by Lemma 3 and is equivalent to the identity map on the boundary of A_a. By Lemma 2, the index of the map satisfies $degree(\eta, A_a, \alpha) = degree(Identity, A_a, \alpha) = 1$ for any interior point α in A_a. This implies that η is onto, and thus ψ is onto. Hence, there exists an $\alpha_a > 0$ in A_a such that $\psi(\alpha_a) = 0$. $\square$

As emphasized earlier, we shall make use of a certain type of non-existence result; namely, we require the following, which is a special case of a general non-existence result in [28] (Theorem 1 in that paper).

Lemma 6. *Consider the boundary value problem*

$$\begin{cases} -\Delta \mathbf{u} = \mathbf{F}(|x|, \mathbf{u}) & \text{in } B_R, \\ \mathbf{u} = 0 & \text{on } \partial B_R, \end{cases} \tag{2.3}$$

where **F** *is cooperative, Lipschitz continuous in* **u**, *and has supercritical growth. Then system* (2.3) *has no positive solution for any $R > 0$.*

2.3 Proof of Theorem 2

Fix a number $a > 0$ and consider the initial value problem (2.1). We proceed by contradiction and assume that there does not exist any $\alpha > 0$ for which a global solution of (2.1) exists. In view of Lemma 5, we know there is some $\alpha_a > 0$ such that $\psi(\alpha_a) = 0$, and thus we denote by $\mathbf{w} = \mathbf{w}(r, \alpha_a)$ the unique positive solution of (2.1) with initial value α_a. So by definition of the target map, there exists a smallest positive number $r_0 = r_0(\alpha_a)$ such that $w_{i_0}(r_0, \alpha_a) = 0$ for some $i_0 \in \{1, 2, \ldots, L\}$. Indeed, this implies that $\mathbf{w}(r_0, \alpha) = \psi(\alpha_a) = 0$. Then there holds that $\mathbf{u}(x) := \mathbf{w}(|x|, \alpha)$ is a positive solution of (2.3) with $R = r_0$, but this contradicts with Lemma 6. This completes the proof of the theorem. $\square$

2.4 Proof of Lemma 3

Choose any $\overline{\alpha} \in \mathbb{R}_+^L$ and we show ψ is continuous at $\overline{\alpha}$. To do so, there are two cases to consider. Case (1): When $\overline{\alpha} > 0$ and the corresponding solution $\mathbf{u}(r, \overline{\alpha})$ touches the wall. Case (2): When $\overline{\alpha} \in \partial\mathbb{R}_+^L$.

Case (1): This case is simple. Namely, since the nonlinearity $\mathbf{F}$ is non-negative, $u'_{i_0}(r_0, \overline{\alpha}) < 0$ by basic calculations or simply by Hopf's Lemma. This transversality condition along with ODE stability imply that for α sufficiently close to $\overline{\alpha}$, the solution to the perturbed initial value problem with initial condition α must be close to $\psi(\overline{\alpha})$. This proves that ψ is continuous at $\overline{\alpha}$ for this case.

Case (2): The continuity of ψ at $\overline{\alpha} = 0$ follows easily from Lemma 1 since $|\psi(\alpha) - \psi(\overline{\alpha})| = |\psi(\alpha)| \leq |\alpha| \longrightarrow 0$ as $\alpha \longrightarrow \overline{\alpha}$. Therefore, we can assume $\overline{\alpha} \in \partial\mathbb{R}_+^L$ is non-zero. In fact, Condition **A** ensures that we may assume

$$\overline{\alpha}_1 = \overline{\alpha}_2 = \cdots = \overline{\alpha}_j = 0 \;\text{ and }\; \overline{\alpha}_{j+1}, \ldots, \overline{\alpha}_L > 0, \;\text{ for some }\; 1 \leq j < L.$$

Of course, the first j-components of ψ are continuous at $\overline{\alpha}$ since

$$|\psi_k(\alpha) - \psi_k(\overline{\alpha})| \leq |\alpha_k| \longrightarrow 0 \;\text{ as }\; |\alpha - \overline{\alpha}| \longrightarrow 0 \quad \text{for }\; k = 1, 2, \ldots, j.$$

However, we must verify this holds for all the solution components, and to circumvent this issue we will make use of Condition **A**. Particularly, we first prove

Lemma 7. *There exists a suitably small $\delta_1 \in (0, \delta)$ such that for any $\delta_2 \in (0, \delta_1)$*

$$|\overline{\alpha} - \mathbf{u}(r, \alpha)| < 2C(\overline{\alpha})\delta_2 \;\textit{ for }\; 0 \leq r \leq r_0(\alpha) \;\textit{ whenever }\; |\alpha - \overline{\alpha}| < \delta_2, \quad (2.4)$$

where $C(\overline{\alpha})$ is the same constant found in Condition **A**.

If we momentarily assume this lemma holds, then it is clear we can then set $r = r_0(\alpha)$ in (2.4) to get

$$|\psi(\overline{\alpha}) - \psi(\alpha)| = |\overline{\alpha} - \psi(\alpha)| \longrightarrow 0 \quad \text{as } \alpha \longrightarrow \overline{\alpha},$$

and this would complete the proof. Therefore, it only remains to prove the last lemma.

Proof of Lemma 7 Assume the contrary; that is, for some suitably small $\delta_2 > 0$, there exist $\beta > 0$ with $|\beta - \overline{\alpha}| < \delta_2$ and $R \in (0, r_0(\beta))$ for which

$$|\overline{\alpha} - \mathbf{u}(r,\beta)| < 2C(\overline{\alpha})\delta_2 \ \text{ in } \ (0,R) \ \text{ but } \ |\overline{\alpha} - \mathbf{u}(R,\beta)| = 2C(\overline{\alpha})\delta_2.$$

Without loss of generality, we assume $C(\overline{\alpha}) > 2(L+1)$ and $2C(\overline{\alpha})\delta_2 < \delta$. Then, for $0 < r < R$,

$$\begin{aligned} |\overline{\alpha} - \mathbf{u}(r,\beta)| &\le |\overline{\alpha} - \beta| + |\beta - \mathbf{u}(r,\beta)| \\ &\le \delta_2 + \sum_{k=1}^{j} |\beta_k - u_k(r,\beta)| + \sum_{k=j+1}^{L} |\beta_k - u_k(r,\beta)|. \end{aligned} \tag{2.5}$$

From Lemma 4 and part (b) of Condition **A**, we get

$$\begin{aligned} \sum_{k=j+1}^{L} |\beta_k - u_k(r,\beta)| &= \sum_{k=j+1}^{L} G_k(r,\beta) \\ &\le \frac{C(\overline{\alpha})}{L} \sum_{k=1}^{j} G_k(r,\beta) \\ &= \frac{C(\overline{\alpha})}{L} \sum_{k=1}^{j} |\beta_k - u_k(r,\beta)|. \end{aligned}$$

Inserting this estimate into (2.5) yields

$$\begin{aligned} |\overline{\alpha} - \mathbf{u}(r,\beta)| &< \delta_2 + \sum_{k=1}^{j} |\beta_k - u_k(r,\beta)| + \frac{C(\overline{\alpha})}{L} \sum_{k=1}^{j} |\beta_k - u_k(r,\beta)| \\ &\le \delta_2 + \left(1 + \frac{C(\overline{\alpha})}{L}\right) \sum_{k=1}^{j} |\beta_k| \le (1 + L + C(\overline{\alpha}))\delta_2 \\ &< \frac{3C(\overline{\alpha})}{2}\delta_2. \end{aligned}$$

We arrive at a contradiction once we send $r \longrightarrow R$ in the last estimate. This proves the lemma and thus completes the proof of the continuity of the target map.

$\square$

2.5 Further Applications

We conclude this paper with some brief remarks on simple extensions of our earlier results. First, notice carefully that Lemma 6 does not require $\mathbf{F}$ to be non-negative. Therefore, the assumption that $\mathbf{F} \geq 0$ in Theorem 2 can be lifted so long as the continuity of the target map holds. Fortunately, the continuity of the map persists provided that Condition **A** is slightly modified. More precisely, we replace (1.9) in Condition **A** part (b) with

$$\sum_{k=j+1}^{L} |f_{i_k}(|x|, \mathbf{u})| \leq \frac{C(\overline{\alpha})}{L} \sum_{k=1}^{j} f_{i_k}(|x|, \mathbf{u}), \quad x \in \Omega \backslash \{0\},$$

and assume additionally that

$$\sum_{i=1}^{L} f_i(|x|, \mathbf{u}) \geq 0 \text{ in } \Omega \backslash \{0\} \times \mathbb{R}_+^L.$$

The proof of the continuity in this setting follows a similar process as our proof above but with some obvious adjustments. It more or less follows the arguments developed recently in [9], and so we refer the reader to that paper for the details. Nevertheless, the conclusion of Theorem 2 remains true under these changes.

Remark 3. *Note carefully that the continuity of the target map remains true without imposing that* $\mathbf{F}$ *is cooperative and has supercritical growth as their requirement was only because of Lemma 6.*

Although the continuity of the target map holds under the modified conditions, Lemma 6 does not appear general enough to include some interesting examples with nonlinearities containing weighted coefficients and sign-changing components. For instance, consider the weighted system with sign-changing sources:

$$\begin{cases} -\Delta u = |x|^\sigma (v^p + \lambda_1 v^q - \lambda_2 u^p) & \text{in } \Omega, \\ -\Delta v = \lambda_2 |x|^\sigma u^p & \text{in } \Omega, \end{cases} \tag{2.6}$$

where $\lambda_1, \lambda_2 > 0$,

$$\sigma \in (-2, \infty) \;\text{ and }\; p,\, q \geq \frac{N+2+2\sigma}{N-2}. \tag{2.7}$$

Indeed, $\mathbf{F}$ in this cooperative system is continuous and locally Lipschitz continuous in $\mathbf{u}$ in the proper region, satisfies the modified version of Condition $\mathbf{A}$, and the sum of its sources is non-negative thereby ensuring the continuity of the target map. On the other hand, a Pohozaev type identity (similar to those derived in [9]) can be established. The calculations are standard and so we only sketch the main steps (the reader is referred to [9, 21, 27] for detailed steps in deriving similar identities). In particular, if $\Omega = B_R$ and (u, v) is a positive solution of class $C^2(\Omega\backslash\{0\}) \cap C^1(\bar{\Omega})$ to (2.6) satisfying (2.7) with zero Dirichlet data, then elementary calculations yield the Pohozaev type identity

$$\begin{aligned}
-R\int_{\partial B_R} \nabla u \cdot \nabla v \, dS &+ (2-N)\int_{B_R} \nabla u \cdot \nabla v \, dx \\
&= \frac{R}{2}\int_{\partial B_R} |\nabla v|^2 \, dS - \frac{N+\sigma}{1+p}\int_{B_R} |x|^\sigma v^{p+1} \, dx - \frac{N+\sigma}{1+q}\int_{B_R} \lambda_1 |x|^\sigma v^{q+1} \, dx \\
&+ \frac{N-2}{2}\int_{B_R} |\nabla v|^2 \, dx - \frac{N+\sigma}{1+p}\int_{B_R} \lambda_2 |x|^\sigma u^{p+1} \, dx.
\end{aligned}$$

Since (u, v) is a solution, we can use this fact to further simplify the previous identity to get

$$\begin{aligned}
\Big(\frac{N+\sigma}{1+p} - \frac{N-2}{2}\Big)\int_{B_R} \Big(|x|^\sigma v^{p+1} + \lambda_2 |x|^\sigma u^{p+1}\Big) \, dx& \\
+\Big(\frac{N+\sigma}{1+q} - \frac{N-2}{2}\Big)\int_{B_R} \lambda_1 |x|^\sigma v^{q+1} \, dx& \\
= R\int_{\partial B_R} \Big(\nabla u \cdot \nabla v + \frac{1}{2}|\nabla v|^2\Big) \, dS.&
\end{aligned} \tag{2.8}$$

By noticing that the right-hand side of identity (2.8) is strictly positive, we arrive at a contradiction. Hence, system (2.6) with zero Dirichlet data satisfying (2.7) has no positive solution in $\Omega = B_R$ for any $R > 0$. As a result of our method, this is indeed enough to conclude that system (2.6) admits a positive solution in $\Omega = \mathbb{R}^N$.

Remark 4. *There are plenty of works in the literature dedicated to non-existence results to various elliptic problems and some examples can be found in [21, 22, 23, 26, 27, 28] just to list a few. It is therefore natural to seek* *other applications of our degree theoretic approach in view of these non-existence results. This can certainly be addressed in future investigations.*

Remark 5. *We should add that our results also include variants of system* (1.4) *with different power exponents. Although the existence of positive solutions for a more general version of system* (1.4) *was covered in [16], it is not difficult to adapt our ideas and methods here to recover and improve those results by allowing for weighted coefficients, for example.*

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