University of Texas Rio Grande Valley

ScholarWorks @ UTRGV

Theses and Dissertations

5-2016

Lie symmetry to nonlinear oscillator systems and applications

Xiaoyan Li
The University of Texas Rio Grande Valley

Follow this and additional works at: https://scholarworks.utrgv.edu/etd

Part of the Mathematics Commons

Recommended Citation

Li, Xiaoyan, "Lie symmetry to nonlinear oscillator systems and applications" (2016). *Theses and Dissertations*. 54.
https://scholarworks.utrgv.edu/etd/54

This Thesis is brought to you for free and open access by ScholarWorks @ UTRGV. It has been accepted for inclusion in Theses and Dissertations by an authorized administrator of ScholarWorks @ UTRGV. For more information, please contact justin.white@utrgv.edu, william.flores01@utrgv.edu.

LIE SYMMETRY TO NONLINEAR OSCILLATOR SYSTEMS AND APPLICATIONS

A Thesis

by

XIAOYAN LI

Submitted to the Graduate College of
The University of Texas Rio Grande Valley
In partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

May 2016

Major Subject: Mathematics

LIE SYMMETRY TO NONLINEAR OSCILLATOR SYSTEMS AND APPLICATIONS

A Thesis
by
XIAOYAN LI

COMMITTEE MEMBERS

Dr. Zhaosheng Feng
Chair of Committee

Dr. Tim Huber
Committee Member

Dr. Zhijun Qiao
Committee Member

Dr. Jasang Yoon
Committee Member

May 2016

Copyright 2016 Xiaoyan Li
All Rights Reserved

ABSTRACT

Li, Xiaoyan, Lie Symmetry to Nonlinear Oscillator Systems and Applications. Master of Science (MS), May, 2016, 68 pp., 41 references, 23 titles.

In this paper, we apply the theory of Lie symmetry to study a generalized second-order nonlinear differential equation, which includes several physical nonlinear oscillators such as force-free Helmholtz oscillator, force-free Duffing and Duffing-van der Pol oscillators, modified Emden-type equation and its hierarchy etc, and investigate the dynamical properties of this rather general equation. We identify and classify several new integrable cases for arbitrary values of exponents, which determine the tangent vector as well as the infinitesimal generator. Using the Lie point symmetry, we find the useful infinitesimal generators and canonical coordinates, and obtain the first integrals of the second-order nonlinear systems under certain parametric conditions. It shows that many classical integrable nonlinear oscillators can be derived as subcases of the obtained results and significantly enlarge the list of integrable equations that exists in the contemporary literature. Applications to wave equations of Korteweg-de Vries- Burgers-type equations are also presented.

DEDICATION

I dedicate my thesis work to my family and many friends. And I would like to extend my sincere gratitude to my supervisor, Dr. Zhaosheng Feng, for his instructive advice and useful suggestions on my thesis. I am deeply grateful of his help in the completion of this thesis.

ACKNOWLEDGMENTS

I would like to express my gratitude to all those who helped me during the writing of this thesis. My deepest gratitude goes first and foremost to Professor Zhaosheng Feng, my supervisor, for his constant encouragement and guidance. Without his consistent and illuminating instruction, this thesis could not have reached its present form. And I would like to thank all my committee members for their kind help and support. Then I am extremely grateful for School of Statistical and Mathematical Sciences to give me a job being a GTA. Besides, I am pleased to acknowledge my classmates for their invaluable assistance throughout the preparation of the original manuscript.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

Many phenomena in physics and engineering are described by nonlinear differential equations. Nonlinear differential equations have a wide array of applications in many fields. They could describe the motion of isolated waves, localized in a small part of space. Their applications could extend to magnetofluid dynamics, water surface gravity waves, electromagnetic radiation reactions, and ion acoustic waves in plasmas. Looking for exact solitary wave solutions to nonlinear evolution equations has long been a major concern for both mathematicians and physicists. These solutions may well describe various phenomena in physics and other fields, such as solitons and propagation with a finite speed, and thus they may give more insight into the physical aspects of the problems. In order to obtain the periodic wave and soliton solutions of nonlinear evolution equations, a number of methods have been proposed, such as tanh-sech function method, extended tanh function method , hyperbolic function method, sine-cosine method, Jacobi elliptic function expansion method, F-expansion method, transformed rational function method and the first integral method. The first integral method is a powerful solution method for the computation of exact traveling wave solutions [23]. The study of non-linear oscillators has been important in the development of the theory of dynamical systems. Van der Pol and Van der Mark (1927) studying a simple non-linear electronic circuit (a neon tube was the non-linear element) experimentally found, but were not much interested in,"noisy behavior" that we would now identify as chaos, and Carwright and Littlewood (1945) studied chaos like behavior in a non-linear oscillator, predating Lorenz's work by decades. Nonlinearity is ubiquitous and all pervading in the physical world. For a long time nonlinear systems were essentially studied under linear approximations, barring a few exceptions. However, the famous Fermi-Pasta-Ulam numerical experiments of the year 1955 on energy sharing between modes in anharmonic lattices triggered the golden era of modern nonlinear dynamics. Several path-breaking discoveries followed in the subsequent decades.[18]. For many nonlinear problems, it is not always possible and sometimes not even advantageous to express exact solutions of nonlinear differential equations explicitly in terms of elementary functions, but it is possible to find elementary functions that are constant on solution curves, that is , elementary first integrals. These first integrals allow us to occasionally deduce properties that an explicit solution would not necessarily reveal [22].

In the previous work, Prelle and Singer proposed a procedure for solving first-order ODEs that admits solutions in terms of elementary functions if such solutions exists [22]. Duarte et al. modified the technique developed by Prelle and Singer and applied it to second-order ODEs. Their approach was based on the conjecture that if an elementary solution exists for the given second-order ODE then there exists as least one elementary first integral $I(x,y,y')$ whose derivatives are all rational functions of x,y and y' [3, 4]. Chandrasekar et al. [3] used the generalized extended Prelle-Singer procedure applicable to identify integrable nonlinear oscillators and systems and construct integrating factors. Another powerful technique for seeking the first integrals of various differential equations is the Lie symmetry reduction method [10, 12, 14, 17]. Feng [11] applied Lie symmetry reduction method to find the first integrals of nonlinear second-order ODEs. Special types of first integrals and dynamical behaviors are of fundamental importance to our understanding of physical, chemical and biological phenomena modelled differential equations.

Considering the force-free Duffing-van der Pol (DVP) oscillator,

$$y'' + (\alpha + \beta y^2)y' - \gamma y + y^3 = 0, \tag{1.1}$$

where α, β and γ are arbitrary parameters. It is integrable for the parametric condition $\alpha = 4/\beta$ and $\gamma = -3/\beta^2$. Under the transformation

$$w = -ye^{(1/\beta)x}, \qquad z = e^{-(2/\beta)x}, \tag{1.2}$$

Eq. (1.1) with restriction $\alpha = 4/\beta$ and $\gamma = -3/\beta^2$ was shown to be transformable to the form

$$w'' - \frac{\beta^2}{2} w^2 w' = 0,$$

which can then be integrated[3]. Eq. (1.1) arises in a model describing the propagation of voltage pulses along a neuronal axon and has recently received much attention from many authors. A large amount of literature exists on this equation[16].

In a parallel direction, while performing the invariance analysis of a similar kind of problem, we find that not only the Eq. (1.1) but also its generalized version ,

$$y'' + \left(\frac{4}{\beta} + \beta y^2\right) y' + \frac{3}{\beta^2} y + y^3 + \delta y^5 = 0, \qquad \delta = \text{arbitrary parameter}, \tag{1.3}$$

is invariant under the same set of Lie point symmetries. As a consequence one can use the same transformation (1.2) to integrate Eq. (1.3) to the form

$$w'' - \frac{\beta^2}{2} w^2 w' + \delta w^5 = 0,$$

which is not so simple to integrate straightforwardly. However, we observe that this equation coincides with the second equation in the so-called modified Emden equation(MEE) hierarchy, investigated by Feix et al.[5],

$$y'' + y^l y' + g y^{2l+1} = 0, \qquad l = 1, 2, \ldots, n,$$

where g is an arbitrary parameter. In fact, Feix et al. have shown that through a direct transformation to a third-order equation the above equation can be integrated to obtain the general solution for the specific choice of the parameter g, namely, for $g = 1/(l+2)^2$[3, 5]. However, one can also expect that there should be a number of integrable equations which also admits solutions which are both oscillatory and nonoscillatory types in the class

$$y'' + (k_1 y^q + k_2) y' + k_3 y^{2q+1} + k_4 y^{q+1} + \lambda_1 y = 0, q \in R. \tag{1.4}$$

where $k_i's$, $i = 1, 2, 3, 4$, and λ_1 are arbitrary parameters. In this paper, we focus on this equation. Eq. (1.4) is a unified model for several ground-breaking physical systems which includes simple harmonic oscillator, anharmonic oscillator, force-free Helmholtz oscillator, force-free Duffing oscillator, MEE hierarchy, generalized DVP hierarchy, and so on [3]. If $k_3 = 0$, then Eq. (1.4) is the Duffing-van der Pol oscillator. When $q = 1$, Eq. (1.4) becomes the generalized MEE

$$y'' + (k_1 y + k_2)y' + k_3 y^3 + k_4 y^2 + \lambda_1 y = 0,$$

which provides us the force-free Helmholtz oscillator. When $q = 2$, Eq. (1.4) gives us the force-free Duffing-van der Pol oscillator.

Now, let us consider how to use Lie symmetry reduction to obtain the first integrals of nonlinear ODEs. Symmetry is the key to solve differential equations. There are many well-known techniques for obtaining exact solutions, but most of them are merely special cases of a few powerful symmetry methods. These methods can be applied to differential equations of an unfamiliar type. Instead, a given differential equation can be made to reveal its symmetries, which are then used to construct exact solutions [17]. The usual procedure in the solution of any ordinary differential equation is to search for some method whereby the order of the equation is depressed. The two standard methods are reduction of order using a Lie point symmetry and determination of a first integral. These are not equivalent processes [19]. Second-order ODEs whose Lie algebra is three-dimensional cannnot be solved by using a two-dimensional solvable subalgebra, for no such subalgebra exists. However these ODEs can be solved by a different approach. We can derive a simple way of using Lie point symmetries to determine first integrals of a given ODE. This technique relies upon the dimension of the Lie algebra being higher than the order of the ODE. Remarkably, the method enables us to solve some ODEs without having to carry out any quadrature whatsoever[17].

In this paper, we study Eq. (1.4) to obtain its first integrals under certain parametric conditions by applying the Lie point symmetry reduction method. In fact, the exponent q determines the tangent vector, which induces the infinitesimal generator, see Chapter 2. As a result of this, we can classify the integrable cases by the different values of q. Particular cases are presented in Chapter 4. For certain value of q, we can derive certain parametric conditions by the determining equations for the infinitesimal generator, which can be reduced to canonical coordinates. Through the inverse transformation, we obtain the first integrals of Eq. (1.4).

CHAPTER II

PRELIMINARIES

This paper refers to Peter E. Hydon's Symmetry Methods for Differential equations: A Beginner's Guide [17]. Materials presented in this section include the determining equations for Lie point symmetries and the infinitesimal generator.

2.1 How to obtain Lie point symmetries of ODEs

Let us consider ODEs of the form

$$y^{(n)} = \omega\left(x, y, y', \cdots, y^{(n-1)}\right), \qquad y^{(k)} \equiv \frac{d^k y}{dx^k}. \tag{2.1}$$

It is assumed that ω is (locally) a smooth function of all its arguments. We first state the symmetry condition. A symmetry condition of (2.1) is a diffeomorphism that maps the set of solutions of the ODE to itself. Any diffeomorphism,

$$\Gamma : (x, y) \mapsto (\hat{x}, \hat{y}),$$

maps smooth planar curves to smooth planar curves. On the plane, the diffeomorphism Γ generates a mapping on the derivatives $y^{(k)}$

$$\Gamma : \left(x, y, y', \cdots, y^{(n)}\right) \mapsto \left(\hat{x}, \hat{y}, \hat{y}', \cdots, \hat{y}^{(n)}\right),$$

where

$$\hat{y}^{(k)} \equiv \frac{d^k \hat{y}}{d\hat{x}^k}, \qquad k = 1, 2, \cdots, n.$$

Using the chain rule, the function $\hat{y}^{(k)}$ can be written as

$$\hat{y}^{(k)} = \frac{d\hat{y}^{(k-1)}}{d\hat{x}} = \frac{D_x \hat{y}^{k-1}}{D_x \hat{x}}, \qquad k = 1, 2, \cdots, n, \tag{2.2}$$

$$y^{(0)} \equiv \hat{y},$$

where $D(x)$ is the total derivative with respect to x:

$$D_x = \partial_x + y' \partial_y + y'' \partial_{y'} + \cdots .$$

We obtain the symmetry condition for ODE (2.1)

$$\hat{y}^{(n)} = \omega\left(\hat{x}, \hat{y}, \hat{y}', \cdots, \hat{y}^{(n-1)}\right), \tag{2.3}$$

where the function $\hat{y}^{(k)}$ is given by equation (2.2).

The action of a Lie symmetry maps every point on an orbit to a point on the same orbit. Now consider the orbit through a noninvariant point (x, y). The **tangent vector** to the orbit at the point $(\hat{x}, \hat{y})$ is $(\xi(\hat{x}, \hat{y}), \eta(\hat{x}, \hat{y}))$, where

$$\frac{d\hat{x}}{d\varepsilon} = \xi(\hat{x}, \hat{y}), \qquad \frac{d\hat{y}}{d\varepsilon} = \eta(\hat{x}, \hat{y}).$$

In particular, the tangent vector at (x, y) is

$$(\xi(x,y), \eta(x,y)) = \left(\left.\frac{d\hat{x}}{d\varepsilon}\right|_{\varepsilon=0}, \left.\frac{d\hat{y}}{d\varepsilon}\right|_{\varepsilon=0}\right).$$

For almost all ODEs, the symmetry condition (2.3) is nonlinear. Lie symmetries are obtained by linearizing (2.3) about $\varepsilon = 0$. It is usually easy to check whether or not a given diffeomorphism is a symmetry of a particular ODE. Since the trivial symmetry condition corresponding to $\varepsilon = 0$ leaves every point unchanged, then for ε sufficiently close to 0, the prolonged Lie symmetries are of the form

$$\begin{aligned}\hat{x} &= x + \varepsilon\xi + O(\varepsilon^2),\\ \hat{y} &= y + \varepsilon\eta + O(\varepsilon^2),\\ \hat{y}^{(k)} &= y^{(k)} + \varepsilon\eta^{(k)} + O(\varepsilon^2), \quad k \geq 1.\end{aligned} \tag{2.4}$$

Note that the superscript in $\eta^{(k)}, (k = 1, 2, \cdots, n)$ is merely an index; it does not denote a derivative of η. Substituting (2.4) into the symmetry condition (2.3); the $O(\varepsilon)$ terms yield the linearized symmetry condition

$$\eta^{(n)} = \xi\omega_x + \eta\omega_y + \eta^{(1)}\omega_{y'} + \cdots + \eta^{(n-1)}\omega_{y^{(n-1)}} \qquad \text{when (2.1) holds.} \tag{2.5}$$

The function $\eta^{(k)}(k=1,2,\cdots,n)$ can be obtained from equation (2.2). For $k \geq 1$, we have

$$\hat{y}^{(k)} = \frac{D_x \hat{y}^{(k-1)}}{D_x \hat{x}} = \frac{y^{(k)} + \varepsilon D_x \eta^{(k-1)} + O(\varepsilon^2)}{1 + \varepsilon D_x \xi + O(\varepsilon^2)}$$
$$= y^{(k)} + \varepsilon \left(D_x \eta^{(k-1)} - y^{(k)} D_x \xi \right) + O(\varepsilon^2).$$

From (2.4), we obtain

$$\eta^{(k)} \left(x, y, y', \cdots, y^{(k)} \right) = D_x \eta^{(k-1)} - y^{(k)} D_x \xi. \tag{2.6}$$

In this paper, we consider second-order ODEs of the form

$$y'' = \omega \left(x, y, y' \right). \tag{2.7}$$

The diffeomorphism of the form

$$(\hat{x}, \hat{y}) = (\hat{x}(x,y), \hat{y}(x,y))$$

is called a **point symmetry**. To find the Lie point symmetry of a second-order ODE, we just need to calculate $\eta^{(1)}$ and $\eta^{(2)}$ first. Since the functions ξ and η depend upon x and y only, equation (2.6) gives

$$\eta^{(1)} = \eta_x + (\eta_y - \xi_x) y' - \xi_y y'^2, \tag{2.8}$$

$$\eta^{(2)} = \eta_{xx} + (2\eta_{xy} - \xi_{xx}) y' + (\eta_{yy} - 2\xi_{xy}) y'^2 - \xi_{yy} y'^3 + (\eta_y - 2\xi_x - 3\xi_y y') y''. \tag{2.9}$$

The **linearized symmetry condition** of (2.7) is obtained by substituting (2.8) and (2.9) into (2.5) and then replacing y'' by $\omega(x,y,y')$. This gives

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx}) y' + (\eta_{yy} - 2\xi_{xy}) y'^2 - \xi_{yy} y'^3$$
$$= (-\eta_y + 2\xi_x + 3\xi_y y') \omega + \xi \omega_x + \eta \omega_y + \{ \eta_x + (\eta_y - \xi_x) y' - \xi_y y'^2 \} \omega_{y'}, \tag{2.10}$$

which can be solved in some cases. Since ξ and η are independent of y', then (2.10) can be decomposed into a system of PDEs, which are **the determining equations** for the Lie point symmetries. Similarly, for higher-order ODEs, we can also obtain the linearized symmetry condition, which is complicated.

2.2 Infinitesimal Generator and Canonical Coordinates

Suppose that a first-order ODE has a one-parameter Lie group of symmetries, whose tangent vector at (x,y) is (ξ,η). Then the partial differential operator

$$X = \xi(x,y)\partial_x + \eta(x,y)\partial_y$$

is called **the infinitesimal generator** of the Lie group. To deal with the action of Lie symmetries on derivatives of order n or smaller, we introduce the **prolonged infinitesimal generator**

$$X^{(n)} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'} + \cdots + \eta^{(n)}\partial_{y^{(n)}}.$$

The coefficient of $\partial_{y^{(n)}}$ is the $O(\varepsilon)$ term in the expansion of $\hat{y}^{(k)}$, and so $X^{(n)}$ is associated with the tangent vector in the space of variables $(x,y,y',\cdots,y^{(n)})$. We can use the prolonged infinitesimal generator to write the linearized symmetry condition (2.5) in a compact form:

$$X^{(n)}\left(y^{(n)} - \omega(x,y,y',\ldots,y^{(n-1)})\right) = 0 \qquad \text{when (2.1) holds.}$$

Let $\mathscr{L}$ denotes the set of all infinitesimal generators of one-parameter Lie groups of point symmetries of an ODE of order $n \geq 2$. The linearized symmetry condition is linear in ξ and η, and so

$$X_1, X_2 \in \mathscr{L} \Rightarrow c_1X_1 + c_2X_2 \in \mathscr{L} \qquad \forall c_1, c_2 \in R.$$

Hence $\mathscr{L}$ is a vector space. The dimension of this vector space is the number of arbitrary constants that appear in the general solution of the linearized symmetry condition.

We know that if an ordinary differential equation admits an infinitesimal generator, then there exists a pair of variables

$$r = r(x,y) \qquad \text{and} \qquad s = s(x,y),$$

which are called **canonical coordinates**, with r and $s(s \neq 0)$ being arbitrary particular solutions of the first-order linear partial equations

$$\xi(x,y)\frac{\partial r}{\partial x} + \eta(x,y)\frac{\partial r}{\partial y} = 0,$$

$$\xi(x,y)\frac{\partial s}{\partial x}+\eta(x,y)\frac{\partial s}{\partial y}=1.$$

The change of coordinates should be invertible in some neighbourhood of (x,y), so we impose the nondegeneracy condition

$$r_x s_y - r_y s_x \neq 0.$$

Suppose that $\xi(x,y)\neq 0$. The invariant canonical coordinate $r(x,y)$ is a first integral of

$$\frac{dx}{\xi(x,y)}=\frac{dy}{\eta(x,y)}.$$

The coordinate $s(x,y)$ is obtained by quadrature

$$s(x,y)=\int\frac{dx}{\xi(x,y(r,x))},$$

here the integral is evaluated with r being treated as a constant. Similarly, if $\xi(x,y)=0$ and $\eta(x,y)\neq 0$ then

$$r=x \qquad \text{and} \qquad s=\int\frac{dy}{\eta(x,y)},$$

are canonical coordinates.

2.3 How to obtain Lie point symmetries of PDEs

Considering PDEs with one dependent variable, u and two independent variables, x and t. A point transformation is a diffeomorphism

$$\Gamma:(x,t,u)\mapsto(\hat{x}(x,t,u),\hat{t}(x,y,u),\hat{u}(x,t,u)).$$

This transformation maps the surface $u=u(x,t)$ to the following surface:

$$\begin{aligned}\hat{x}&=\hat{x}(x,t,u(x,t)),\\ \hat{t}&=\hat{t}(x,t,u(x,t)),\\ \hat{u}&=\hat{u}(x,t,u(x,t)).\end{aligned}\tag{2.11}$$

To calculate the prolongation of a given transformation, we need to differentiate (2.11) with respect to each parameters x and t. To do this, we introduce the following derivatives:

$$D_x=\partial_x+u_x\partial_u+u_{xx}\partial_{u_x}+u_{xt}\partial_{u_t}+\cdots,$$

$$D_t=\partial_t+u_t\partial_u+u_{xt}\partial_{u_x}+u_{tt}\partial_{u_t}+\cdots.$$

the first two equations of (2.11) may be inverted (locally) to give x and t in terms of $\widehat{x}$ and $\widehat{t}$, provided that the Jacobian is nonzero, that is,

$$\mathscr{J} = \begin{vmatrix} D_x\hat{x} & D_x\hat{t} \\ D_t\hat{x} & D_t\hat{t} \end{vmatrix} \neq 0 \qquad \text{when} \quad u = u(x,t). \tag{2.12}$$

If (2.12) is satisfied, then the last equation of (2.11) can be rewritten as

$$\hat{u} = \hat{u}(\hat{x}, \hat{t}). \tag{2.13}$$

Applying the chain rule to (2.13), we obtain

$$\begin{bmatrix} D_x\hat{u} \\ D_t\hat{u} \end{bmatrix} = \begin{bmatrix} D_x\hat{x} & D_x\hat{t} \\ D_t\hat{x} & D_t\hat{t} \end{bmatrix} \begin{bmatrix} \hat{u}_{\hat{x}} \\ \hat{u}_{\hat{t}} \end{bmatrix},$$

and therefor by Cramer's rule

$$\hat{u}_{\hat{x}} = \frac{1}{\mathscr{J}} \begin{vmatrix} D_x\hat{u} & D_x\hat{t} \\ D_t\hat{u} & D_t\hat{t} \end{vmatrix}, \quad \hat{u}_{\hat{t}} = \frac{1}{\mathscr{J}} \begin{vmatrix} D_x\hat{x} & D_x\hat{u} \\ D_t\hat{x} & D_t\hat{u} \end{vmatrix}. \tag{2.14}$$

Similarly, we can obtain the second-order prolongations by repeating the above argument.

$$\begin{gathered} \hat{u}_{\hat{x}\hat{x}} = \frac{1}{\mathscr{J}} \begin{vmatrix} D_x\hat{u}_{\hat{x}} & D_x\hat{t} \\ D_t\hat{u}_{\hat{x}} & D_t\hat{t} \end{vmatrix}, \quad \hat{u}_{\hat{t}\hat{t}} = \frac{1}{\mathscr{J}} \begin{vmatrix} D_x\hat{x} & D_x\hat{u}_{\hat{t}} \\ D_t\hat{x} & D_t\hat{u}_{\hat{t}} \end{vmatrix}, \\ \hat{u}_{\hat{x}\hat{t}} = \frac{1}{\mathscr{J}} \begin{vmatrix} D_x\hat{u}_{\hat{t}} & D_x\hat{t} \\ D_t\hat{u}_{\hat{t}} & D_t\hat{t} \end{vmatrix} = \frac{1}{\mathscr{J}} \begin{vmatrix} D_x\hat{x} & D_x\hat{u}_{\hat{x}} \\ D_t\hat{x} & D_t\hat{u}_{\hat{x}} \end{vmatrix}. \end{gathered} \tag{2.15}$$

Define point symmetries of an nth order PDE:

$$\Delta(x, t, u, u_x, u_t, ...) = 0. \tag{2.16}$$

For simplicity, we shall restrict attention to PDEs of the form

$$\Delta = u_\sigma - \omega(x, t, u, u_x, u_t, \dots) = 0,$$

where u_σ is one of the nth order derivatives of u and ω is independent of u_σ. And the point transformation Γ is a point symmetry of (2.16) if

$$\Delta(\hat{x}, \hat{t}, \hat{u}, \hat{u}_{\hat{x}}, \hat{u}_{\hat{t}}, ...) = 0 \qquad \text{when (2.16) holds.} \tag{2.17}$$

Typically, the symmetry condition (2.17) is extremely complicated, so we shall not try to solve it directly. Nevertheless, it is quite easy to check whether or not a given point transformation is a symmetry of a particular PDE.

Generally speaking, we do not know a prior what form the point symmetries of a given PDE will take. However, it is usually possible to carry out a systematic search for one-parameter Lie groups of point symmetries. We seek point symmetries of the form

$$\begin{aligned} \hat{x} &= x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \\ \hat{t} &= t + \varepsilon\tau(x,t,u) + O(\varepsilon^2), \\ \hat{u} &= u + \varepsilon\eta(x,t,u) + O(\varepsilon^2). \end{aligned} \tag{2.18}$$

Just as for Lie point transformations of the plane, each one-parameter (local) Lie group of point transformation is obtained by exponentiating its infinitesimal generator, which is

$$X = \xi\partial_x + \tau\partial_t + \eta\partial_u.$$

Equivalently, we can obtain $(\hat{x},\hat{t},\hat{u})$ by solving

$$\frac{d\hat{x}}{d\varepsilon} = \xi(\hat{x},\hat{t},\hat{u}), \quad \frac{d\hat{t}}{d\varepsilon} = \tau(\hat{x},\hat{t},\hat{u}), \quad \frac{d\hat{u}}{d\varepsilon} = \eta(\hat{x},\hat{t},\hat{u}),$$

subject to the initial conditions

$$(\hat{x},\hat{t},\hat{u})\big|_{\varepsilon=0} = (x,t,u).$$

A surface $u = u(x,t)$ is mapped to itself by the group of transformations generated by X if

$$X(u - u(x,t)) = 0 \quad \text{when} \;\; u = u(x,t). \tag{2.19}$$

This condition can be expressed neatly by using the characteristic of the group, which is

$$Q = \eta - \xi u_x - \tau u_t.$$

From (2.19), the surface $u = u(x,t)$ is invariant provided that

$$Q = 0 \quad \text{when} \quad u = u(x,t). \tag{2.20}$$

Equation (2.20) is called the **invariant surface condition**; it is central to some of the main techniques for finding exact solutions of PDEs. The prolongation of the point transformation (2.18) to first derivatives is

$$\hat{u}_{\hat{x}} = u_x + \varepsilon\eta^x(x,t,u,u_x,u_t) + O(\varepsilon^2),$$
$$\hat{u}_{\hat{t}} = u_t + \varepsilon\eta^t(x,t,u,u_x,u_t) + O(\varepsilon^2),$$

where, form (2.14),

$$\eta^x(x,t,u,u_x,u_t) = D_x - u_x D_x\xi - u_t D_x\tau,$$
$$\eta^t(x,t,u,u_x,u_t) = D_t - u_x D_t\xi - u_t D_t\tau.$$

The transformation is prolonged to higher-order derivatives recursively, using (2.15). Suppose that

$$\hat{u}_J = u_J + \varepsilon\eta^J + O(\varepsilon^2),$$

where

$$u_J = \frac{\partial^{j_1+j_2}u}{\partial x^{j_1}\partial t^{j_2}}, \qquad \widehat{u}_J = \frac{\partial^{j_1+j_2}\widehat{u}}{\partial\widehat{x}^{j_1}\partial\widehat{t}^{j_2}},$$

for some numbers j_1 and j_2. Then (2.15) yields

$$\hat{u}_{J\hat{x}} = u_{Jx} + \varepsilon\eta^{Jx} + O(\varepsilon^2),$$
$$\hat{u}_{J\hat{t}} = u_{Jt} + \varepsilon\eta^{Jt} + O(\varepsilon^2),$$

where

$$\begin{aligned} \eta^{Jx} &= D_x^J - u_{Jx}D_x\xi - u_{Jt}D_x\tau, \\ \eta^{Jt} &= D_t^J - u_{Jx}D_t\xi - u_{Jt}D_t\tau. \end{aligned} \tag{2.21}$$

The infinitesimal generator is prolonged to derivatives by adding all terms of the form $\eta^J\partial_{u_J}$ up to the desired order. For example,

$$X^{(1)} = \xi\partial_x + \tau\partial_t + \eta\partial_u + \eta^x\partial_{u_x} + \eta^t\partial_{u_t} = X + \eta^x\partial_{u_x} + \eta^t\partial_{u_t},$$
$$X^{(2)} = X^{(1)} + \eta^{xx}\partial_{u_{xx}} + \eta^{xt}\partial_{u_{xt}} + \eta^{tt}\partial_{u_{tt}}.$$

From now on, we adopt the convention that the generator is pronged as many times as is needed to describe the group's action on all variables. To find the Lie point symmetries, we need explicit expression for (2.21). Then Lie point symmetries are obtained by differentiating the symmetry condition (2.17) with respect to ε at $\varepsilon = 0$. We obtain the **linearized symmetry condition**

$$X\triangle = 0 \qquad \text{when} \qquad \triangle = 0. \tag{2.22}$$

The restriction (2.16) enables us to eliminate u_σ from (2.22); then we split the remaining terms (according to their dependence on derivatives of u) to obtain a linear system of determining equations for ξ, τ and η. The vector space $\mathscr{L}$ of all Lie point symmetry generators of a given PDE is a Lie algebra, although it may not be finite dimensional.

Since the late 1980s, quite a few mathematicians and physicists have obtained explicit exact solutions to the Burgers-KdV equation by various methods. Feng[7, 9, 8, 6] proposed a new approach by applying the first integral method to study the compound Burgers-KdV equation and Burger-KdV equation. Applying Lie point symmetry method, we can get the first integrals of PDEs. The first integral method is a powerful solution method for the computation of exact traveling wave solutions. This method is one of the most direct and effective algebraic methods for finding exact solutions of nonlinear partial differential equations. Different from other traditional methods, the first integral method has many advantages, which is the avoidance of a great deal of complicated and tedious calculations resulting in more exact and explicit traveling solitary solutions with high accuracy[23]. In this paper, we focus on the first integrals of ODEs, for PDEs, we will present them in the future.

CHAPTER III

NONLINEAR OSCILLATORS AND SYSTEMS

3.1 Determining systems for the infinitesimal generator

Considering a second-order nonlinear ordinary differential equation of the form

$$y'' + (k_1 y^q + k_2)y' + k_3 y^{2q+1} + k_4 y^{q+1} + \lambda_1 y = 0, q \in R. \tag{3.1}$$

where $k_i's$, $i = 1,2,3,4$, and λ_1 are arbitrary parameters. In this chapter, we assume that q is arbitrary. For the general cases, we will present in the following sections. Chandrasekar et al. [3] used the generalized extended Prelle-Singer procedure to identify the first integrals of Eq. (3.1). Feng [11] applied Lie symmetry to find the first integrals of Eq. (3.1) when the exponent q is arbitrary. Now, let us apply the method of Lie point symmetry [21, 17] to find new first integrals of Eq. (3.1) under some certain parametric conditions. Following procedures to determine the symmetries of a differential equation introduced in Chapter 2, we can get the linearized symmetry condition concerning Eq. (3.1). Although (2.10) looks complicated, it is not difficult to solve $\xi(x,y)$ and $\eta(x,y)$. Since the unknown functions do not depend on the derivative y', after setting the coefficients of the powers $(y')^i (i = 0,1,2,3)$ in (2.10) to zero, the linearized symmetry condition (2.10) can be decomposed into the determining equations as follows

$$[y']^3 : \xi_{yy} = 0, \tag{3.2}$$

$$[y']^2 : \eta_{yy} - 2\xi_{xy} = -2\xi_y k_1 y^q - 2\xi_y k_2, \tag{3.3}$$

$$[y']^1 : 2\eta_{xy} - \xi_{xx} = -\xi_x(k_1 y^q + k_2) - 3k_3 \xi_y y^{2q+1} - 3k_4 \xi_y y^{q+1} - 3\lambda_1 \xi_y y \\ -qk_1 \eta y^{q-1}, \tag{3.4}$$

$$[y']^0 : \eta_{xx} = -2k_3\xi_x y^{2q+1} - 2k_4\xi_x y^{q+1} - 2\lambda_1\xi_x y + k_3\eta_y y^{2q+1} + k_4\eta_y y^{q+1}$$

$$+\lambda_1\eta_y y - (2q+1)k_3\eta y^{2q} - (q+1)k_4\eta y^q - \lambda_1\eta - (k_1 y^q + k_2)\eta_x. \tag{3.5}$$

The first equation (3.2) gives

$$\xi = a(x)y + b(x). \tag{3.6}$$

Substituting (3.6) into (3.3), we have

$$\eta = -\frac{2a(x)k_1}{(q+1)(q+2)}y^{q+2} + \{a'(x) - a(x)k_2\}y^2 + c(x)y + d(x), \tag{3.7}$$

where $q \neq -1$ and $q \neq -2$. And $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are functions of x to be determined. Plugging (3.6) and (3.7) into (3.4) leads to

$$\begin{aligned}
&[y^{2q+1}] : -3k_3 a + \frac{2k_1^2 q}{(q+1)(q+2)}a = 0, \\
&[y^{q+1}] : \frac{(q-1)(q+3)}{q+1}k_1 a' + 3k_4 a - k_1 k_2 q a = 0, \\
&[y^q] : k_1 b' + k_1 q c = 0, \\
&[y^{q-1}] : k_1 q d = 0, \\
&[y] : a'' - k_2 a' + \lambda_1 a = 0, \\
&[y^0] : 2c' - b'' + k_2 b' = 0.
\end{aligned} \tag{3.8}$$

Similarly, plugging (3.6) and (3.7) into (3.5), we find that

$$\begin{aligned}
&[y^{3q+2}] : k_1k_3a = 0,\\
&[y^{2q+2}] : (2q-1)k_2k_3a - k_3(2q+1)a' + \frac{2k_1^2}{(q+1)(q+2)}a' - \frac{2k_1k_4}{(q+1)(q+2)}a = 0,\\
&[y^{2q+1}] : k_3b' + qk_3c = 0,\\
&[y^{2q}] : k_3(2q+1)d = 0,\\
&[y^{q+2}] : \frac{q(q+3)}{(q+1)(q+2)}k_1a'' - (q-1)k_2k_4a + (q+1)k_4a' + \frac{2k_1\lambda_1}{q+2}a\\
&\qquad - \frac{q^2+3q+4}{(q+1)(q+2)}k_1k_2a' = 0,\\
&[y^{q+1}] : 2k_4b' + qk_4c + k_1c' = 0,\\
&[y^{q}] : (q+1)k_4d + k_1d' = 0,\\
&[y^{2}] : a''' + k_2\lambda_1a + \lambda_1a' - k_2^2a' = 0,\\
&[y] : c'' + 2\lambda_1b' + k_2c' = 0,\\
&[y^{0}] : d'' + \lambda_1d + k_2d' = 0.
\end{aligned} \tag{3.9}$$

In this section, we also assume that $q \neq 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, 1$ or 2. Since we can combine the coefficients of y with the same power, this assumption can avoid this case. In following sections, we will state these cases in detail.

3.2 Integrable cases under certain parametric conditions

Case 1: k_1, k_2, k_3, k_4 and λ_1 are arbitrary.

The first equation in (3.9) gives

$$k_1k_3a = 0,$$

which means $a(x) = 0$. The forth equation in (3.8) gives

$$k_1qd = 0,$$

which means $d(x)=0$. Then the determining system for $b(x)$ and $c(x)$ is reduced to

$$2c'-b''+k_2b'=0, \tag{3.10}$$

$$qc+b'=0, \tag{3.11}$$

$$c''+2\lambda_1b'+k_2c'=0, \tag{3.12}$$

$$2k_4b'+qk_4c+k_1c'=0. \tag{3.13}$$

Substituting (3.11) into (3.10), we get

$$c(x)=c_0e^{\frac{k_2q}{q+2}x}, \qquad b(x)=-\frac{(q+2)c_0}{k_2}e^{\frac{k_2q}{q+2}x}+c_1,$$

where c_0 and c_1 are constants. Substituting $b(x)$ and $c(x)$ into (3.12) and (3.13), we obtain parametric conditions

$$\lambda_1=\frac{q+1}{(q+2)^2}k_2^2, \qquad k_4=\frac{k_1k_2}{q+2}, \tag{3.14}$$

respectively. Hence, the general solution of the linearized symmetry condition is

$$\xi=-\frac{(q+2)c_0}{k_2}e^{\frac{k_2q}{q+2}x}+c_1, \qquad \eta=c_0e^{\frac{k_2q}{q+2}x}y.$$

Using ξ and η, every infinitesimal generator is of the form

$$\chi=c_0\chi_0+c_1\chi_1,$$

where

$$\chi_0=-\frac{(q+2)}{k_2}e^{\frac{k_2q}{q+2}x}\partial_x+e^{\frac{k_2q}{q+2}x}y\partial_y,$$
$$\chi_1=\partial_x.$$

For the generator χ_1, it is a homothety operator. Generally, it is hard to use this operator to find the first integrals of complicated second-order nonlinear ODEs. So we choose χ_0, which is a translation operator, as a generator to get canonical coordinates. Note that in the following cases, we assume that $c_0=1, c_1=0$ to get the generator χ_0.

The invariant canonical coordinate $r(x,y)$ is a first integral of

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)}.$$

Then we obtain

$$r(x,y) = \frac{q+2}{k_2} e^{\frac{k_2}{q+2}x} y. \tag{3.15}$$

The coordinate $s(x,y)$ is obtained by quadrature:

$$s(x,y) = \int \frac{dx}{\xi(x, y(r,x))}.$$

It is readily to get

$$s(x,y) = \frac{1}{q} e^{-\frac{qk_2}{q+2}x}. \tag{3.16}$$

Equations (3.15) and (3.16) can be rewritten to the parametric form [12]

$$x = -\frac{q+2}{qk_2} \ln(qs), \qquad y = \frac{k_2}{q+2}(qs)^{\frac{1}{q}} r. \tag{3.17}$$

Using the nonlinear transformations (3.17) yields [12]

$$\frac{\partial y}{\partial x} = -\frac{k_2^2}{(q+2)^2} q^{\frac{q+1}{q}} \left(r_s s^{\frac{q+1}{q}} + \frac{1}{q} r s^{\frac{1}{q}} \right), \tag{3.18}$$

$$\frac{\partial^2 y}{\partial^2 x} = \frac{k_2^3}{(q+2)^3} q^{\frac{2q+1}{q}} \left(r_{ss} s^{\frac{2q+1}{q}} + \frac{q+2}{q} r_s s^{\frac{q+1}{q}} + \frac{1}{q^2} r s^{\frac{1}{q}} \right). \tag{3.19}$$

Substituting equations (3.18) and (3.19) into equation (3.1), under parametric condition (3.14), we obtain

$$r_{ss} = \frac{k_1 k_2^{q-1}}{(q+2)^{q-1}} r_s r^q - \frac{k_3 k_2^{2q-2}}{(q+2)^{2q-2}} r^{2q+1}, \tag{3.20}$$

which is integrated as

$$\frac{2k_1}{\sqrt{k_1^2 - 4(q+1)k_3}} \tanh^{-1} \left[\frac{k_1(q+2)k_2^q r^{q+1} - 2k_2(q+1)(q+2)^q r_s}{(q+2)k_2^q r^{q+1} \sqrt{k_1^2 - 4(q+1)k_3}} \right]$$
$$+ \ln \left[k_3 k_2^{2q-2} r^{2q+2} - k_1 k_2^{q-1} (q+2)^{q-1} r_s + (q+1)(q+2)^{2q-2} r_s^2 \right] = I, \tag{3.21}$$

where I is an arbitrary constant. Using the inverse transformation of (3.17), we have

$$r_s = -\frac{q+2}{k_2} e^{\frac{(q+1)k_2}{q+2}x} y - \frac{(q+2)^2}{k_2^2} e^{\frac{(q+1)k_2}{q+2}x} y', \tag{3.22}$$

where $k_2y+(q+2)y'\neq 0$.

(i): If $k_3\neq\frac{k_1^2}{4(q+1)}$, substituting equation (3.22) into (3.21), we obtain the first integral of equation (3.1) as [3, 11]

$$\frac{2k_1}{\sqrt{k_1^2-4(q+1)k_3}}\tanh^{-1}\left[\frac{k_1y^{q+1}+\frac{2k_2(q+1)}{q+2}y+2(q+1)y'}{\sqrt{k_1^2-4(q+1)k_3}y^{q+1}}\right]+\frac{2q+2}{q+2}k_2x+$$

$$\ln\left[(q+2)y^{q+1}[k_3(q+2)y^{q+1}+k_1k_2y+k_1(q+2)y']+(q+1)[k_2y+(q+2)y']^2\right]=I.$$

(ii): If $k_3=\frac{k_1^2}{4(q+1)}$, equation (3.20) can be integrated as

$$\frac{1}{1-\frac{2k_2(q+2)^q(q+1)r_s}{k_1k_2^q(q+2)r^{q+1}}}+\ln\left[1-\frac{2k_2(q+2)^q(q+1)r_s}{k_1k_2^q(q+2)r^{q+1}}\right]+\ln(r^{q+1})=I. \tag{3.23}$$

Substituting equation (3.22) into (3.23), we obtain the first integral of equation (3.1) as [3, 11]

$$\frac{q+1}{q+2}k_2x+\ln\left[k_1y^{q+1}+2(q+1)y'+\frac{2(q+1)}{q+2}k_2y\right]$$

$$+\frac{k_1(q+2)y^{q+1}}{k_1(q+2)y^{q+1}+2(q+1)k_2y+2(q+1)(q+2)y'}=I.$$

Case 2: $k_3=0$, k_1,k_2,k_4 and λ_1 are arbitrary.

In this case, Eq. (3.1) becomes the Duffing van der Pol type oscillator. The first integral of this kind of oscillator is presented in [3, 12].

By the first equation and the forth equation in (3.8), we obtain $a(x)=0$ and $d(x)=0$. Then the determining system for $b(x)$ and $c(x)$ is same as **case 1**. So we obtain

$$c(x)=c_0e^{\frac{k_2q}{q+2}x},\qquad b(x)=-\frac{(q+2)c_0}{k_2}e^{\frac{k_2q}{q+2}x}+c_1,$$

where c_0 and c_1 are arbitrary constants and same parametric conditions

$$\lambda_1=\frac{q+1}{(q+2)^2}k_2^2,\qquad k_4=\frac{k_1k_2}{q+2}. \tag{3.24}$$

Using

$$\xi=-\frac{q+2}{k_2}e^{\frac{k_2q}{q+2}x},\qquad \eta=e^{\frac{k_2q}{q+2}x}y,$$

and following procedures (3.15) to (3.20), under parametric condition (3.24), we obtain

$$r_{ss} = \frac{k_1 k_2^{q-1}}{(q+2)^{q-1}} r_s r^q,$$

which is integrated as

$$r_s = \frac{k_1 k_2^{q-1}}{(q+1)(q+2)^{q-1}} r^{q+1} + I_1, \tag{3.25}$$

where I_1 is an arbitrary constant. Substituting equation (3.22) into (3.25), we obtain the first integral of equation (3.1) as

$$e^{\frac{(q+1)k_2}{q+2}x}\left(y' + \frac{k_2}{q+2}y + \frac{k_1}{q+1}y^{q+1}\right) = I_1.$$

Note that this first integral is same as the results in [3, 12].

Case 3: $k_1 = 0, k_3 = 0$ **and** k_2, k_4, λ_1 **are arbitrary.**

In this case, Eq. (3.1) becomes the Duffing-type oscillator. Similarly, the first integral of this oscillator is presented in [3, 12].

The second equation in (3.8) gives $a(x) = 0$. And the seventh equation in (3.9) gives $d(x) = 0$. The determining system for $b(x)$ and $c(x)$ is reduced to

$$2c' - b'' + k_2 b' = 0, \tag{3.26}$$

$$qc + 2b' = 0, \tag{3.27}$$

$$c'' + 2\lambda_1 b' + k_2 b' = 0. \tag{3.28}$$

Substituting (3.27) into (3.26), we get

$$c(x) = c_0 e^{\frac{qk_2}{q+4}x}, \qquad b(x) = -\frac{(q+4)c_0}{2k_2} e^{\frac{qk_2}{q+4}x} + c_1,$$

where c_0 and c_1 are arbitrary constants. In this case, we assume $q \neq -4$. If $q = -4$, then we get $c(x) = 0$, the equation (3.1) is partially integrable. Substituting $b(x)$ and $c(x)$ into (3.28), we obtain one parametric condition

$$\lambda_1 = \frac{2(q+2)}{(q+4)^2} k_2^2. \tag{3.29}$$

Assuming $c_0 = 1, c_1 = 0$ yields

$$\xi = -\frac{q+4}{2k_2} e^{\frac{qk_2}{q+4}x}, \qquad \eta = e^{\frac{qk_2}{q+4}x} y.$$

Using ξ and η, we have

$$r(x,y) = \frac{q+4}{2k_2} e^{\frac{2k_2}{q+4}x} y, \tag{3.30}$$

$$s(x,y) = \frac{2}{q} e^{-\frac{qk_2}{q+4}x}. \tag{3.31}$$

Formulas (3.30) and (3.31) can be rewritten to the parametric form

$$x = -\frac{q+4}{qk_2} \ln\left(\frac{qs}{2}\right), \qquad y = \frac{2k_2}{q+4} \left(\frac{qs}{2}\right)^{\frac{2}{q}} r. \tag{3.32}$$

Using the nonlinear transformations (3.32) yields

$$\frac{\partial y}{\partial x} = -\frac{2qk_2^2}{(q+4)^2} \left(\frac{q}{2}\right)^{\frac{2}{q}} \left(r_s s^{\frac{q+2}{q}} + \frac{2}{q} r s^{\frac{2}{q}}\right), \tag{3.33}$$

$$\frac{\partial^2 y}{\partial^2 x} = \frac{2q^2 k_2^3}{(q+4)^3} \left(\frac{q}{2}\right)^{\frac{2}{q}} \left[r_{ss} s^{\frac{2q+2}{q}} + \left(1 + \frac{4}{q}\right) r_s s^{\frac{q+2}{q}} + \frac{4}{q^2} r s^{\frac{2}{q}}\right]. \tag{3.34}$$

Substituting equations (3.33) and (3.34) into equation (3.1), under parametric condition (3.29), we obtain

$$r_{ss} = -k_4 \left(\frac{2k_2}{q+4}\right)^{q-2} r^{q+1},$$

which is integrated as

$$r_s^2 = -\frac{2k_4}{q+2} \left(\frac{2k_2}{q+4}\right)^{q-2} r^{q+2} + I_2, \tag{3.35}$$

where I_2 is an arbitrary constant. Using the inverse transformation of (3.32), we have

$$r_s = -\frac{q+4}{2k_2} e^{\frac{k_2(q+2)}{q+4}x} \left(y + \frac{q+4}{2k_2} y'\right), \tag{3.36}$$

where $2k_2 y + (q+4)y' \neq 0$. Substituting equation (3.36) into (3.35), we obtain the first integral of equation (3.1) as [3, 12]

$$e^{\frac{2(q+2)k_2}{q+4}x} \left[\frac{y'^2}{2} + \frac{2k_2}{(q+4)} yy' + \frac{2k_2^2}{(q+4)^2} y^2 + \frac{k_4}{q+2} y^{q+2}\right] = I_2.$$

Case 4: $k_1 = 0$**,** $k_4 = 0$ **and** k_2, k_3, λ_1 **are arbitrary.**

The first equation in (3.8) and the forth equation in (3.9) gives $a(x) = 0$ and $d(x) = 0$. Then we can solve for $b(x)$ and $c(x)$ by following system

$$2c' - b'' + k_2 b' = 0, \tag{3.37}$$

$$qc + b' = 0, \tag{3.38}$$

$$c'' + 2\lambda_1 b' + k_2 c' = 0. \tag{3.39}$$

Substituting (3.38) into (3.37), we get

$$c(x) = c_0 e^{\frac{qk_2}{q+2}x}, \qquad b(x) = -\frac{(q+2)c_0}{k_2} e^{\frac{qk_2}{q+2}x} + c_1,$$

where c_0 and c_1 are constants. Substituting $b(x)$ and $c(x)$ into (3.39), we obtain one parametric condition

$$\lambda_1 = \frac{q+1}{(q+2)^2} k_2^2. \tag{3.40}$$

We assume $c_0 = 1, c_1 = 0$, then

$$\xi = -\frac{q+2}{k_2} e^{\frac{qk_2}{q+2}x}, \qquad \eta = e^{\frac{qk_2}{(q+2)}x} y.$$

Following procedures (3.15) to (3.20), under parametric condition (3.40), we obtain

$$r_{ss} = -\frac{k_3 k_2^{2q-2}}{(q+2)^{2q-2}} r^{2q+1},$$

which is integrated as

$$r_s^2 = -\frac{k_3 k_2^{2q-2}}{(q+1)(q+2)^{2q-2}} r^{2q+2} + I_3, \tag{3.41}$$

where I_3 is an arbitrary constant. Substituting equation (3.22) into (3.41), we obtain the first integral of equation (3.1) as

$$e^{\frac{(2q+2)k_2}{q+2}x} \left[\frac{k_2^2}{2(q+2)^2} y^2 + \frac{k_2}{q+2} y y' + \frac{y'^2}{2} + \frac{k_3}{2q+2} y^{2q+2} \right] = I_3.$$

Note that the first integrals in this chapter are identical to the results in [3]. For other cases of k_1, k_2, k_3, k_4 and λ_1, the original equation is partially integrable or the first integral can be derived directly without applying Lie point symmetry. So it is not meaningful to mention them. But the value of q may effect the first integrals of equation (3.1). For example, in this chapter, if $q=-1$ or -2, the tangent vector $\eta(x,y)$ is undefined. However, we can still obtain the first integrals of equation (3.1) when $q=-1,-2$. And when $q=1$, if we combine the coefficients of y^q with y and other y with same power, then the determining system for the infinitesimal generator may change. But if $q=2, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}$, or -$\frac{1}{3}$, the integrable cases are same as this chapter. In the following chapter, we will consider these values of q which are not included in this chapter.

CHAPTER IV

PARTICULAR CASES

4.1 Integrable cases of q=1

In the proceeding section, if $q = 1$, the determining system for the infinitesimal generator will change. Equation (3.1) becomes

$$y'' + (k_1 y + k_2)y' + k_3 y^3 + k_4 y^2 + \lambda_1 y = 0. \tag{4.1}$$

By (3.6) and (3.7), we obtain

$$\xi = a(x)y + b(x), \tag{4.2}$$

and

$$\eta = -\frac{a(x)k_1}{3} y^3 + \{a'(x) - a(x)k_2\} y^2 + c(x)y + d(x), \tag{4.3}$$

where $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are functions of x to be determined. Same as Chapter 3, substituting (4.2) and (4.3) into (3.4) and (3.5), we obtain two systems

$$\begin{aligned}
&[y^3] : k_1^2 a - 9k_3 a = 0,\\
&[y^2] : k_1 k_2 a - 3k_4 a = 0,\\
&[y] : 3k_2 a' - 3a'' - 3\lambda_1 a - k_1 b' - k_1 c = 0,\\
&[y^0] : b'' - 2c' - k_2 b' - k_1 d = 0.
\end{aligned} \tag{4.4}$$

and

$$[y^4] : 3k_2k_3a - k_1k_4a - 9k_3a' + k_1^2a' = 0,$$

$$[y^3] : -6k_4a' - 2k_1\lambda_1 a - 2k_1a'' + 4k_1k_2a' - 6k_3c - 6k_3b' = 0,$$

$$[y^2] : -k_4c - 2k_4b' - 3k_3d - k_1c' + k_2^2a' - a''' - \lambda_1 a' - \lambda_1 k_2 a = 0, \tag{4.5}$$

$$[y] : 2\lambda_1 b' + 2k_4d + k_1d' + k_2c' + c'' = 0,$$

$$[y^0] : d'' + k_2d' + \lambda_1 d = 0.$$

Clearly, these two systems are different from Chapter 3, since we combine some equations together.

Case 1: k_1, k_2, k_3**,** k_4 **and** λ_1 **are arbitrary.**

In order to solve $a(x), b(x), c(x)$ and $d(x)$ easily, we assume following conditions.

Condition (1):

$$k_1^2 = 9k_3, \qquad k_1k_2 = 3k_4. \tag{4.6}$$

From above parametric condition, the first and second equation in system (4.4) and the first equation in system (4.5) are always true. In addition, by the parametric condition (4.6), we obtain the third equation in system (4.4) and the second equation in system (4.5) are equivalent, which can be simplified as

$$3(a'k_2 - a'' - \lambda_1 a) - k_1(b' + c) = 0.$$

Let $b' + c = 0$, then the determining system for $a(x)$, $b(x)$, $c(x)$ and $d(x)$ is reduced to

$$a'' - k_2a' + \lambda_1 a = 0, \tag{4.7}$$

$$a''' + (\lambda_1 - k_2^2)a' + k_2\lambda_1 a + (k_4c - 3k_3d - k_1c') = 0, \tag{4.8}$$

$$-3c' + k_2c - k_1d = 0, \tag{4.9}$$

$$d'' + k_2d' + \lambda_1 d = 0, \tag{4.10}$$

$$2\lambda_1 c - k_2c' - c'' - 2k_4d - k_1d' = 0. \tag{4.11}$$

Under the parametric condition (4.6), in equation (4.8), we obtain

$$k_4c - 3k_3d - k_1c' = \frac{k_1}{3}(-3c' + k_2c - k_1d) = 0.$$

By equations (4.7) and (4.10), we get

$$a(x) = c_0 e^{p+k_2} x, \qquad d(x) = c_1 e^p x, \tag{4.12}$$

where $p = \frac{-k_2 \pm \sqrt{k_2^2 - 4\lambda_1}}{2}$ and c_0, c_1 are constants. Then from (4.9), we obtain

$$c(x) = c_2 e^{\frac{k_2}{3}x} - \frac{k_1 c_1}{3p - k_2} e^{px}, \tag{4.13}$$

where $p \neq \frac{k_2}{3}$, c_2 is a constant. If $p = \frac{k_2}{3}$, then $d(x) = 0$. Plugging (4.12) and (4.13) into (4.11), we can get parametric conditions

$$\lambda_1 = \frac{2k_2^2}{9}, \qquad p = -\frac{k_2}{3}. \tag{4.14}$$

As a result of this, we obtain

$$a(x) = c_0 e^{\frac{2k_2}{3}x}, \qquad b(x) = -\frac{3c_2}{k_2} e^{\frac{k_2}{3}x} + \frac{3k_1 c_1}{2k_2^2} e^{-\frac{k_2}{3}x} + c_3,$$
$$c(x) = c_2 e^{\frac{k_2}{3}x} + \frac{k_1 c_1}{2k_2} e^{-\frac{k_2}{3}x}, \qquad d(x) = c_1 e^{-\frac{k_2}{3}x},$$

which satisfy equations (4.7) – (4.11).

Every infinitesimal generator is of the from

$$\chi = c_0 \chi_0 + c_1 \chi_1 + c_2 \chi_2 + c_3 \chi_3,$$

where

$$\chi_0 = e^{\frac{2k_2}{3}x} y \partial_x, \qquad \chi_1 = \frac{3k_1}{2k_2^2} e^{-\frac{k_2}{3}x} \partial_x + \left(\frac{k_1}{2k_2} e^{-\frac{k_2}{3}x} y + e^{-\frac{k_2}{3}x} \right) \partial_y,$$
$$\chi_2 = -\frac{3}{k_2} e^{\frac{k_2}{3}x} \partial_x + e^{\frac{k_2}{3}x} \partial_y, \qquad \chi_3 = \partial_x,$$

which is a summary for two cases: $d(x) = 0$ and $d(x) \neq 0$. In general, for second-order ODEs, we can only get 0, 1, 2, 3 or 8 infinitesimal generators for one case. Moreover, we can get 8 infinitesimal generators if and only if the ODE either is linear, or is linearizable by a point transformation.

Part (a): If we choose χ_1 as the infinitesimal generator, then assuming

$$\xi = \frac{3}{k_2} e^{-\frac{k_2}{3}x}, \qquad \eta = e^{-\frac{k_2}{3}x} \left(y + \frac{2k_2}{k_1} \right)$$

to make it simper to get canonical coordinates. We find that

$$r(x,y) = e^{-\frac{k_2}{3}x}\left(y + \frac{2k_2}{k_1}\right), \qquad s(x,y) = e^{\frac{k_2}{3}x}.$$

Then $r(x,y)$ and $s(x,y)$ can be written as

$$x = \frac{3}{k_2}\ln(s), \qquad y = rs - \frac{2k_2}{k_1}.$$

Using the nonlinear transformations of x and y yields

$$\frac{\partial y}{\partial x} = \frac{k_2}{3}\left(r_s s^2 + rs\right), \tag{4.15}$$

$$\frac{\partial^2 y}{\partial^2 x} = \frac{k_2^2}{9}\left(r_{ss}s^3 + 3r_s s^2 + rs\right). \tag{4.16}$$

Substituting equations (4.15) and (4.16) into equation (4.1), under the parametric condition (4.14), we obtain

$$r_{ss} = \frac{3k_1}{k_2}r_s r - \frac{k_1^2}{k_2^2}r^3,$$

which is integrated as

$$\frac{(k_2 r_s + k_1 r^2)^2}{2k_2 r_s + k_1 r^2} = I_4, \tag{4.17}$$

where I_4 is a constant. Using the inverse transformation of x and y, we obtain

$$r_s = e^{-\frac{2k_2}{3}x}\left(-y - \frac{2k_2}{k_1} + \frac{3}{k_2}y'\right), \tag{4.18}$$

where

$$-y - \frac{2k_2}{k_1} + \frac{3}{k_2}y' \neq 0.$$

Plugging (4.18) into (4.17), the first integral of equation (4.1) is

$$e^{-\frac{2k_2}{3}x}\left[\frac{\left(3k_1k_2y + 3k_1y' + k_1^2y^2 + 2k_2^2\right)^2}{k_1y^2 + 2k_2y + 6y'}\right] = I_4.$$

Part (b): If we choose χ_2 to get canonical coordinates, we obtain

$$\xi = -\frac{3}{k_2}e^{\frac{k_2}{3}x}, \qquad \eta = e^{\frac{k_2}{3}x}y.$$

The canonical coordinates are

$$r(x,y) = \frac{3}{k_2}e^{\frac{k_2}{3}x}y, \qquad s(x,y) = e^{-\frac{k_2}{3}x}.$$

Formulas of $r(x,y)$ and $s(x,y)$ can be rewritten to the parametric form

$$x = -\frac{3}{k_2}\ln(s), \qquad y = \frac{k_2}{3}sr. \tag{4.19}$$

Using the nonlinear transformations (4.19) yields

$$\frac{\partial y}{\partial x} = -\frac{k_2^2}{9}\left(r_s s^2 + rs\right), \tag{4.20}$$

$$\frac{\partial^2 y}{\partial^2 x} = \frac{k_2^3}{27}r_{ss}s^3 + \frac{k_2^3}{9}r_s s^2 + \frac{k_2^3}{27}sr. \tag{4.21}$$

Substituting equations (4.20) and (4.21) into equation (4.1), under parametric conditions

$$k_1^2 = 9k_3, \qquad k_1k_2 = 3k_4, \qquad \lambda_1 = \frac{2}{9}k_2^2,$$

we obtain

$$r_{ss} = k_1 r_s r - \frac{k_1^2}{9}r^3,$$

which is integrated as

$$\frac{9r_s^2 + k_1^2r^4 - 6k_1r_sr^2}{k_1^2r^2 - 6k_1r_s} = I_5, \tag{4.22}$$

where I_5 is an arbitrary constant. Using the inverse transformation of (4.19) yields

$$r_s = -\frac{3}{k_2}e^{\frac{2k_2}{3}x}y - \frac{9}{k_2^2}e^{\frac{2k_2}{3}x}y'. \tag{4.23}$$

where $k_2y + 3y' \neq 0$. Substituting equation (4.23) into (4.22), we obtain the first integral of equation (4.1) as

$$e^{\frac{2k_2}{3}x}\left[\frac{(k_2y + k_1y^2 + 3y')^2}{k_1y^2 + 2k_2y + 6y'}\right] = I_5.$$

Condition (2):

$$3k_4 = k_1k_2, \qquad k_1^2 \neq 9k_3.$$

If $k_1^2 \neq 9k_3$, then from the first equation in system (4.4), we can get $a(x) = 0$. Then by the third equation in system (4.4), we obtain

$$b' + c = 0. \tag{4.24}$$

Plugging (4.24) into the forth equation in system (4.4) and the third equation in system (4.5), after subtraction, we have

$$(9k_3 - k_1^2)d = 0.$$

Clearly, we obtain $d(x) = 0$. The determining system for $b(x)$ and $c(x)$ is reduced to

$$b'' - 2c' - k_2 b' = 0, \tag{4.25}$$

$$c + b' = 0, \tag{4.26}$$

$$2\lambda_1 b' + k_2 c' + c'' = 0. \tag{4.27}$$

Plugging (4.26) into (4.25), we get

$$c(x) = c_0 e^{\frac{k_2}{3}x}, \qquad b(x) = -\frac{3c_0}{k_2} e^{\frac{k_2}{3}x} + c_1,$$

where c_0 and c_1 are constants. Substituting $b(x)$ and $c(x)$ into (4.27), we obtain one parametric condition

$$\lambda_1 = \frac{2}{9} k_2^2. \tag{4.28}$$

Hence, the general solution of the linearized symmetry condition is

$$\xi = -\frac{3c_0}{k_2} e^{\frac{k_2}{3}x} + c_1, \qquad \eta = c_0 e^{\frac{k_2}{3}x} y.$$

Assuming $c_0 = 1, c_1 = 0$, then following procedures of **part (b)** in **Condition (1)**, under the parametric condition (4.28), we obtain

$$r_{ss} = k_1 r r_s - k_3 r^3, \tag{4.29}$$

which is integrated as

$$\frac{2k_1}{\sqrt{k_1^2 - 8k_3}} \tanh^{-1}\left[\frac{k_1 r^2 - 4r_s}{\sqrt{k_1^2 - 8k_3 r^2}}\right] + \ln\left(k_3 r^4 - k_1 r^2 r_s + 2r_s^2\right) = I_6, \tag{4.30}$$

where I_6 is an arbitrary constant.

(i): If $k_1^2 \neq 8k_3$, substituting equation (4.23) into (4.30), we obtain the first integral of equation (4.1) as

$$\frac{2k_1}{\sqrt{k_1^2-8k_3}}\tanh^{-1}\left[\frac{3k_1y^2+4k_2y+12y'}{3\sqrt{k_1^2-8k_3y^2}}\right]+\frac{4k_2}{3}x$$
$$+\ln\left[3y^2(3k_3y^2+k_1k_2y+3k_1y')+2(k_2y+3y')^2\right]=I_6.$$

(ii): If $k_1^2 = 8k_3$, equation (4.29) can be integrated as

$$\frac{1}{1-\frac{4r_s}{k_1r^2}}+\ln\left(1-\frac{4r_s}{k_1r^2}\right)+\ln(r^2)=I_6. \tag{4.31}$$

Substituting equation (4.23) into (4.31), we obtain the first integral of equation (4.1) as

$$\frac{2k_2}{3}x+\ln\left(4k_2y+3k_1y^2+12y'\right)+\frac{3k_1y^2}{4k_2y+3k_1y^2+12y'}=I_6.$$

Note that this first integral is identical to the results in [3].

Condition (3): If $k_4 \neq 3k_1k_2$, $9k_3 = k_1^2$, we can only find the infinitesimal generator $\chi = \partial_x$, which means equation (4.1) is partially integrable.

Case 2: $k_1 = 0$**,** k_2, k_3, k_4 **and** λ_1 **are arbitrary.**

The first equation in (4.4) gives $a(x) = 0$. Then the determining system for $b(x), c(x)$ and $d(x)$ yields

$$b'+c=0, \tag{4.32}$$

$$b''-2c'-k_2b'=0, \tag{4.33}$$

$$-k_4c-2k_4b'-3k_3d=0, \tag{4.34}$$

$$2\lambda_1b'+2k_4d+k_2c'+c''=0, \tag{4.35}$$

$$d''+k_2d'+\lambda_1d=0. \tag{4.36}$$

Plugging (4.32) into (4.33), we obtain

$$c(x)=c_0e^{\frac{k_2}{3}x}, \qquad b(x)=-\frac{3c_0}{k_1}e^{\frac{k_2}{3}x}+c_1,$$

where c_0, c_1 are constants. Similarly, plugging $b(x)$ and $c(x)$ into (4.34), we can find

$$d(x) = \frac{k_4}{3k_3}c(x) = \frac{k_4 c_0}{3k_3}e^{\frac{k_2}{3}x}.$$

Then from (4.36), we can conclude a parametric condition

$$\lambda_1 = -\frac{4k_2^2}{9}. \tag{4.37}$$

Substituting (4.37) into (4.35), we obtain another parametric condition

$$k_4^2 = -2k_2^2 k_3. \tag{4.38}$$

Assuming $c_0 = 1, c_1 = 0$ yields

$$\xi = -\frac{3}{k_2}e^{\frac{k_2}{3}x}, \qquad \eta = e^{\frac{k_2}{3}x}\left(y + \frac{k_4}{3k_3}\right).$$

It is readily to get

$$r(x,y) = e^{\frac{k_2}{3}x}\left(y + \frac{k_4}{3k_3}\right), \qquad s(x,y) = e^{-\frac{k_2}{3}x}.$$

Then we can find

$$x = -\frac{3}{k_2}\ln(s), \qquad y = rs - \frac{k_4}{3k_3},$$

which induce

$$\frac{\partial y}{\partial x} = -\frac{k_2}{3}\left(r_s s^2 + rs\right), \tag{4.39}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{k_2^2}{9}\left(r_{ss}s^3 + 3r_s s^2 + rs\right). \tag{4.40}$$

Substituting equations (4.39) and (4.40) into equation (4.1), under the parametric conditions (4.37) and (4.38), we have

$$r_{ss} = -\frac{9k_3}{k_2^2}r^3,$$

which is integrated as

$$r_s^2 = -\frac{9k_3}{2k_2^2}r^4 + I_7, \tag{4.41}$$

where I_7 is an arbitrary constant. Taking $r(x,y)$ and $s(x,y)$ into account yields

$$r_s = -e^{\frac{2k_2}{3}x}\left(\frac{3}{k_2}y' + y + \frac{k_4}{3k_3}\right). \tag{4.42}$$

where $9k_3y' + 3k_2k_3y + k_2k_4 \neq 0$. Plugging (4.42) into (4.41), we obtain the first integral of equation (4.1) as

$$e^{\frac{4k_2}{3}x}\left(-5y^2 + \frac{9k_3}{2k_2^2}y^4 + \frac{6k_4}{k_2^2}y^3 - \frac{2k_4}{3k_3}y + \frac{9}{k_2^2}y'^2 + \frac{2k_4}{k_2k_3}y' + \frac{6}{k_2}yy'\right) = I_7.$$

Case 3: $k_1 = 0, k_4 = 0$ **and** k_2, k_3 **and** λ_1 **are arbitrary.**

Same as section (4.1.2), we have $a(x) = 0$. By (4.34), we have $d(x) = 0$. Then the determining system for $b(x)$ and $c(x)$ is reduced to

$$b'' - 2c' - k_2b' = 0, \tag{4.43}$$

$$c + b' = 0, \tag{4.44}$$

$$2\lambda_1 b' + k_2c' + c'' = 0. \tag{4.45}$$

Plugging (4.44) into (4.43), we get

$$c(x) = c_0e^{\frac{k_2}{3}x}, \qquad b(x) = -\frac{3c_0}{k_2}e^{\frac{k_2}{3}x} + c_1,$$

where c_0 and c_1 are constants. Substituting $b(x)$ and $c(x)$ into (4.45), we obtain one parametric condition

$$\lambda_1 = \frac{2}{9}k_2^2. \tag{4.46}$$

Then, we obtain

$$\xi = -\frac{3c_0}{k_2}e^{\frac{k_2}{3}x} + c_1, \qquad \eta = c_0e^{\frac{k_2}{3}x}y.$$

Let $c_0 = 1$, $c_1 = 0$ and following procedures of **part (b)** in **case 1**, under parametric condition (4.46), we obtain

$$r_{ss} = -k_3r^3,$$

which is integrated as

$$r_s^2 = -\frac{k_3r^4}{2} + I_8, \tag{4.47}$$

where I_8 is an arbitrary constant. Substituting equation (4.23) into (4.47), we obtain the first integral of equation (4.1) as

$$e^{\frac{4k_2}{3}x}\left(\frac{k_2^2}{18}y^2 + \frac{k_2}{3}yy' + \frac{y'^2}{2} + \frac{k_3}{4}y^4\right) = I_8.$$

Note that this first integral is presented in [3].

Case 4: $k_1 = 0, k_3 = 0$**,** k_2, k_4 **and** λ_1 **are arbitrary.**

In this case, Eq. (4.1) is the Helmholtz Oscillator[12]. The second equation in (4.4) yields $a(x) = 0$. Then the determining system for $b(x)$ and $c(x)$ is reduced to

$$b'' - 2c' - k_2 b' = 0, \tag{4.48}$$

$$c + 2b' = 0. \tag{4.49}$$

$$2\lambda_1 b' + 2k_4 d + k_2 c' + c'' = 0, \tag{4.50}$$

$$d'' + k_2 d' + \lambda_1 d = 0. \tag{4.51}$$

Substituting (4.49) into (4.48), we obtain

$$c(x) = c_0 e^{\frac{k_2}{5}x}, \qquad b(x) = -\frac{5c_0}{2k_2} e^{\frac{k_2}{5}x} + c_1,$$

where c_0 and c_1 are constants. Substituting $b(x)$ and $c(x)$ into (4.50), we obtain one parametric condition

$$d = \frac{1}{2k_4}\left(\lambda_1 - \frac{6}{25}k_2^2\right)c. \tag{4.52}$$

Similarly, substituting (4.52) into (4.51), we obtain another parametric condition

$$\frac{1}{2k_4}\left(\lambda_1 - \frac{6}{25}k_2^2\right)\left(\lambda_1 + \frac{6}{25}k_2^2\right)c = 0.$$

Part (c): $\lambda_1 = \frac{6}{25}k_2^2$, then by equation (4.52), we have $d(x) = 0$. Assuming $c_0 = 1, c_1 = 0$ yields

$$\xi = -\frac{5}{2k_2} e^{\frac{k_2}{5}x}, \qquad \eta = e^{\frac{k_2}{5}x} y.$$

Using ξ and η, we obtain the invariant canonical coordinates are

$$r(x,y) = \frac{5}{2k_2} e^{\frac{2k_2}{5}x} y, \qquad s(x,y) = 2e^{-\frac{k_2}{5}x}. \tag{4.53}$$

The canonical coordinates $r(x,y)$ and $s(x,y)$ can be rewritten to the parametric form

$$x = -\frac{5}{k_2}\ln\left(\frac{s}{2}\right), \qquad y = \frac{k_2}{10}s^2 r. \tag{4.54}$$

Using the nonlinear transformations (4.54) yields

$$\frac{\partial y}{\partial x} = -\frac{k_2^2}{50}\left(r_s s^3 + 2rs^2\right), \tag{4.55}$$

$$\frac{\partial^2 y}{\partial^2 x} = \frac{k_2^3}{250} r_{ss} s^4 + \frac{k_2^3}{50} r_s s^3 + \frac{2k_2^3}{125} s^2 r. \tag{4.56}$$

Substituting equations (4.55) and (4.56) into equation (4.1), under the parametric condition $\lambda_1 = 6k_2^2/25$, we obtain

$$r_{ss} = -\frac{5k_4}{2k_2} r^2,$$

which is integrated as

$$r_s^2 = -\frac{5k_4}{3k_2} r^3 + I_9, \tag{4.57}$$

where I_9 is an arbitrary constant. Using the inverse transformation of (4.54), we have

$$r_s = -\frac{25}{4k_2^2} e^{\frac{3k_2}{5}x}\left(y' + \frac{2}{5} k_2 y\right). \tag{4.58}$$

where $5y' + 2k_2 y \neq 0$. Substituting equation (4.58) into (4.57), we obtain the first integral of equation (4.1) as

$$e^{\frac{6k_2}{5}x}\left(\frac{y'^2}{2} + \frac{2k_2}{5} yy' + \frac{2k_2^2}{25} y^2 + \frac{k_4}{3} y^3\right) = I_9.$$

Part (d): $\lambda_1 = -\frac{6}{25}k_2^2$, then by equations $b(x), c(x)$ and (4.52), we can get

$$c(x) = c_0 e^{\frac{k_2}{5}x}, \qquad b(x) = -\frac{5c_0}{2k_2} e^{\frac{k_2}{5}x} + c_1, \qquad d = -\frac{6k_2^2 c_0}{25k_4} e^{\frac{k_2}{5}x}.$$

Similarly, we assume $c_0 = 1, c_1 = 0$, then

$$\xi = -\frac{5}{2k_2} e^{\frac{k_2}{5}x}, \qquad \eta = e^{\frac{k_2}{5}x}\left(y - \frac{6k_2^2}{25k_4}\right).$$

Using ξ and η, we obtain

$$r(x,y) = e^{\frac{2k_2}{5}x}\left(y - \frac{6k_2^2}{25k_4}\right), \qquad s(x,y) = 2e^{-\frac{k_2}{5}x}. \tag{4.59}$$

Following procedures in **part (c)**, under the parametric condition $\lambda_1 = -6k_2^2/25$, we have

$$r_{ss} = -\frac{25k_4}{4k_2^2} r^2,$$

which is integrated as

$$r_s^2 = -\frac{25k_4}{6k_2^2} r^3 + I_{10}. \tag{4.60}$$

By equation (4.59), we can get

$$r_s = -e^{\frac{3k_2}{5}x} \left(y - \frac{6k_2^2}{25k_4} + \frac{5}{2k_2} y' \right), \tag{4.61}$$

where

$$y - \frac{6k_2^2}{25k_4} + \frac{5}{2k_2} y' \neq 0.$$

Plugging equation (4.61) into (4.60), we obtain the first integral of equation (4.1) as

$$e^{\frac{6k_2}{5}x} \left(-2y^2 + \frac{25}{4k_2^2} y'^2 + \frac{6k_2^2}{25k_4} y + \frac{5}{k_2} y y' - \frac{6k_2}{5k_4} y' + \frac{25k_4}{6k_2^2} y^3 \right) = I_{10}.$$

These two first integrals of equation (4.1) are presented in [12, 1, 2, 13, 14].

Case 5: $k_3 = 0$, k_1, k_2, k_4 and λ_1 are arbitrary.

Taking the first equation in (4.4) into account, we find $a(x) = 0$. The determining system for $b(x)$ and $c(x)$ is reduced to

$$k_4 c + 2k_4 b' + k_1 c' = 0, \tag{4.62}$$

$$c + b' = 0, \tag{4.63}$$

$$b'' - 2c' - k_2 b' - k_1 d = 0, \tag{4.64}$$

$$2\lambda_1 b' + 2k_4 d + k_1 d' + k_2 c' + c'' = 0, \tag{4.65}$$

$$d'' + k_2 d' + \lambda_1 d = 0. \tag{4.66}$$

Substituting (4.63) into (4.62), we find that

$$c(x) = c_0 e^{\frac{k_4}{k_1}x}, \qquad b(x) = -\frac{c_0 k_1}{k_4} e^{\frac{k_4}{k_1}x} + c_1,$$

where c_0 and c_1 are constants. Substituting $b(x)$ and $c(x)$ into (4.64), we obtain a parametric condition

$$d = \frac{k_1 k_2 - 3k_4}{k_1^2} c. \tag{4.67}$$

Plugging $b(x)$ and $c(x)$ into (4.65), we obtain

$$2\left(\lambda_1+\frac{4k_4^2}{k_1^2}-\frac{2k_2k_4}{k_1}\right)c=0. \tag{4.68}$$

Plugging (4.67) into (4.66), we have

$$\frac{k_1k_2-3k_4}{k_1^2}\left(\frac{k_4^2}{k_1^2}+\frac{k_2k_4}{k_1}+\lambda_1\right)c=0. \tag{4.69}$$

Combining (4.68) with (4.69), we have

$$\frac{3k_4}{k_1^4}(k_1k_2-3k_4)(k_1k_2-k_4)=0.$$

Part (e): $k_1k_2=3k_4$, by equation (4.68), we obtain a parametric condition

$$\lambda_1=\frac{2k_2^2}{9}.$$

And by equations $b(x),c(x)$ and (4.67), we have

$$d(x)=0, \qquad c(x)=c_0e^{\frac{k_2}{3}x}, \qquad b(x)=-\frac{3c_0}{k_2}e^{\frac{k_2}{3}x}+c_1.$$

Then, by letting $c_0=1,c_1=0$, following procedures of **part (b)** in **case 1**, under parametric conditions

$$k_1k_2=3k_4, \qquad \lambda_1=2k_2^2/9,$$

we obtain

$$r_{ss}=k_1rr_s,$$

which is integrated as

$$r_s=\frac{k_1}{2}r^2+I_{11}, \tag{4.70}$$

where I_{11} is an arbitrary constant. Substituting equation (4.23) into (4.70), we obtain the first integral of equation (4.1) as

$$e^{\frac{2k_2}{3}x}\left(y'+\frac{k_2}{3}y+\frac{k_1}{2}y^2\right)=I_{11},$$

which is identical to the results in [3].

Part (f): $k_1k_2 = k_4$, from equation (4.68), we have

$$\lambda_1 = -2k_2^2.$$

From equations $b(x), c(x)$ and (4.67), we have

$$b(x) = -\frac{c_0}{k_2}e^{k_2x} + c_1, \qquad c(x) = c_0e^{k_2x}, \qquad d(x) = -\frac{2k_2c_0}{k_1}e^{k_2x}.$$

Assuming $c_0 = 1$ and $c_1 = 0$, we get

$$\xi = -\frac{1}{k_2}e^{k_2x}, \qquad \eta = e^{k_2x}y - \frac{2k_2}{k_1}e^{k_2x}.$$

Taking ξ and η into account, we obtain

$$r(x,y) = e^{k_2x}\left(y - \frac{2k_2}{k_1}\right), \qquad s(x,y) = e^{-k_2x}. \tag{4.71}$$

Equation (4.71) can be rewritten to the parametric form

$$x = -\frac{1}{k_2}\ln(s), \qquad y = rs + \frac{2k_2}{k_1}. \tag{4.72}$$

Using the nonlinear transformations (4.72) yields

$$\frac{\partial y}{\partial x} = -k_2\left(r_ss^2 + rs\right), \tag{4.73}$$

$$\frac{\partial^2 y}{\partial^2 x} = k_2^2\left(r_{ss}s^3 + 3r_ss^2 + rs\right). \tag{4.74}$$

Substituting equations (4.73) and (4.74) into equation (4.1), under parametric conditions

$$k_1k_2 = k_4, \qquad \lambda_1 = -2k_2^2,$$

we obtain

$$r_{ss} = \frac{k_1}{k_2}rr_s,$$

which is integrated as

$$r_s = \frac{k_1}{2k_2}r^2 + I_{12}, \tag{4.75}$$

where I_{12} is an arbitrary constant. Using the inverse transformation of (4.72) yields

$$r_s = -e^{2k_2 x}\left(y + \frac{y'}{k_2} - \frac{2k_2}{k_1}\right), \tag{4.76}$$

where $k_1k_2y + k_1y' - 2k_2^2 \neq 0$. Substituting equation (4.76) into (4.75), we obtain the first integral of equation (4.1) as

$$e^{2k_2 x}\left(2y' - 2k_2y + k_1y^2\right) = I_{12}.$$

If $k_3 = k_4 = 0$ and k_1, k_2, λ_1 are arbitrary, it is easy to check equation (4.1) is partially integrable.

Case 6: $k_2 = 0$**,** k_1, k_3, k_4 **and** λ_1 **are arbitrary.**

The second equation in (4.4) yields $a(x) = 0$. Then the determining system for $b(x), c(x)$ and $d(x)$ yields

$$b' + c = 0, \tag{4.77}$$

$$b'' - 2c' - k_1 d = 0, \tag{4.78}$$

$$-k_4 c - 2k_4 b' - 3k_3 d - k_1 c' = 0, \tag{4.79}$$

$$2\lambda_1 b' + 2k_4 d + k_1 d' + c'' = 0, \tag{4.80}$$

$$d'' + \lambda_1 d = 0. \tag{4.81}$$

Substituting (4.77) into (4.78), we have

$$d(x) = -\frac{3}{k_1}c'. \tag{4.82}$$

Similarly, substituting (4.77) and (4.82) into (4.79) gives

$$c = c_0 e^{\frac{k_1k_4}{k_1^2 - 9k_3}x},$$

where c_0 is a constant and $k_1^2 - 9k_3 \neq 0$. Then from (4.81), we can get

$$\left[\frac{k_1^2k_4^2}{(k_1^2 - 9k_3)^2} + \lambda_1\right] d = 0. \tag{4.83}$$

Taking (4.83) and (4.80) into account yields

$$-\frac{6k_4^2}{k_1^2 - 9k_3}c = 0.$$

Since k_4 is arbitrary, we obtain $c(x) = d(x) = 0$ and $b(x) = c_0$. Assuming $c_0 = 1$, then there is one infinitesimal generator

$$\chi = \partial_x.$$

This indicates that only one infinitesimal generator is found, which means the differential equation is partially integrable. If $k_2 = k_3 = 0$, k_1, k_4, λ_1 are arbitrary, (4.1) is partially integrable. But if $k_1 = k_2 = 0$, k_3, k_4, λ_1 are arbitrary, we can get the first integral of (4.1) directly, which is

$$\frac{y'^2}{2} + \frac{k_3}{4}y^4 + \frac{k_4}{3}y^3 + \frac{\lambda_1}{2}y^2 = I_{13}.$$

where I_{13} is an arbitrary constant.

Case 7: $k_2 = k_4 = 0$**,** k_1, k_3 **and** λ_1 **are arbitrary.**

The determining system for $a(x)$, $b(x)$, $c(x)$, and $d(x)$ is reduced to

$$-3a'' - k_1 b' - 3\lambda_1 a - k_1 c = 0, \tag{4.84}$$

$$b'' - 2c' - k_1 d = 0, \tag{4.85}$$

$$-6k_3 c - 2k_1\lambda_1 a - 2k_1 a'' - 6k_3 b' = 0, \tag{4.86}$$

$$-\lambda_1 a' - 3k_3 d - k_1 c' - a''' = 0, \tag{4.87}$$

$$2\lambda_1 b' + k_1 d' + c'' = 0, \tag{4.88}$$

$$d'' + \lambda_1 d = 0. \tag{4.89}$$

From (4.84), we find that

$$a'' + \lambda_1 a = -\frac{k_1}{3}(b' + c).$$

Similarly, by (4.86), we have

$$a'' + \lambda_1 a = -\frac{3k_3}{k_1}(b' + c).$$

We assume that $b' + c = 0$, then from (4.85), we obtain

$$c' = -\frac{k_1}{3}d. \tag{4.90}$$

Since $a'' + \lambda_1 a = 0$, then (4.87) becomes

$$3k_3 d + k_1 c' = 0. \tag{4.91}$$

By (4.90) and (4.91), we get a parametric condition

$$9k_3 = k_1^2.$$

Taking (4.89) and (4.90) into account, we have

$$c'' + \lambda_1 c = 0. \tag{4.92}$$

Then by (4.88), (4.90) and (4.92), we conclude another parametric condition

$$\lambda_1 = -\frac{k_1^2}{9},$$

and

$$c(x) = d(x) = c_0 e^{-\frac{k_1}{3}x}, \qquad b(x) = \frac{3c_0}{k_1} e^{-\frac{k_1}{3}x} + c_1,$$

where c_0 and c_1 are constants. As a result, we can get

$$a(x) = c_2 e^{-\frac{k_1}{3}x},$$

where c_2 is constant. The infinitesimal generator is

$$\chi = c_0\chi_0 + c_1\chi_1 + c_2\chi_2,$$

where

$$\chi_0 = \frac{3}{k_1} e^{-\frac{k_1}{3}x}\partial_x + e^{-\frac{k_1}{3}x}(y+1)\partial_y,$$
$$\chi_1 = \partial_x, \qquad \chi_2 = y e^{-\frac{k_1}{3}x}\partial_x.$$

We choose χ_0 as the generator to get canonical coordinates. Here

$$\xi = \frac{3}{k_1} e^{-\frac{k_1}{3}x}, \qquad \eta = e^{-\frac{k_1}{3}x}(y+1).$$

Using ξ and η, we obtain

$$r(x,y) = e^{-\frac{k_1}{3}x}(y+1), \qquad s(x,y) = e^{\frac{k_1}{3}x}. \tag{4.93}$$

From (4.93), we can solve x and y as follows

$$x = \frac{3}{k_1}\ln(s), \qquad y = rs - 1.$$

From above equations, we have

$$\frac{\partial y}{\partial x} = \frac{k_1}{3}\left(r_s s^2 + rs\right), \tag{4.94}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{k_1^2}{9}\left(r_{ss}s^3 + 3r_s s^2 + rs\right). \tag{4.95}$$

Plugging equations (4.94) and (4.95) into equation (4.1), under parametric conditions $9k_3 = k_1^2$ and $9\lambda_1 = -k_1^2$, we obtain

$$r_{ss} = -3r_s r - r^3,$$

which is integrated as

$$\frac{(r_s + r^2)^2}{2r_s + r^2} = I_{14}, \tag{4.96}$$

where I_{14} is an arbitrary constant. Using the inverse transformation of (4.93) yields

$$r_s = e^{-\frac{2k_1}{3}x}\left(\frac{3}{k_1}y' - y - 1\right), \tag{4.97}$$

where $3y' - k_1 y - k_1 \neq 0$. Substituting (4.97) into (4.96), we get the first integral of equation (4.1) as

$$e^{-\frac{2k_1}{3}x}\left[\frac{(k_1 y^2 + k_1 y + 3y')^2}{k_1^2 y^2 + 6k_1 y' - k_1^2}\right] = I_{14}.$$

Case 8: $k_4 = 0$ **and** k_1, k_2, k_3, λ_1 **are arbitrary.**

By the second equation in (4.94), we obtain $a(x) = 0$. Then the determining equations for $b(x), c(x)$ and $d(x)$ is reduced to

$$b' + c = 0, \tag{4.98}$$

$$b'' - 2c' - k_2 b' - k_1 d = 0, \tag{4.99}$$

$$3k_3 d + k_1 c' = 0, \tag{4.100}$$

$$2\lambda_1 b' + k_1 d' + k_2 c' + c'' = 0, \tag{4.101}$$

$$d'' + k_2 d' + \lambda_1 d = 0. \tag{4.102}$$

Substituting (4.98) and (4.100) into (4.99), we have

$$c(x) = c_0 e^{\frac{3k_2k_3}{9k_3-k_1^2}x},$$

where c_0 is a constant. We assume that $9k_3 - k_1^2 \neq 0$, because if $9k_3 - k_1^2 = 0$, then $c(x)$=0, which is a partially integrable case. By (4.100), we find that

$$d(x) = \frac{-k_1k_2}{9k_3 - k_1^2} c(x).$$

Plugging $d(x)$ into (4.102), we obtain

$$\left[\frac{36k_2^2k_3^2 - 3k_1^2k_2^2}{(9k_3 - k_1)^2} + \lambda_1\right] d(x) = 0.$$

If $d(x) = 0$, which means $c(x) = 0$, then we can not obtain the first integral. Thus we assume that $d(x) \neq 0$, which means

$$\lambda_1 = -\frac{36k_2^2k_3^2 - 3k_1^2k_2^2}{(9k_3 - k_1)^2}. \tag{4.103}$$

Substituting (4.103), $b(x), c(x)$ and $d(x)$ into (4.101), we have

$$\frac{12k_2^2k_3}{9k_3 - k_1^2} c(x) = 0,$$

which is a contradiction with k_2, k_3 are arbitrary. Hence, we conclude Eq. (4.1) in this case is partially integrable. For other cases, we can also obtain the partially integrable equations or we can get the first integral directly.

4.2 Integrable cases identical to Chapter 3

If $q = 2$, the determining system for $a(x), b(x), c(x)$ and $d(x)$ is different from the case q is arbitrary. For example, we can combine the coefficients for y^q and y^2 together. But we can check that the parametric conditions for $q = 2$ and q is arbitrary to determine the first integral is identical, same as in [3]. So we just need plugging $q = 2$ into the first integrals in Chapter 3 to get the first integrals for q=2, which are as follows:

(1) $k_1 = k_4 = 0$, k_2, k_3, λ_1 are arbitrary.

$$e^{\frac{3k_2}{2}x}\left(\frac{k_2^2}{32}y^2 + \frac{k_2}{4}yy' + \frac{y'^2}{2} + \frac{k_3}{6}y^6\right) = I, \qquad \text{with} \quad \lambda_1 = \frac{3}{16}k_2^2.$$

(2) $k_1=0, k_3=0$ and k_2, k_4, λ_1 are arbitrary. In this case, Eq. (3.1) is the Damped duffing equation.

$$e^{\frac{4k_2}{3}x}\left(\frac{y'^2}{2}+\frac{k_2}{3}yy'+\frac{k_2^2}{18}y^2+\frac{k_4}{4}y^4\right)=I_1, \qquad \text{with} \quad \lambda_1=\frac{2}{9}k_2^2.$$

(3) $k_3=0$ and k_1, k_2, k_4, λ_1 are arbitrary. In this case, Eq. (3.1) is the force-free Duffing-van der Pol oscillator.

$$e^{\frac{3k_2}{4}x}\left(y'+\frac{k_2}{4}y+\frac{k_1}{3}y^3\right)=I_2, \qquad \text{with} \quad \lambda_1=\frac{3}{16}k_2^2, \quad k_4=\frac{k_1k_2}{4}.$$

The results on the first integral established in [12, 15] agree with this case.

(4) $k_1, k_2, k_3, k_4, \lambda_1$ are arbitrary.

$$\frac{2k_1}{\sqrt{k_1^2-12k_3}}\tanh^{-1}\left[\frac{k_1y^3+\frac{3k_2}{2}y+6y'}{\sqrt{k_1^2-12k_3y^3}}\right]+\frac{3k_2}{2}x$$
$$+\ln\left[4y^3(4k_3y^3+k_1k_2y+4k_1y')+3(k_2y+4y')^2\right]=I_3,$$

where $k_3\neq\frac{k_1^2}{12}$.

$$\frac{3k_2}{4}x+\ln\left(2k_1y^3+3k_2y+12y'\right)-\frac{3(k_2y+4y')}{2k_1y^3+3k_2y+12y'}=I_3,$$

where $k_3=\frac{k_1^2}{12}$. In this case,

$$\lambda_1=\frac{3}{16}k_2^2, \quad k_4=\frac{k_1k_2}{4}.$$

Similarly, if $q=\frac{1}{2}$, $-\frac{1}{2}$, $\frac{1}{3}$, or -$\frac{1}{3}$, we can substitute the values of q into the first integrals in chapter 3, respectively to get the first integrals for different values of q.

If $q=0$, the determining system for ξ and η is different from Chapter 3, but it is easy to get first integral in this case. Then the equation (3.1) becomes

$$y''+(k_1+k_2)y'+(k_3+k_4+\lambda_1)y=0.$$

Assume that

$$k_1+k_2=\alpha, \qquad k_3+k_4+\lambda_1=\beta.$$

(i): If $\alpha^2=4\beta$, then the first integral is

$$\frac{\alpha y}{\alpha y+2y'}+\ln(\alpha y+2y')=I_{15}.$$

(ii): If $\alpha^2 \neq 4\beta$, the first integral is

$$\frac{1}{2}\ln\left[\frac{\alpha y'}{y}+\beta+\frac{y'^2}{y^2}\right]+\ln(y)-\frac{\alpha}{\sqrt{4\beta-\alpha^2}}\tan^{-1}\left[\frac{\alpha+\frac{2y'}{y}}{\sqrt{4\beta-\alpha^2}}\right]=I_{15}.$$

In Chapter 3, if $q=-1$ or $q=-2$, the function $\eta(x,y)$ is undefined. But if $q=-1$ or $q=-2$, we can get new infinitesimal generators, which means new first integrals will be generated. In the following two sections, we will show the new results induced by the cases $q=-1$ and $q=-2$.

4.3 Integrable cases of q=-1

If $q=-1$, the equation (3.1) becomes

$$y''+\left(\frac{k_1}{y}+k_2\right)y'+\frac{k_3}{y}+\lambda_1 y+k_4=0. \tag{4.104}$$

Plugging $q=-1$ into (3.6) and (3.7), we obtain

$$\xi(x,y)=a(x)y+b(x),$$
$$\eta(x,y)=-2k_1a(x)y\ln(y)+\{a'(x)-k_2a(x)\}y^2+\{2k_1a(x)+c(x)\}y+d(x),$$

where $a(x),b(x),c(x)$ and $d(x)$ are functions of x to be determined. Substituting $\xi(x,y)$ and $\eta(x,y)$ into (3.4) yields

$$\begin{aligned}
&[y^0]: 2c'-b''+k_2b'+3k_4a+k_1k_2a=0,\\
&[y]: a''-k_2a'+\lambda_1a=0,\\
&[y^{-1}]: k_1b'+3k_3a-2k_1^2a-k_1c=0,\\
&[y^{-2}]: k_1d=0,\\
&[\ln(y)]: k_1a'=0,\\
&[y^{-1}\ln(y)]: k_1^2a=0.
\end{aligned} \tag{4.105}$$

Similarly, substituting $\xi(x,y)$ and $\eta(x,y)$ into (3.5) yields

$$
\begin{aligned}
&[y^0] : d'' + 2k_4 b' + 3k_2 k_3 a - k_4 c - k_3 a' + 2k_1^2 a' + k_1 c' + k_2 d' + \lambda_1 d = 0,\\
&[y] : 3k_1 a'' + 2k_2 k_4 a + 2\lambda_1 k_1 a + k_1 k_2 a' + c'' + 2\lambda_1 b' + k_2 c' = 0,\\
&[y^2] :, k_2 \lambda_1 a + \lambda_1 a' - k_2^2 a' + a''' = 0,\\
&[y^{-1}] : 2k_3 b' - 2k_3 c - 2k_1 k_3 a + k_1 d' = 0,\\
&[y \ln(y)] : k_1 a'' + k_1 k_2 a' = 0,\\
&[\ln(y)] : k_1 k_4 a - k_1^2 a' = 0,\\
&[y^{-1}\ln(y)] : k_1 k_3 a = 0.
\end{aligned}
\tag{4.106}
$$

Case 1: k_1, k_2, k_3, k_4 **and** λ_1 **are arbitrary**

By the forth equation and the last equation in (4.105) , we have $d(x) = 0$ and $a(x) = 0$. Then the determining equations for $b(x)$ and $c(x)$ is

$$2c' - b'' + k_2 b' = 0, \tag{4.107}$$

$$b' - c = 0, \tag{4.108}$$

$$2k_4 b' - k_4 c + k_1 c' = 0, \tag{4.109}$$

$$c'' + 2\lambda_1 b' + k_2 c' = 0. \tag{4.110}$$

Plugging (4.108) into (4.107), we have

$$c(x) = c_0 e^{-k_2 x}, \qquad b(x) = -\frac{c_0}{k_2} e^{-k_2 x} + c_1,$$

where c_0 and c_1 are constants. Substituting $b(x)$ and $c(x)$ into (4.110), we have $\lambda_1 = 0$. If $\lambda_1 \neq 0$, then we get $c(x)$=0, which is a partially integrable case. Similarly, from (4.109), we get $k_4 = k_1 k_2$. Let $c_0 = 1, c_1 = 0$, we obtain

$$\xi = -\frac{1}{k_2} e^{-k_2 x}, \qquad \eta = e^{-k_2 x} y.$$

Using ξ and η, we can get canonical coordinates

$$r(x,y) = e^{k_2 x} y, \qquad s(x,y) = -e^{k_2 x}.$$

From $r(x,y)$ and $s(x,y)$, we can solve x and y as follows

$$x = \frac{1}{k_2}\ln(-s), \qquad y = -rs^{-1}.$$

Using the nonlinear transformation of x and y yields

$$\frac{\partial y}{\partial x} = -k_2\left(r_s - rs^{-1}\right), \tag{4.111}$$

$$\frac{\partial^2 y}{\partial^2 x} = -k_2^2\left(r_{ss}s - r_s + rs^{-1}\right). \tag{4.112}$$

Substituting (4.111) and (4.112) into equation (4.104), under parametric conditions $\lambda_1 = 0$ and $k_4 = k_1k_2$, we get

$$r_{ss} = \frac{k_1}{k_2}r^{-1}r_s - \frac{k_3}{k_2^2}r^{-1}, \tag{4.113}$$

which is integrated as

$$r_s = \frac{k_3}{k_1k_2}\left[W\left(\frac{e^{\frac{I_{16}k_1^2k_2^2}{k_3}-1}r^{\frac{k_1^2}{k_3}}}{k_3}\right)+1\right], \tag{4.114}$$

where I_{16} is an arbitrary constant. **W** is the Lambert W function, also called the omega function or product logarithm, is a set of functions, namely the branches of the inverse relation of the function $f(z) = ze^z$ where e^z is the exponential function and z is any complex number. Using the inverse transformation of $r(x,y)$ and $s(x,y)$ yields

$$r_s = -y - \frac{1}{k_2}y', \tag{4.115}$$

where $k_2y + y' \neq 0$. Substituting (4.115) into (4.114), we obtain

$$\frac{k_3}{k_1k_2}\left[W\left(\frac{e^{\frac{I_{16}k_1^2k_2^2}{k_3}-1}e^{\frac{k_1^2k_2}{k_3}x}y^{\frac{k_1^2}{k_3}}}{k_3}\right)+1\right] + y + \frac{1}{k_2}y' = 0.$$

If $k_3 = 0$, then (4.113) can be integrated as

$$r_s = \frac{k_1}{k_2}\ln(r) + I_{17}, \tag{4.116}$$

where I_{17} is an arbitrary constant. Substituting (4.115) into (4.116), we have the first integral of equation (4.104) as

$$y + \frac{1}{k_2}y' + \frac{k_1}{k_2}\ln(y) + k_1x = I_{17}.$$

Case 2: $k_1 = 0$, k_2, k_3, k_4 **and** λ_1 **are arbitrary.**

By the third equation in (4.105), we obtain $a(x) = 0$. Then the determining equations for $b(x), c(x)$ and $d(x)$ is reduced as

$$2c' - b'' + k_2 b' = 0, \tag{4.117}$$

$$b' = c, \tag{4.118}$$

$$c'' + 2\lambda_1 b' + k_2 c' = 0, \tag{4.119}$$

$$2k_4 b' - k_4 c + d'' + k_2 d' + \lambda_1 d = 0. \tag{4.120}$$

Substituting (4.118) into (4.117), we get

$$c(x) = c_0 e^{-k_2 x}, \qquad b(x) = -\frac{c_0}{k_2} e^{-k_2 x} + c_1,$$

where c_0 and c_1 are constants. From (4.119), we find that $\lambda_1 = 0$. Then, plugging $b(x)$ and $c(x)$ into (4.120), we obtain

$$d(x) = \frac{c_0 k_4}{k_2} x e^{-k_2 x} + c_2 e^{-k_2 x} + c_3.$$

(i): If $k_4 = 0$, taking $b(x), c(x)$ and $d(x)$ into account, we get

$$\chi_0 = -\frac{1}{k_2} e^{-k_2 x} \partial_x + e^{-k_2 x} y \partial_y, \qquad \chi_1 = \partial_y, \qquad \chi_2 = e^{-k_2 x} \partial_y.$$

We choose χ_0 to get canonical coordinates, following procedures of **case 1** in this section, under parametric condition $k_4 = \lambda_1 = 0$, we can get

$$r_{ss} = -\frac{k_3}{k_2^2} r^{-1},$$

which can be integrated as

$$\frac{r_s^2}{2} + \frac{k_3}{k_2^2} \ln(r) = I_{18}, \tag{4.121}$$

where I_{18} is a constant. Substituting (4.115) into (4.121), we get the first integral of equation (4.104)

$$\frac{y^2}{2} + \frac{1}{k_2} y y' + \frac{1}{2k_2^2} y'^2 + \frac{k_3}{k_2} x + \frac{k_3}{k_2^2} \ln(y) = I_{18}.$$

(ii): If $k_4 \neq 0$, we can get three infinitesimal generators.

$$\chi_0 = -\frac{1}{k_2} e^{-k_2 x}\partial_x + e^{-k_2 x}\left(y + \frac{k_4}{k_2}x\right)\partial_y, \qquad \chi_1 = \partial_x, \qquad \chi_2 = e^{-k_2 x}\partial_y.$$

In this case, considering χ_0, it is hard to apply Lie point symmetry to get the first integral. And if we taking χ_2 into account, we find that y is a constant.

Case 3: $k_1 = k_3 = 0$, k_2, k_4 **and** λ_1 **are arbitrary.**

In this case, the system of determining equations of $a(x), b(x), c(x)$ and $d(x)$ is as following

$$a'' - k_2 a' + \lambda_1 a = 0, \tag{4.122}$$

$$2c' - b'' + k_2 b' + 3k_4 a = 0, \tag{4.123}$$

$$k_2 \lambda_1 a + \lambda_1 a' - k_2^2 a' + a''' = 0, \tag{4.124}$$

$$c'' + 2\lambda_1 b' + k_2 c' + 2k_2 k_4 a = 0, \tag{4.125}$$

$$d'' + k_2 d' + \lambda_1 d + 2k_4 b' - k_4 c = 0. \tag{4.126}$$

Considering (4.126), we assume that $2b' - c = 0$, then we get

$$d'' + k_2 d + \lambda_1 d = 0. \tag{4.127}$$

From (4.122) and (4.127), we find that

$$a(x) = c_0 e^{px}, \qquad d(x) = c_1 e^{(p-k_2)x},$$

where $p = \frac{k_2 \pm \sqrt{k_2^2 - 4\lambda_1}}{2}$ and c_0, c_1 are arbitrary constants.. From the parametric condition $2b' - c = 0$ and (4.123), we obtain

$$c(x) = c_2 e^{-\frac{k_2}{3}x} - \frac{6k_4 c_0}{3p + k_2} e^{px}, \qquad b(x) = -\frac{3c_2}{2k_2} e^{-\frac{k_2}{3}x} - \frac{3k_4 c_0}{(3p + k_2)p} e^{px} + c_3,$$

where c_2 and c_3 are arbitrary constants. If $p = -k_2/3$, we obtain $d(x) = 0$. Substituting $a(x), b(x)$, $c(x)$ into (4.125), we obtain parametric conditions

$$\lambda_1 = \frac{2}{9}k_2^2, \qquad p = \frac{k_2}{3}, \tag{4.128}$$

which satisfies (4.124). Under parametric conditions, we obtain four infinitesimal generators

$$\chi_0 = e^{\frac{k_2}{3}x}\left(-\frac{9k_4}{2k_2^2}+y\right)\partial_x + e^{\frac{k_2}{3}x}\left(-\frac{3k_4}{k_2}y\right)\partial_y, \qquad \chi_1 = e^{-\frac{k_2}{3}x}\partial_y,$$
$$\chi_2 = -\frac{3}{2k_2}e^{-\frac{k_2}{3}x}\partial_x + e^{-\frac{k_2}{3}x}y\partial_y, \qquad \chi_3 = \partial_x.$$

Note that these four infinitesimal generators are concluded by two cases: $d(x) = 0$ or $d(x) \neq 0$. Same as section (4.1), in order to get the first integral easily, we choose χ_2 to get canonical coordinates as

$$r(x,y) = e^{\frac{2k_2}{3}x}y, \qquad s(x,y) = -2e^{\frac{k_2}{3}x}.$$

Then we can solve for x and y as

$$x = \frac{3}{k_2}\ln\left(-\frac{s}{2}\right), \qquad y = 4rs^{-2}.$$

Using the nonlinear transformation yields

$$\frac{\partial y}{\partial x} = \frac{4k_2}{3}\left(r_s s^{-1} - 2rs^{-2}\right), \tag{4.129}$$
$$\frac{\partial^2 y}{\partial^2 x} = \frac{4k_2^2}{9}\left(r_{ss} - 3r_s s^{-1} + 4rs^{-2}\right). \tag{4.130}$$

Substituting (4.129) and (4.130) into equation (4.104) , under parametric condition (4.128), we obtain

$$r_{ss} = -\frac{9k_4}{4k_2^2},$$

which can be integrated as

$$2r_s^2 + \frac{9k_4}{4k_2^2}r = I_{19}, \tag{4.131}$$

where I_{19} is a constant. Then by $r(x,y)$ and $s(x,y)$, we obtain

$$r_s = -e^{\frac{k_2}{3}x}\left(y + \frac{3}{2k_2}y'\right), \tag{4.132}$$

where $2k_2y+3y' \neq 0$. Substituting (4.132) into (4.131), we get the first integral of equation (4.104) as

$$e^{\frac{2k_2}{3}x}\left(2y^2 + \frac{6}{k_2}yy' + \frac{18}{4k_2^2}y'^2 + \frac{9k_4}{k_2^2}y\right) = I_{19}.$$

Similarly, in this chapter, we will not concern the partially integrable cases or the cases in which at least three parameters are equal to zero.

4.4 Integrable cases of q=-2

If $q = -2$, the equation (3.1) becomes

$$y'' + (k_1 y^{-2} + k_2)y' + k_3 y^{-3} + k_4 y^{-1} + \lambda_1 y = 0. \tag{4.133}$$

Similarly to section (4.4), we can get

$$\xi = a(x)y + b(x), \qquad \eta = 2k_1 a(x)\ln(y) + \{a'(x) - k_2 a(x)\}y^2 + c(x)y + d(x).$$

And there are two systems to derive functions $a(x), b(x), c(x)$ and $d(x)$. The first one is

$$\begin{aligned}
&[y^0] : 2c' - b'' + k_2 b' = 0,\\
&[y] : a'' - k_2 a' + \lambda_1 a = 0,\\
&[y^{-1}] : 3k_1 a' + 3k_4 a - 2k_1 k_2 a = 0,\\
&[y^{-2}] : 2k_1 c - k_1 b' = 0,\\
&[y^{-3}] : 3k_3 a + 2k_1 d = 0,\\
&[y^{-3}\ln(y)] : k_1^2 a = 0.
\end{aligned} \tag{4.134}$$

The second one is

$$\begin{aligned}
&[y^0] : d'' + k_2 d' + \lambda_1 d + 3k_2 k_4 a - 2k_1 \lambda_1 a - k_4 a' + k_1 a'' - k_1 k_2 a' = 0,\\
&[y] : c'' + 2\lambda_1 b' + k_2 c' = 0,\\
&[y^2] :, k_2 \lambda_1 a + \lambda_1 a' - k_2^2 a' + a''' = 0,\\
&[y^{-1}] : -2k_4 b' + 2k_4 c - k_1 c' = 0,\\
&[y^{-2}] : 3k_3 a' - 5k_2 a + 2k_1 k_4 a + k_4 d - k_1 d' = 0,\\
&[y^{-3}] : 2k_3 c - k_3 b' = 0,\\
&[y^{-4}] : 2k_1 k_3 a + 3k_3 d = 0,\\
&[\ln(y)] : k_1 a'' + k_1 k_2 a' + k_1 \lambda_1 a = 0,\\
&[y^{-2}\ln(y)] : k_1 k_4 a - k_1^2 a = 0,\\
&[y^{-4}\ln(y)] : k_1 k_3 a = 0.
\end{aligned} \tag{4.135}$$

Case1: k_1, k_2, k_3, k_4 **and** λ_1 **are arbitrary.**

By the last equation and the seventh euqtion in (4.135), we have $a(x) = d(x) = 0$. Then the determining equations for $b(x)$ and $c(x)$ is reduced to

$$b' = 2c,$$

$$2c' - b'' + k_2 b' = 0,$$

$$c'' + 2\lambda_1 b' + k_2 c' = 0,$$

$$-2k_4 b' + 2k_4 c - k_1 c' = 0.$$

Substituting $b' = 2c$ into $2c' - b'' + k_2 b' = 0$, we obtain

$$k_2 b' = 0,$$

which means $b' = c = 0$. Then the original equation is partially integrable. But under some parametric conditions, we can generate a Lie point symmetry to get the first integral.

Case 2: $k_2 = 0$**,** k_1, k_3, k_4 **and** λ_1 **are arbitrary.**

In this case, we can get $a(x) = d(x) = 0$. Similarly, the determining system is reduced to

$$b' = 2c, \tag{4.136}$$

$$c'' + 2\lambda_1 b' = 0, \tag{4.137}$$

$$-2k_4 b' + 2k_4 c - k_1 c' = 0. \tag{4.138}$$

Substituting (4.136) into (4.137), we obtain

$$c(x) = c_0 e^{\frac{-2k_4}{k_1} x}, \qquad b(x) = -\frac{c_0 k_1}{k_4} e^{\frac{2k_4}{k_1} x},$$

where c_0 and c_1 are arbitrary constants. Similarly, plugging $b(x)$ and $c(x)$ into (4.138), we get a parametric condition

$$\lambda_1 = -\frac{k_4^2}{k_1^2}. \tag{4.139}$$

Assume that $c_0 = 1, c_1 = 0$, we have

$$\xi = -\frac{k_1}{k_4} e^{-\frac{2k_4}{k_1} x}, \qquad \eta = e^{-\frac{2k_4}{k_1} x} y.$$

Then, it is readily to get

$$r(x,y) = e^{\frac{k_4}{k_1}x}y, \qquad s(x,y) = -\frac{1}{2}e^{\frac{2k_4}{k_1}x}.$$

Using $r(x,y)$ and $s(x,y)$, we have

$$x = \frac{k_1}{2k_4}\ln(-2s), \qquad y = r(-2s)^{-\frac{1}{2}}.$$

Then by the nonlinear transformation, we obtain

$$\frac{\partial y}{\partial x} = \frac{2k_4}{k_1}(-2)^{-\frac{1}{2}}\left(r_s s^{\frac{1}{2}} - \frac{1}{2}rs^{-\frac{1}{2}}\right), \tag{4.140}$$

$$\frac{\partial^2 y}{\partial^2 x} = \frac{4k_4^2}{k_1^2}(-2)^{-\frac{1}{2}}\left(r_{ss}s^{\frac{3}{2}} + \frac{1}{4}rs^{-\frac{1}{2}}\right). \tag{4.141}$$

Substituting (4.140) and (4.141) into equation (4.133), under parametric condition (4.139), we obtain

$$r_{ss} = \frac{k_1^2}{k_4}r_s r^{-2} - \frac{k_1^2 k_3}{k_4^2}r^{-3}. \tag{4.142}$$

(i): If $k_3 \neq 0$, (4.142) can be integrated as

$$\frac{1}{2}\ln\left[\frac{k_1^2(k_3 - k_4 r r_s) - k_4^2 r^2 r_s^2}{k_3}\right] + \frac{k_1}{\sqrt{k_1^2 + 4k_3}}\tanh^{-1}\left[\frac{k_1^2 + 2k_4 r r_s}{k_1\sqrt{k_1^2 + 4k_3}}\right] - \ln(r) = I_{20}, \tag{4.143}$$

where I_{20} is a constant.

If $k_1^2 + 4k_3 \neq 0$, using $r(x,y)$ and $s(x,y)$, we can get

$$r_s = -e^{-\frac{k_4}{k_1}x}\left(y + \frac{k_1}{k_4}y'\right), \tag{4.144}$$

where $k_4 y + k_1 y' \neq 0$. Substituting (4.144) into (4.143), we get the first integral of equation (4.133) is

$$\frac{1}{2}\ln\left[\frac{k_1^2(k_3 + k_4 y^2 + k_1 y y') - (k_4 y^2 + k_1 y')^2}{k_3}\right] +$$

$$\frac{k_1}{\sqrt{k_1^2 + 4k_3}}\tanh^{-1}\left[\frac{k_1^2 - 2k_4 y^2 - 2k_1 y y'}{k_1\sqrt{k_1^2 + 4k_3}}\right] - \ln(y) - \frac{k_4}{k_1}x = I_{20}.$$

If $k_1^2 + 4k_3 = 0$, then (4.142) can be integrated as

$$\frac{1}{\frac{2k_4 r r_s}{k_1^2} + 1} + \ln\left[\frac{2k_4 r r_s}{k_1^2} + 1\right] - \ln(r) = I_{20}. \tag{4.145}$$

Substituting (4.144) into (4.145), we have the first integral of equation (4.133) is

$$\frac{k_1^2}{k_1^2-2k_4y^2-2k_1yy'}+\ln\left[\frac{k_1^2-2k_4y^2-2k_1yy'}{k_1^2}\right]-\ln(y)-\frac{k_4}{k_1}x=I_{20}.$$

(ii): If $k_3=0$, (4.142) can be integrated as

$$r_s=-\frac{k_1^2}{k_4}r^{-1}+I_{21}, \tag{4.146}$$

where I_{21} is a constant. Substituting (4.144) into (4.146), we obtain the first integral of equation (4.133) is

$$e^{-\frac{k_4}{k_1}x}\left(\frac{k_1^2}{k_4}y^{-1}-y-\frac{k_1}{k_4}y'\right)=I_{21}.$$

Case 3: $k_1=k_3=0$**,** k_2,k_3,k_4,λ_1 **are arbitrary.**

By the third equation in (4.134) and the fifth equation in (4.135), we have $a(x)=d(x)=0$. Then we obtain

$$2c'-b''+k_2b'=0, \tag{4.147}$$

$$c''+2\lambda_1b'=0, \tag{4.148}$$

$$b'=c. \tag{4.149}$$

Substituting (4.149) into (4.147), we get

$$c=c_0e^{-k_2x}, \qquad b=-\frac{c_0}{k_2}e^{-k_2x}+c_1,$$

where $c_0=1,c_1=0$. Substituting $b(x)$ and $c(x)$ into (4.148), we obtain λ_1=0. Following **case 1** in section (4.4), we get

$$r_{ss}=-\frac{k_4}{k_2^2}r^{-1},$$

which can be integrated as

$$r_s^2=-\frac{2k_4}{k_2^2}\ln(r)+I_{22}. \tag{4.150}$$

Substituting (4.115) into (4.150), we get the first integral of equation (4.133) is

$$y^2+\frac{2}{k_2}yy'+\frac{1}{k_2^2}y'^2+\frac{2k_4}{k_2}x+\frac{2k_4}{k_2^2}\ln(y)=I_{22}.$$

For other cases, we can only get the infinitesimal generator as $\chi=\partial_x$ or we can get the first integral directly. So we will not mention them in the paper.

CHAPTER V

APPLICATION OF FIRST INTEGRALS

Nonlinear oscillator

$$y'' + (k_1 y^q + k_2)y' + k_3 y^{2q+1} + k_4 y^{q+1} + \lambda_1 y = 0, q \in R.$$

can be transformed from a partial differential equation. In this chapter, we choose two integrable equations from above class to study their exact solutions and the dynamics, which are presented in [20].

During the past few decades, a great number of works have been published to deal with traveling solitary wave solutions of the compound Burgers-Korteweg-de Vries(compound Burgers-KdV) equation [7, 9]

$$u_t + \alpha_1 u u_x + \beta_1 u^2 u_x + \mu_1 u_{xx} - s_1 u_{xxx} = 0, \tag{5.1}$$

and the Burgers-KdV equation

$$u_t + \alpha u u_x + \beta u_{xx} + s u_{xxx} = 0. \tag{5.2}$$

Assume that (5.1) and (5.2) have traveling solutions of the form

$$u = \phi(\xi), \qquad \xi = x - vt. \tag{5.3}$$

Substituting (5.3) into (5.1) and (5.2), then performing one integration yields [7, 9]

$$\phi'' + k_2\phi' + k_3\phi^3 + k_4\phi^2 + \lambda_1\phi + d = 0, \tag{5.4}$$

where $k_2 = -\frac{\mu}{s_1}$, $k_3 = -\frac{\beta}{3s_1}$, $k_4 = -\frac{\alpha_1}{2s_1}$, $\lambda_1 = -\frac{v}{s_1}$, d is an arbitrary constant and

$$\phi'' + k_2\phi' + k_4\phi^2 + \lambda_1\phi + d = 0, \tag{5.5}$$

where $k_2=-\frac{\beta}{s}$, $k_4=-\frac{\alpha}{2s}$, $\lambda_1=-\frac{v}{s}$, d is an arbitrary constant. By a coordinate translation of ϕ, for simplicity, we can take $d=0$ in (5.4) and (5.5).

Cubic and quadratic nonlinear oscillator with damping can be changed to an equivalent dynamical system.

5.1 Exact solutions and dynamics of the integrable quadratic oscillator with damping

Eq. (5.5) is a quadratic nonlinear oscillator with damping, which is equivalent to the planar dynamical system

$$\begin{cases} \dfrac{d\phi}{d\xi}=y, \\ \dfrac{dy}{d\xi}=-k_2y-k_4\phi^2-\lambda_1\phi. \end{cases} \tag{5.6}$$

If $k_4\neq 0$, there exists two regular equilibrium points of system (5.6): $O(0,0)$ and $A(-\frac{\lambda_1}{k_4},0)$. Let $M(\phi_j,0)$ be the coefficient matrix of the linearized system of (5.6) at the equilibrium point $(\phi_j,0)$. Then we have

$$\det M(0,0)=\lambda_1, \qquad \det M(-\frac{\lambda_1}{k_4},0)=-\lambda_1, \qquad \mathrm{Trace}M(0,0)=-k_2,$$

and

$$(\mathrm{Trace}\ M(0,0))^2-4J(0,0)=k_2^2-4\lambda_1.$$

Following Chapter 4.1, if $\lambda_1=\frac{6}{25}k_2^2$, then system (5.6) is integrable, which has the first integral depending on ξ as follows:

$$H_1(\phi,y,\xi)=e^{\frac{6}{5}k_2\xi}\left[\frac{1}{2}\left(y+\frac{2}{5}k_2\phi\right)^2+\frac{1}{3}k_4\phi^3\right]=h, \tag{5.7}$$

where h is a real constant. For an equilibrium point of a planar integrable system, if $J<0$, then the equilibrium point is a saddle point; if $J>0$ and $(\mathrm{Trace}M)^2-4J<0$, then it is a focus point. if $J>0$ and $(\mathrm{Trace}M)^2-4J\geq 0$, then it is a node point. Clearly, the equilibrium point $O(0,0)$ is a node point and if $k_2>0$ ($k_2<0$), it is stable (unstable). And the other equilibrium point $A(-\frac{\lambda_1}{k_4},0)$ is a saddle point.

(i): The case of $h=0$

Obviously, when $h = 0$, $H_1(\phi, y, \xi)$ implies that there are two invariant curve solutions of system (5.6) given by

$$y = -\frac{2}{5}k_2\phi \pm \sqrt{-\frac{2}{3}k_4\phi^3},$$

where consists of three orbits of system (5.6). In this paper, we consider the case of $k_2 > 0$ and $k_4 > 0$. For other cases, the results are similar. For the heteroclinic orbit connecting equilibrium points O and A defined by $y = -\frac{2}{5}k_2\phi - \sqrt{-\frac{2}{3}k_4\phi^3}$ and using the first equation of system (5.6), we have

$$\phi(\xi) = \frac{6k_2^2}{-25k_4\left(1 + u_{01}\exp\left(\frac{1}{5}k_2\xi\right)\right)^2}, \tag{5.8}$$

where

$$u_{01} = 1 + \frac{2k_2}{5\sqrt{-\frac{2}{3}k_4\phi_{01}}} > 0 \qquad \text{and} \qquad -\frac{6k_2^2}{25k_4} < \phi_{01} < 0.$$

Formula (5.8) gives rise to a kink-profile wave solution of equation (5.5), because when $\xi \to \infty$ it has $\phi(\xi) \to 0$, and when $\xi \to -\infty$ it has $\phi(\xi) \to -\frac{6k_2^2}{25k_4}$.

For the unstable manifold of the saddle A defined by $y = -\frac{2}{5}k_2\phi - \sqrt{-\frac{2}{3}k_4\phi^3}$, by using the first equation of system (5.6), we have

$$\phi(\xi) = -\frac{6k_2^2}{25k_4\left(1 + u_{02}\exp\left(\frac{1}{5}k_2\xi\right)\right)^2}, \tag{5.9}$$

where

$$u_{02} = -1 - \frac{2k_2}{5\sqrt{-\frac{2}{3}k_4\phi_{02}}} < 0 \;\; \text{and} \;\; -\infty < \phi_{02} < -\frac{6k_2^2}{25k_4} < 0.$$

Formula (5.9) gives rise to a unbounded wave solution of equation (5.5).

For the stable manifold of the node point O defined by $y = -\frac{2}{5}k_2\phi + \sqrt{-\frac{2}{3}k_4\phi^3}$ and using the first equation of system (5.5), we have

$$\phi(\xi) = -\frac{6k_2^2}{25k_4\left(1 - u_{03}\exp\left(\frac{1}{5}k_2\xi\right)\right)^2}, \tag{5.10}$$

where

$$u_{03} = 1 + \frac{2k_2}{5\sqrt{-\frac{2}{3}k_4\phi_{03}}} > 0 \;\; \text{and} \;\; -\infty < \phi_{03} < 0.$$

Formula (5.10) gives rise to another unbounded wave solution of equation (5.5).

(ii): The case of $h \neq 0$

By making the coordinate transformations as

$$w = \frac{1}{\sqrt{2}}\phi \exp\left(\frac{2}{5}k_2\xi\right), \quad z = -\frac{5}{k_2}\exp\left(-\frac{1}{5}k_2\xi\right),$$

the first integral H_1 defined by (5.7) can be converted into the form

$$\left(\frac{dw}{dz}\right)^2 + \frac{2\sqrt{2}k_4}{3}w^3 = h. \tag{5.11}$$

From (5.11), for $h > 0$ we have

$$\begin{aligned} z - z_0 &= \int_{-\infty}^{w} \frac{dw}{\sqrt{h}\sqrt{1 - \left(\frac{2\sqrt{2}k_4}{3h}w^3\right)}} \\ &= \left(-\frac{2\sqrt{2}k_4}{3}\right)^{-\frac{1}{3}} h^{-\frac{1}{6}} \int_{-\infty}^{\hat{w}} \frac{d\hat{w}}{\sqrt{1-\hat{w}^3}}. \end{aligned}$$

It follows that

$$\begin{aligned} w(z) &= -\left(\frac{3h}{-2\sqrt{2}k_4}\right)^{\frac{1}{3}} \hat{w} \\ &= \left(-\frac{3h}{2\sqrt{2}k_4}\right)^{\frac{1}{3}} \left(\frac{2\sqrt{3}}{1 - \mathrm{cn}(\omega_1(h)(z-z_0), k_{01})} - (1+\sqrt{3})\right), \end{aligned}$$

where z_0 is an arbitrary constant, and

$$\omega_1(h) = h^{\frac{1}{6}} 3^{\frac{1}{4}} \left(\frac{-2\sqrt{2}k_4}{3}\right)^{\frac{1}{3}}, \quad k_{01} = \frac{\sqrt{2+\sqrt{3}}}{2}.$$

Hence, from $\phi(\xi) = \sqrt{2}w\exp\left(-\frac{2}{5}k_2\xi\right)$ we obtain

$$\phi(\xi) = \left(-\frac{3h}{k_4}\right)^{\frac{1}{3}} \left(\frac{2\sqrt{3}}{1 - \mathrm{cn}\left(\omega_1(h)\left(-\frac{5}{k_2}\exp\left(-\frac{1}{5}k_2\xi\right) - z_0\right), k_{01}\right)} - \left(1+\sqrt{3}\right)\right)\exp\left(-\frac{2}{5}k_2\xi\right). \tag{5.12}$$

From (5.11), for $h < 0$ we have

$$\begin{aligned} z - z_0 &= \int_{w}^{\infty} \frac{dw}{\sqrt{|h|}\sqrt{\left(-\frac{2\sqrt{2}k_4}{3|h|}w^3\right) - 1}} \\ &= \left(\frac{-2\sqrt{2}k_4}{3}\right)^{-\frac{1}{3}} |h|^{-\frac{1}{6}} \int_{\tilde{w}}^{\infty} \frac{d\tilde{w}}{\sqrt{1-\tilde{w}^3}}. \end{aligned}$$

Hence, it further gives

$$\begin{aligned} w(z) &= \left(-\frac{3|h|}{2\sqrt{2}k_4}\right)^{\frac{1}{3}} \tilde{w} \\ &= \left(-\frac{3|h|}{2\sqrt{2}k_4}\right)^{\frac{1}{3}} \left(\left(1-\sqrt{3}\right) + \frac{2\sqrt{3}}{1-\mathrm{cn}(\omega_2(h)(z-z_0),k_{02})}\right), \end{aligned}$$

where z_0 is an arbitrary constant, and

$$\omega_2(h) = |h|^{\frac{1}{6}} 3^{\frac{1}{4}} \left(\frac{-2\sqrt{2}k_4}{3}\right)^{\frac{1}{3}}, \quad k_{02} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

It implies that

$$\phi(\xi) = \left(-\frac{3|h|}{k_4}\right)^{\frac{1}{3}} \left(\frac{2\sqrt{3}}{1-\mathrm{cn}\left(\omega_2(h)\left(-\frac{5}{k_2}\exp\left(-\frac{1}{5}k_2\xi\right)-z_0\right),k_{02}\right)} + \left(1-\sqrt{3}\right)\right) \exp\left(-\frac{2}{5}k_2\xi\right). \tag{5.13}$$

Formulas (5.12) and (5.13) give rise to the parametric representations of all orbits of system (5.6), which approach to the invariant curve solution determined by $y = -\frac{2}{5}k_2\phi - \sqrt{-\frac{2}{3}k_4\phi^3}$ and the node point O. These parametric representations are not associated with any periodic solution of equation (5.5).

To sum up, we have the following results. For system (5.6), under the parametric conditions $k_2 > 0,\ k_4 > 0$ and $\lambda_1 = \frac{6k_2^2}{25}$, we obtain that

(a) equation (5.5) with $d = 0$ has a kink-profile wave solution given by formula (5.8);

(b) equation (5.5) with $d = 0$ has two unbounded wave solutions given by formulas (5.9) and (5.10); and

(c) corresponding to the level curves defined by system (5.6) with $h \neq 0$, equation (5.5) with $d = 0$ has two families of exact solutions given by formulas (5.12) and (5.13), which are not periodic solutions.

5.2 Exact solutions and dynamics of the integrable cubic nonlinear oscillator with damping

Equation (5.4) is equivalent to the planar dynamical system

$$\begin{cases} \frac{d\phi}{d\xi} = y, \\ \frac{dy}{d\xi} = -k_2 y - k_3\phi^3 - k_4\phi^2 - \lambda_1\phi - d. \end{cases} \tag{5.14}$$

By making a translation $\phi = \tilde{\phi} - \frac{k_4}{3k_3}$ and taking $d = \frac{k_4\lambda_1}{3k_3} - \frac{2k_4^3}{27k_3^2}$, system (5.14) becomes (still denote $\tilde{\phi}$ by ϕ):

$$\begin{cases} \frac{d\phi}{d\xi} = y, \\ \frac{dy}{d\xi} = -k_2 y - k_3\phi^3 - m\phi, \end{cases} \tag{5.15}$$

where $m = \lambda_1 - \frac{k_4^2}{3k_3}$.

Following Chapter 4.1, we know that when $m = \frac{2k_2^2}{9}$, system (5.15) is integrable, which has the first integral depending on ξ as follows:

$$H_2(\phi, y, \xi) = \left[\frac{1}{2}\left(y + \frac{1}{3}k_2\phi\right)^2 + \frac{1}{4}k_3\phi^4\right]\exp\left(\frac{4}{3}k_2\xi\right) = h, \tag{5.16}$$

where h is a real constant.

Clearly, for any $k_3 < 0$ system (5.15) has three equilibrium points at $A_\mp\left(\mp\frac{1}{3}\sqrt{\frac{2}{-k_3}}|k_2|, 0\right)$ and $O(0,0)$. Let $M(\phi_j, 0)$ be the coefficient matrix of the linearized system of (5.15) at the equilibrium point $(\phi_j, 0)$. Then, we have

$$\det M(0,0) = \frac{2}{9}k_2^2, \quad \det M\left(\mp\frac{1}{3}\sqrt{\frac{2}{-k_3}}|k_2|, 0\right) = -\frac{4}{9}k_2^2$$

and

$$(\mathrm{Trace}M(0,0))^2 - 4J(0,0) = \frac{1}{9}k_2^2. \tag{5.17}$$

From (5.17), we see that when $m = \frac{2k_2^2}{9}$, the equilibrium point $O(0,0)$ is a node point when $k_2 > 0$ (< 0), it is stable (unstable)), while $A_\mp\left(\mp\frac{1}{3}\sqrt{\frac{2}{-k_3}}|k_2|, 0\right)$ are two saddle points.

When $h = 0$, $H_2(\phi, y, \xi) = 0$ implies that there are four invariant curve solutions of system (5.15) given by

$$y = -\frac{1}{3}k_2\phi \pm \sqrt{-\frac{1}{2}k_3}\phi^2,$$

which consist of six orbits of system (5.15).

Let us consider the explicit parametric representations of orbits defined by $H_2(\phi, y, \xi) = h$. Here we only investigate the case of $k_2 > 0$ and $k_3 < 0$. Discussions for other cases are closely similar.

(i): The case of $h = 0$

With regard to the heteroclinic orbit connecting the equilibrium points O and A_- defined by $y = -\frac{1}{3}k_2\phi - \sqrt{-\frac{1}{2}k_3}\phi^2$, by using the first equation of system (5.15) we obtain

$$\phi(\xi) = -\frac{k_2}{3\left(\sqrt{-\frac{k_3}{2}} + v_{01}\exp\left(\frac{1}{3}k_2\xi\right)\right)}, \tag{5.18}$$

where

$$v_{01} = \frac{1}{\phi_{01}}\left(-\frac{1}{3}k_2 - \sqrt{-\frac{1}{2}k_3}\phi_{01}\right) > 0, \quad -\frac{1}{3}\sqrt{-\frac{2}{k_3}}|k_2| < \phi_{01} < 0.$$

Formula (5.18) gives rise to a kink-profile wave solution of system (5.15), because when $\xi \to \infty$ it has $\phi(\xi) \to 0$, and when $\xi \to -\infty$ it has $\phi(\xi) \to -\frac{1}{3}\sqrt{-\frac{2}{k_3}}|k_2|$.

Regarding the heteroclinic orbit connecting the equilibrium points O and A_+ defined by $y = -\frac{1}{3}k_2\phi + \sqrt{-\frac{1}{2}k_3}\phi^2$, we have

$$\phi(\xi) = \frac{k_2}{3\left(\sqrt{\frac{-k_3}{2}} + v_{01}\exp\left(\frac{1}{3}k_2\xi\right)\right)}, \tag{5.19}$$

where

$$v_{01} = \frac{1}{\phi_{01}}\left(-\frac{1}{3}k_2 - \sqrt{-\frac{1}{2}k_3}\phi_{01}\right) > 0, \quad -\frac{1}{3}\sqrt{-\frac{2}{k_3}}|k_2| < \phi_{01} < 0.$$

Formula (5.19) gives rise to an anti-kink wave solution of system (5.15), because when $\xi \to \infty$, it has $\phi(\xi) \to 0$, and when $\xi \to -\infty$ it has $\phi(\xi) \to \frac{1}{3}\sqrt{-\frac{2}{k_3}}|k_2|$.

Regarding the two stable manifolds to the node point $O(0,0)$ defined by $y = -\frac{1}{3}k_2\phi \mp \sqrt{-\frac{1}{2}k_3}\phi^2$, we obtain

$$\phi(\xi) = \pm\frac{-k_2}{3\left(v_{02}\exp\left(\frac{1}{3}k_2\xi\right) - \sqrt{\frac{-k_3}{2}}\right)}, \tag{5.20}$$

where

$$v_{02} = \frac{1}{\phi_{02}}\left(-\frac{1}{3}k_2 + \sqrt{-\frac{1}{2}k_3\phi_{02}}\right) > 0, \quad -\infty < \phi_{02} < 0.$$

Formula (5.20) gives rise to two unbounded wave solution of system (5.15).

Regarding the two unstable manifolds to the saddle points $A_{\mp}$ defined by $y = -\frac{1}{3}k_2\phi \mp \sqrt{-\frac{1}{2}k_3\phi^2}$, we have

$$\phi(\xi) = \pm\frac{-k_2}{3\left(\sqrt{\frac{-k_3}{2}} + v_{03}\exp\left(\frac{1}{3}k_2\xi\right)\right)}, \tag{5.21}$$

where

$$v_{03} = \frac{1}{\phi_{03}}\left(-\frac{1}{3}k_2 - \sqrt{-\frac{1}{2}k_3\phi_{03}}\right) < 0, \quad -\infty < \phi_{03} < -\frac{1}{3}\sqrt{-\frac{2}{k_3}}|k_2|.$$

Formula (5.21) gives rise to two unbounded wave solution of system (5.15).

(ii): The case of $h \neq 0$

By making the transformations

$$w = \frac{1}{\sqrt{2}}\phi\exp\left(\frac{1}{3}k_2\xi\right), \quad z = -\frac{3}{k_2}\exp\left(-\frac{1}{3}k_2\xi\right),$$

the first integral H_2 defined by (5.16) can be converted into the form

$$\left(\frac{dw}{dz}\right)^2 + k_3w^4 = h. \tag{5.22}$$

From (5.22), for $h > 0$ we have

$$\begin{aligned} z - z_0 &= \int_w^\infty \frac{dw}{\sqrt{h}\sqrt{1-\left(\frac{k_3}{h}w^4\right)}} \\ &= (-k_3h)^{-\frac{1}{4}}\int_{\hat{w}}^\infty \frac{d\hat{w}}{\sqrt{1+\hat{w}^4}}. \end{aligned}$$

It further gives

$$w(z) = \left(-\frac{h}{k_3}\right)^{\frac{1}{4}}\hat{w} = \left(-\frac{h}{k_3}\right)^{\frac{1}{4}}\left(\frac{1+\mathrm{cn}(\Omega_1(h)(z-z_0),k_1)}{1-\mathrm{cn}(\Omega_1(h)(z-z_0),k_1)}\right)^{\frac{1}{2}},$$

where $\Omega_1(h) = 2(-k_3h)^{\frac{1}{4}}$, $k_1 = \frac{1}{\sqrt{2}}$ and z_0 is an arbitrary constant. Thus, we obtain

$$\phi(\xi) = \sqrt{2}\left(-\frac{h}{k_3}\right)^{\frac{1}{4}}\left(\frac{1+\mathrm{cn}\left(\Omega_1(h)\left(-\frac{3}{k_2}\exp\left(-\frac{1}{3}k_2\xi\right)-z_0\right),k_1\right)}{1-\mathrm{cn}\left(\Omega_1(h)\left(-\frac{3}{k_2}\exp\left(-\frac{1}{3}k_2\xi\right)-z_0\right),k_1\right)}\right)^{\frac{1}{2}}\exp\left(-\frac{1}{3}k_2\xi\right). \tag{5.23}$$

From (5.22), for $h<0$ we have

$$\begin{aligned} z-z_0 &= \int_{w_0}^{w} \frac{dw}{\sqrt{|h|}\sqrt{\left(-\frac{k_3}{|h|}w^4-1\right)}} \\ &= (-k_3|h|)^{-\frac{1}{4}} \int_1^{\tilde{w}} \frac{d\hat{w}}{\sqrt{\tilde{w}^4-1}}, \end{aligned}$$

which implies that

$$\phi(\xi)=\sqrt{2}\left(-\frac{|h|}{k_3}\right)^{\frac{1}{4}} \mathrm{nc}\left\{\Omega_2(h)\left[-\frac{3}{k_2}\exp\left(-\frac{1}{3}k_2\xi\right)-z_0\right], k_1\right\}, \tag{5.24}$$

where $\Omega_2(h)=2(-k_3|h|)^{\frac{1}{4}}$ and z_0 is an arbitrary constant.

Formulas (5.23) and (5.24) give rise to the parametric representations of all orbits of system (5.15), which approach to the invariant curve solution determined by $y=-\frac{1}{3}k_2\phi \pm \sqrt{-\frac{1}{2}k_3}\phi^2$ and the node point O. These parametric representations are not associated with any periodic orbit.

To sum up, we have the following results.

For system (5.15), under the parametric conditions $k_2>0,\ k_3<0$ and $m=\frac{2k_2^2}{9}$, we obtain that

(a) system (5.15) has a kink-profile wave solution given by (5.18) and an anti-kink wave solution given by (5.19);

(b) system (5.15) has two unbounded wave solution given by formulas (5.20) and (5.21);

(c) corresponding to the level curves defined by (5.16) with $h\neq 0$, system (5.15) has two families of exact solutions given by (5.23) and (5.24), which are not periodic solutions.

CHAPTER VI

CONCLUSION

The first integral method is widely used in recent papers. In this paper, we identified the first integrals of second-order nonlinear ODEs, which include several physically important nonlinear oscillators and classify the integrable cases by certain parametric conditions. The choices of variables in Eq. (3.1) provided several oscillators. First, we introduced how to obtain Lie point symmetries of ODEs and PDEs. The important thing for symmetry was to obtain the linearized symmetry condition. Since the exponent q in Eq. (3.1) determined the tangent vector, we obtained various integrable cases. Then, to find the first integrals of Eq. (3.1), we derived the determining equations, which induced the infinitesimal generator. As a result, canonical coordinates were constructed which changed Eq. (3.1) to a simple integrable equation. By the inverse transformation, the first integrals were obtained under certain parametric conditions. We also discussed some special integrable cases of nonlinear oscillator systems. At last, applications to wave equations of Korteweg-de Vries- Burgers-type equations were also presented.

REFERENCES

[1] J. A. Almendral and M. F. Sanjuán. Integrability and symmetries for the helmholtz oscillator with friction. *J. Phys. A (Math. Gen.)*, 36:695–710, 2003.

[2] V. K. Chandrasckar, M. Senthilvelan, and M. Lakshmanan. On the complete integrability and linearization of certain second-order nonlinear ordinary differential equations. *Proc. R. Soc. Lond. Ser.A*, 461:2451–2476, 2005.

[3] V. K. Chandrasekar, S. N. Pandey, M. Senthilvelan, and M. Lakshmanan. A simple and unified approach to identify integrable nonlinear oscillators and systems. *Journal of Mathematics Physics*, 47, 2006.

[4] L. G. S. Duarte, S. E.S. Duarte, A. C. P. da Mota, and J. E. F. Skeca. Solving second-order ordinary differential equations by extending the prelle-singer method. *J. Phys. A(Math. Gen.)*, 34:3015–3024, 2001.

[5] M. R. Feix, C. Geronimi, L. Cairo, P.G.L. Leach, R.L. Lemmer, and S. Bouquet. On the singularity analysis of ordinary differential equation invariant under time translation and rescaling. *J. Phys. A*, 30:7347–7461, 1997.

[6] Z. Feng. The first integral method to study the burgers-korteweg-de vries equation. *J. Phys. A Math: Gen.*, 35:343–349, 2002.

[7] Z. Feng. On explicit exact solutions to the compound burgers-kdv equation. *Phys. Lett. A*, 293:57–66, 2002.

[8] Z. Feng. The first integral method to the two-dimensional burgers-korteweg-de vries equation. *Physics Letter A*, 308:173–178, 2003.

[9] Z. Feng. A note on 'explicit exact solutions to the compound burgers-kdv equation'. *Phys. Lett. A*, 312:65–70, 2003.

[10] Z. Feng. On traveling wave solutions of the burgers-korteweg-de vries equation. *Nonlinearity*, 20:343–356, 2007.

[11] Z. Feng. Duffing-van der pol-type oscillator systems. *DCDS-S*, 7:1231–1257, 2014.

[12] Z. Feng, G. Gao, and J. Cui. Duffing-van der pol-type oscillator system and its first integrals. *Communications on Pure and Applied Analysis*, 10:1377–1391, 2011.

[13] Z. Feng and Q. G. Meng. Exact solution for a two-dimensional kdv-burgers-type equation with nonlinear terms of any order. *Discrete Contin. Dyn. Syst. Scr. B*, 7:285–291, 2007.

[14] Z. Feng, S. Zheng, and D. Y. Gao. Traveling wave solutions to a reaction-diffusion equation,. *Z.angew. Math. Phys.*, 60:756–773, 2009.

[15] G. Gao and Z. Feng. First integrals for the duffing-van der pol-type osillator. *E. J. Diff. Equs.*, 2010:1–12, 2010.

[16] P. Holmes and D. Rand. Phase portraits and bifurcations of the non-linear oscillator: $\ddot{x}+(\alpha+\gamma x^2)\dot{x}+\beta x+\delta x^3=0$. *Int. J. Non-Linear Mech.*, 15:449–458, 1980.

[17] P. E. Hydon. *Symmetry Methods for differential Equations*. Cambridge University Press, 2000.

[18] M. Lakshmanan and S. Rajasekar. *Nonlinear Dynamics*. Springer Verlag, 2003.

[19] P. G. L. Leach, K. S. Govinder, and B. Abraham-Shrauner. Symmetries of first integrals and their associated differential equations. *Journal of Mathematical Analysis and Applications*, 235:58–83, 1999.

[20] J. Li and Z. Feng. Quadratic and cubic nonlinear oscillators with dampling and their applications. *IJBC*, 26:1–11, 2016.

[21] P. J. Olver. *Applications of Lie Groups to Differential Equations*. Springer Verlag, 1991.

[22] M. Prelle and M. Singer. Elementary first integrals of differential equations. *Trans. Am. Math. Soc.*, 279:215–229, 1983.

[23] N. Taghizadeh, M. Mirzazadeh, and A. Samiei Paghaleh. The first integral method to nonlinear partial differential equations. *Applications and Applied Mathematics*, 7:117–132, 2012.

BIOGRAPHICAL SKETCH

Xiaoyan Li was born on May 05, 1991 in China. She obtained her Bachelor degree of in Mathematics at Tianjin University of Technology and Education in 2014. After graduation, she studied for the Master degree in University of Texas Rio Grande Valley. Her research background was Differential Equations. She was awarded Master of Science in Mathematics in May 2016. Her permanent mailing address is Chicheng Town, Zhangjiakou City, Hebei Province, China, 075545.