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Paul Bracken
The University of Texas Rio Grande Valley, paul.bracken@utrgv.edu

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Research Article

Paul Bracken*

Delaunay surfaces expressed in terms of a Cartan moving frame

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Abstract: Delaunay surfaces are investigated by using a moving frame approach. These surfaces correspond to surfaces of revolution in the Euclidean three-space. A set of basic one-forms is defined. Moving frame equations can be formulated and studied. Related differential equations which depend on variables relevant to the surface are obtained. For the case of minimal and constant mean curvature surfaces, the coordinate functions can be calculated in closed form. In the case in which the mean curvature is constant, these functions can be expressed in terms of Jacobi elliptic functions.

Keywords: Metric, geometry, manifold, fundamental form, structure equations, Delaunay

MSC 2010: 55R10

1 Introduction

Delaunay surfaces constitute a basic class of constant mean curvature surface in the Euclidean three-space. Although Delaunay surfaces have been investigated before, it is intended here to give a complete formulation of the subject in terms of Cartan's idea of a moving frame. It will be shown that by defining a few fundamental differential forms, the moving frame approach provides a general framework for studying such surfaces. New types of differential equations also result from this type of approach and can be solved in closed form when the surface mean curvature is zero or constant. It is possible to derive the coordinates explicitly for these cases of surfaces of revolution of constant mean curvature. Some examples are found in which the coordinate functions are expressed in terms of Jacobi elliptic functions.

To begin with, let us introduce the two main topics which will be discussed, that of Delaunay surfaces and the method of moving frames [1–3, 14]. The method of moving frames is used to show that the first fundamental form and the second fundamental form constitute a complete invariant system on hypersurfaces in $\mathbb{R}^{m+1}$. From the involution surface condition, a basic system of one-forms can be defined. These are substituted back into the Cartan structure equations. Consequences with regard to surfaces can be developed by doing so, such as determination of the curvatures.

1.1 Delaunay surfaces

Classically, a parametrized surface is locally the image of an immersion

$$(u, v) \to \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)), \tag{1.1}$$

defined on an open set $D \subset \mathbb{R}^2$. In these coordinates the pullback of the Riemannian metric on the surface

***Corresponding author: Paul Bracken,** Department of Mathematics, University of Texas, Edinburg, TX 78540, USA,
e-mail: paul.bracken@utrgv.edu

can be expressed as

$$I = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

which is known as the metric or first fundamental form. The coefficients in I are given by $E = \mathbf{r}_u \cdot \mathbf{r}_u$, $F = \mathbf{r}_u \cdot \mathbf{r}_v$ and $G = \mathbf{r}_v \cdot \mathbf{r}_v$. One would like to find the most appropriate coordinates in which E, F and G are as simple as possible. If the set of coordinates is such that $F = 0$, so the first fundamental form is diagonal, the system of coordinates is called orthogonal. If, in addition, $E = G = \xi(u, v)$, the coordinate system is called conformal. The angle between any two directions on the surface is equal to the angle of their pre-images in the Euclidean plane D. Orthogonal coordinates can always be found. Conformal coordinates always exist, but it is a much harder problem to find them in explicit form.

Surfaces of Delaunay were defined originally [4] as surfaces obtained by revolving profile curves which themselves arise from rolling conics on a line [5–9]. Such surfaces are called roulettes of conics. Sturm characterizes Delaunay surfaces from the variational perspective as those surfaces of revolution having a minimal lateral area at a fixed volume. This lets us understand why these surfaces appear in the discussion of soap bubbles and liquid drops. The complete list of Delaunay surfaces is given by: planes, spheres, catenoids, cylinders, nodoids and unduloids. The main subject of the paper is the rotational surface

$$\mathbf{r}(u, v) = \Big(e^{\sigma(u)} \cos(v), e^{\sigma(u)} \sin(v), \int_0^u e^{\sigma(t)} \sin(\Omega(t))\,dt\Big). \tag{1.2}$$

In this coordinate chart, the first fundamental form has components

$$E = e^{2\sigma(u)}, \quad F = 0, \quad G = e^{2\sigma(u)}. \tag{1.3}$$

1.2 Differential forms and moving frames

Over a two-dimensional submanifold $\Sigma \subset \mathbb{R}^3$, there exists a system of orthogonal frames $\{p, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ which is well defined at each point $p \in \Sigma$. The unit normal vector at p is $\mathbf{e}_3$ and $\mathbf{e}_1$, $\mathbf{e}_2$ are located along principal directions tangent to Σ. The orientation of the frame is the same as a chosen orientation of $\mathbb{R}^3$, and the first two vectors determine the orientation of Σ. Suppose the forms ω_i and ω_{ij} are the corresponding components for the frame field so that

$$d\mathbf{p} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2, \quad \omega_3 = 0. \tag{1.4}$$

The set of vectors $\mathbf{e}_i$ of the frame satisfy the following system of equations:

$$d\mathbf{e}_1 = \omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e}_3, \quad d\mathbf{e}_2 = \omega_{21}\mathbf{e}_1 + \omega_{23}\mathbf{e}_3, \quad d\mathbf{e}_3 = \omega_{21}\mathbf{e}_1 + \omega_{23}\mathbf{e}_2, \tag{1.5}$$

where the ω_{ij} satisfy

$$\omega_{ij} + \omega_{ji} = 0.$$

Thus, $\mathbf{r} : \Sigma \to \mathbb{R}^3$ represents a smooth surface or manifold in $\mathbb{R}^3$. A system of local coordinates u, v can be chosen in a coordinate neighborhood U of Σ so the surface can be expressed in the form

$$x^i = x^i(u, v), \quad i = 1, 2, 3. \tag{1.6}$$

The ω_i and ω_{ij} are differential one-forms which depend on coordinates (u, v), once they have been specified. These equations satisfy the following system of structure equations for the manifold:

$$d\omega_1 = \omega_2 \wedge \omega_{21}, \quad d\omega_2 = \omega_1 \wedge \omega_{12}, \tag{1.7}$$

$$d\omega_{12} = \omega_{13} \wedge \omega_{32}, \quad d\omega_{13} = \omega_{12} \wedge \omega_{23}, \quad d\omega_{23} = \omega_{21} \wedge \omega_{13}, \tag{1.8}$$

$$\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} = 0. \tag{1.9}$$

Cartan's Lemma and (1.9) imply that the forms ω_{13} and ω_{23} can be written as the linear combinations

$$\omega_{13} = a\omega_1 + b\omega_2, \quad \omega_{23} = b\omega_1 + c\omega_2.$$

Once this set of differential forms has been explicitly defined, it is then possible to write down the fundamental forms for Σ in terms of them, which are defined by

$$I = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2, \tag{1.10}$$

$$II = \omega_1 \otimes \omega_{13} + \omega_2 \otimes \omega_{23} = a\omega_1 \otimes \omega_1 + 2b\omega_1 \otimes \omega_2 + c\omega_2 \otimes \omega_2.$$

$$III = \omega_{13} \otimes \omega_{13} + \omega_{23} \otimes \omega_{23}. \tag{1.11}$$

2 One-forms and curvatures

To study this type of manifold, the one-forms for the moving frame need to be defined. The most appropriate coordinates are sought in order to ensure the coefficients of the fundamental forms are as simple as possible. The metric has to be diagonal and these diagonal components are equal to each other and positive. The metric is then called conformal. The angle between any two directions on the surface is equal to the angle of their pre-images in the Euclidean (u, v)-plane.

The results in (1.2), (1.3) can be used to start the process of defining the forms. The forms ω_1 and ω_2 are defined to be

$$\omega_1 = e^{\sigma(u)}\, du, \quad \omega_2 = e^{\sigma(u)}\, dv. \tag{2.1}$$

Using (1.7) with u, v as subscripts denoting partial differentiation, we obtain

$$\omega_{12} = \sigma_u\, dv.$$

Combining (1.10) with (2.1), the first fundamental form is given by

$$I = e^{2\sigma(u)}\, (du^2 + dv^2). \tag{2.2}$$

Let $\Omega(u)$ be at least a C^2 function of u. Then in terms of $\Omega(u)$, the differential forms ω_{13} and ω_{23} are

$$\omega_{13} = -\Omega_u(u)\, du, \quad \omega_{23} = -\sin(\Omega(u))\, dv. \tag{2.3}$$

Guided by (1.1) again, the components of the second fundamental form are given by

$$a = -e^{-\sigma(u)}\Omega_u(u), \quad b = 0, \quad c = -e^{-\sigma(u)}\sin(\Omega(u)).$$

The second fundamental form is diagonal in this case and can be written

$$II = -e^{-\sigma(u)}\Omega_u(u)\omega_1 \otimes \omega_1 - e^{-\sigma(u)}\sin(\Omega(u))\omega_2 \otimes \omega_2.$$

Finally, by (1.11), the form III is given by

$$III = (\Omega_u(u))^2\, du \otimes du + \sin^2(\Omega(u))\, dv \otimes dv.$$

These can now be substituted into the structure equations to see what results. The second pair of equations in (1.8) is clearly satisfied since

$$d\omega_{13} = 0 = \omega_{12} \wedge \omega_{23}, \quad d\omega_{23} = -\big(\sin(\Omega(u))_u\, du \wedge dv\big) = \omega_{21} \wedge \omega_{13}.$$

An expression for the Gaussian curvature K of Σ follows from (1.8) since

$$d\omega_{12} = \sigma_{uu}(u)\, du \wedge dv = \sigma_{uu}(u)e^{-2\sigma(u)}\omega_1 \wedge \omega_2 = -K\omega_1 \wedge \omega_2. \tag{2.4}$$

The Gaussian curvature of Σ implied by (2.4) is given by

$$K = -\sigma_{uu}(u)e^{-2\sigma(u)}. \tag{2.5}$$

If $\mathcal{F}_I$ and $\mathcal{F}_{II}$ are the matrix representations of the fundamental forms I and II, then the shape operator, or Weingarten map, is defined to be

$$\mathcal{W} = \mathcal{F}_I^{-1}\mathcal{F}_{II}.$$

The two most important characteristics of Σ, the Gaussian curvature K and the mean curvature H, follow directly as the invariants of $\mathcal{W}$:

$$K = \det(\mathcal{W}), \quad H = \frac{1}{2}\,\mathrm{Tr}(\mathcal{W}). \tag{2.6}$$

It follows from the expressions for the fundamental forms I and II that

$$\mathcal{F}_I^{-1}\mathcal{F}_{II} = \begin{pmatrix} -e^{-\sigma(u)}\Omega_u(u) & 0 \\ 0 & -e^{-\sigma(u)}\sin(\Omega(u)) \end{pmatrix}.$$

The Gauss and mean curvatures can also be calculated by means of (2.6):

$$K = e^{-2\sigma(u)}\sin(\Omega(u))\Omega_u(u), \quad H = -\frac{1}{2}e^{-\sigma(u)}(\Omega_u(u) + \sin(\Omega(u))). \tag{2.7}$$

This has provided a second way to evaluate K, and so by equating equations (2.5) and (2.7), the following equation results:

$$\sigma_{uu}(u) = -\sin(\Omega(u))\Omega_u(u) = \cos(\Omega(u))_u. \tag{2.8}$$

One way of looking at (2.8) is to regard it as an integrability condition. Integrating it once yields

$$\sigma_u(u) = \cos(\Omega(u)). \tag{2.9}$$

All the one-forms needed to define the problem have now been specified explicitly. Equations (1.5) corresponding to this case can now be written down. They consist of the following three equations:

$$\begin{cases} d\mathbf{e}_1 = \sigma_u(u)\,dv\mathbf{e}_2 - \Omega_u(u)\,du\mathbf{e}_3, \\ d\mathbf{e}_2 = -\sigma_u(u)\,dv\mathbf{e}_1 - \sin(\Omega(u))\,dv\mathbf{e}_3, \\ d\mathbf{e}_3 = \Omega_u(u)\,du\mathbf{e}_1 - \sin(\Omega(u))\,dv\mathbf{e}_2. \end{cases} \tag{2.10}$$

It can be verified by differentiation that the following set constitutes a three-dimensional representation of the set $\{\mathbf{e}_i\}$ which satisfy (2.10):

$$\begin{cases} \mathbf{e}_1 = (\cos(\Omega(u))\cos(v), \cos(\Omega(u))\sin(v), \sin(\Omega(u))), \\ \mathbf{e}_2 = (-\sin(v), \cos(v), 0), \\ \mathbf{e}_3 = (\sin(\Omega(u))\cos(v), \sin(\Omega(u))\sin(v), -\cos(\Omega(u))). \end{cases} \tag{2.11}$$

The set of vectors $\mathbf{e}_i$ satisfies the orthogonality relation $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ with $i, j \in \{1, 2, 3\}$ under the usual inner product. Given the representation (2.11), functions (1.6) can be obtained in explicit form by integrating (1.4). Since

$$\begin{aligned} \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 = (&e^{\sigma(u)}\cos(\Omega(u))\cos(v)\,du - e^{\sigma u)}\sin(v)\,dv, e^{\sigma(u)}\cos(\Omega(u))\sin(v)\,du + e^{\sigma(u)}\cos(v)\,dv, \\ &e^{\sigma(u)}\sin(\Omega(u))\,du), \end{aligned}$$

the components can be matched up. Doing so, the following system of equations results:

$$\begin{cases} x_u^1\,du + x_v^1\,dv = (e^{\sigma(u)})_u\cos(v)\,du + e^{\sigma(u)}(\cos(v))_v\,dv, \\ x_u^2\,du + x_v^2\,dv = (e^{\sigma(u)})_u\sin(v)\,du + e^{\sigma(u)}(\sin(v))_v\,dv, \\ \qquad\qquad x_u^3\,du = e^{\sigma(u)}\sin(\Omega(u))\,du. \end{cases} \tag{2.12}$$

This system can be integrated by equating the coefficients of du and dv on both sides. A parametrized coordinate system for Σ is locally found to be the image of the immersion defined by

$$\mathbf{r}(u, v) = \Big(e^{\sigma(u)}\cos(v), e^{\sigma(u)}\sin(v), \int_0^u e^{\sigma(t)}\sin(\Omega(t))\,dt\Big). \tag{2.13}$$

This agrees with what is expected for this class of surface (1.2). Moreover, (2.12) implies that $\Omega(u)$ denotes the polar angle of the unit normal vector to the surface Σ. Let us now apply what has been developed to some special cases.

3 Minimal surfaces

Minimal surfaces are those for which $H = 0$ (see [7–9]). These are investigated by substituting $H = 0$ in the equation for the mean curvature

$$\Omega_u(u) = -\sin(\Omega(u)).$$

This equation is separable and may be integrated in closed form by means of the substitution

$$\cos(\Omega(u)) = \Big(\frac{1-t^2}{1+t^2}\Big).$$

It is found that

$$\cos(\Omega(u)) = \tanh(u), \quad \sin(\Omega(u)) = (1-\tanh^2(u))^{1/2} = \operatorname{sech}(u).$$

The function $\sigma(u)$ can be calculated by integrating (2.9):

$$\sigma(u) = \int \cos(\Omega(u))\, du = \int \tanh(u)\, du = \log(\cosh(u)).$$

The constant of integration is omitted since it contributes only a scale factor upon exponentiation, so

$$e^{\sigma(u)} = \cosh(u).$$

Substituting $e^{\sigma(u)}$ and $\cos(\Omega(u))$ into the third component of (2.13), one obtains $x^3(u,v) = u$.

4 Constant mean curvature surfaces

The case in which $H = \beta/2$, where β is a nonzero constant, is also studied [1, 2]. The second equation of (2.7) is given by

$$(\Omega_u(u) + \sin(\Omega(u)))e^{-\sigma(u)} = -\beta. \tag{4.1}$$

Differentiating both sides of (4.1) with respect to u, we obtain

$$(\cos(\Omega(u))\Omega_u(u) + \Omega_{uu}(u))e^{-\sigma(u)} - (\Omega_u(u) + \sin(\Omega(u)))\sigma_u(u)e^{-\sigma(u)} = 0.$$

This simplifies to the second-order equation

$$\Omega_{uu}(u) = \sin(\Omega(u)) \cdot \cos(\Omega(u)). \tag{4.2}$$

To integrate this, multiply both sides first by $\Omega_u(u)$. A nonlinear first-order equation results:

$$(\Omega_u(u))^2 = \sin^2(\Omega(u)) + \alpha. \tag{4.3}$$

The characteristics of the solution spaces to this equation are greatly altered by the sign of the integration constant α. Each of the separate cases $\alpha = -a^2, 0, a^2$ are examined in turn.

(i) The first case to be considered is

$$(\Omega(u))^2 = \sin^2(\Omega(u)) - a^2. \tag{4.4}$$

Into (4.4) introduce the variable $\tau = \sin(\Omega(u))$ so that $\tau_u = \cos(\Omega(u)) \cdot \Omega_u(u)$, and hence τ satisfies

$$\tau_u(u) = -\sqrt{(1-\tau^2)(\tau^2 - a^2)}.$$

This first-order equation is integrated by means of Jacobi's elliptic function $\operatorname{dn}(u,k)$, where the elliptic modulus k is related to the constant a by means of identification of a with the complementary elliptic modulus $a = \tilde{k}$ so we have (see [5])

$$\tau = \operatorname{dn}(u,k), \quad k^2 = 1 - a^2 = 1 - \tilde{k}^2.$$

This implies that $\Omega(u)$ given by

$$\Omega(u) = \pi - \arcsin(\mathrm{dn}(u, k)) \tag{4.5}$$

and satisfies (4.2). Constraint (2.9) requires that

$$\sigma_u(u) = \cos(\Omega(u)) = k\mathrm{sn}(u, k).$$

The integral of $\mathrm{sn}(u, k)$ is evaluated by using successive substitutions. First set $\mathrm{sn}(u, k) = \xi$ and then $\xi^2 = t$ so we obtain that

$$\int \mathrm{sn}(u, k)\, du = \frac{1}{2} \int \frac{dt}{\sqrt{(1 - t^2)(1 - kt^2)}} = \frac{1}{k} \log(\sqrt{1 - k^2 t} + k\sqrt{1 - t}).$$

Since $t = \mathrm{sn}^2(u, k)$, the following form for $\sigma(u)$ is obtained:

$$\sigma(u) = \log(\mathrm{dn}(u, k) + k\mathrm{cn}(u, k)) \tag{4.6}$$

Exponentiating (4.6) gives the result

$$e^{\sigma(u)} = \mathrm{dn}(u, k) + k\mathrm{cn}(u, k). \tag{4.7}$$

To obtain the third component of (2.13), we integrate the equation

$$x_u^3(u, v) = e^{\sigma(u)} \sin(\Omega(u)) = (\mathrm{dn}(u, k) + k\mathrm{cn}(u, k)) \cdot \mathrm{dn}(u, k).$$

The result of doing this is

$$x^3(u, v) = E(\mathrm{am}(u, k), k) + k\mathrm{sn}(u, k). \tag{4.8}$$

In (4.8), $E(\eta, k)$ is the incomplete elliptic integral of the second kind and $\mathrm{am}(u, k)$ is the Jacobi amplitude function.

(ii) The second case concerns $\alpha = 0$. Hence (4.3) is

$$\Omega_u(u) = \sin(\Omega(u)).$$

This is separable, and so by integrating we find that

$$\cos(\Omega(u)) = -\tanh(u), \quad \sin(\Omega(u)) = \mathrm{sech}(u), \quad \Omega(u) = \pi - \arcsin(\mathrm{sech}(u)).$$

The function $\sigma(u)$ is given by

$$\sigma(u) = -\int \tanh(u)\, du = -\log(\cosh(u)), \quad e^{\sigma(u)} = \mathrm{sech}(u).$$

The third component of (2.13) is obtained by integration of

$$x_u^3(u, v) = \mathrm{sech}^2(u), \quad x^3(u, v) = \tanh(u).$$

Therefore, the resulting surface is simply a sphere in this case.

(iii) The last case is the one with $\alpha = a^2$. Then (4.3) takes the form

$$(\Omega_u(u))^2 = \sin^2(\Omega(u)) + a^2. \tag{4.9}$$

The substitution $\tau = \sin(\Omega(u))$ transforms (4.9) into a separable equation

$$\tau_u = \sqrt{(1 - \tau^2)(\tau^2 + a^2)}.$$

Therefore, $\tau(u)$ is an elliptic function, namely

$$\tau(u) = \mathrm{cn}\Big(\frac{u}{k}, k\Big), \quad k^2 = \frac{1}{1 + a^2}, \quad \tilde{k}^2 = \frac{a^2}{1 + a^2}.$$

From these we deduce that

$$\Omega(u) = \pi - \arcsin\Big(\mathrm{cn}\Big(\frac{u}{k}, k\Big)\Big).$$

The function $\sigma_u(u)$ satisfies the equation

$$\sigma_u(u) = \cos(\Omega(u)) = \mathrm{sn}\Big(\frac{u}{k}, k\Big).$$

Upon integration, it yields

$$\sigma(u) = \log\Big(\mathrm{dn}\Big(\frac{u}{k}, k\Big) + k\mathrm{cn}\Big(\frac{u}{k}, k\Big)\Big), \quad e^{\sigma(u)} = \mathrm{dn}\Big(\frac{u}{k}, k\Big) + k\mathrm{cn}\Big(\frac{u}{k}, k\Big).$$

The third coordinate of (2.13) is obtained by integrating

$$(x^3)_u = e^{\sigma(u)} \sin(\Omega(u)) = \Big(\mathrm{dn}\Big(\frac{u}{k}, k\Big) + k\mathrm{cn}\Big(\frac{u}{k}, k\Big)\Big) \cdot \mathrm{cn}\Big(\frac{u}{k}, k\Big),$$

and therefore

$$x^3(u, v) = k\mathrm{sn}\Big(\frac{u}{k}, k\Big) + E\Big(\mathrm{am}\Big(\frac{u}{k}, k\Big), k\Big) - \frac{1-k^2}{k} u,$$

This completes the study of all three cases.

5 Dual surfaces

The classical conformal immersions have another important property which is the phenomenon of dual surfaces. It is said that two immersions $\mathbf{x}$ and $\mathbf{y}$ are dual to each other if they share the same tangent plane at corresponding points.

This definition follows from the observation that if the first fundamental form which is induced by $\mathbf{r}$ is given by (2.2), then the one form

$$\omega = e^{-2\sigma(u,v)}(\mathbf{x}_u\, du - \mathbf{x}_v\, dv)$$

is closed. It can locally be considered a differential of the immersion $\mathbf{y}$ given by

$$d\mathbf{y} = e^{-2\sigma(u,v)}(\mathbf{x}_u\, du - \mathbf{x}_v\, dv).$$

For surfaces of revolution, it is simpler as in this case there exists an explicit expression for their dual immersions. If such a surface is parametrized in terms of isothermal coordinates (u, v) in the form

$$\mathbf{x}(u, v) = (r(u)\cos(v), r(u)\sigma(u)\sin(v), z(u)),$$

then the dual is given by

$$\tilde{\mathbf{y}}(u, v) = \Big(-\frac{\cos(v)}{r(u)}, -\frac{\sin(v)}{r(u)}, \tilde{z}(u)\Big). \tag{5.1}$$

The function in the third entry $\tilde{z}(u)$ satisfies the following separable ordinary differential equation:

$$\frac{d\tilde{z}}{du} = \frac{1}{r^2}\frac{dz(u)}{du}.$$

Provided that we have

$$\frac{1}{r^2}\frac{dz}{du} = e^{-\sigma(u)} \sin(\Omega(u)),$$

the last component of the dual in (5.1) is

$$\tilde{z}(u) = \int_0^u e^{-\sigma(t)} \sin(\Omega(t))\, dt.$$

Therefore, the dual surface is determined by means of the representation

$$\mathbf{y}(u, v) = \Big(-e^{-\sigma(u)}\cos(v), -e^{-\sigma(u)}\sin(v), \int_0^u e^{-\sigma(t)}\sin(\Omega(t))\, dt\Big).$$

This result serves to determine the dual surface in explicit form. As an example, by using (4.5), (4.7) and (4.8), for the unduloids we obtain

$$\tilde{z} = \int_0^u \frac{\mathrm{dn}(t,k)}{\mathrm{dn}(t,k) + k\mathrm{cn}(t,k)}\,dt = \frac{1}{\tilde{k}^2} \cdot [E(\mathrm{am}(u,k),k) - k\mathrm{sn}(u,k)].$$

6 Summary

It has been seen that the moving frame is extremely effective in providing a framework for calculating all of the relevant functions for the class of surface defined by the forms (2.1) and (2.3). In terms of $\sigma(u)$ and $\Omega(u)$, the Gaussian and mean curvatures are given by formulas (2.7). The function for H in (2.7) can be integrated in the cases of minimal and constant mean curvature. The functions $\sigma(u)$ and $\Omega(u)$ can be used in the equation for K to give the Gaussian curvature. The coordinate functions for these cases can be calculated so that a graphical representation may be developed. Numerous practical applications of this work exist as well [6].

It is of interest to say that the differential equations might be solved for other choices of the function H. For example, if H is an arbitrary function of u such that

$$H = -\frac{1}{2}e^{-\sigma(u)}g(\sin(\Omega(u))),$$

then (2.7) can be expressed in the form of a quadrature to give $\Omega(u)$ implicitly:

$$u = \int_c^{\Omega(u)} \frac{dt}{g(\sin(t)) - \sin(t)}. \tag{6.1}$$

Using the $\Omega(u)$ given in (6.1), the function $\sigma(u)$ is given by (2.9); see [10–13].

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