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Political Design Meets Policy Complexity

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Abstract

The rules that are employed to pass policies in legislative bodies vary widely. It is generally argued that policies that differ in complexity or importance level should be decided under different kinds of voting rules. While this question has been examined for *static* legislative mechanisms, an analysis of the precise relationship between the level of policy complexity and the type of voting rule is still missing for *dynamic* mechanisms. We address this problem from the perspective of a *preference-blind* political designer. Given the level of complexity of the decision that is to be made, the political designer's goal is to select the supermajority rule that avoids (1) policy instability; (2) guarantees efficiency; and (3) minimizes institutional status quo bias. We provide an answer to this objective problem, deriving a closed-form relationship between voting rule and policy complexity. Our analysis rationalizes the use of different rules to adopt different types of policy only when preferences are weak. When preferences are strong, the optimal rule is unique, and it does not vary by level of policy complexity. These findings significantly differ from those obtained for static mechanisms. Our study also implies that more complex policies are more likely to be persistent, even after a change in political preferences.

JEL codes: P16, D72, C7, H41.

Keywords: Legislative Institutions; Dynamic Decision Making; Political Design; Voting Rules; Policy Complexity; Policy Importance.

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1 Introduction

"There are two general rules. First, the more the grave and important the questions discussed the nearer should the opinion that is to prevail approach to unanimity. Second, the more the matter in hand calls for speed, the smaller the prescribed difference in the number of votes may be allowed to become: when an immediate decision has to be reached, a majority of one should suffice." (Rousseau, 1762, n.a.)

The rules that are employed to adopt policies in legislative bodies vary widely. For example, the United States House of Representatives may, by a simple majority vote, impeach a federal official, whereas his or her removal from office requires a two-thirds majority in the Senate. The United Nations Security Council requires the approval of a three-fifths supermajority of all of its members on substantive matters, whereas procedural matters require a simple majority of those present and voting. The legislature of Nebraska can enact property-tax increases reflecting changes in the Consumer Price Index by a simple majority, while larger increases, of up to 5%, require a three-quarters majority. The European Union applies the unanimity rule for particularly "sensitive" issues, and either a simple or a qualified majority rule for "technical" matters. This phenomenon of variation in majority requirements within the same legislative institution raises the obvious question of why these requirements differ so much across policy types.

It has been argued that the size of the supermajority needed to pass a policy should depend on the complexity (and/or on the importance) of that policy.[1] However, the question of how voting rules can be rationalized in terms of policy complexity has not been sufficiently studied. A partial answer to this question comes from a few early studies (Craven (1971), Ferejohn & Grether (1974), Greenberg (1979), Caplin & Nalebuff (1988), Austen-Smith & Banks (2000)). This pioneering and influential literature, however, has focused exclusively on *static* decision-making mechanisms. While static mechanisms are used in real-life institutions, most institutions also employ *dynamic* (or *sequential*) procedures, wherein a proposed policy generally goes through a sequence of amendments prior to its final adoption (see, e.g., Harsanyi (1974), Baron & Ferejohn (1989), Chwe (1994), Eguia & Shepsle (2015), among many others). We are not aware of any study that analyzes the relationship between the level of policy complexity and the type of voting rule in a dynamic legislative mechanism. Addressing this gap in the literature

[1]The introductory quote from Jean-Jacques Rousseau is evidence of this view. More recent works expressing this basic intuition are Erlenmaier & Gersbach (2001), Barberà & Jackson (2004), Maggi & Morelli (2006), and Rogoff (2016), among others.

has practical implications for the optimal design and functioning of political institutions.

In this paper, we analyze this question from the perspective of a *preference-blind* political designer who in the present has to choose a supermajority rule for the selection of policies in the future. Indeed, the political designer has no knowledge of individual political preferences at the time that he or she is choosing the voting rule (as the voting rules are chosen prior to the actual votes). He or she only has general information about *policy complexity* (or *policy importance*), the latter being measured by the social dimensionality of each class of policy in question (e.g., tax increases above a pre-determined percentage threshold, impeachment of a senior official, conviction in a capital punishment case by a jury)—with *social dimensionality* being the number of dimensions or sectors of the society's life and well-being that that class of policy is most likely to affect.[2] In other words, policy complexity represents the various non-correlated dimensions of citizens' preferences that may be affected by a policy change (Harris & Sutton (1983); Ehrlich (2011); Krijnen et al. (2015)). For instance, if withdrawing from a union (e.g., Brexit) affects a country in four dimensions, namely its diplomatic relations, its ability to attract talented immigrants, its national security, and its economy, then we say that the level of complexity (or importance) or the social dimensionality of this decision is four.

Given the level of complexity of the decision that might have to be dealt with in the future, the political designer's goal is to choose a supermajority rule that avoids policy instability and guarantees efficiency, while promoting fair competition among the various policy alternatives. More precisely, the supermajority rule chosen by the political designer should:

(P1): avoid policy instability by ensuring that a *stable policy* exists regardless of the extent to which individual preferences, which might be unknown to the political designer, are antagonistic;

(P2): lead to all *stable* policies being *Pareto-efficient*; and

(P3): minimize institutional status quo bias.[3]

[2]It naturally follows, as is made clear in section 2.1, that the *complexity* of a policy is measured by the dimension of the space to which this policy belongs (for continuous policies) or by the number of competing policy alternatives (for discrete policies).

[3]Properties **(P1)** and **(P2)** are natural. Property **(P3)** is equivalent to minimizing the size of the supermajority required to change the incumbent policy. **(P3)** has both a normative and a positive interpretation. From a normative point of view, it is equivalent to choosing a rule that promotes fair competition among the competing policy alternatives, as supermajority rules are generally biased towards preserving the status quo. From a positive point of view, **(P3)** can be justified on the ground of cost minimization, especially in a context in which the formation of larger political coalitions is more costly, as it is generally argued in the political science

The legislative procedure under which decision making takes place is a natural *dynamic mechanism*. The latter is a simplification of well-known legislative procedures used in democratic societies, where decision making generally follows a succession of stages. The main feature of these real-life mechanisms is that a policy introduced to challenge the status quo can be amended or possibly retracted by its sponsors prior to the final vote. The mechanism that we use in this paper only retains this feature, and omits procedural details that might be specific to each society (see section 2.3).

It follows from the natural requirements set out above that the political designer's objective problem is a constrained minimization problem. This problem involves finding the minimum majority size (as a result of **(P3)**) that satisfies the constraints **(P1)** and **(P2)** under the sequential mechanism. This problem is solved both for continuous and for discrete decisions. A continuous problem involves a choice set consisting of a continuum of competing policy alternatives, while a discrete problem involves a finite number of policy alternatives. A closed-form solution is derived, which provides the exact relationship between the level of policy complexity and the voting rule when individual preferences are *weak* (Theorem 1 and Theorem 2) or *strong* (Theorem 3). This relationship is depicted for discrete decisions in figure 1 (for *weak* preferences) and figure 2 (for *strong* preferences). In these figures, we also represent this relationship for *static* mechanisms to show how our findings differ from related studies.[4]

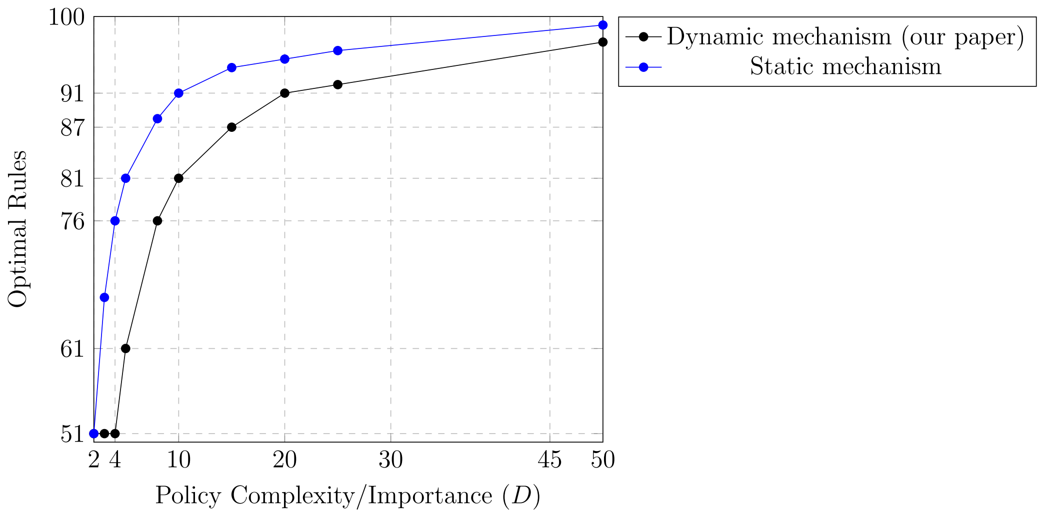

Figure 1: Optimal rules for discrete decisions under *weak* preferences

Clearly, when preferences are *weak*, the optimal voting rule is a weakly increasing function

literature.

[4]In both figures 1 and 2, the y-axis represents the number of votes needed to pass a policy in a legislature where the number of legislators is normalized to 100. The representation of this relationship for continuous decisions is shown in figures 4 and 5 for *weak* preferences and figure 6 for *strong* preferences (section 4).

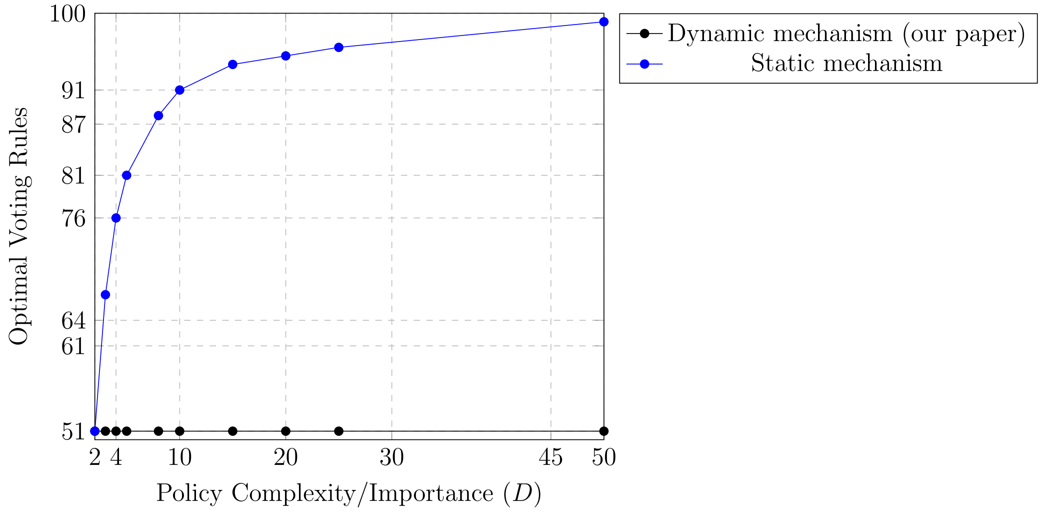

Figure 2: Optimal rules for discrete policies under *strong* preferences

of policy complexity. The aforementioned solution to the political designer's problem rests on the assumption that legislators generally have *weak preferences*, even if these preferences are unknown to the political designer. In real-life politics, preferences of this nature can be justified by the fact that voters are not sufficiently aware of the different implications of alternative policies to be able to discriminate between them. If we assume that this information is available, which will lead to legislators having *strong preferences*, then we show that the *majority rule* is the unique voting rule that solves the political designer's objective problem both for continuous and discrete decisions (Theorem 3 in section 3.3).

Closely related literature. Our paper is the first to address the problem of how policy complexity affects political design in a *dynamic* setting. Early influential studies (e.g., Craven (1971), Ferejohn & Grether (1974), Greenberg (1979), Austen-Smith & Banks (2000)) have exclusively focused on the static mechanism wherein a challenger is pitted against the status quo and the winner is enforced as the policy. Clearly, figures 1 and 2 show that our findings differ significantly from those obtained for the static mechanism (see section 4 for a full comparison). In the static mechanism, the relationship between policy complexity and the voting rule does not vary depending on whether preferences are *weak* or *strong*, whereas in the dynamic setting, it does. In general, there are three main differences between our findings. First, when preferences are *weak*, the optimal voting rule is an increasing function of the level of policy complexity for both the static and the dynamic mechanisms, but the functional form differs significantly. For a given level of policy complexity, the size of the supermajority rule is generally much *smaller* under the dynamic mechanism compared to the static mechanism. For example, when policies are discrete and preferences are *weak*, if the level of policy complexity is 3, the optimal rule is the *majority rule* under our sequential mechanism whereas it is the *two-thirds majority rule* under the static mechanism. Second, when preferences are *strong*, the optimal rule in the dynamic mechanism is the *majority rule* regardless of policy complexity; it is therefore a constant function, whereas it is an increasing function of policy complexity in the static mechanism.[5]

These findings have significantly different implications for political design. In particular, when preferences are strong, the majority rule should be chosen by the political designer to select policies in the dynamic framework, regardless of whether policies are discrete or continuous, and regardless of whether policy complexity is known with certainty or not. In the static framework, the political designer will choose the majority rule only for the simplest policies, and a rule that approaches the unanimity rule for the most complex policies. Interestingly, this also shows that Rousseau's intuition is mainly confirmed in the static case. Our analysis demonstrates that the optimal choice of political rules ultimately depends on whether the legislative procedure is static or dynamic.

The rest of the paper is organized as follows. Section 2 presents the basic framework that describes the political designer's problem. Section 3 presents the solution to this problem when preferences are *weak* and when preferences are *strong*. In section 4, we show how our results differ from some previous works on supermajority rules and dimensionality in static mechanisms. In section 5, we provide some implications of our results for the design of political institutions, and discuss the Brexit case. Section 6 reviews other related literature, and section 7 concludes. For clarity, we collect all the proofs in an appendix.

2 The Preference-blind Political Designer's Problem

This section formalizes the political designer's objective problem. It considers the fact that the designer is blind to legislators' preferences. Given the level of policy complexity, his or her goal is to determine the supermajority rule that avoids policy instability, guarantees policy efficiency, and ensures fair competition between the different policy alternatives. These requirements are described as **(P1)**, **(P2)**, and **(P3)** in the Introduction. Decision making follows a dynamic procedure that, in most of its essentials, describes the legislative mechanisms encountered in several democratic countries. We also assume that voters are fully rational. The various notions needed for the formalization of this objective problem are provided below.

[5] Table 1 in section 4 summarizes the analysis comparing our optimal voting rules to those obtained in static mechanisms.

2.1 The Measurement of Policy Complexity

In this study, the *complexity* of a policy is measured by the *dimension* of the space to which this policy belongs (for continuous policies) and by the *number* of competing policy alternatives (for discrete policies). More precisely, we say that a continuous policy is of complexity (or importance) level D if the space to which this policy belongs is a compact and convex subset of $\mathbb{R}^D$ (e.g., $[0,1]^D$). Similarly, we say that a discrete policy is of complexity level D if the cardinality of the set to which this policy belongs is D. It follows that, in both cases, the complexity level of a decision is measured by the richness of the policy space. The intuition underlying this definition is provided below.

In the literature on decision theory, a decision is generally considered complex (or important) when it involves a choice set with many conflicting possibilities. The multiplicity of conflicting alternatives also means that the decision is difficult to make. Consistent with this view, Krijnen et al. (2015) argue that "people assume difficult decisions to be important and important decisions to be difficult," and Ehrlich (2011) writes that "the more points of access provided to interest groups, the more complex policy will be". In a political context, we define a complex decision by its social dimensionality, D, which can be interpreted as the number of a society's sectors that the decision is likely to affect. It is clear that this definition is consistent with Harris & Sutton (1983), Krehbiel (2004), Ehrlich (2011), and Krijnen et al. (2015) in the sense that an increase in D expands the decision's choice set and therefore the number of competing policies.

As an illustration, assume that a particular class of decision is likely to affect individuals along two non-correlated dimensions captured by the variables x_1 and x_2. Consider that these variables are discrete, with each taking the values 0 and 1. Then the policy space or the choice set envisioned by the political designer is $\{(0,0),(0,1),(1,0),(1,1)\}$ (or a subset of this set if some alternatives in it are not feasible). If instead, the variables x_1 and x_2 are continuous, with each taking values in the interval $[0,1]$, for example, then the policy space is $[0,1]^2$. Importantly, a class of decision that appears to be discrete, such as the decision on whether or not to withdraw from a union, might in actual fact be a continuous decision, especially if each of the social implications of such a decision is captured by a continuous variable. In answering the question of how decision complexity determines voting rules, we are mostly concerned about the social dimensionality of a class of decision, and not about its mathematical dimension. This point is illustrated in the Brexit example set out in section 5.2. Despite the fact that the Brexit vote was apparently a binary decision, experts believed that withdrawal from the European

Union would have consequences for immigration, national security, international trade, and living standards. This implies that these experts viewed the Brexit vote as a vote on a four-dimensional policy space.

It is important to bear in mind that a political designer who is faced with the problem of choosing the voting rule that is most appropriate for selecting a class of policy of a given level of complexity cannot anticipate individual preferences over the choice set induced by the complexity parameter. Our paper only assumes that the designer has enough resources to help her or him identify the main sectors of the society that are likely to be affected by the adoption of a particular class of policy. This information could be obtained by, for instance, gathering the opinions of various experts and specialists (including legislators). Indeed, assessing decision complexity is one of the major problems addressed in the field of *informational politics*. Krehbiel (2004) argues that "many of the public policies that legislatures address are complex, and legislators differ in their inclinations and abilities to sort through the uncertain consequences of many proposals that fall onto the legislative agenda" (Krehbiel, 2004, p.10). For this reason, consulting multiple experts in order to gauge the complexity of a decision is essential. The analysis will prove that this information suffices for the derivation of the optimal voting rule—that is, a supermajority rule that satisfies the properties of policy stability, efficiency, and minimal status quo bias as outlined in the Introduction.

2.2 Policy Complexity and Induced Legislature

Given the complexity level D of a given class of decision, the political designer's goal is to choose the supermajority rule $r(D)$ such that any legislature voting over the decision using $r(D)$ selects a stable and efficient policy regardless of the diversity of individual preferences within the legislature.

Formally, a legislature legislating on a decision of importance (or complexity) level D is a tuple $\mathcal{L} = (N, D, r(D), (\succeq_i))$ where:

1. $N = \{1, 2, ..., n\}$ is a finite number of legislators of size $n > 1$. A set $S \subseteq N$, $S \neq \emptyset$, is called a coalition of N. 2^N denotes the set of all coalitions.

2. $r(D)$ is the supermajority rule used to select a decision of complexity level D, and it is an integer that is strictly greater than $n/2$. Under the rule $r(D)$, a coalition of voters $S \subseteq N$ is a decisive or winning coalition if it consists of at least $r(D)$ legislators. The condition $r(D) > \frac{n}{2}$ also rules out the possibility of having two non-overlapping winning

coalitions under the rule $r(D)$. $\mathfrak{C}$ denotes the set of winning coalitions in the legislature $\mathcal{L}$. Formally, $\mathfrak{C} = \{S \in 2^N : |S| \geq r(D)\}$, where $|X|$ denotes the size of a given set X.

3. $(\succeq_i)$ represents the profile of individual preferences, that might not be known to the political designer. Each preference relation $\succeq_i$ is *weak*, i.e., it is reflexive, complete, and transitive over the choice set or policy space A induced by the decision of complexity level D. These are the only assumptions that we impose on preferences for *discrete* decisions. $\mathcal{U}$ denotes the set of all such preferences, and $\mathcal{U}^n$ denotes the set of all of the preference profiles. If the decision is *continuous*, we also assume that each preference relation is continuous and convex (to be defined in section 3.1). $\mathcal{U}_{con}$ denotes the set of all such preferences, and $\mathcal{U}^n_{con}$ denotes the set of all the preference profiles. We recall that a continuous and convex preference relation is a relation that can be represented by a quasi-concave utility function. Consequently, the class of continuous and convex preferences is extremely large.[6]

Note that, because individual preferences are not known to the political designer, the goal of the latter is to choose the rule $r(D)$, given D, to ensure that, for any preference profile $(\succeq_i)$, the legislature $\mathcal{L} =(N, D, r(D), (\succeq_i))$ selects a *stable* and *efficient* policy under the sequential mechanism described below.

2.3 The Dynamic Political Mechanism

Decisions of any given complexity level are made under a sequential legislative procedure which simplifies the legislative procedures employed in most democratic societies. Different legislative mechanisms exist in real life, but most share a common feature. In most countries, legislative decision making follows a succession of stages, whereby, in the first stage, a bill is introduced, possibly followed by several amendments. In addition, in most systems, a bill introduced to challenge the incumbent policy might be withdrawn by its sponsor(s) if it encounters opposition. The mechanism, depicted in figure 3, only retains this main feature, leaving out procedural details that might be specific to each jurisdiction. It is therefore meant to be as broadly applicable as possible. It comprises three stages described as follows:

[6]Remark that the requirements imposed on preferences are minimal. This implies that our findings still hold if we assume cardinal utilities (with no restrictions) for the case of discrete decisions, and cardinal utilities that are quasi-concave in the case of continuous decisions. Evidently, in each case, these utilities form a very large class.

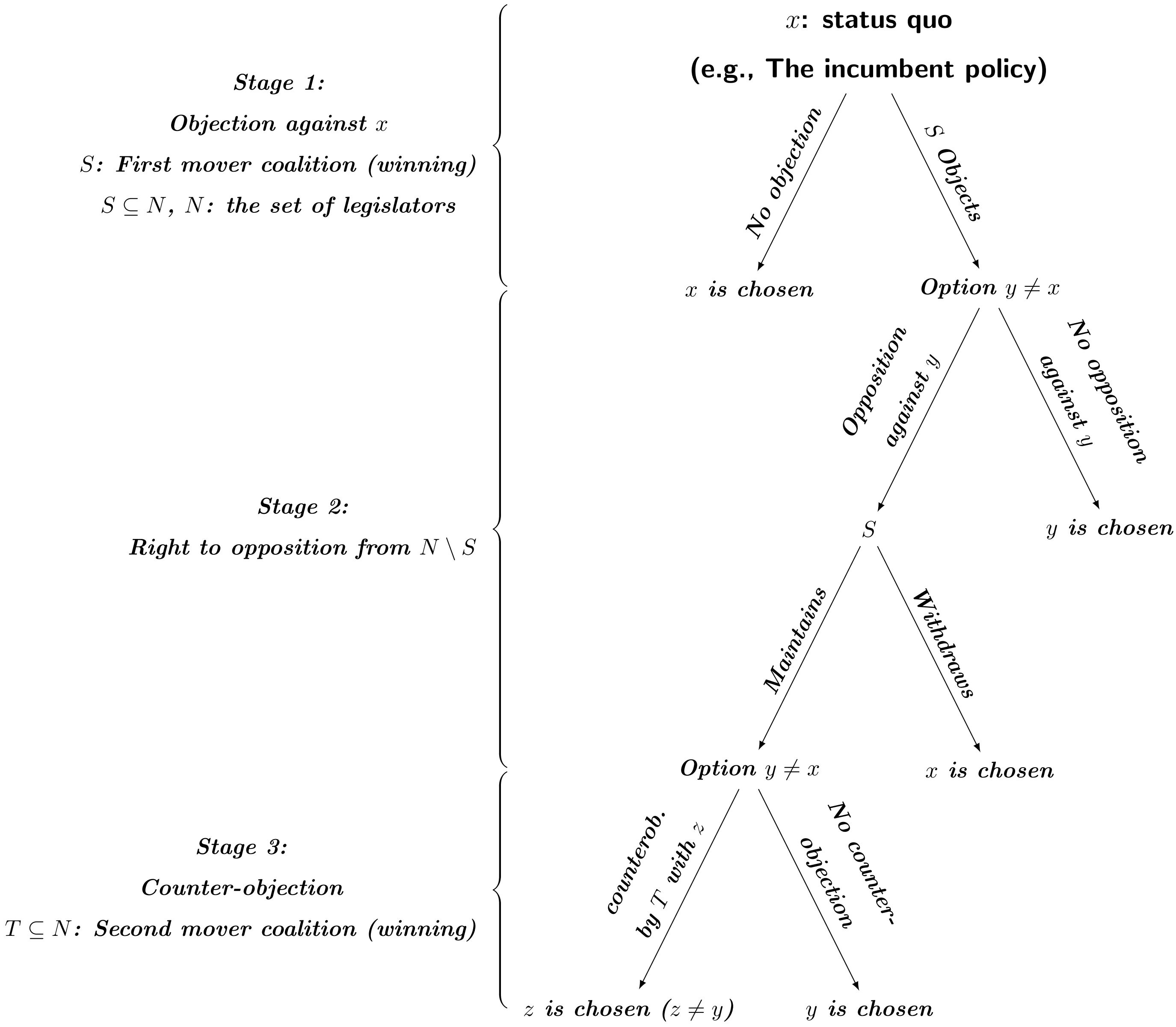

Figure 3: The Dynamic Political Mechanism

- *Stage 1 (Objection)*: Let x be the status quo policy in a decision (or in a class of decision) of complexity level D.[7] Any legislator has the right to seek the replacement of x by proposing a motion y. If no motion is submitted, then x remains in place and the process ends. If a motion y is proposed and is supported by a sponsoring coalition that is winning under the rule $r(D)$, y is submitted with the possibility of an amendment.

- *Stage 2 (Right to opposition)*: Any legislator who is not part of the sponsoring coalition has the right to oppose bill y. If there is no opposition, y is adopted as the new policy and the process ends. If there is any opposition, the sponsoring coalition is given the

[7]The status quo is either the incumbent policy or any alternative that is chosen from the policy space that is induced by the decision of complexity level D.

opportunity to withdraw bill y. If it chooses to withdraw y, the status quo policy x remains in place and the process ends. But, if it chooses to maintain y, the process moves to the third stage.

- *Stage 3 (Counter-objection)*: The opposition has the right to propose an amendment z. If it does not propose any amendment, y becomes the new policy and the process ends. If it proposes an amendment z, a vote is organized between y and z. If a majority of legislators (under the rule $r(D)$) supports z against y, z becomes the new policy and the process ends. Otherwise y is the new policy and the process ends.

A *stable* policy is an alternative such that, if it is the status quo, no winning coalition will seek to replace it. We denote by $\mathcal{R}(\mathcal{L})$ the set of *stable* policies in a legislature $\mathcal{L} = (N, D, r(D), (\succeq_i))$. This solution concept is formally defined in the next section.

2.4 Equilibrium Concept

This section formalizes the rational behavior of legislators under the mechanism described in the preceding section.

Definition 1. Let $\mathcal{L} = (N, D, r(D), (\succeq_i))$ be a legislature, S be a winning coalition, and $x,\ y \in A(D)$ be two competing policies, with $A(D)$ being the choice set induced by D.

1. **Objection**: $(y, S) \in A(D) \times \mathfrak{C}$ is said to be an objection against x if $y \succ_S x$ (i.e., each individual in S prefers y to x).

2. **Counter-objection**: let (y, S) be an objection against x. A pair $(z, T) \in A(D) \times \mathfrak{C}$ is said to be a counter-objection against (y, S) if $z \succ_T y$ and *not*$(y \succeq_T x)$.[8]

3. **Unfriendly counter-objection**: let (y, S) be an objection against x and $(z, T) \in A(D) \times \mathfrak{C}$ be a counter-objection against (y, S). The counter-objection (z, T) is said to be unfriendly if *not*$(z \succeq_S x)$.[9]

[8] *not*$(y \succeq_T x)$ means that a counter-objection can be formulated by a winning coalition T only if some members of T prefer the status quo x to the motion y. This follows from the clause stated in stage 2 of the sequential mechanism that only an *opponent* of a motion y can initiate the formation of a counter-objecting coalition.

[9] A counter-objection (z, T) against the objection (x, S) is unfriendly if some members of S prefer the status quo policy x to z. The phrase "unfriendly counter-objection" is inspired by the phrase "unfriendly amendment" used in the legislative jargon.

4. **Justified objection**: an objection (y, S) against x is said to be justified if there is no unfriendly counter-objection against (y, S).

5. **Stable policy**: a *stable* policy is an alternative in $A(D)$ against which no justified objection exists.

A stable policy is a policy such that, if it is the incumbent policy, no winning coalition will have an incentive to deviate from it under the sequential mechanism. In this regard, the solution concept, $\mathcal{R}(\mathcal{L})$, is defined like the (strong) Nash equilibrium or the core, with the difference that the latter concepts are most appropriate to capture rationality in one-shot games. One-shot games, however, are not widely used in legislative settings, which is the reason why we do not analyze them in this paper. We also remark that, notwithstanding the dynamic nature of the decision-making procedure, it is not an extensive-form game of perfect information. The reason is that there is no predetermined order in which coalitions move, and that the mechanism is also silent about how coalitions form. For this reason, the notion of subgame perfect Nash equilibrium cannot be used to capture the rational behavior of agents in this sequential framework. Indeed, the solution concept, $\mathcal{R}(\mathcal{L})$, follows the "blocking approach" (Ray & Vohra (2014)). This approach has been used in a number of other studies to formalize rational behavior in games (see, e,g., Harsanyi (1974), Chwe (1994), Ray & Vohra (2015), Dutta & Vohra (2017), amomg others).

Our approach to formalizing rational behavior shares common features with several bargaining models (see, e.g., Aumann & Maschler (1964) and the literature that this pioneering paper has inspired). For instance, the definition of an objection is classical. This is a simple explanation of the solution concept presented in Definition 1. If, following an objection (y, S) against the status quo x, a coalition T formulates a counter-objection or an amendment (z, T), then S should weakly prefer z over x only if certain members of T strictly prefer x over y. In fact, if all the members of T prefer y over x, under the mechanism presented above, they will not have any incentive to formulate a counter-objection unless this counter-objection does not hurt S. This is because doing so will cause S to withdraw the bill y, thus allowing x to remain in place and undermining the interests of all the members of T since they all prefer y over x. If, on the contrary, some members of T are harmed by the objection (y, S) against the status quo x, these members have the right to oppose this objection and to form T in order to formulate a counter-objection. This occurs only if the objection is not withdrawn. In this case, the members of S should prefer the outcome z of any amendment (z, T) when sponsoring a bill in order to not regret the final outcome of the vote relative to the status quo.

It is important to note that our notion of rationality does not require that individuals forming a winning coalition in order to object against a policy or counter an objection have a *binding* agreement that would, for instance, prevent a member from undertaking a future action without the prior consent of the other members. An individual who decides to enter a coalition is motivated only by his or her self-interests, and once an action is undertaken by a coalition, each member is free to leave and join a different coalition.

The set $\mathcal{R}(\mathcal{L})$ is also called the reciprocity set in Pongou & Tondji (2018). This is because they prove that, under the above-described sequential legislative procedure, any second-stage movers cannot rationally free-ride on the actions of first-stage movers and first-stage movers cannot take actions that harm second-stage movers who can successfully retaliate. In other words, this procedure induces legislators to take reciprocal actions. In particular, it encourages positive reciprocity.

2.5 Stating the Political Designer's Objective Problem

This section states the political designer's problem. Given the level of policy complexity D, the objective is to find the supermajority rule $r(D)$ such that:

(P1): there exists a stable policy in the legislature $\mathcal{L} = (N, D, r(D), (\succeq_i))$ for any preference profile $(\succeq_i)$: $\mathcal{R}(\mathcal{L}) \neq \varnothing$ for any $(\succeq_i) \in \mathcal{U}^n$ (or $\mathcal{U}^n_{con}$ for a continuous decision)[10];

(P2): any stable policy $x \in \mathcal{R}(\mathcal{L})$ is Pareto-efficient; and

(P3): $r(D)$ minimizes status quo bias[11]—that is, $r(D)$ minimizes the size of the supermajority needed to change the status quo.

[10] The stability requirement is the only requirement considered in studies that analyze the relationship between policy complexity and voting rule in the static framework. This requirement is important because the existence of a stable policy implies that a permanent decision can be achieved regardless of the diversity of individual preferences. This property is similar to the concepts of decisiveness (Dasgupta & Maskin (2008)) in the literature. It can also be justified from a positive point of view since policy instability can be economically costly.

[11] As explained in the Introduction, this property has both a normative and a positive interpretation. From a normative point of view, it is equivalent to choosing a rule that promotes fair competition among policy alternatives. As explained by Dasgupta & Maskin (2008), if "the candidates are, say, various amendments to a nation's constitution, then one might want to give special treatment to the status quo—namely, to no change—and so ensure that constitutional change occurs only with overwhelming support" (p.2). Given the fact that such a bias can be viewed as being unfair to potential challengers, we assume that the political designer seeks to minimize it. From a positive point of view, the minimal status quo bias can be justified on the ground of cost minimization, especially in a context in which the formation of larger political coalitions is more costly, as it is generally argued in the political science literature.

The preference-blind political designer's objective problem is summarized by the following optimization problem:

$$\begin{aligned} \min \quad & r(D) \\ \text{s.t.} \quad & (\mathbf{P1}) \text{ and } (\mathbf{P2}) \quad \text{are satisfied.} \end{aligned} \tag{1}$$

The minimization problem comes from **(P3)**, with **(P1)** and **(P2)** being the constraints. In the next section, we solve this problem for both discrete and continuous policies.

3 The Solution: Supermajority Rule as a Function of Policy Complexity

The solution to problem (1) is simplified by the following result according to which any stable alternative under the political mechanism described in section 2.3 is Pareto-efficient regardless of the nature of the policy space.

Lemma 1. *Let $\mathcal{L} = (N, D, r(D), (\succeq_i))$ be a legislature. Any policy in $\mathcal{R}(\mathcal{L})$ is Pareto-efficient.*

The intuition behind this result follows from the fact that the dynamic mechanism induces legislators to adopt a reciprocal behavior. No potential second mover will react against a first mover who replaces a status quo x with a policy y that Pareto-dominates x. Opposing such a move would cause the sponsoring coalition to withdraw y, thus allowing x to remain in place and inducing the persistence of an alternative that is less preferred by all the legislators.

This lemma is important because it reduces the political designer's objective problem to an optimization problem that only requires a supermajority rule to guarantee that a stable policy exists regardless of the extent to which legislators diverge in their political views. The simplified problem is formulated as follows:

$$\begin{aligned} \min \quad & r(D) \\ \text{s.t.} \quad & \mathcal{R}(\mathcal{L}) \neq \varnothing, \text{ for any } (\succeq_i) \in \mathcal{U}^n \text{ (or } \mathcal{U}^n_{con}). \end{aligned} \tag{2}$$

We start by addressing the case of continuous decisions in the next section.

3.1 Continuous Decisions

We recall that a decision of complexity level D is continuous when it involves a continuum of competing policy alternatives forming a compact and convex subset of the multidimensional vector space $\mathbb{R}^D$ (e.g., $[0,1]^D$). Moreover, D is the number of society's sectors that are likely to be affected by a change in policy. The political designer lacks full information about individual preferences when choosing a rule. He or she only assumes that preferences belong to the class of preferences denoted $\mathcal{U}_{con}$, which is the class of reflexive, transitive, complete, continuous and convex preferences. Indeed, continuity and convexity are classical assumptions that are imposed on preferences when the choice set is a continuum. The definitions of these notions are presented below.

Definition 2. Let $\succeq$ be a preference relation over a policy space $A(D)$.

1. $\succeq$ is said to be continuous if, for any policies $x, y \in A(D)$ such that $x \succ y$, there exists a neighborhood $S(x)$ of x and a neighborhood $S(y)$ of y such that $z \succ t$ for every $z \in S(x)$ and $t \in S(y)$.

2. $\succeq$ is said to be convex if for any policies $x, y \in A(D)$ such that $x \succeq y$ and $x \succeq z$, $x \succeq \lambda y + (1 - \lambda)z$ for every $\lambda \in [0, 1]$.

To state the main result for this section, the following notation is needed. For a real number x, $\lfloor x \rfloor$ denotes the greatest integer that is smaller than or equal to x. The exact relationship between policy complexity and voting rule is given below.

Theorem 1. *Let D ($2 \leq D < \infty$) be the level of complexity of a continuous decision in a legislature of size n. Then, the solution to problem (1) is given by:*

$$r_1^*(D) = \left\lfloor \frac{D}{D+1} n \right\rfloor + 1.$$

The proof of this result is provided in the appendix. We would like to give the intuition of this proof. First, we prove that if a voting rule r is such that $r > (\frac{D}{D+1})n$, then, there always exists a stable policy in a legislature deciding on a policy of complexity level D (see Lemma 3 in the appendix). This result is quite intuitive. Certainly, if r is the unanimity rule, then a stable policy always exists, since there exists at least one Pareto-efficient policy and since any Pareto-efficient policy is stable under unanimity. If r is the majority rule, then, a stable policy might not exist if D is sufficiently high. Following an argument that resembles that of the Intermediate Value Theorem, there should be a threshold such that if r is greater than that threshold, a stable alternative always exists, and if r is smaller than that threshold, a stable policy may not exist. This threshold is $\frac{D}{D+1}n$. We also prove that if r is smaller than $\frac{D}{D+1}n$, then, there exists a preference profile under which a stable policy does not exist (see Lemma 6 in the appendix). Therefore, the optimal rule is $r_1^*(D) = \lfloor \frac{D}{D+1}n \rfloor + 1$. We illustrate the relationship between policy complexity and voting rule in figure 4.

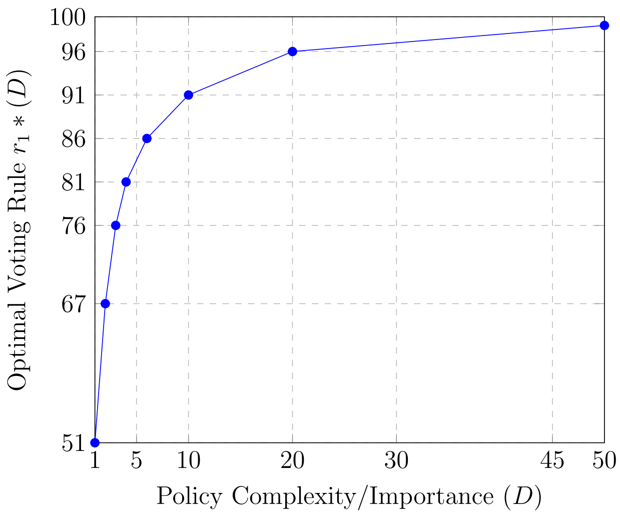

Figure 4: Optimal supermajority rule as a function of the level of policy complexity for continuous decisions

Theorem 1 has some practical implications. If a policy change is likely to affect only one sector of the society or only one dimension of individual preferences (that is, $D = 1$), even if we do not know how preferences will be affected, the optimal voting rule is the majority rule ($r_1^*(1) = \lfloor \frac{1}{2}n \rfloor + 1$). For instance, if the decision is to choose a bus stop along a straight road, then the majority rule is the optimal voting rule, especially if the only factor that matters is the distance separating each household from the bus stop, regardless of whether some people like to be closer and others like to be farther, and regardless of whether others have even more complex preferences. The same majority rule is the optimal solution in cases in which legislators have single-peaked preferences over the induced policy set considered as a real line (i.e., the dimension of individuals' preferences is 1). If the social dimensionality of a decision is two, then the optimal rule is the two-thirds majority rule ($r_1^*(2) = \lfloor \frac{2n}{3} \rfloor + 1$). For much more complex (or important) decisions, that is, when D is large enough, the optimal voting rule approaches the unanimity rule (remark that as D tends to infinity, $r_1^*(D)$ tends to n). These findings confirm the basic intuition of Rousseau (1762) and are consistent with the variation in majority requirements observed in legislative bodies around the world, as is noted in the first paragraph of the Introduction. However, as we show in section 3.3, Rousseau's intuition is not confirmed when preferences are *strong*.

3.2 Discrete Decisions

A decision of complexity level D is discrete when it involves D competing policy alternatives. Here, individual preferences over the policy space induced by the level of decision complexity are reflexive, transitive, and complete. The main finding set out below establishes the precise relationship between policy complexity and voting rule.

Theorem 2. *Let D ($2 \leq D < \infty$) be the level of complexity of a discrete decision in a legislature of size n. Then, the solution to the political designer's problem (1) is given by:*

$$r_2^*(D) = \max\left\{\left\lfloor \frac{D-2}{D}n \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor\right\} + 1.$$

We provide some intuition of the proof of Theorem 2, which is presented in the appendix. Due to the discrete nature of the policy space, this proof is different from that of Theorem 1, but the logical steps are the same. First, if a voting rule r is such that $r > (\frac{D-2}{D})n$, there always exists a stable policy in a legislature deciding on a policy of complexity level D (see Lemma 8 in the appendix). Second, if a voting rule r is smaller than or equal to to the number $(\frac{D-2}{D})n$, then, there exists a preference profile under which a stable policy does not exist (see Lemma 9 in the appendix). Consequently, the optimal rule sought by the political designer is $r_2^*(D) = \max\left\{\left\lfloor \frac{D-2}{D}n \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor\right\} + 1$, given the fact that, by definition, a supermajority rule is strictly greater than $\frac{n}{2}$.

The relationship between policy complexity and voting rule in the discrete case can be visualized in figure 1 in the Introduction. Figure 1 illustrates the finding that the optimal voting rule for decisions that involve fewer than five competing options is the majority rule. If the social dimensionality of a decision is 16, then the 87.5% supermajority rule is the most appropriate rule. The unanimity rule should be used for sufficiently important policies. In general, the size of the supermajority needed to adopt a policy weakly increases with its level of complexity (or importance). These findings rationalize the fact that, when preferences are weak, different rules are used to adopt policies of a different nature in most real-life political institutions.

The findings of this section also have practical implications for the design of political institutions. Consider, for instance, the democratic choice of the official language of a country. The analysis implies that such a choice should be made using the majority rule in a country like Rwanda and using a rule that approaches the unanimity rule in a country like Cameroon or Nigeria (Rwanda has 2 local languages whereas Cameroon and Nigeria have over 250 local languages each). Interestingly, the result also implies that the status quo is more likely to persist in more fractionalized societies. Indeed, in ethnically fractionalized societies, the number of policy alternatives generally reflects the number of ethnic groups, and our analysis prescribes a greater supermajority size when the number of competing policies is higher. Clearly, the status quo is more likely to persist under a rule that is closer to the unanimity rule than under a rule that is closer to the majority rule.

3.3 The Solution to the Political Designer's Problem when Preferences are Strong

The solution to the political designer's problem in sections 3.1 and 3.2 (Theorem 1 and Theorem 2) assumes that legislators have weak preferences. In real-life politics, weak preferences can be justified in an environment where information on policy alternatives is not sufficient enough to allow voters to discriminate among them. In advanced societies, access to the media might make it possible for the public to be sufficiently aware of differences between policy options, potentially leading to *strong* individual preferences (that is, complete, transitive, and antisymmetric preferences). In this section, we solve the political designer's problem when individuals display such preferences. The problem is stated similarly as in section 2.5, except that in (**P1**), the set $\mathcal{U}^n$ (or $\mathcal{U}^n_{con}$ for continuous and convex decisions) is replaced by $\mathcal{V}^n$ ($\mathcal{V}^n_{con}$ for continuous and convex decisions), where $\mathcal{V}$ (resp. $\mathcal{V}_{con}$) is the set of strict (resp. strict, continuous, and convex) preferences over the set of alternatives $A(D)$. It is stated as follows. Given the level of policy complexity D of the decision that is to be made, the political designer's problem is to find the supermajority rule $r(D)$ such that:

(P1'): there exists a stable policy in the legislature $\mathcal{L} = (N, D, r(D), (\succeq_i))$ for any preference profile $(\succeq_i)$: $\mathcal{R}(\mathcal{L}) \neq \varnothing$ for any $(\succeq_i) \in \mathcal{V}^n$ (or $\mathcal{V}^n_{con}$ for a continuous decision);

(P2): any stable policy $x \in \mathcal{R}(\mathcal{L})$ is Pareto-efficient; and

(P3): $r(D)$ minimizes status quo bias.

Equivalently, the preference-blind political designer's objective problem is summarized by the following optimization problem:

$$\begin{aligned} \min \quad & r(D) \\ \text{s.t.} \quad & (\mathbf{P1'}) \text{ and } (\mathbf{P2}) \quad \text{are satisfied.} \end{aligned} \tag{3}$$

The finding below shows that the *majority rule* is the unique solution to the political designer's problem, regardless of the level of decision complexity D.

Theorem 3. *Let D $(2 \leq D < \infty)$ be the level of complexity of a decision that is either continuous or discrete in a legislature of size n. Then, the solution to problem (3) is given by the majority rule:* $r_3^*(D) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

To prove Theorem 3, we first establish that, with the majority rule, the stability condition is satisfied under the sequential mechanism when legislators display strong preferences over the set of feasible policies in the legislature (see Lemmas 10 and 11 in the appendix). We know, from Lemma 1 that each stable policy is Pareto-efficient. Since the majority rule is the minimal supermajority rule by definition, we can conclude that the majority rule is the unique solution of problem (1).

Theorem 3 can be seen as providing a new perspective on the role that access to information (possibly through the media) can play in the design of democratic rules. According to Dasgupta & Maskin (2008) and Maskin & Sen (2017a,b), the majority rule embodies the very idea of democracy. Therefore, if the role of the media is to supply the necessary information that allows each member of the society to discriminate among the competing policies, then our analysis implies that the majority rule is the unique optimal rule when information access is sufficiently high.[12]

4 Comparison to Static Mechanisms

In this section, we compare our findings to the early literature on the relationship between voting rule and stability in static mechanisms. This literature examines this question in both deterministic (e.g., Craven (1971), Ferejohn & Grether (1974), Greenberg (1979)) and probabilistic setups (e.g., Caplin & Nalebuff (1988)). Conceptually, it differs from our analysis in two major respects. First, it primarily focuses on policy stability, whereas we consider efficiency and the minimization of institutional status quo bias in addition. Second, it exclusively addresses static (or one-shot) mechanisms, whereas we use a dynamic (or sequential) mechanism. As a result, our findings completely differ. Those differences are summarized in Table 1 below.

[12] In this paper, we show how policy complexity determines the voting rule that guarantees policy stability and efficiency, and minimizes institutional status quo bias. Note that if we relax the requirement of minimal status quo bias in the formalization of the political designer's objective problem (1), then for a policy of complexity level D, any supermajority rule of a size greater than or equal to the threshold $r_1^*(D)$ for continuous policies, and the threshold $r_2^*(D)$ for discrete policies, when preferences are weak, or any supermajority rule of a size greater than or equal to majority rule ($r_3^*(D)$) when preferences are strong, will induce the selection of a stable and efficient policy.

Table 1: Optimal supermajority rule: dynamic versus static mechanisms

Authors	Continuous policies	Discrete policies	Preferences	Optimal supermajority rule
Craven (1971)	No	Yes	Strong	$\left\lfloor \frac{D-1}{D} n \right\rfloor + 1$
Ferejohn & Grether (1974)	No	Yes	Acyclic	$\left\lfloor \frac{D-1}{D} n \right\rfloor + 1$
Greenberg (1979)	No	Yes	Weak	$\left\lfloor \frac{D-1}{D} n \right\rfloor + 1$
Greenberg (1979)	Yes	No	Weak	$\left\lfloor \frac{D}{D+1} n \right\rfloor + 1$
Our paper	Yes	No	Weak	$\left\lfloor \frac{D}{D+1} n \right\rfloor + 1$
Our paper	No	Yes	Weak	$\max \left\{ \left\lfloor \frac{D-2}{D} n \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor \right\} + 1$
Our paper	Yes	No	Strong	$\left\lfloor \frac{n}{2} \right\rfloor + 1$ (majority rule)
Our paper	No	Yes	Strong	$\left\lfloor \frac{n}{2} \right\rfloor + 1$ (majority rule)

Note: In Table 1, D ($2 \leq D < \infty$) represents the complexity level of a decision, and n represents the number of voters in a legislature. The political designer derives the optimal supermajority rule as a function of D and n, without any prior knowledge of individual preferences.

Craven (1971) shows that with a finite policy space of size D, the minimum majority size needed to ensure the existence of a stable policy is $(D-1)/D$ when preferences are strong. Ferejohn & Grether (1974) and Greenberg (1979) find similar results for finite policy space when preferences are acyclic or weak, respectively. Greenberg (1979) further shows that with an D-dimensional policy space, the minimum majority size needed to guarantee the existence of a stable policy is $D/(D+1)$.[13] Caplin & Nalebuff (1988) show that when individual preferences are Euclidean and the distribution of types (range of preferences) on most preferred policies are represented by a concave density function, the majority size needed to ensure the existence of a stable policy is no greater than $1-[D/(D+1)]^D$, with a limit of $1+(1/e)$ or just under 64%, as D becomes very large. Because we use a deterministic framework like Craven (1971), Ferejohn & Grether (1974) and Greenberg (1979), it makes sense to compare our findings. Figures 1 and 2 in the Introduction, and figures 5 and 6 below show the main differences (or similarities) between our results. Our findings are less comparable to Caplin & Nalebuff (1988) because their setup is probabilistic. Clearly, our predictions are very different from those of Craven (1971), Ferejohn & Grether (1974) and Greenberg (1979) for finite policy space and weak preferences. For instance, when the number of competing policies is four, the majority rule is the optimal rule under the dynamic mechanism whereas the other studies in static mechanisms predict that the optimal rule is the 76%-supermajority rule (see figure 1). Similarly, when preferences are strong, our analysis shows that the majority rule is the optimal rule, whereas each of the aforementioned studies in static mechanisms predicts a majority threshold that is much larger

[13] Interestingly, our result coincides with Greenberg (1979) when policies are continuous and preferences are weak (see figure 5), but we differ significantly when preferences are strong (see Table 1 or figure 6).

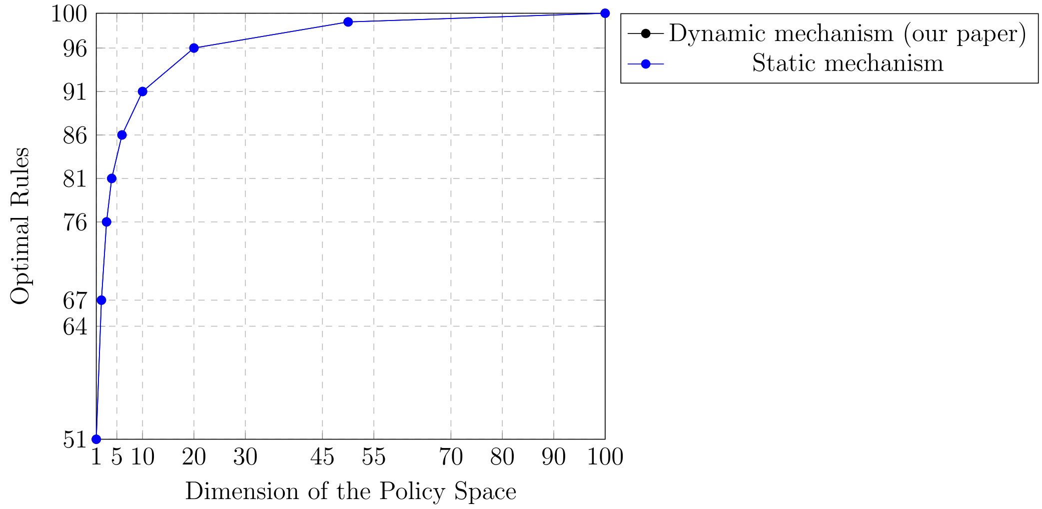

Figure 5: Optimal rules for continuous policies under weak preferences

and is increasing with policy complexity (see figures 2 and 6).

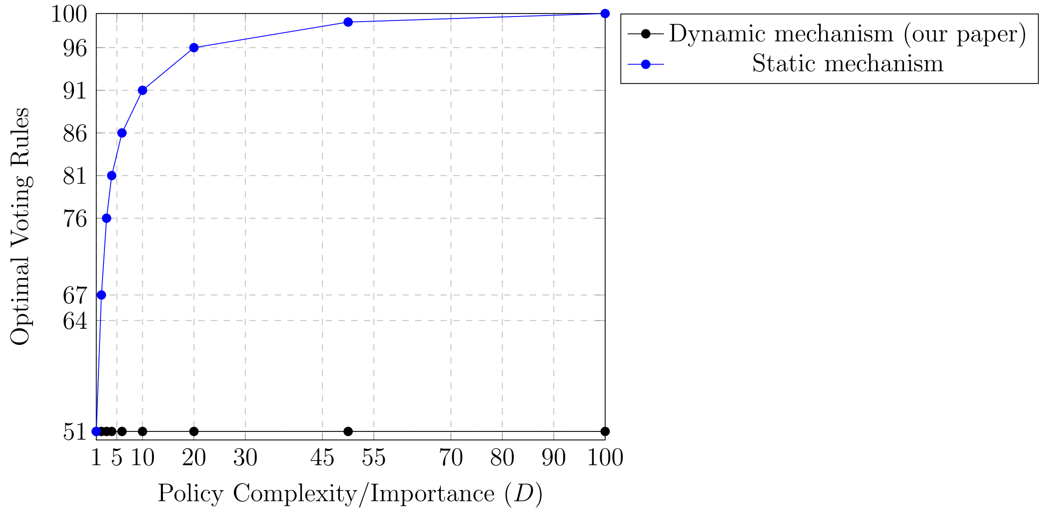

Figure 6: Optimal rules for continuous policies under strong preferences

5 Implications For The Design of Political Institutions

In this section, we discuss some practical implications of our findings for political design, and we use the Brexit vote as an illustration.

5.1 Five Implications for Political Design

Our findings have implications for the optimal design of political institutions in a *dynamic* setting. We briefly discuss five implications. First, in choosing the rule that should be used to select a particular class of policy, the political designer should pay attention to the complexity level of this policy.[14] If the complexity level of a policy is sufficiently low, it should be selected using the majority rule. Our analysis shows that the only continuous policies that should be selected using the majority rule are one-dimensional policies. As for the selection of discrete policies, the majority rule is the most appropriate rule if their dimension is at most equal to four. Our analysis also implies that policies of sufficiently high importance should be selected using a rule that approaches the unanimity rule. These findings provide an important theoretical foundation for Rousseau's prescriptions when preferences are weak.

Second, our findings provide a rationale for using different rules to pass different kinds of policies within the same legislative institution. In fact, the level of policy complexity might vary depending on whether the policy concerns education, health care, national defense, international trade, terrorism, impeachment, removal from office, human rights, taxes, or land property rights. It therefore follows from our analysis that policies falling within the scope of these categories could be selected using different voting rules, each of which would be associated with a particular category.

Third, our findings provide a rationale for why *majoritarian democracy* might be more effective in certain societies than in others. According to Dasgupta & Maskin (2008), the majority rule embodies the principle of "one man, one vote", and can be viewed as the most robust democratic rule. Nevertheless, our analysis implies that the majority rule is not optimal for fractionalized or multiethnic societies. The reason is that, since in these societies, the number of competing policies generally reflects the number of major ethnic groups, the optimal voting rule has to deviate from the majority rule when preferences are weak, especially if there are more than four competing policies. In contrast, the majority rule is the most appropriate option in societies with few ethnic groups or in those characterized by a left-right political spectrum. These societies are therefore more likely to be stable under majoritarian democracy.

Fourth, our findings provide a rationale for why certain policies are more likely to be more persistent than others, even when political preferences change. Indeed, figures 1, 2, and 4 imply that the size of the supermajority rule that is optimal for the selection of a more complex policy should be greater when legislators have weak preferences. Given that a greater supermajority size implies a greater bias towards the status quo, it follows that more complex status quo policies have to be more persistent. For example, the fact that changing the constitution of a country generally requires an "overwhelming support" (Dasgupta & Maskin (2008)) could

[14] The literature on informational politics argues that, in real life, information about the complexity level or likely consequences of a policy is gathered with the help of experts and specialists (e.g., Harris & Sutton (1983), Krehbiel (2004), Krijnen et al. (2015)).

explain why constitutions are more persistent than, for instance, environmental or tax policies, most of which are selected using the majority rule.

Finally, our findings have implications for how the media could affect the design of political rules. Indeed, if the role of the media is to supply the necessary information that will allow each individual to sharply discriminate among the competing policies, then our analysis implies that the majority rule is the unique optimal rule when media access is sufficiently high. This result holds regardless of the nature of the policy space, or the level of policy complexity. Therefore, our findings suggest that freedom of the press is vital for enhancing the consistency and the optimality of the majority rule, which embodies the very notion of democracy (see, e.g., Dasgupta & Maskin (2008), Maskin & Sen (2017a,b)). Thus, our analysis provides a different perspective on the role of media in the functioning of a democratic society.

5.2 An Illustration: The Brexit Vote

On June 23, 2016, the British people voted on whether to remain in or leave the European Union. If this vote is considered a *binary decision* and if it takes place under the sequential mechanism described in this paper, then our analysis (figure 1 or figure 2) implies that it should be decided using the majority rule, which was the actual rule under which the vote took place. Several scholars, however, did not regard the Brexit referendum to be a simple decision, consequently raising some concerns about the voting process (see, e.g., Rogoff (2016), Stiglitz (2016)).

If the Brexit vote was not a simple issue, what could its *social dimensionality* be? Scholars and political observers have argued that the decision to leave the European Union is likely to have a significant impact on important sectors of the United Kingdom (see, e.g., Dhingra et al. (2016b), Dhingra et al. (2016a), Rogoff (2016), Stiglitz (2016), Sampson (2017)). The most frequently mentioned sectors are immigration, international trade, national security, and living standards. Assume that each of these four sectors is represented by a continuous variable defined over the interval $[0, 1]$. Then, an alternative is a vector (x_1, x_2, x_3, x_4), with each component taking its values in $[0, 1]$. The analysis(figure 4), then suggests that the 80% supermajority rule was the optimal voting rule since the complexity or social dimensionality (D) of this decision is equal to four. Importantly, it should be observed that this conclusion is reached without any knowledge of individual preferences.

6 Other Related Literature

Rousseau (1762) was among the first to suggest that the size of the supermajority needed to pass a policy should be related to the importance of that policy. He argued that decisions that are "straightforward" should be made using the majority rule, whereas "more grave and important" matters should be decided using a rule that approaches the unanimity rule. Our study is the first to provide a theory that shows conditions under which Rousseau's intuition is valid when decisions are made in a *dynamic* framework. We find that, if the level of policy importance or complexity is sufficiently small, then the majority rule is the appropriate voting rule. However, highly complex or important issues should be decided using a rule that approaches the unanimity rule, which is consistent with Rousseau's view. More importantly, we make Rousseau's argument even more precise by deriving the exact relationship between policy importance and voting rule under a sequential decision-making mechanism. However, we also find that, irrespective of the level of complexity and nature of the decision to be made, the *majority rule* is the unique optimal rule when individual preferences are strong. This latter finding shows that, in sequential decision-making mechanisms, Rousseau's intuition is valid only when preferences are not sufficiently discriminating. It also highlights one of the key differences between our results and those obtained for static decision-making mechanisms (see section 4).[15]

This study also contributes to a small literature on the rationalization of political rules. In their seminal book, *The Calculus of Consent*, Buchanan & Tullock (1962) adopt the rational individualistic approach to the collective-choice decision-making process to explain the existence of various voting rules in society. Inspired by the works of Buchanan & Tullock, studies like Erlenmaier & Gersbach (2001), Barberà & Jackson (2004), Holden (2005), and Maggi & Morelli (2006) derive various rationales for the variation of decision rules across various models based on incomplete social contracts.

Erlenmaier & Gersbach (2001) introduce the concept of "flexible" majority rules in public projects to improve the lack of efficiency in democratic processes. The main objective of their study is to propose constitutional rules and principles that lead to the implementation of socially efficient outcomes under conditions of uncertainty. Using either a fixed majority rule or a flexible majority rule, they derive some constitutional principles that might be embedded in a well-defined incomplete social contract to achieve the first-best allocation. Barberà & Jackson

[15]In general, even when individual preferences have a sufficiently general structure, for any given level of decision complexity, the optimal supermajority rule is smaller under the dynamic mechanism than under the static mechanism (see Table 1).

(2004) introduce a model in which, "self-stable" constitutions are derived. These are rules that would be immune to change in a society in which they are used to change the extant rules. They demonstrate that these rules include supermajority rules. Maggi & Morelli (2006) determine the optimal size of the supermajority in international organizations when there is imperfect enforcement. Holden (2005) analyzes the optimal supermajority requirement determined by multilateral bargaining behind the veil of ignorance, whereby the policy space is unidimensional, and a decision is assessed according to the level of risk aversion.

The research question assessed in our paper significantly differs from the above-mentioned studies in terms of its scope, analyses, and findings. The goal is to analyze the relationship between policy complexity (or importance) and supermajority rule. This relationship is revealed under a typical sequential legislative procedure. The analysis differentiates between discrete and continuous decisions, and ultimately rationalizes the use of different rules to pass policies that differ in their nature and in their level of complexity. Our analysis can also explain why more complex status quo policies are more persistent, even after a change in political preferences.

7 Conclusion

Voting rules vary widely across policy types within legislative bodies. It is argued that these variations are related to the fact that certain decisions are more complex (or more important) than others. While this intuition is sensible, a formal analysis of how policy complexity should determine the most appropriate voting rule in a decision-making process has not been studied in the economics or political science literature when the electoral framework is *dynamic*. In this paper, we address this question from the perspective of a *preference-blind* political designer who in the present has to choose a voting rule that will be used to select policies in the future. Given the level of policy complexity, his or her goal is to determine the supermajority rule that (1) avoids policy instability regardless of the extent to which individual political opinions diverge; (2) ensures that any chosen policy is efficient; and (3) promotes a fair competition among the different policy alternatives by minimizing institutional status quo bias. Policies are selected under a natural dynamic mechanism that, in most of its essentials, retains the main features of legislative procedures used in various democratic societies, leaving out procedural details that might be specific to each society. A closed-form solution to the designer's objective problem for both continuous and discrete decisions is derived. The analysis shows that, when preferences are weak, the size of the majority needed to adopt a policy increases with the level of policy complexity, with the functional form varying across continuous and discrete decisions. However, when preferences are strong, the majority rule is the unique solution to the political designer's problem for continuous and discrete decisions, regardless of the level of policy complexity. Our findings differ significantly from those obtained for the static mechanism.

The findings have practical implications for the design of political institutions. Indeed, the analysis sheds light on how voting rules should be selected based only on policy complexity in order to reach permanent and optimal decisions, while at the same time ensuring that the selection process is as fair as possible to competing policy alternatives. It also suggests a new perspective on the role that access to information (or the media) might play in the design of democratic rules. Indeed, if the role of the media is to supply the necessary information that makes it possible to be sufficiently aware of differences between competing policy alternatives and to discriminate among them, then our analysis suggests that the *majority rule* is the unique optimal rule when access to information is sufficiently high.

Our analysis also has testable implications for the functioning of political institutions. It ultimately rationalizes the use of different voting rules to adopt policies of differing degrees of complexity (or importance) within legislative bodies around the world. It also enables the researcher and the political observer to objectively infer the legislator's perspective on the importance of different policies within a legislature. For example, if one assumes that the impeachment of a United States president and his removal from office are discrete policies, then our findings imply that, from the legislator's perspective, the social dimensionality of the former policy is between 2 and 4 whereas that of the latter policy is 6. If one considers these policies to be continuous, the analysis indicates that, from the legislators' perspective, the social dimensionality of the former is 1, and that of the latter is 2. The legislator's perspective might not be consistent with that of ordinary political observers. A recent real-life example of this sort of discrepancy is the Brexit vote, which, as noted in the Introduction, Rogoff (2016) thought was a more important decision than the legislators had originally anticipated.

Finally, the results have testable implications for the functioning of democratic institutions in ethnically fractionalized societies. If ethnic affiliation determines political preferences, then the findings clearly imply that the size of the majority needed to obtain stable policies should increase with the number of ethnic groups. This in turn implies that incumbent policies are more likely to persist in more fractionalized societies. This could explain why the ruling leaders of such societies tend to stay in power for longer periods.

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Appendix: Proofs of Results

In order to facilitate the exposition of the proofs, we state in Lemma 2 set out below a necessary and sufficient condition for a policy to be stable under the dynamic mechanism described in section 2.3.

Lemma 2. *Let $\mathcal{L} = (N, D, r(D), (\succeq_i))$ be a legislature, and $x \in A(D)$ be an alternative. Statements 1. and 2. below are equivalent:*

1. *$x \in \mathcal{R}(\mathcal{L})$.*

2. *There does not exist an alternative $y \in A(D)$ such that $y \gtrdot x$, where $y \gtrdot x$ if there exists a winning coalition S such that:*
   a) $y \succ_S x$ and;
   b) $[\forall (z, T) \in A(D) \times \mathfrak{C},\ S \neq T,\ z \succ_T y$ and $not(y \succeq_T x)]$ implies $[z \succeq_S x]$.

Proof. Let $M(\mathcal{L}, \gtrdot)$ denote the set of all maxima elements of binary relation $\gtrdot$.

1. First, we prove that $\mathcal{R}(\mathcal{L}) \subseteq M(\mathcal{L}, \gtrdot)$. Let $x \in A(D)$ be an option such that $x \notin M(\mathcal{L}, \gtrdot)$. Then, there exists a policy $y \in A(D)$ and a winning coalition $S \in \mathfrak{C}$ such that $y \gtrdot_S x$. It follows that a couple (y, S) is an objection against policy x. The objective is to demonstrate that this objection against x is justified. If there is no counter-objection against (y, S), the process is complete. Assuming that there exists a counter-objection (z, T) against (y, S), the following assertions hold:

i) $z \succ_T y$; and

ii) *not*$(y \succeq_T x)$.

Given the fact that $y \gtrdot_S x$, then 1.i) and 1.ii) entail $z \succeq_S x$, and the counter-objection (z, T) against (y, S) is friendly. Therefore, a pair (y, S) is a justified objection against x by definition, meaning that $x \notin \mathcal{R}(\mathcal{L})$. We conclude that $\mathcal{R}(\mathcal{L}) \subseteq M(\mathcal{L}, \gtrdot)$.

2. Second, we prove that $M(\mathcal{L}, \gtrdot) \subseteq \mathcal{R}(\mathcal{L})$. Let $x \in A(D)$ be a policy such that $x \notin \mathcal{R}(\mathcal{L})$. Then, there exists a justified objection $(y, S) \in A(D) \times \mathfrak{C}$ against policy x.

iii) If there is no winning coalition T and no policy z such that $z \succ_T y$, then, $y \succ_S x$ and implication b) is satisfied since the right-hand side of this implication is false. It follows that $y \gtrdot x$ and $x \notin M(\mathcal{L}, \gtrdot)$.

iv) Assume that there exists $(z, T) \in A(D) \times \mathfrak{C}$ such that $z \succ_T y$. We have two possibilities described hereunder.

- If $not(y \succeq_T x)$, then (z, T) is a counter-objection against (y, S). Since (y, S) is a justified objection, it follows that any counter-objection against (y, S) is friendly, leading to $z \succeq_S x$. Then, $y \gtrdot x$ and $x \notin M(\mathcal{L}, \gtrdot)$.
- If $y \succeq_T x$, so $not(y \succeq_T x)$ is false, then the implication $[not(y \succeq_T x) \Rightarrow z \succeq_S x]$ is true. Therefore, $y \gtrdot_S x$ and $x \notin M(\mathcal{L}, \gtrdot)$. We can conclude that $M(\mathcal{L}, \gtrdot) \subseteq \mathcal{R}(\mathcal{L})$.

We conclude the proof that $\mathcal{R}(\mathcal{L}) = M(\mathcal{L}, \gtrdot)$. □

Proof of Lemma 1. Let $\mathcal{L} = (N, D, r, (\succeq_i))$ be a legislature, and $x \in A(D)$ be a policy such that $x \in \mathcal{R}(\mathcal{L})$, where $A(D)$ is the policy space induced by D. If x is Pareto-dominated, then there exists $y \in A(D)$ such that $y \succ_N x$. It follows that, for any $z \in A(D)$ such that $z \succ_T y$ with $T \in \mathfrak{C}$, we have $y \succ_T x$ because $T \subset N$. Then, not $(y \succeq_T x)$ is always false. Therefore the implication $[not\ (y \succeq_T x) \Rightarrow z \succeq_N x]$ is always true—that is y defeats x via N or (y, N) is a justified objection against x, which is a contradiction. Thus, x is Pareto-efficient.

Proof of Theorem 1. In order to establish this proof, we need to prove some preliminary results. First, the proof illustrates the finding that if a voting rule r is such that $r > (\frac{D}{D+1})n$, there always exists a stable policy in a legislature legislating on a policy of complexity level D. This result is provided below:

Lemma 3. *Let $\mathcal{L} = (N, D, r, (\succeq_i))$ be a legislature such that $r > (\frac{D}{D+1})n$ and $(\succeq_i) \in \mathcal{U}^n_{con}$. Then, there exists a stable policy.*

Proof. Let $A(D)$ be the policy space induced by the decision of complexity D in the legislature. Convex and continuous preferences satisfy the following properties: for each legislator $i \in N$, the binary relation $\succeq_i$ has an open graph in $A(D) \times A(D)$ (a binary relation $\succeq_i$ defined on $A(D)$ is a subset of $A(D) \times A(D)$; the binary relation $\succeq_i$ is said to have an open graph if $\succeq_i$ is open in the product topology on $A(D) \times A(D)$, where $A(D)$ is a topological space); and for each legislator $i \in N$ and for each alternative $x \in A(D)$, $x \notin C[A(D)_i(x)]$, where $A(D)_i(x) = \{y \in A(D),\ y \succ_i x\}$, and $C[X]$ denotes the convex hull of X. In order to prove Lemma 3, we use the result set out below.

Lemma 4. *(Greenberg, 1979, Lemma 1, p.6) Let N be a finite set with cardinality n. Consider the collection C^s of subsets of N of cardinality of at least s. Then, all intersections of $m+1$ elements of C^s are non-empty if and only if $s > (\frac{m}{m+1})n$.*

We prove that there exists a stable outcome in the legislature $\mathcal{L}$, whatever the minimum size of majority required (r) is strictly greater than the number $(\frac{D}{D+1})n$. Consider a legislator $i \in N$, and an alternative $x \in A(D)$. Define the sets $A(D)^{(x,y)} = \{i \in N, y \in A(D)_i(x)\}$, and $A(D)^r(x) = \left\{y \in A(D), |A(D)^{(x,y)}| \geq r\right\}$.

1. There exists an alternative $x^* \in A(D)$ such that $A(D)^r(x^*)$ is a non-empty set.

a) $A(D)^r(x)$ is an open set of $A(D)$ (the topology here is the usual topology in the multi-dimensional set $\mathbb{R}^D$).

Let $y \in A(D)^r(x)$ be an alternative. Then $y \in A(D)_i(x)$ for any legislator $i \in A(D)^{(x,y)}$. Given the fact that legislators' preferences are continuous, for each legislator $i \in A(D)^{(x,y)}$, there exists a neighborhood $A(D)_i(y)$ of alternative y such that option $y' \succ_i x$ for any option $y' \in A(D)_i(y)$. Consider the set $\overline{A(D)} = \bigcap_{i \in A(D)^{(x,y)}} A(D)_i(y)$, and let $\overline{y} \in \overline{A(D)}$ be an alternative. It follows that the option $\overline{y} \in A(D)_i(y)$ for any legislator $i \in A(D)^{(x,y)}$. This means $\overline{y} \succ_i x\ \forall i \in A(D)^{(x,y)}$, and then option $\overline{y} \in A(D)^r(x)$. Therefore, $A(D)^r(x)$ contains an open set $\overline{A(D)}$, which is a neighborhood of alternative y, for $y \in A(D)^r(x)$. Hence, the set $A(D)^r(x)$ is an open set of the policy space $A(D)$.

b) Let $C[A(D)^r(x)]$ be the convex hull of $A(D)^r(x)$. Demonstrate that, for each alternative $x \in A(D)$, $x \notin C[A(D)^r(x)]$. The following result proved to be useful.

Lemma 5. *(Nikaido, 1968, Theorem 2.4, p.19) The convex hull $C[X]$ of a set X in $\mathbb{R}^n$ equals the set of all points represented by $\sum_{i=1}^{n+1} \alpha_i x^i$, $\sum_{i=1}^{n+1} \alpha_i = 1$, $\alpha_i \geq 0$ $(i = 1, 2, ..., n+1)$ as the $n+1$ independently range over X and the weight of α_i take on all possible values.*

Given the premise that preferences satisfy the convex property, for each alternative $x \in A(D)$ and for each legislator $i \in N$, $x \notin C[A(D)_i(x)]$. Assuming by negation that the alternative $\overline{x} \in C[A(D)^r(\overline{x})]$, then following Lemma 5, there exists alternatives $x_1, x_2, ..., x_{D+1}$ and real values $\lambda_1, \lambda_2, ..., \lambda_{D+1}$ with $\sum_{j=1}^{D+1} \lambda_j = 1$, $\lambda_j \geq 0$ such that $\overline{x} = \sum_{j=1}^{D+1} \lambda_j x_j$, and each alternative $x_j \in A(D)^r(\overline{x})$. Alternative $x_j \in A(D)^r(\overline{x})$ implies that $x_j \in A(D)_i(\overline{x})$ for each legislator $i \in A(D)^{(\overline{x},x_j)}$. Since the number $r > (\frac{D}{D+1})n$, and $|A(D)^{(\overline{x},x_j)}| \geq r$ for each $j = 1, 2, ..., D+1$ (given that $x_j \in A(D)^r(\overline{x})$), then $|A(D)^{(\overline{x},x_j)}| > (\frac{D}{D+1})n$. According to Lemma 4, there exists a legislator $i_0 \in N$ such that option $x_j \in A(D)_{i_0}(\overline{x})$ for each $j = 1, 2, ..., D+1$. It follows that the alternative $\overline{x}$ is equal $\sum_{j=1}^{D+1} \lambda_j x_j$, with $x_j \in A(D)_{i_0}(\overline{x})$. Consequently, the option $\overline{x} \in C[A(D)_{i_0}(\overline{x})]$, which is a contradiction by assumption. Thus, for each alternative $x \in A(D)$, $x \notin C[A(D)^r(x)]$ and the set $A(D)^r(x)$ is an open set of the policy space $A(D)$. It follows that there exists an alternative $x^* \in A(D)$ such that the set $A(D)^r(x^*) = \emptyset$. The alternative x^* is not strictly preferred by another alternative in the policy space $A(D)$.

2. The alternative x^* is stable. Assuming the contrary, then there exists a policy $y \in A(D)$, and a coalition $C \subseteq N$ such that the alternative $y \succ_i x$ for each legislator $i \in C$. The coalition $C \subseteq A(D)^{(x,y)}$ and the size of C, $|C| \geq r$. It follows that the number $|A(D)^{(x,y)}| \geq r$ and the alternative $y \in A(D)^r(x^*)$, which is a contradiction, since the set $A(D)^r(x^*) = \emptyset$. $\square$

The result set out below stipulates that, if the size of the supermajority needed to pass a policy is less than or equal to the number $(\frac{D}{D+1})n$, then there might exist a preference profile where there is no stable policy.

Lemma 6. *If $r \leq (\frac{D}{D+1})n$, there exists a preferences' profile $(\succeq_i) \in \mathcal{U}^n_{con}$ such that the legislature $\mathcal{L} = (N, D, r, (\succeq_i))$ cannot enact a stable policy.*

Proof. We prove that, if the minimum size of the supermajority required r is less than or equal to the number $(\frac{D}{D+1})n$, then there exists a profile $(\succeq_i) \in \mathcal{U}^n_{con}$ which leads the legislature to political cycles, meaning that there exists no stable policy.

Consider the policy space $A(D)$ induced by the decision of complexity D in the legislature $\mathcal{L}$. The dimension of $A(D)$ is D. It contains a D-dimensional simplex S_D. Denote by b_i, $i = 1, 2, ..., D+1$ the $(D+1)$ affinely independent vertices of S_D. For alternatives x and y in the policy space $A(D)$, define the distance from x to y, denoted as $d(x, y)$, by the real number: $d(x, y) = \|x - y\|$. Also, define the function:

$$\begin{array}{rlc} f_i: & A(D) \to & \mathbb{R} \\ & x \mapsto & f_i(x) = -d(x, b_i), \end{array}$$

where f_i is continuous and strictly quasi-concave. The proof differentiates three cases with respect to the number of legislators, n.

1. $n = D + 1$ and consider that the preference of legislator i is represented by the function f_i, $i = 1, 2, ..., D + 1$. Legislator i's ideal point is b_i corresponding to a vertex of S_D. Consider a policy $x \in A(D)$.

   a) The alternative $x \notin S_D$. Given the premise that S_D is a compact and convex set, there exists a unique alternative $y(x) \in S_D$ close to x. Then, $d(y(x), b_i) < d(x, b_i)$ for each $i = 1, 2, ..., D + 1$ or $f_i(y(x)) > f_i(x)$ for each $i \in N$ i.e., $y(x) \succ_N x$. If there exists an alternative $y' \in A(D)$ such that $y' \succ_T y(x)$, with $T \in \mathfrak{C}$, then $f_i(y') > f_i(y(x))$ for each $i \in T$. Since, $y(x) \succ_N x$, it follows that $f_i(y') > f_i(x)$ for each $i \in T$ and $y(x)$ dominates x ($y(x) \gtrdot x$).

   b) The alternative $x \in S_D$. There exists an $(D - 1)$-dimensional face F of S_D such that the alternative $x \notin F$. The set F is also a simplex; therefore there exists a unique alternative $y(x) \in F$ close to x. It follows that $d(y(x), b_i) < d(x, b_i)$ with b_i, $i = 1, 2, ..., D$ the D-vertices affinely independent of F. Thus, there exists a coalition C of cardinality D such that $y(x) \succ_C x$. Since the number $D \geq \frac{r}{n-r}$ and $n = D + 1$, then $r \leq D$ and C is a winning coalition. Assume that there exists an alternative $y' \in A(D)$ such that $d(y'_i, b_i) < d(y(x), b_i)$ for t vertices b_i of the policy space $A(D)$ (i.e.,y' is strictly preferred to alternative $y(x)$ by a coalition T with $|T| = t \geq r$).

      - If b_i, $i = 1, 2, ..., l$ are the vertices of the set F, then by the definition of $y(x)$, we have $d(y(x), b_i) < d(x, b_i)$ for each $i = 1, 2, ..., t$. It follows that $not(y(x) \succeq_T x)$ is false and the implication $[not(y(x) \succeq_T x) \Rightarrow y' \succeq_C x]$ is true, and therefore $y(x)$ defeats x ($y(x) \gtrdot x$).
      - Assume that there exists an alternative $h_0 \in \{1, 2, ..., l\}$ such that the vertex $b_{h_0} \notin F$ and $d(y'_{h_0}, b_{h_0}) < d(y(x), b_{h_0})$. If *not* $(y(x) \succeq_T x)$ is false, then the implication [*not* $(y(x) \succeq_T x) \Rightarrow y' \succeq_C x$] is true, and so $y(x)$ defeats x via C. Assume *not* $(y(x) \succeq_T x)$ and *not*$(y' \succeq_C x)$, then there exists a legislator $i_0 \in C$ such that $x \succ_{i_0} y'$ or $d(x, b_{i_0}) < d(y'_{i_0}, b_{i_0})$, with b_{i_0} being a vertex of F. By definition of the alternative $y(x)$, $d(y(x), b_{i_0}) < d(x, b_{i_0})$, then $d(y(x), b_{i_0}) < d(y'_{i_0}, b_{i_0})$, and the legislator $i_0 \notin \{1, 2, ..., t\}$. Given the fact that F contains D vertices, then $h_0 = i_0$, and therefore b_{i_0} is not a vertex of F, which is a contradiction. Therefore, the alternative $y(x)$ defeats x via C.

2. $n < D+1$. Let F be the face of S_D generated by the vertices b_i, $i = 1, 2, ..., n$. Legislators' preferences are represented by the functions f_i, $i = 1, 2, ..., n$. F is a simplex of dimension

$n-1$; then following **point 1.** of this proof, each alternative that belongs to F can be defeated by a coalition that consists of $n-1$ legislators, and each alternative that does not belong to F can be defeated by the coalition N. It follows that each alternative can be defeated by a coalition that consists of at least $n-1$ legislators. Since $D \geq \frac{r}{n-r}$ and $n < D+1$, then $r < D$. Furthermore, $\frac{D}{D+1} < 1$ and $r \leq (\frac{D}{D+1})n$; it therefore follows that $r < n$. Since r is an integer, then $r \leq n-1$, which means that every coalition of cardinality of at least $n-1$ is a winning coalition. In the same manner as in **point 1.** of this proof, each alternative is defeated in the sense of the binary relation $\gtrdot$.

3. $n > D+1$. There exists two integers s and k, $s \geq 1$ and $0 \leq k < D+1$ such that $n = s(D+1)+k$. Consider a partition of population N ($|N| = n$) in $(D+1)$ sets where legislators belong to the same set if and only if they have the same preferences, and hence the same ideal point. Assume that $s+1$ legislators belong to each of the first k sets and that s other legislators belong to the remaining $(D+1-k)$ sets. In each set, the preferences are given by f_j, $j = 1, 2, ..., D+1$.

   c) If $k = 0$, then $n = s(D+1)$. This means that, there are s legislators in each set. According to **point 1.** of this proof, each alternative $x \in A(D)$ is defeated by a coalition of cardinality that is greater or equal to $n-r$. Since the number $D \geq \frac{r}{n-r}$ and $n = s(D+1)$, then $r \leq Ds = n-s$.

   Each alternative is defeated. Let $x \in A(D)$ be an alternative policy. Then, there exists a unique alternative $y(x) \in A(D)$ such that $y(x)$ defeats x via a coalition C with $|C| \geq n-s$. Assume that there exists an alternative $y' \in A(D)$ such that y' is preferred to alternative $y(x)$ by a coalition T. It follows that T contains at least $n-s$ legislators.

   - If $T = C$, then the option $y' \succ_C y(x) \succ_C x$, so $y' \succ_C x$, and so $y(x)$ dominates x.
   - If $T \neq C$, consider that alternative $y(x) \succeq_T x$. The implication $[not(y(x) \succeq_T x) \Rightarrow y' \succeq_C x]$ is satisfied, then $y(x) \gtrdot_C x$. If *not* $(y(x) \succeq_T x)$, then there exists $i \in T$ such that $x \succ_i y(x)$. The alternative $y(x)$ is a unique closed point of x and legislator i has b_i as the ideal point. It follows that $d(y(x), b_i) < d(x, b_i)$. Therefore, $not(y(x) \succeq_T x)$, and $y(x) \gtrdot_C x$.

   d) Assume $k \geq 1$. Each vertex of the simplex corresponds to the ideal point of at most $s+1$ legislators. According to **point 1.** of this proof, each alternative can be defeated by at least $n-(s+1)$ legislators. Since the number $D \geq \frac{r}{n-r}$ and $n = s(D+1)+k$, then $r \leq D(s+\frac{k}{D+1})$ (i.e., $r < Ds+k$) and then $r \leq Ds+k-1 = n-(s+1)$, given the fact that r and $Ds+k$ are integers. Following the proof in case $k=0$ (**point 3.c**), we can conclude that each policy is defeated.

It suffices to consider the legislators' preferences profile $(\succeq_i) = (\succeq_1, ..., \succeq_n)$ such that f_i is a cardinal representation of $\succeq_i$ for each legislator $i \in N$. Each preference $\succeq_i$ is represented by a quasi-concave function f_i; then $\succeq_i$ is equivalent to an ordinal convex preference. We can conclude that $(\succeq_i) \in \mathcal{U}^n_{con}$ and that there is no stable policy.

Finally, we have all the tools necessary to conclude the proof of this theorem. Let $\mathcal{L} = (N, D, r(D), (\succeq_i))$ be a legislature. Consider $A(D)$ to be a compact and convex policy space, a subset of Euclidean space $\mathbb{R}^D$, of dimension D, and $(\succeq_i) \in \mathcal{U}^n_{con}$. Lemmas 3 and 6 reveal that there exists a stable outcome if and only if $r(D) > (\frac{D}{D+1})n$. Since r is an integer, the smallest value that satisfies the latter inequality and is strictly greater than $n/2$ is $\lfloor \frac{D}{D+1}n \rfloor + 1$. $\square$

Proof of Theorem 2. This proof also involves several steps. First, we prove a lemma set out below that illustrates the structure of the voting process in a legislature under which there is no stable policy. Throughout this proof, $A(D)$ denotes the policy space induced by the discrete decision of complexity D.

Lemma 7. *Let $\mathcal{L} = (N, D, r, (\succeq_i))$ be a legislature with $(\succeq_i) \in \mathcal{U}^n$ such that there is no stable policy. Then, there exists a voting cycle: $x_1 \gtrdot_{S_1} x_2 \gtrdot_{S_2} x_3 \gtrdot_{S_3} ... \gtrdot_{S_{p-1}} x_p \gtrdot_{S_p} x_1$ of length p ($p \leq D,\ p > 2$) such that there is no individual $i \in N$ belonging to more than $p-2$ winning coalitions among the set $\mathcal{C} = \{S_1, S_2, ..., S_p\}$.*

Proof. Suppose that there exists a legislator i who belongs to $p-1$ winning coalitions among $\mathcal{C} = \{S_1, S_2, ..., S_p\}$. Prove that the legislator $i \notin S_j \cap S_{j+1(modp)}$ for each $j = 1, 2, ..., p$.[16] Without loss of generality, consider that the legislator $i \in S_1 \cap S_p$. By definitions,

$$\{i \in S_1 \text{ and } x_1 \gtrdot_{S_1} x_2\} \Rightarrow \left\{x_1 \succ_i x_2 \textit{ and } [not(x_1 \succeq_{S_p} x_2) \Rightarrow x_p \succeq_i x_2]\right\}.$$

Also,

$$\left\{i \in S_p \text{ and } x_p \gtrdot_{S_p} x_1\right\} \Rightarrow \left\{x_p \succ_i x_1 \textit{ and } [not(x_p \succeq_{S_{p-1}} x_1) \Rightarrow x_{p-1} \succeq_i x_1]\right\}.$$

1. Legislator $i \in S_{p-1} \cup S_{p-2}$, since he or she belongs to $p-1$ winning subgroups among $C = \{S_1, S_2, ..., S_p\}$. The legislator $i \in S_m \in \{S_{p-1}, S_{p-2}\}$. Then,

$$\{i \in S_m \text{ and } x_m \gtrdot_{S_m} x_{m+1}\}.$$

[16] The notation "*mod p*" means "modulo p". In our context, x $(modp)$ y if $x - y = kp$, where k is an integer.

This implies that:

$$\left\{x_m \succ_i x_{m+1} \text{ and } [not(x_m \succeq_{S_{m-1}} x_{m+1}) \Rightarrow x_{m-1} \succeq_i x_{m+1}]\right\}.$$

The alternative $x_2 \in \{x_m, x_{m-1}\}$, and therefore $x_2 \succeq_i x_{m+1}$. Since the coalition $S_m \in \{S_{p-1}, S_{p-2}\}$, it follows that $x_{m+1} \succeq_i x_1$. By transitivity, alternative $x_2 \succeq_i x_1$, which is a contradiction, because the option $x_1 \succ_i x_2$. Thus, $x_2 \notin \{x_m, x_{m-1}\}$.

2. Legislator i belongs to coalition S_m. Then, as previously mentioned, legislator i belongs to a coalition S_l such that $S_l \in \{S_{m-1}, S_{m-2}\}$ and alternative $x_2 \notin \{x_l, x_{l-1}\}$. Certainly,

$$\{i \in S_l \text{ and } x_l \gtrdot_{S_l} x_{l+1}\} \Rightarrow \left\{x_l \succ_i x_{l+1} \text{ and } [not(x_l \succeq_{S_{l-1}} x_{l+1}) \Rightarrow x_{l-1} \succeq_i x_{l+1}]\right\}.$$

If alternative $x_2 \in \{x_l, x_{l-1}\}$, then $x_2 \succeq_i x_{l+1}$ and the option $x_{l+1} \succeq_i x_{m+1} \succeq_i x_1$. Therefore, $x_2 \succeq_i x_1$, which is a contradiction.

3. At each step of this process, some alternatives that differ from x_2 are eliminated. Given the fact that the number of alternatives is finite, there exists an alternative $x_t \in A(D)$ such that $x_2 \in \{x_t, x_{t-1}\}$. Alternative $x_t \gtrdot_{S_t} x_{t+1}$, and the legislator $i \in S_t$. Hence, $x_t \succ_i x_{t+1}$, and $[not(x_t \succeq_{S_{t-1}} x_{t+1}) \Rightarrow x_{t-1} \succeq_i x_{t+1}]$. It follows that:

$$x_2 \succeq_i x_{t+1} \succeq_i \ldots \succeq_i x_{l+1} \succeq_i x_{m+1} \succeq_i x_1.$$

By transitivity, the option $x_2 \succeq_i x_1$, which is a contradiction. Consequently, the legislator $i \notin S_j \cap S_{j+1(mod p)}$ $\forall j = 1, 2, \ldots, p$.

4. At this point, the legislator i belongs to at most $\frac{p}{2}$ winning coalitions. Therefore, $p-1 \leq \frac{p}{2}$ (the legislator i belongs to $p-1$ winning coalitions). It follows that, $p \leq 2$. This is absurd, since $p > 2$ by hypothesis. □

The next result indicates that, if a voting rule r is such that $r > (\frac{D-2}{D})n$, there exists a stable policy in a legislature deciding on a discrete policy of complexity level D.

Lemma 8. *There exists a stable policy in a legislature $\mathcal{L} = (N, D, r, (\succeq_i))$, $2 < D < \infty$, for any preference profile $(\succeq_i) \in \mathcal{U}^n$ if $r > (\frac{D-2}{D})n$.*

Proof. Let $\mathcal{L} = (N, D, r, (\succeq_i))$ be a legislature, $(\succeq_i) \in \mathcal{U}^n$ be the legislators' preferences profile, and $A(D)$ be the policy space of size D, with $2 < D < \infty$. The objective is to clarify that, if $r > (\frac{D-2}{D})n$, then there exists a stable outcome.

It is already proved in Lemma 7 that, if there is no stable policy for the legislators' preferences profile $(\succeq_i) \in \mathcal{U}^n$, then there exists a voting cycle $x_1 \gtrdot_{S_1} x_2 \gtrdot_{S_2} x_3 \gtrdot_{S_3} \ldots \gtrdot_{S_{p-1}} x_p \gtrdot_{S_p} x_1$ of length p ($p \leq D$, $p > 2$), such that there is no legislator $i \in N$ belonging to more than $p-2$ winning coalitions among the set $\mathcal{C} = \{S_1, S_2, \ldots, S_p\}$.

1. First, assume that, $n/2 < r < n-1$. Suppose that the set $\mathcal{R}(\mathcal{L})$ is empty for a profile of weak order preferences $(\succeq_i) \in \mathcal{U}^n$. From Lemma 7, there exists a cycle $x_1 \gtrdot_{S_1} x_2 \gtrdot_{S_2} x_3 \gtrdot_{S_3} ... \gtrdot_{S_{p-1}} x_p \gtrdot_{S_p} x_1$ of length p ($p \leq D,\ p > 2$). Consider $\overline{s} = \frac{\sum_{j=1}^{p} |S_j|}{n}$ the average number of times that a legislator appears in the winning coalitions. According to Lemma 7, there is no legislator $i \in N$ belonging to more than $p-2$ winning coalitions among $\mathcal{C} = \{S_1, S_2, ..., S_p\}$, thus the number $p - 2 \geq \overline{s}$. Let $s_j = |S_j| - r$, s_j is non-negative. $|S_j| \geq r$ implies that S_j contains at least r number of legislators. It follows that $\overline{s} = \frac{\sum_{j=1}^{p} s_j + pr}{n}$, and $\overline{s} \leq p-2$. By a simple arrangement, $p \geq \frac{2n}{n-r} + \frac{\sum_{j=1}^{p} s_j}{n-r}$. Since $\frac{\sum_{j=1}^{p} s_j}{n-r}$ is non-negative, then $D \geq p \geq \frac{2n}{n-r}$, which is a contradiction.

2. Second, assume that, $r = n-1$. Let $(\succeq_i) \in \mathcal{U}^n$ such that $\mathcal{R}(\mathcal{L})$ is empty. The proof proceeds by induction on the length of the cycle. Assume that, there are no cycles of length D and prove that, there are no cycles of length $D+1$.

   Assume in the contrary that, there is a cycle of length $D+1$—that is, $x_1 \gtrdot_{S_1} x_2 \gtrdot_{S_2} x_3 \gtrdot_{S_3} ... \gtrdot_{S_D} x_{D+1} \gtrdot_{S_{D+1}} x_1$.

   Case 1: there exists $j \in \{1, 2, ..., D\}$ such that $S_j = S_{j+1}$.

   By using transitivity of $\gtrdot_{S_j}$, we obtain $x_j \gtrdot_{S_j} x_{j+2}$ and thus reduce the length of the cycle to D. This contradicts our induction assumption.

   Case 2: for each $j \in \{1, 2, ..., D\}$, $S_j \neq S_{j+1}$. Assume without loss of generality that $1 \in S_1 \cap S_{D+1}$. $x_{D+1} \gtrdot_{S_{D+1}} x_1$, then $x_{D+1} \succ_{S_{D+1}} x_1$, and [*not* $(x_{D+1} \succeq_{S_D} x_1) \Rightarrow x_D \succeq_{S_{D+1}} x_1$]. Given the fact that $1 \in S_{D+1}$ and $S_{D+1} \neq S_D$, then $x_{D+1} \succ_1 x_1$ and $x_D \succeq_1 x_1$. Since $S_D \neq S_{D-1}$, then $S_D \cup S_{D-1} = N$; it therefore follows that $1 \in S_D$ or (and) $1 \in S_{D-1}$. If $1 \in S_D$, given the fact that $x_D \gtrdot_{S_D} x_{D+1}$, we have $x_D \succ_1 x_{D+1}$ and $x_{D-1} \succeq_1 x_{D+1}$. It follows that $x_{D-1} \succeq_1 x_{D+1}$ and $x_{D+1} \succ_1 x_1$, then $x_{D-1} \succ_1 x_1$. If $1 \in S_{D-1}$, given the fact that $x_{D-1} \gtrdot_{S_{D-1}} x_D$, we have $x_{D-1} \succ_1 x_D$ and $x_{D-2} \succeq_1 x_D$ because $S_{D-1} \neq S_{D-2}$. It follows that $x_D \succeq_1 x_1$ and $x_{D-1} \succ_1 x_D$, then $x_{D-1} \succ_1 x_1$. Thus, in each case, we have $x_{D-1} \succ_1 x_1$.

   We proved that $x_j \succeq_1 x_1$ for any j greater than or equal to l (i.e., $x_j \succeq_1 x_1\ \forall j \geq l$). If $1 \in S_l$, since $x_l \gtrdot_{S_l} x_{l+1}$, then $x_l \succ_1 x_{l+1}$ and $x_{l-1} \succeq_1 x_{l+1}$ given the fact that $S_l \neq S_{l-1}$. Since $x_{l+1} \succeq_1 x_1$, then, with $x_{l-1} \succeq_1 x_{l+1}$, it follows that $x_{l-1} \succeq_1 x_1$. If $1 \notin S_l$, then $1 \in S_{l-1}$, because $S_l \cup S_{l-1} = N$. Given the fact that $x_{l-1} \gtrdot_{S_{l-1}} x_l$, and $S_{l-1} \neq S_{l-2}$, it follows that $x_{l-1} \succ_1 x_l$ and $x_{l-2} \succeq_1 x_l$. $x_l \succeq_1 x_1$; then $x_{l-1} \succ_1 x_l \succeq_1$

x_1. By transitivity, $x_{l-1} \succ_1 x_1$. It has been verified that, in each case $x_{l-1} \succeq_1 x_1$. It follows that $x_2 \succeq_1 x_1$. Since, $1 \in S_1$ and $x_1 \gtrdot_{S_1} x_2$, then it follows that $x_1 \succ_1 x_2$. Hence, $x_1 \succ_1 x_2 \succeq_1 x_1$. By contradiction, $x_1 \succ_1 x_1$, which is a contradiction.

3. Third, assume that, $r = n$. Let $(\succeq) \in \mathcal{U}^n$, and $\mathcal{R}(\mathcal{L}) = \emptyset$. Then, there exists a cycle $x_1 \gtrdot_N x_2 \gtrdot_N x_3 \gtrdot_N ... \gtrdot_N x_p \gtrdot_N x_1$ of length p ($p \leq D,\ p > 2$). It follows that $x_1 \succ_N x_2 \succ_N x_3 \succ_N ... \succ_N x_p \succ_N x_1$. By transitivity, $x_1 \succ_N x_1$, which is a contradiction. Therefore, $\mathcal{R}(\mathcal{L}) \neq \emptyset$.

Summary: if $r > (\frac{D-2}{D})n$, then for each legislators' preferences profile $(\succeq_i) \in \mathcal{U}^n$, there exists a stable outcome. $\square$

Now, the result set out below reveals that, if a voting rule r is smaller than or equal to the number $(\frac{D-2}{D})n$, then there exists a preference profile under which a stable policy does not exist.

Lemma 9. *Let $\mathcal{L} = (N, D, r, (\succeq_i))$ be a legislature, where $2 < D < \infty$. If $r \leq (\frac{D-2}{D})n$, then there exists a profile $(\succeq_i) \in \mathcal{U}^n$ that leads to voting cycles, i.e., $\mathcal{R}(\mathcal{L}) = \emptyset$.*

Let $\overline{D} = \lceil \frac{2n}{n-r} \rceil$ and $D \geq \overline{D} > 2$ (for a real number x, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x). The scope of this proof is to construct a preference profile $(\succeq_i) \in \mathcal{U}^n$ such that:

$$x_1 \gtrdot_{S_1} x_2 \gtrdot_{S_2} x_3 \gtrdot_{S_3} ... \gtrdot_{S_{\overline{D}-1}} x_p \gtrdot_{S_{\overline{D}}} x_1,$$

with $|S_j| \geq r$ for any $j = 1, 2, ..., \overline{D}$. We have to show that:

$$x_l \succ_i x_{l+1} \text{ and } [not(x_l \succeq_{S_{l-1}} x_{l+1}) \Rightarrow x_{l-1} \succeq_{S_l} x_{l+1}]$$

for any $l = 1, 2, ..., \overline{D}$ (for $l = \overline{D}$, $l + 1 = 1$).

We enumerate four cases:

<u>**Case 1**</u>: the integer $n - r$ is even.

We partition the population of n legislators into $\overline{D}$ sets P_j, $j = 1, 2, ..., \overline{D}$ where, in each set, legislators have the same preferences:

$P_1 : x_1\ x_{\overline{D}}\ x_2\ x_3\ ...\ x_{\overline{D}-1}\ (x_{\overline{D}+1}\ ...\ x_D)$

$P_2 : x_2\ x_1\ x_3\ x_4\ ...\ x_{\overline{D}}\ (x_{\overline{D}+1}\ ...\ x_D)$

$P_3 : x_3\ x_2\ x_4\ x_5\ ...\ x_1\ (x_{\overline{D}+1}\ ...\ x_D)$

$P_4 : x_4 \ x_3 \ x_5 \ x_6 \ ... \ x_2 \ (x_{\overline{D}+1} \ ... \ x_D)$

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$P_{\overline{D}-3} : x_{\overline{D}-3} \ x_{\overline{D}-4} \ x_{\overline{D}-2} \ x_{\overline{D}-1} \ ... \ x_{\overline{D}-5} \ (x_{\overline{D}+1} \ ... \ x_D)$

$P_{\overline{D}-2} : x_{\overline{D}-2} \ x_{\overline{D}-3} \ x_{\overline{D}-1} \ x_{\overline{D}} \ ... \ x_{\overline{D}-4} \ (x_{\overline{D}+1} \ ... \ x_D)$

$P_{\overline{D}-1} : x_{\overline{D}-1} \ x_{\overline{D}-2} \ x_{\overline{D}} \ x_1 \ ... \ x_{\overline{D}-3} \ (x_{\overline{D}+1} \ ... \ x_D)$

$P_{\overline{D}} : x_{\overline{D}} \ x_{\overline{D}-1} \ x_1 \ x_2 \ ... \ x_{\overline{D}-2} \ (x_{\overline{D}+1} \ ... \ x_D)$

Consider:

$$p_j = |P_j| = \begin{cases} \frac{n-r}{2} & if \quad j \neq \overline{D} \\ n - \sum_{k=1}^{\overline{D}} p_k = n - \frac{\overline{D}-1}{2}(n-r) & if \quad j = \overline{D} \end{cases} .$$

Given that $N = \bigcup_{j=1}^{\overline{D}} P_j$, consider $S_l = N - (P_{l+1} \cup P_{l+2})$, for any $l = 1, 2, ..., \overline{D}$. (for $l = \overline{D}$, $l + 1 = 1$ and $0 < p_{\overline{D}} \leq \frac{n-r}{2}$). Since $p_j = \frac{n-r}{2}$ for $j \neq D$, then $p_{l+1} + p_{l+2} \leq n - r$ and $|S_l| \geq r$.

- For any $i \in S_l$, $x_l \succ_i x_{l+1}$; hence $x_l \succ_{S_l} x_{l+1}$.
- For any $i \in S_l$; $x_{l-1} \succ_i x_{l+1}$ (i.e., $x_{l-1} \succ_{S_l} x_{l+1}$) and according to the profile, there exists a player $i' \in S_{l-1}$ such that $x_{l+1} \succ_{i'} x_l$ (i.e., $not(x_l \succeq_{S_{l-1}} x_{l+1})$). It follows that $x_l \gtrdot_{S_l} x_{l+1}$ for any $l = 1, 2, ..., \overline{D}$.

Case 2: the integer $n - r$ and $\overline{D}$ are odd.

The same profile of the preference's structure as in **Case 1** is used here, with the exception being that the size of each partition P_j changes. Consider:

$$p_j = |P_j| = \begin{cases} \lceil \frac{n-r}{2} \rceil & if & j \ is \ even \\ \lfloor \frac{n-r}{2} \rfloor & if & j \ is \ odd \ and \ j \neq \overline{D} \\ n - \frac{\overline{D}-1}{2}(n-r) & if & j = \overline{D} \end{cases} .$$

Note that $0 < p_{\overline{D}} \leq \lfloor \frac{n-r}{2} \rfloor$. Using the same reasoning as in **Case 1**, $x_l \gtrdot_{S_l} x_{l+1}$ for any $l = 1, 2, ..., \overline{D}$.

Case 3: the integer $n - r$ is odd, $\overline{D}$ is even and $\overline{D} = \frac{2n}{n-r}$.

Consider the same profile as in the **Case 1** and **Case 2**) and assume that:

$$p_j = \begin{cases} \lfloor \frac{n-r}{2} \rfloor & if & j \ is \ odd \\ \lceil \frac{n-r}{2} \rceil & if & j \ is \ even \end{cases} .$$

The reasoning is the same as in the previous **Cases**.

__Case 4__: the integer $n-r$ is odd, $\overline{D}$ is even and $\overline{D} \neq \frac{2n}{n-r}$ (remember that $\overline{D} = \lceil \frac{2n}{n-r} \rceil$) .

The population of n legislators is partitioned in $\overline{D}+1$ sets described hereunder.

$P_1 : x_1 \quad x_{\overline{D}}\ x_2\ x_3\ ...\ x_{\overline{D}-1}\ (x_{\overline{D}+1}\ ...\ x_D)$

$P_2 : x_2 \quad x_1\ x_3\ x_4\ ...\ x_{\overline{D}}\ (x_{\overline{D}+1}\ ...\ x_D)$

$P_3 : x_3 \quad x_2\ x_4\ x_5\ ...\ x_1\ (x_{\overline{D}+1}\ ...\ x_D)$

$P_4 : x_4 \quad x_3\ x_5\ x_6\ ...\ x_2\ (x_{\overline{D}+1}\ ...\ x_D)$

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$P_{\overline{D}-3} : x_{\overline{D}-3} \quad x_{\overline{D}-4}\ x_{\overline{D}-2}\ x_{\overline{D}-1}\ ...\ x_{\overline{D}-5}\ (x_{\overline{D}+1}\ ...\ x_D)$

$P_{\overline{D}-2} : x_{\overline{D}-2} \quad x_{\overline{D}-3}\ x_{\overline{D}-1}\ x_{\overline{D}}\ ...\ x_{\overline{D}-4}\ (x_{\overline{D}+1}\ ...\ x_D)$

$P_{\overline{D}-1} : x_{\overline{D}-2}\ x_{\overline{D}}\ x_{\overline{D}-3}\ x_{\overline{D}-1}\ x_1\ ...\ x_{\overline{D}-4}\ (x_{\overline{D}+1}\ ...\ x_D)$

$P_{\overline{D}} : \quad x_{\overline{D}-1} \quad x_{\overline{D}-2}\ x_{\overline{D}}\ x_1\ ...\ x_{\overline{D}-3}\ (x_{\overline{D}+1}\ ...\ x_D)$

$P_{\overline{D}+1} : x_{\overline{D}} \quad x_{\overline{D}-1}\ x_1\ x_2\ ...\ x_{\overline{D}-2}\ (x_{\overline{D}+1}\ ...\ x_D)$

Consider:

$$p_j = |P_j| = \begin{cases} \lfloor \frac{n-r}{2} \rfloor & if \quad j \text{ is odd and } j \notin \{\overline{D}-1, \overline{D}+1\} \\ \lfloor \frac{n-r}{2} \rfloor & if \quad j \in \{\overline{D}-2, \overline{D}\} \\ \lceil \frac{n-r}{2} \rceil & if \quad j \text{ is even and } j < \overline{D}-2 \\ 1 & if \quad j = \overline{D}-1 \\ n - \sum_{i=1}^{\overline{D}} p_i & if \quad j = \overline{D}+1 \end{cases} .$$

Demonstrate that $0 < p_{\overline{D}+1} = n - 1 - 3(\frac{n-r-1}{2}) - \frac{\overline{D}-4}{2}(n-r) \leq \lfloor \frac{n-r}{2} \rfloor$. If $n-r = 2x+1$, with x being an integer, then $\lfloor \frac{n-r}{2} \rfloor = x = \frac{n-r-1}{2}$, and $\lceil \frac{n-r}{2} \rceil = x+1 = \frac{n-r+1}{2}$. Given the fact that $N = \bigcup_{j=1}^{\overline{D}+1} P_j$, consider:

1. $S_l = N - (P_{l+1} \cup P_{l+2})$, for $l = 1, 2, ..., \overline{D}-5$, then $|S_l| \geq r$, and S_l is a winning coalition;
2. For any $l \in \{\overline{D}-4, \overline{D}-3, \overline{D}-2, \overline{D}-1, \overline{D}\}$, $|S_l| \geq r$, since:

- $S_{\overline{D}-4} = N - (P_{\overline{D}-3} \cup P_{\overline{D}-2} \cup P_{\overline{D}-1})$;
- $S_{\overline{D}-3} = N - (P_{\overline{D}-2} \cup P_{\overline{D}-1} \cup P_{\overline{D}})$;
- $S_{\overline{D}-2} = N - (P_{\overline{D}} \cup P_{\overline{D}+1})$;
- $S_{\overline{D}-1} = N - (P_{\overline{D}-1} \cup P_{\overline{D}+1} \cup P_1)$;
- $S_{\overline{D}} = N - (P_1 \cup P_2)$;

3. For any $l \in \{1, 2, ..., \overline{D} - 5\}$, $x_l \gtrdot_{S_l} x_{l+1}$ as in **Case 1** and **Case 2**.
4. According to the new partition: $x_{\overline{D}-4} \succ_{S_{\overline{D}-4}} x_{\overline{D}-3}$, $not(x_{\overline{D}-4} \succeq_{S_{\overline{D}-5}} x_{\overline{D}-3})$, and $x_{\overline{D}-5} \succ_{S_{\overline{D}-4}} x_{\overline{D}-3}$. It follows that $x_{\overline{D}-4} \gtrdot_{S_{\overline{D}-4}} x_{\overline{D}-3}$. In the same manner, $x_{\overline{D}-3} \gtrdot_{S_{\overline{D}-3}} x_{\overline{D}-2}$, $x_{\overline{D}-2} \gtrdot_{S_{\overline{D}-2}} x_{\overline{D}-1}$, $x_{\overline{D}-1} \gtrdot_{S_{\overline{D}-1}} x_{\overline{D}}$, $x_{\overline{D}} \gtrdot_{S_{\overline{D}}} x_1$, and a cycle of length $\overline{D}$ occurs.

For each **Case** studied, and for each legislator's preferences in each set P_j, $j = 1, 2, ..., \overline{D}$ or $\overline{D}+1$, the alternatives $x_{\overline{D}+1}$, $x_{\overline{D}+2}, ..., x_D$ do not belong to the set $\mathcal{R}(\mathcal{L})$, because they are Pareto-dominated. Moreover, each alternative x_l, $l = 1, 2, ..., \overline{D}$ is dominated by only one alternative x_{l-1}. Therefore, we have constructed a profile $(\succeq_i) \in \mathcal{U}^n$ such that there is no stable outcome.

Now, we complete the proof of the theorem. Let $\mathcal{L} = (N, D, r(D), (\succeq_i))$ be a legislature, with $2 \leq D < \infty$, and let $(\succeq_i) \in \mathcal{U}^n$ be a legislator's preferences profile. Lemmas 8 and 9 give necessary and sufficient conditions that a supermajority rule should satisfy in order to fulfill the constraint in the political designer minimization problem (1). We show that, if the supermajority size needed to pass a policy of complexity level D in a legislature of size n is greater than $(\frac{D-2}{D})n$, then a stable policy exists regardless of the extent to which legislators' preferences are antagonistic. Conversely, if a stable policy of complexity level D exists at any preference profile in a legislature of size n, then the supermajority size needed to pass a policy should be greater than $(\frac{D-2}{D})n$. Formally, these results illustrate the fact that the set $\mathcal{R}(\mathcal{L})$ is non-empty, for each $(\succeq_i) \in \mathcal{U}^n$ if and only if $r > (\frac{D-2}{D})n$. Since r is an integer number, the smallest value that satisfies the latter inequality is $\lfloor \frac{D-2}{D} n \rfloor + 1$. Moreover $r > n/2$; then the voting rule $\max\left\{\lfloor \frac{D-2}{D} n \rfloor, \lfloor \frac{n}{2} \rfloor\right\} + 1$ is the optimal solution.

Proof of Theorem 3. For each $i \in N$, $\succeq_i$ is a strong preference relation over the induced policy space $A(D)$. Assume that legislators enact decisions under the dynamic mechanism (figure 3) endowed with the majority rule. To prove Theorem 3, we need to prove the lemmas below.

Lemma 10. *If $A(D)$ is a finite and discrete policy space ($|A(D)| < \infty$) and preferences are strong, then, a stable policy exists.*

Proof. For each $x \in A(D)$, define $f(x) = |\{y \in A(D) : \exists\ S \in \mathfrak{C},\ y \succ_S x\}|$ and let $x_0 \in A(D)$ such that $f(x_0) = \min\limits_{x \in A(D)} \{f(x)\}$. Prove that x_0 is stable. If this assertion is not true, then there exists $y \in A(D)$ and a majority $S \in \mathfrak{C}$ such that $y \gtrdot_S x_0$. It follows that:

$$\begin{cases} (\alpha)\ y \succ_S x,\ |S| > \frac{n}{2} \\ (\beta)\ \forall (z, T):\ T \neq S,\ z \succ_T y, |T| > \frac{n}{2} \text{ and not}(y \succeq_T x) \Rightarrow z \succeq_S x \end{cases} \tag{4}$$

By definition of x_0 and given that the preference relation $\succ$ is asymmetric, there exists $c \in A(D)$ such that $c \succ y$ and *not* ($c \succ x_0$). Thus, there exists a coalition T, with $|T| > \frac{n}{2}$ such that $c \succ_T y$.

- If $y \succ_T x_0$, then we have $c \succ_T y$ and $y \succ_T x_0$; it follows that $c \succ_T x_0$ by transitivity, which is a contradiction, because *not* ($c \succ x_0$).
- If *not* ($y \succ_T x_0$), then according to the assertion (β) in (4), $c \succ_S x_0$, which is a contradiction.

In conclusion, $x_0 \in \mathcal{R}(\mathcal{L})$, which means that x_0 is stable. □

Lemma 11. *If $A(D)$ is compact and convex, and each preference relation $\succeq_i$ is continuous, strong, and endowed with the topology of closed convergence, then a stable policy exists.*

Proof. Let f be the function defined over the policy $A(D)$ by:

$$f(x) = \mathfrak{LB}(\{y \in A(D) : \exists\ S \in \mathfrak{C},\ y \succ_S x\}),$$

where $\mathfrak{LB}$ denotes the Lebesgue measure on the manifold spanned by $A(D)$. Given that preferences are continuous, and $A(D)$ is compact and convex, there exists $x_1 \in A(D)$ such that $f(x_1) = \min\limits_{x \in A(D)} \{f(x)\}$. Following the same reasoning as in Lemma 10, we prove that $x_1 \in \mathcal{R}(\mathcal{L})$. □

Lemmas 10 and 11 reveal the existence of a stable policy when $A(D)$ is either a continuum or a discrete space. Given that each stable policy is efficient and the majority rule is the minimal supermajority rule by definition, we conclude the proof.