2019

Locating an Optimal Site for a Controversial Facility

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Abstract

We consider a situation in which policymakers in a local community have to choose an optimal site for a controversial and essential project in a democratic setting. Policymakers have either single-dipped or multi-dipped preferences over a Euclidean space of possible locations. We provide two existence results for this issue. There exists at most two optimal sites if the size of policymakers is odd, and they have single-dipped preferences over a one-dimensional site space.

***JEL**: C70, D02, D72.*

***Keywords**: Public bads, Single-dipped preferences, Multi-dipped preferences, Stability, Efficiency.*

I. Introduction

Scholars and policymakers have demonstrated a growing interest in locating controversial facilities, as surveyed by Aldrich (2010) in his recent book. Governments often face fundamental siting issues as they attempt to build or expand

*A special thanks to Victoria Barham, Edouard Djeutem, Diego Escobari, Lars Ehlers, Anthony Heyes, Jason Garred, Vikram Manjunath, Zephirin Nganmeni, Roland Pongou, Aggey Semenov, Wilfried Youmbi, and Stanley Winer for their keen insights and useful remarks. I am also grateful to Tatiana Nouya for reading and editing an earlier version of this paper.

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public projects that serve the needs of residents but potentially bring negative externalities into their targeted communities. Recent examples include, among others, the Enbridge Northern Gateway Pipelines, and the Trans Mountain Expansion Projects in Canada. States must choose among a myriad of possible policy instruments to handle local resistance, ranging from strong-arm tactics of coercion and hard social control to more peaceful incentives and inducements. Addressing issues on *where to locate* and *how to overcome civil society* have significant implications for the design and implementation of public policies towards protecting both environment and human wellbeing.

In this paper, we look at how residents of a local community interact peacefully in a democratic institution to address the problem of locating a *stable* and *efficient* site for a public facility, such as a windmill park, a garbage dumping ground, or a nuclear plant. Such public projects are called *public bads*: residents agree about their value but typically do not want them in their backyards. We assume that the public bad is to be located either within a one-dimensional space or within a multi-dimensional space. In the former, each resident has a *single-dipped* preference over the set of possible (or feasible) sites: there is the worst point, called the *dip*, and the resident's satisfaction increases with the Euclidean distance from this dip. In the latter, we assume that each resident may have a *multi-dipped* preference: a resident may have multiple local dips over the site space. In the community, the *Council* (a finite group of political representatives, henceforth called, legislators) adopts the policy through voting. A vote then consists of reporting one's dip. We assume that legislators are rational, truthful, and they vote according to the preferences of their constituents. A location for the public bad is adopted if (1) there is an endless cycle of voting: legislators agree on the public bad's site, and (2) the chosen site is *efficient*: there is no other site which would be preferred by all legislators to the chosen site. When (1) and (2) are satisfied, the council has chosen an *optimal* site for the project. The main issue here is that legislators may have antagonistic tastes upon where to locate the public facility, then render uncertain the outcome of this social decision by voting.

Legislators make decisions under a sequential mechanism. Indeed, most of the legislative mechanisms that exist in real life share a common feature: decision making follows a succession of stages, whereby, in the first stage, a bill is introduced, possibly followed by several amendments. Furthermore, in some of these legislatures, a bill introduced to challenge the incumbent policy might be withdrawn by its sponsor(s) if it encounters opposition. The legislative mechanism considered in this paper only retains this main feature, leaving out procedural details that might be specific to each constitutional entity. Formally, the council decides on the choice of a site under an agenda setting that comprises three stages described as follows:

Stage 1 (Objection): Let x be the status quo. The latter is any alternative that is chosen from the location space (or the set of feasible locations). Any legislator has the right to seek the replacement of x by proposing a motion y ($y \neq x$). If no motion is submitted, then x remains in place and the process ends. If a motion y is proposed and is supported by a sponsoring coalition that is a *majority*, y is submitted with the possibility of an amendment.

Stage 2 (Right to opposition): Any legislator who is not part of the sponsoring coalition has the right to oppose bill y. If there is no opposition, y is adopted as the new site and the process ends. If there is any opposition, the sponsoring coalition is given the opportunity to withdraw bill y. If it chooses to withdraw y, the status quo site x remains in place and the process ends. But, if it chooses to maintain y, the process moves to the third stage.

Stage 3 (Counter-objection): The opposition has the right to propose an amendment z. If it does not propose any amendment, y becomes the new site and the process ends. If it proposes an amendment z, a vote is organized between y and z. If a majority of legislators supports z against y, z becomes the new policy and the process ends. Otherwise y is the new site and the process ends.

We use *majority rule* to allocate decisive power among group (or coalition) of legislators. Thus, if $n(n \geq 2)$ is the size of the council, then any group of legislators that consists of more than $n/2$ is a *majority*. We say that a given site x is *stable* if it cannot be overruled by a majority of legislators if it is submitted to a possible reform or amendment under the three-stage mechanism described above. If the decision-making process ends with a site x, and the latter is efficient, then x is optimal and will be adopted for the community. Otherwise, the council adjourns.

We prove that there exists an optimal (stable and efficient) site of the public bad when the location space is one-dimensional (Theorem 1) or multi-dimensional (Theorem 2). In addition, if the size of legislators is odd and preferences over a one-dimensional site space are single-dipped, then the council can select at most two optimal sites for the public bad (Theorem 1).

In this paper, we ask the following question: where to locate a public bad or a controversial facility when individuals have conflicting preferences over the location space? Addressing this issue has already been of interest to scholars in social sciences. Our work must be viewed as an immediate successor of their contributions. Early theory suggests a deep involvement of citizen participation through local referenda allowing residents to vote for or against the public facility (e.g., Mitchell & Carson (1986)). In his book on siting issues and civil society, Aldrich (2010) debates about the patterns by which most controversial projects are sited in France, Japan, and United States, and the policy instruments used by these countries to handle civil opposition, when it arises. Other studies assume that individual preferences over the set of possible sites are single-dipped and focus on defining and characterizing collective rules, in which each agent has the incentives to reveal his or her sincere preferences (e.g., Ehlers (2002), Barberà et al. (2012), Öztürk et al. (2013), Manjunath (2014), Öztürk et al. (2014), Lahiri et al. (2016), Peters et al. (2017), and the references therein). In a direct revelation mechanism setting, Yamamura (2016) analyzes the choice of locating a public bad from a coalitional standpoint using the equilibrium concepts of strong Nash equilibrium and coalition proof Nash equilibrium. In contrast with the preceding studies, we determine the existence of an *efficient* and *stable* site for a controversial facility under a sequential legislative mechanism with perfect information when legislators display specific preferences over the set of feasible sites.

The paper is organized as follows. In Section II, we introduce our notations and the main definitions. In Section III, we provide the existence of an optimal location for a controversial facility when legislators have either single-dipped or multi-dipped preferences over a Euclidean location space. In Section IV, we conclude.

II. Preliminaries

In this section, we provide formal and brief notations of the basic concepts that we use throughout the paper.

1. The Council

The council is labeled as $\mathfrak{C} = (N, A, (\succeq_i)_{i \in N}, \mathcal{M})$, where A is the Euclidean site space, N is the finite set of legislators (with $|N| = n \geq 2$), $(\succeq_i)_{i \in N}$ is the profile of legislators' preferences over the set A, and $\mathcal{M}$ is the majority rule. For simplicity, we will also consider $\mathcal{M}$ as the set of *majority coalitions*. Specifically, any subgroup which consists of more than half of the council's members belongs to $\mathcal{M}$. The site space A represents the set of feasible sites at stake. We assume that A is one-dimensional space. When appropriate, additional structure will be added to A.

2. Preferences

Given the nature of the site space, we assume that preferences are *continuous*. Preference continuity is a natural assumption which means that a legislator who prefers a site x over another site y prefers any site close enough to x over any site close enough to y. This implies that a small perturbation of the site space does not radically change preferences. This structure of preferences is generally encountered in economic theory. Formally, let $\succeq$ be a preference relation over A. The binary relation $\succeq$ is said to be continuous if for any site $x, y \in A$, such that x is strictly preferred to y ($x \succ y$), there exists a neighborhood $S(x)$ of x and a neighborhood $S(y)$ of y such that site z is strictly preferred to site t ($z \succ t$) for every sites $z \in S(x)$ and $t \in S(y)$. In this paper, we use two classes of continuous preferences: *single-dipped* and *multi-dipped* preferences.

3. Optimal Sites

The council decides on the choice of a site under a sequential voting that we presented in the Introduction. Now, we describe the rational behavior of legislators within the mechanism.

Let's assume that in the council, a status quo site $x \in A$ is at stake. A group of legislators will gather together in a majority coalition, say S and *object* to the status quo if there exists another alternative, say y which is better than x, i.e. $y \succ_S x$. After the *objection* (y, S), a legislator from the remaining group of council $(N \setminus S)$ may be worse off at y, and decides to join another majority coalition T which could propose another policy z to the council, i.e. $z \succ_T y$. In that case, a pair $(z, T) \in Z \times \mathcal{M}$ is a *counter-objection* against the objection (y, S) if $z \succ_T y$ and not$(y \succeq_T x)$. Also, legislators of T have to consider the fact that the first majority coalition S may withdraw their support for the change of the status quo x in case the new proposition z could hurt some of its legislators, i.e. not$(z \succeq_S x)$. If the latter occurs, then the counter-objection is *unfriendly* against

S. In the council, a majority coalition S will initiate the first move against the status quo x by proposing a new option y if there is no unfriendly counter-objection against (y, S), i.e. $(z \succeq_S x)$; then the objection (y, S) is considered to be *justified*.

A *stable* site is an option in A against which no justified objection exists. In other words, a stable site is an alternative such that if it is the status quo, no majority coalition will seek to replace it. More formally, a site, say $y \in A$ *defeats* another site x, labeled by $y \gg x$, if there exists a majority coalition $S \in \mathcal{M}$ such that:

1. $y \succ_S x$ and;
2. $[\forall (z, T) \in A \times \mathcal{M},\ S \neq T,\ z \succ_T y$ and $not(y \succeq_T x)]$ implies $[z \succeq_S x]$.

The set of stable sites, denoted by $O(\mathfrak{C})$, contains options that can't be defeated in the council $\mathfrak{C}$. A decision is adopted if legislators reach an agreement in the end of legislative procedure, i.e, $O(\mathfrak{C}) \neq \emptyset$, and the outcome of this agreement is efficient. It is straightforward to show that any stable site is efficient. Therefore, the optimal decision of the council is guaranteed if stability succeeds. It follows that $O(\mathfrak{C})$ denotes the set of optimal sites for the public bad. The set $O(\mathfrak{C})$ is also called the *reciprocity set* in Pongou & Tondji (2018). By addressing a different question, they prove that the legislative procedure that we use in this study induces legislators to take reciprocal actions. In the next section, we derive our main results.

III. Results

We first establish the existence of a stable site when preferences are *single-dipped*. A preference relation over a space is said to be *single-dipped* if; the sites can be ordered as points on a line; the preference relation has a least preferred point–dip or worse point; and, points further away from the least point are more preferred. Formally, assume that all the sites are ordered by a binary relation denoted θ, and all individuals perceive them as being arranged in this order. An individual m of the council has a *single-dipped* preference $\succeq_m$, if there exists a site x_m such that: (1) for any other site $x \neq x_m$, $x \succ_m x_m$; and (2) for any site $x, y \in A$, it holds that: if $(x_m \; \theta \; y \; \theta \; x)$ or $(x \; \theta \; y \; \theta \; x_m)$, then $x \succ_m y$.

Theorem 1 *Let $\mathfrak{C} = (N, A, (\succeq_i)_{i \in N}, \mathcal{M})$ be a council where, A is a one-dimension site space, and where preferences are single-dipped. Then, an optimal site exists. In addition, if $|N|$ is odd, there are at most two optimal sites; and if $|N|$ is even, any feasible site could be optimal.*

Theorem 1 is consistent with other studies on the site of a public bad under *strategy-proof* and efficient rules. Strategy-proofness avoids members' improvement of satisfaction or gains by misrepresenting their true preferences, and efficiency ensures that the satisfaction of all members cannot improve by changing the chosen site of a public bad. As mentioned above, any stable site under the three-stage mechanism used in the Council is efficient. Manjunath (2014) studies the site of a public bad when agents have single-dipped preferences along a closed interval $[0, T]$. He proves that the range of an efficient and strategy-proof rule is the set $\{0, T\}$, the two extremes of the domain. In another study, Barberà et al. (2012) proves that the range of strategy-proof rules under single-dipped preferences contains two alternatives at most. Theorem 1 provides the same characterization when the size of the council is odd. In fact, each legislator has at most two *peak* (the most preferred) points and, those sites coincide with the minimum and maximum elements of A. Therefore, these two bounds are the optimal sites when preferences are single-dipped and the size of the council is odd. However, Theorem 1 also shows that the size of $O(\mathfrak{C})$ may be larger than two. Barberà et al. (2012) also provide examples of different sub-classes of single-dipped preferences admitting strategy-proof rules with larger ranges. For simplicity, we prove Theorem 1 for $n = 3$ and $n = 4$ below. We relegate the general proof in the appendix.

Proof. Let $x \in A$ be a site, and define the number $S(x)$ as follows: $S(x)$ is a number of legislators whom has x as a peak point (the most preferred point). If there is no legislator who has x as a peak point, then $S(x)$ takes the value 0. Let $V(q, q')$ denote the set of legislators who vote for q against q' in a pairwise opposition between q and q'. Two cases will be differentiated: each member has only one peak over the set A or there exists at least one member who has two peak sites. In what follows, q_m is the dip point for legislator $m, m = 1, 2, 3, 4$.

1. Assume $n = 3$. Legislators have single-dipped preferences. Therefore, they prefer to vote for sites far away from their dip points. In this first part of the proof, we consider two cases and determine the stable sites.

Case a: $S(q_1) = 1$, $S(q_2) = 0$, and $S(q_3) = 2$ (see figure 1). The pairwise

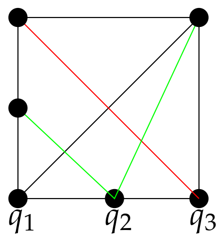

Figure 1: *One legislator has only q_1 as peak ($S(q_1) = 1$); No one has q_2 as peak ($S(q_2) = 0$); and two legislators have q_3 as peak ($S(q_3) = 2$).*

oppositions between sites q_1, q_2 and q_3 give the following results: $q_1 \gtrdot_{23} q_2$ i.e., q_1 defeats q_2 thanks to the majority $\{2, 3\}$; $q_3 \gtrdot_{12} q_1$ or q_3 defeats q_1 thanks to the coalition $\{1, 2\}$; $q_3 \gtrdot_{12} q_2$ or q_3 defeats q_2 thanks to coalition $\{1, 2\}$. There is no site which defeats option q_3. Hence, $q_3 \gtrdot_{12} q_1$, $q_3 \gtrdot_{12} q_2$, and $O(\mathcal{C}) = \{q_3\}$.

Case b: $S(q_1) = 2$, $S(q_2) = 0$, and $S(q_3) = 2$ (see figure 2).

As presented previously, the pairwise oppositions in this case give the following results: q_1 defeats q_2 with the support of coalition $\{2, 3\}$ ($q_1 \succ_{23} q_2$); and q_3 defeats q_2 with the support of majority $\{1, 2\}$ ($q_3 \succ_{12} q_2$). The pairwise opposition between q_1 and q_3 is interesting: $V(q_1, q_3) = \{3\}$ and $V(q_3, q_1) = \{1\}$, then no winner. It follows that the site q_2 is defeated by either q_1 or q_3 and

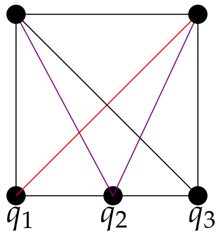

Figure 2: *Two legislators have q_1 as peak ($S(q_1) = 2$); No one has q_2 as peak ($S(q_2) = 0$); and two legislators have q_3 as peak ($S(q_3) = 2$).*

no point defeats either q_1 or q_3. Hence, $O(\mathfrak{C}) = \{q_1, q_3\}$. With three legislators, there are at most two optimal sites if the number of legislators is odd.

2. Assume $n = 4$. We use the same methodology as in the previous situation with $n = 3$.

Case c: Each legislator has only one peak point. This case can be illustrated in either figure 3 or figure 4. We start with the illustration displayed in figure 3.

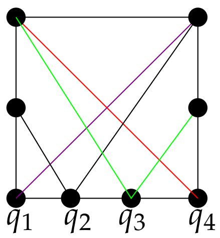

Figure 3: *The Council's size is 4 and each legislator has a unique peak—first scenario.*

The pairwise oppositions between different sites give the following results: $V(q_1, q_2) = \{2, 3, 4\}$ and $V(q_2, q_1) = \{1\}$, then q_1 defeats q_2 with the support of coalition $\{2, 3, 4\}$. $V(q_1, q_3) = \{2, 3, 4\}$ and $V(q_3, q_1) = \{1\}$, then, q_1 defeats q_3 with the support of coalition $\{2, 3, 4\}$. $V(q_1, q_4) = \{3, 4\}$ and $V(q_4, q_1) = \{1, 2\}$, then, q_1 and q_4 are tied. $V(q_3, q_4) = \{4\}$ and $V(q_4, q_3) = \{1, 2, 3\}$, then, q_4 defeats q_3 with the support of majority $\{1, 2, 3\}$. As result, the only non-defeated sites are the two peak points q_1 and q_4. Hence, $q_1 \succ_{234} q_2$ and $q_4 \succ_{123} q_3$, and $O(\mathfrak{C}) = \{q_1, q_4\}$.

Now, we continue our analysis with the illustration in figure 4. The pairwise

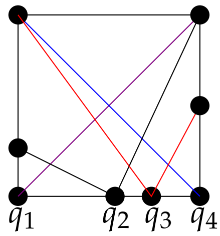

Figure 4: *The Council's size is 4, and each individual has a unique peak—second scenario.*

oppositions between different sites give the following results: $V(q_1, q_2) = \{2, 3, 4\}$ and $V(q_2, q_1) = \{1\}$, then, q_1 defeats q_2 with the support of coalition $\{2, 3, 4\}$. Legislator 2 is indifferent between sites q_1 and q_3, hence $V(q_1, q_3) = \{3, 4\}$ and $V(q_3, q_1) = \{1\}$, so that q_1 and q_3 are tied. Also, $V(q_1, q_4) = \{3, 4\}$ and $V(q_4, q_1) = \{1, 2\}$, then, q_1 and q_4 are tied. As proved above, $V(q_3, q_4) = \{4\}$ and $V(q_4, q_3) = \{1, 2, 3\}$, then, q_4 defeats q_3 with the support of majority $\{1, 2, 3\}$. In the end, the only non-defeated sites are the two peak points q_1 and q_4 even though q_1 and q_3 are tied, $q_4 \succ_{123} q_3$. Given that $q_1 \succ_{234} q_2$, then, $O(\mathfrak{C}) = \{q_1, q_4\}$.

Case d: There exists at least one legislator who has two peak points. This case can be illustrated in figures 5a and 5b.

(a) **(b)**

Figure 5: *The Council's size is 4, and at least one legislator has two peaks.*

The same method (**Case c**) illustrated above can be applied here. The pairwise oppositions between different alternatives give the following results: $V(q_1, q_2) = \{2, 3, 4\}$ and $V(q_2, q_1) = \{1\}$, then q_1 defeats q_2. The only difference is that, from figure 5a, q_1 defeats q_3, since $V(q_1, q_3) = \{2, 3, 4\}$ and $V(q_3, q_1) = \{1\}$. Whereas in figure 5b, legislator 2 is indifferent between candidates q_1 and q_3, leading to a tie between q_1 and q_3 (in fact $V(q_1, q_3) = \{3, 4\}$ and $V(q_3, q_1) = \{1\}$). A pairwise opposition between peaks q_1 and q_4 leads to a tie, because $V(q_1, q_4) = \{4\}$ and $V(q_4, q_1) = \{1, 2\}$. Finally, $V(q_3, q_4) = \{4\}$ and $V(q_4, q_3) = \{1, 2, 3\}$, then q_4 defeats q_3. In conclusion, $O(\mathfrak{C}) = \{q_1, q_4\}$ for the same reasons stated above (**Case c**). With four legislators, the Council can enact only two optimal sites. ■

In real-life situations, legislators may have beliefs (or positions) over social issues that are not always ranked by a one-dimensional spectrum. In that case, the result of Theorem 1 is limited. Therefore, we assume that members in the council may also display multi-dipped continuous preferences over a possibly multi-dimensional site space. In what follows, we assume that the site space A is a compact and convex subset of the k-dimensional Euclidean space ($k < \infty$). We also endow the site space with the topology of closed convergence (see, for example, Hildenbrand (1974)). We start the analysis with the following definition.

Definition. Let $\succeq$ be a binary relation over a site space A. An alternative $x \in A$ is a *local minimizer* of $\succeq$ if there exists a neighborhood $S(x)$ of x such that for every $y \in S(x)$, $y \succeq x$.

Denote by $LM(\succeq, A)$ the set of all the local minimizers of the binary relation $\succeq$. Note that, if the preference $\succeq$ is a continuous binary relation over the space A, then $LM(\succeq, A)$ is a non-empty Borel subset of A. The concepts of Borel set, and the Lebesgue measure (denoted by $\mathfrak{L}$) have proven to be extremely useful in general equilibrium theory. Multi-dipped preferences have one or many local minimizers that form a null set. An interesting subclass of this class of preferences is the set of preferences that have countably many local minimizers. We have the following result.

Theorem 2 *Let $\mathfrak{C} = (N, A, (\succeq_i)_{i \in N}, \mathcal{M})$ be a council where A is a k-dimensional Euclidean site space ($1 \leq k < \infty$), and where each preference $\succeq_i$ is such that $\mathfrak{L}[LM(\succeq_i, A)] = 0$, for $i = 1, 2, ..., n$. Then, an optimal site exists.*

Proof. See appendix. ■

Theorem 2 complements the result of Theorem 1, and proved useful by guaranteeing positive results in a framework where legislators may have several worst sites.

IV. Conclusion

In this paper, we use an agenda formation setting to display two possible theorems for the enactment of an optimal location of a public bad when policy-makers' preferences over the set of competing sites display specific structures. Specifically, we help a council to site a public bad in a local community where residents have either single-dipped or multi-dipped preferences over a set of feasible sites. We show that there exists at most two optimal sites if the council's size is odd and policymakers have single-dipped preferences over a one-dimensional site space.

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Proof of Results

Proof of Theorem 1.

1. Assume that n is odd. Two different scenarios can be distinguished: each legislator has a unique peak point or there exists at least one legislator who has two peaks over A.

Case e: consider the situation where each legislator has a unique peak point over A. Assume q_1^* and q_2^* to be the two possible peak points (this is possible since A is bounded), then $S(q_1^*) + S(q_2^*) = n$, $S(q_1^*), S(q_2^*) \geq 0$. Without loss of generality, assume that $S(q_1^*) > S(q_2^*)$, then $S(q_1^*) > \frac{n}{2}$. Consider the following graph (Figure 6) described below.

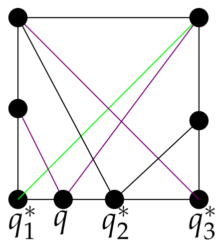

Figure 6: *The Council's size is odd, and each legislator has a unique peak.*

In pairwise opposition between peaks q_1^* and q_2^*, the former wins, because more than half of the council vote for q_1^* against q_2^*, since $S(q_1^*) > S(q_2^*)$ and $S(q_1^*) > \frac{n}{2}$. In a pairwise opposition between q_1^* and any $q \in (q_1^*, q_2^*)$ such that $S(q) = 0$, q loses. Indeed, the number of individuals who have q_1^* as dip point vote for q. It follows that, at least, $S(q_1^*)$ individuals vote for q_1^* against q. Given that $S(q_1^*) > \frac{n}{2}$, then q_1^* defeats all other sites in A and $O(\mathfrak{C}) = \{q_1^*\}$.

Case f: Assume that there exists at least one legislator who has two peaks over A. This situation can be illustrated by a graph similar to the one below (Figure 7).

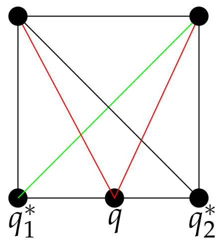

Figure 7: *The Council's size is odd, and at least one legislator has two peaks.*

The legislator for whom q is a dip point has two peak points q_1^* and q_2^*. Thus he or she is indifferent between those peaks during a pairwise opposition when they are candidates.

- Let oppose q_1^* and q_2^* in a pairwise opposition. Then $|V(q_1^*, q_2^*)| + |V(q_2^*, q_1^*)| = n - 1$, because one legislator is indifferent between q_1^* and q_2^*.
  i) If $S(q_1^*) > S(q_2^*)$, and $S(q_1^*) > \frac{n}{2}$, q_1^* defeats q_2^*, since $|V(q_1^*, q_2^*)| \geq S(q_1^*)$.
  ii) If $S(q_1^*) = S(q_2^*)$, then q_1^* and q_2^* are tied.
- Let q be any option between q_1^* and q_2^*. Using the same previous reasoning, q is defeated by either q_1^* or q_2^* in a pairwise opposition. Consequently, if $S(q_1^*) > S(q_2^*)$ with $S(q_1^*) > \frac{n}{2}$, then $O(\mathfrak{C}) = \{q_1^*\}$. Also, if $S(q_1^*) = S(q_2^*)$, then $O(\mathfrak{C}) = \{q_1^*, q_2^*\}$.

2. Assume that n is even. We study two different situations: each legislator has a unique peak point or there exists at least one legislator who has two peaks over A.

Case g: n is even, and each legislator has a unique peak point. Consider q_1^*

and q_2^* as the two possible peak points, then $S(q_1^*) + S(q_2^*) = n$, with $S(q_1^*) \geq 0$ and $S(q_2^*) \geq 0$.

a) Assume that $S(q_1^*) > S(q_2^*)$, then $S(q_1^*) > \frac{n}{2}$, and $S(q_2^*) < \frac{n}{2}$. This item is similar to the situation already demonstrated, where n is odd (see **Case a**). As we proved previously, $O(\mathfrak{C}) = \{q_1^*\}$.

b) Suppose that $S(q_1^*) = S(q_2^*) = \frac{n}{2}$. In a pairwise opposition between q_1^* and q_2^*, there is no winner. Let $q \in A$ be a candidate such that $S(q) = 0$, and oppose this site to the peak point q_1^* in a pairwise opposition. Legislators for whom q_1^* is a dip point will vote for q and individuals for whom q_1^* is a peak point will vote for q_1^*. Also, individuals for whom q is the dip point will vote for q_1^*, although the latter is not their ideal point. Since $S(q_1^*) = \frac{n}{2}$, it follows that $|V(q_1^*, q)| > \frac{n}{2}$, and q_1^* defeats q. Therefore, all other sites between q_1^* and q_2^* are defeated either by q_1^* or q_2^*. Consequently, q_1^* and q_2^* are the only non-defeated sites, and $O(\mathfrak{C}) = \{q_1^*, q_2^*\}$.

Case h: n is even and there exists at least one legislator who has two peaks over A. The individual for whom q is the dip point has two peak points q_1^* and q_2^*, thus he or she is indifferent between those peaks during a pairwise opposition where they are candidates.

a) Assume $S(q_1^*) > S(q_2^*)$, then $S(q_1^*) > \frac{n}{2}$. In a pairwise opposition between q_1^* and q_2^*, the result gives: $|V(q_1^*, q_2^*)| + |V(q_2^*, q_1^*)| = n - 1$, since one individual is indifferent between q_1^* and q_2^*. Since $|V(q_1^*, q_2^*)| \geq S(q_1^*)$ and $S(q_1^*) > \frac{n}{2}$, it follows that q_1^* defeats q_2^*. Any other alternative site q between q_1^* and q_2^*, is defeated. Therefore, $O(\mathfrak{C}) = \{q_1^*\}$.

b) Consider $S(q_1^*) = S(q_2^*) = \frac{n}{2}$. A pairwise opposition between the two peak points q_1^* and q_2^* leads to no winner. Assume a pairwise opposition between any alternative q, such that $S(q) = 0$, and either q_1^* or q_2^*. Legislators for whom q_1^* or q_2^* is a peak point vote for q_1^* or q_2^*. By hypothesis, $S(q_1^*) = S(q_2^*) = \frac{n}{2}$. It follows that neither $|V(q_1^*, q)|$ nor $|V(q, q_1^*)|$ is greater than half of the council, then there is no winner in the voting context. Hence, no site is defeated and $O(\mathfrak{C}) = A$.

Proof of Theorem 2. For any $x \in A$, we denote $P_C(x) = \{y \in A : y \succ_C x\}$, $C \in \mathcal{M}$ and $P(x) = \{y \in A : y \succ x\}$, and $P(x) = \bigcup_{C \in \mathcal{M}} P_C(x)$. Consider h the mapping from A to the real numbers, and define $h(x) = \mathfrak{L}(P(x))$, for any $x \in A$.

1. h is lower semi-continuous.

   Let $x_0 \in A$. Consider a sequence x_k which converges to x_0 when k tends to infinity and show that $\lim_{k \to \infty} \inf h(x_k) \geq h(x_0)$. Let $y \in P(x_0)$, then there exists $C \in \mathcal{M}$ such that $y \in P_C(x_0)$. x_k converges to x_0 and P_C is continuous, thus for k sufficiently large, $y \in P_C(x_k)$. It follows that $y \in \liminf P_C(x_k)$ for some $C \in \mathcal{M}$, and then $y \in \liminf P(x_k)$, so $P(x) \subseteq \liminf P(x_k)$. Since there exists k_0 such that $\mathfrak{L}(\bigcup_{k \geq k_0} P(x_k)) < \infty$, then $\mathfrak{L}(\lim_{k \to \infty} \inf P(x_k)) \leq \lim_{k \to \infty} \inf \mathfrak{L}(P(x_k))$, i.e., $\mathfrak{L}(\lim_{k \to \infty} \inf P(x_k)) \leq \lim_{k \to \infty} \inf h(x_k)$. Since $P(x_0) \subseteq \liminf P(x_k)$, then $\mathfrak{L}(P(x_0)) \leq \mathfrak{L}(\lim_{k \to \infty} \inf P(x_k))$ and therefore $h(x_0) \leq \lim_{k \to \infty} \inf h(x_k)$.

2. h is lower semi-continuous and A compact, then h attains a minimum on A, i.e., there exists $a \in A$ such that $h(a) = \inf_{x \in A} h(x)$.

   Denote by $S = \left\{ a \in A : h(a) = \inf_{x \in A} h(x) \right\}$ and $\overline{LM} = \bigcup_{i=1}^{n} [LM(\succeq_i, A)]$. Consider the **binary relation** $\gg$ defined as follows: (1) for $x, y \in A$ and $C \subset N$, $y \gg_C x$ if $y \succ_C x$ and $z \succ_C x$ for all $z \in A$ such that $z \succ y$ and $z \notin \overline{LM}$; and (2) $y \gg x$ if there exists $C \in \mathcal{M}$ such that $y \gg_C x$. Let $\mathcal{S}(\mathfrak{C}) = \{x \in A : not(y \gg x), y \in A\}$.

3. $\mathcal{S}(\mathfrak{C})$ is non-empty.

   Assume the contrary and consider an arbitrary site $x_0 \in S$. There exists an alternative site $x_1 \in A$ and $C \in \mathcal{M}$ such that $x_1 \gg_C x_0$ i.e., $x_1 \succ_C x_0$ and $z \succ_C x_0$ for any $z \in A$ such that $z \succ x_1$ and $z \notin \overline{LM}$. It follows that $P(x_1) \subseteq P(x_0) \cup (P(x_1) \cap \overline{LM})$. Certainly, if $y \in P(x_1)$, then, there exists $C' \in \mathcal{M}$ such that $y \in P_{C'}(x_1)$, so $y \succ_{C'} x_1$. Since the majority rule satisfies the overlapping condition (i.e., for any $S, T \in \mathcal{M}$, $S \cap T \neq \emptyset$), there exists a member $i \in C \cap C'$ and $y \succ_i x_1 \succ_i x_0$. By transitivity, $y \succ_i x_0$. Since P_C is continuous, there exists a neighborhood $S(y)$ of y such that $y' \succ_i x_0$ for all $y' \in S(y)$, so $y \in LM(\succeq_i, A)$ and $y \in \overline{LM}$. If $C' = C$, then we have $y \succ_C x_1 \succ_C x_0$ i.e., $y \succ_C x_0$ or $y \in P(x_0)$. Hence, $P(x_1) \subseteq P(x_0) \cup (P(x_1) \cap \overline{LM})$, therefore $\mathfrak{L}(P(x_1)) \leq \mathfrak{L}(P(x_0)) + \mathfrak{L}(P(x_1) \cap \overline{LM}) \leq \mathfrak{L}(P(x_0)) + \mathfrak{L}(\overline{LM})$. Since $\overline{LM} = \bigcup_{i=1}^{n} [LM(\succeq_i, A)]$, then $\mathfrak{L}(\overline{LM}) \leq \sum_{i=1}^{n} \mathfrak{L}[LM(\succeq_i, A)]$ and $\mathfrak{L}[LM(\succeq_i, A)] = 0$ for all $i = 1, ..., n$ by hypothesis, therefore $\mathfrak{L}(\overline{LM}) = 0$ and $\mathfrak{L}(P(x_1)) \leq \mathfrak{L}(P(x_0))$ i.e., $h(x_1) \leq h(x_0)$. Since $h(x_0) = \inf_{x \in A} h(x)$, then the inequality $h(x_1) \leq h(x_0)$ implies that $h(x_1) = h(x_0)$ and $x_1 \in S$. We note two different situations described below.

Situation 1: Assume there exists a neighborhood $S(x_1)$ for which there is no $y \in S(x_1)$ satisfying $y \succ x_1$. $x_1 \succ x_0$ and preferences are continuous, then there exists a neighborhood $S'(x_1) \subseteq S(x_1)$ such that $y \succ x_0$ for all $y \in S'(x_1)$. It follows that $S'(x_1) \subseteq P(x_0)$ and $S'(x_1) \subseteq (A - P(x_1))$. Thus, $P(x_1) \subsetneq P(x_0)$ and $h(x_1) < h(x_0)$, which is a contradiction.

Situation 2: For all neighborhood $S(x_1)$ of x_1, there exists an alternative $y \in S(x_1)$ such that, $y \succ x_1$. Let x_2 be a site such that $x_2 \gg_{C'} x_1$. As previously, $P(x_2) \subseteq P(x_1) \cup (P(x_1 \cap \overline{LM}))$ and then $h(x_2) = h(x_1)$. We have $x_2 \succ x_1$ and preferences are continuous, thus there exists a neighborhood $S(x_1)$ such that $x_2 \succ a$ for any $a \in S(x_1)$. By assumption, there exists $y \in S(x_1)$ such that $y \succ x_1$. By continuity of preferences, there exists a neighborhood $S(y)$ of y such that $y' \succ x_1$ for any $y' \in S(y)$. Hence for all $y' \in S(y) \cap S(x_1)$, we have $y' \succ x_1$ and $x_2 \succ y'$, so $h(x_1) > h(y')$ and $h(y') > h(x_2)$, then $h(x_1) > h(x_2)$, a contradiction.

Therefore, $\mathcal{S}(\mathfrak{C})$ is non-empty.

4. Let $x_0 \in \mathcal{S}(\mathfrak{C})$, prove that $x_0 \in O(\mathfrak{C})$.

   Assume that $x_0 \notin O(\mathfrak{C})$, then there exists $x_1 \in A$ and $C \in \mathcal{M}$ such that $x_1 \gtrdot_C x_0$. Let $z \notin \overline{LM}$ such that $z \succ_T x_1$, for $T \in \mathcal{M}$. We would like to demonstrate that $z \succ_C x_0$. Firstly, $z \succ_T x_1$, and $x_1 \gtrdot_C x_0$. For that reason, $z \succeq_C x_0$.

   a) If $z \succ_i x_0$ for any $i \in C$, then $z \succ_C x_0$ and $x_1 \gg x_0$, which is a contradiction.

   b) Assume that there exists $i \in C$ such that $z \sim_i x_0$. $z \succ x_1$ and preferences are continuous, then there exists a neighborhood $S(z)$ such that $z' \succ x_i$ for all $z' \in S(z)$. Because $z \notin \overline{LM}$, then $z \notin LM(\succeq_i, A)$ for any $i \in N$. Hence, there exists $z' \in S(z)$ such that $z \succ_i z'$. Since, $z \sim_i x_0$, then $x_0 \succ_i z'$ by transitivity. Also, $z' \in S(z)$, therefore $z' \succ x_1$ and by transitivity, $x_0 \succ_i x_0$ if $i \in T$. Otherwise, if $i \notin T$, then, given that $T \cap C \neq \emptyset$, there exists $i' \in T \cap C$ such that $x_0 \succ_{i'} z' \succ_{i'} x_1 \succ_{i'} x_0$ or $x_0 \succ_{i'} x_0$ by transitivity, which is a contradiction and $x_0 \notin \mathcal{S}(\mathfrak{C})$. In conclusion, $x_0 \in O(\mathfrak{C})$.