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# Homotopy groups and quantitative Sperner-type lemma

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**Abstract**

We consider a generalization of Sperner's lemma for triangulations of $m$–discs whose vertices are colored in at most $m$ colors. A coloring on the boundary $(m-1)$–sphere defines an element in the corresponding homotopy group of the sphere. Depending on this invariant, a lower bound is obtained for the number of fully colored simplexes. In particular, if the Hopf invariant is nonzero on the boundary of 4-disk, then there are at least 9 fully colored tetrahedra and if the Hopf invariant is $d$, then the lower bound is $3d+3$.

**Keywords:** Hopf invariant, homotopy group of spheres, Sperner lemma, framed cobordism

## 1 Introduction

Sperner's lemma is a discrete analog of the Brouwer fixed point theorem. This lemma states:

*Every Sperner $(n+1)$–coloring of a triangulation $T$ of an $n$–dimensional simplex $\Delta^n$ contains an $n$–simplex in $T$ colored with a complete set of colors* [18].

We found several generalizations of Sperner's lemma [8–15].

Let $K$ be a simplicial complex. Denote by $\mathrm{Vert}(K)$ the vertex set of $K$. Let an $(m+1)$–coloring (labeling) $L$ be a map $L : \mathrm{Vert}(K) \to \{0, 1, \dots, m\}$. Setting

$$f_L(u) := v_k, \text{ where } u \in \mathrm{Vert}(K),\ k = L(u), \text{ and } \{v_0, ..., v_m\} = \mathrm{Vert}(\Delta^m),$$

we have a simpicial map $f_L : K \to \Delta^m$. We say that an $n$–simplex $s$ in $K$ is *fully labeled* if $s$ is labeled with a complete set of labels $\{0, \dots, m\}$.

Suppose there are no fully labeled simplices in $K$. Then $f_L(p)$ lies in the boundary of $\Delta^m$. Since the boundary $\partial\Delta^m$ is homeomorphic to the sphere $S^{m-1}$, we have a continuous map $f_L : K \to S^{m-1}$. Denote the homotopy class of $f_L$ in $[K, S^{m-1}]$ by $[f_L]$.

Let $T$ be a triangulation of a manifold $M$ with boundary $\partial M$. Let $L : \mathrm{Vert}(T) \to \{0, \dots, n+1\}$ be a labeling of $T$. Define

$$\partial L : \mathrm{Vert}(\partial T) \to \{0, 1, \dots, n+1\}, \quad \partial \mathrm{f_L} : \partial T \to \mathrm{Vert}(\Delta^{n+1}).$$

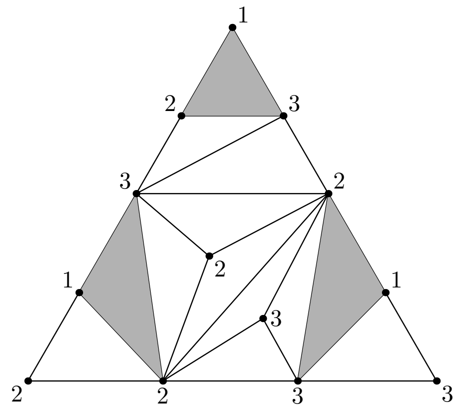


Figure 1: An illustration of Theorem A with $d = 3$

Observe that if the dimension of $M^{n+1}$ is $n + 1$, then $\dim(\partial M) = n$ and the map $\partial \mathrm{f_L} : \partial T \to \partial \Delta^{n+1}$ is well defined. By the Hopf theorem [7, Ch. 7] we have $[\partial M, S^n] = \mathbb{Z}$ and $[\partial \mathrm{f_L}] = \deg(\partial \mathrm{f_L}\,|) \in \mathbb{Z}$.

**Theorem A.** [12, Theorem 3.4] *Let $T$ be a triangulation of an oriented manifold $M^{n+1}$ with nonempty boundary $\partial M$. Let $L : \mathrm{Vert}(T) \to \{1, \ldots, n + 2\}$ be a labeling of $T$. Then $T$ must contain at least $d = |\deg(\partial \mathrm{f_L})|$ fully labelled simplices.*

In Fig.1 is shown an illustration of Theorem A. Here $n = 1$, $M = D^2$ and $d = [\partial \mathrm{f_L}] = 3$. The theorem yields that there are at least three fully labeled triangles.

Observe that for a Sperner labelling we have $d = 1$. Actually, Theorem A can be considered as a quantitative extension of the Sperner lemma.

In [12] with $(n+2)$–covers of a space $X$ we associate certain homotopy classes of maps from $X$ to $n$–spheres. These homotopy invariants can be considered as obstructions for extending covers of a subspace $A \subset X$ to a cover of all of $X$. We are using these obstructions to obtain generalizations of the classic KKM (Knaster–Kuratowski–Mazurkiewicz) and Sperner lemmas. In particular, we proved the following theorem:

**Theorem B.** ([12, Corollary 3.1] & [13, Theorem 2.1]) *Let $T$ be a triangulation of a disc $D^{n+k+1}$. Let $L : \mathrm{Vert}(T) \to \{0, \ldots, n+1\}$ be a labeling of $T$ such that $T$ has no fully labelled $n$–simplices on the boundary $\partial D \cong S^{n+k}$. Suppose $[\partial f_L] \neq 0$ in $\pi_{n+k}(S^n)$. Then $T$ must contain at least one fully labeled $n$–simplex.*

We observe that for $k = 0$ and $M = D$ Theorem A yields Theorem B. However, in this case Theorem A is stronger than Theorem B. In this paper we are going to prove a quantitative extension of Theorem B. First we consider the case $n = 2$ and $k = 1$. In Section 2, the following theorem is proved.

**Theorem 1.1.** *Let $T$ be a triangulation of $D^4$ with a labeling $L : \mathrm{Vert}(T) \to \{A, B, C, D\}$ such that $T$ has no fully labelled 3–simplices on its boundary $\partial T \cong S^3$. Let $\partial \mathrm{f_L}$ on $\partial T$ be of*

*Hopf invariant $d \neq 0$. Then $T$ must contain at least 9 fully labeled 3–simplices and for $d \geq 2$ this number is at least $3d+3$.*

In Section 3 we consider framed cobordisms $\Omega_k^{fr}(X)$ and relative framed cobordisms $\Omega_k^{fr}(X, \partial X)$. In particular, we prove the following extension of Pontryagin's theorem [16].

**Theorem 1.2.** *For all $k \geq 0$ and $n \geq 1$ we have*

$$\Omega_k^{fr}(D^{n+k+1}, S^{n+k}) \cong \pi_{n+k+1}(D^{n+1}, S^n) \cong \pi_{n+k}(S^n) \cong \Omega_k^{fr}(S^{n+k})$$

In Section 4 we prove a simplicial extension of Theorem 1.2 that can be considered as a smooth version of a quantitative Sperner–type lemma.

**Definition 1.1.** Let $T_m$ and $T_n$ with $m \geq n$ be triangulations of spheres $S^m$ and $S^n$. Let $f : T_m \to T_n$ be a simplicial map. Let $s$ be an $n$–simplex of $T_n$ and $s'$ be a smaller $n$–simplex that lies in the the interior of $s$. Let $\Pi(f, s) := f_L^{-1}(s')$. Then $\Pi(f, s)$ is an $m$–dimensional submanifold in $S^m$ and by using orientations of $S^m$ and $S^n$ a natural orientation can be assigned to it. It is clear that under the simplicial homeomorphism $\Pi(f, s)$ does not depend on the choice of $s'$.

We observe that $\partial\Pi(f, s) = f_L^{-1}(\partial s')$ and $t$ is an interior $n$–simplex of $\Pi(f, s)$ if and only if $f(t) = s'$. Denote by $\mu(f, s)$ the number of internal $n$–simplices in $\Pi(f, s)$.

Let $a \in \pi_m(S^n)$. Denote by $\mathcal{F}_a$ the space of all simplicial maps $f : S^m \to S^n$ with $[f] = a$ in $\pi_m(S^n)$. Define

$$\mu(a) := \min_{f \in \mathcal{F}_a, s} \mu(f, s).$$

We obviously have $\mu(0) = 0$ and $\mu(-a) = \mu(a)$.

**Theorem 1.3.** *Let $T$ be a triangulation of $D^{n+k+1}$ and $L : \mathrm{Vert}(T) \to \{0, \ldots, n+1\}$ be a labeling of $T$ such that $T$ has no fully labelled $n$–simplices on its boundary. Suppose $[\partial \mathrm{f_L}] \neq 0$ in $\pi_{n+k}(S^n)$. Then $T$ must contain at least $\mu([\partial \mathrm{f_L}])$ fully labeled $(n+1)$–simplices.*

# 2 Hopf invariant and tetrahedral chains

The *Hopf invariant* of a smooth or simplicial map $f : S^3 \to S^2$ is the linking number

$$H(f) := \mathrm{link}(f^{-1}(x), f^{-1}(y)) \in \mathbb{Z}, \tag{2.1}$$

where $x \neq y \in S^2$ are generic points [3]. Actually, $f^{-1}(x)$ and $f^{-1}(y)$ are the disjoint inverse image circles or unions of circles.

The projection of the *Hopf fibration* $S^1 \hookrightarrow S^3 \to S^2$ is a map $h : S^3 \to S^2$ with Hopf invariant 1. The Hopf invariant classifies the homotopy classes of maps from $S^3$ to $S^2$, i.e. $H : \pi_3(S^2) \to \mathbb{Z}$ is an isomorphism.

We assume that $S^3$ and $S^2$ are triangulated and $f : S^3 \to S^2$ is a simplicial map. Let $s$ be a 2–simplex of $S^2$ with vertices $A, B$ and $C$. We have that $\Pi = \Pi(f, s)$ is a simplicial complex in $S^3$ (see Definition 1.1). Actually, $\Pi$ is the disjoint union of $k \geq 0$ solid tori, another words $\Pi$ consists of $k$ closed chains of 3–simplices with a labeling $L : \mathrm{Vert}(\Pi) \to \{A, B, C\}$. (Here without loss of generality we may assume that $s'$ has the same labels as $s$.)

We observe that the Hopf invariant of $\Pi$ is well defined by (2.1) and $H(\Pi) = H(f)$. Using this fact in [14] is considered a linear algorithm for computing the Hopf invariant.

Since the equality $\pi_3(S^2) = \mathbb{Z}$ allows us to identify integers with elements of the group $\pi_3(S^2)$, *in this section we write* $\mu(d)$ *bearing in mind that* $d$ *is an element of* $\pi_3(S^2)$.

**Lemma 2.1.** $\mu(1) = \mu(2) = 9$ *and* $\mu(d) \geq 3d + 3$ *for all* $d \geq 2$.

*Proof.* Madahar and Sarkaria [6] give the minimal simplicial map $h_1 : \tilde{S}^3_{12} \to S^2_4$ of Hopf invariant one (Hopf map) that has $\mu(h_1, s) = 9$, see [6, Fig. 2]. Madahar [5] gives the minimal simplicial map $h_2 : S^3_{12} \to S^2_4$ of the Hopf invariant two with $\mu(h_2, ABC) = 9$ [5, Fig. 3]. It is clear that $\mu(d) \geq 9$ whenever $d \neq 0$, then we have $\mu(1) = \mu(2) = 9$.

Suppose $H(\Pi) = d$. Let $P$ be a connected component of $\Pi$. Then $P$ is a triangulated solid torus in $S^3$ that is a closed oriented labeled tetrahedral chain. All vertices of $P$ lie on the boundary $\partial P$ and have labels $A$, $B$, and $C$. Moreover, all internal 2–simplices (triangles) are fully labeled, i.e. have all three labels $A, B, C$.

Take any internal triangle $T_1$ of $P$. This triangle is oriented and we assign the order of its vertices $v_1v_2v_3$ in the positive direction. In accordance with the orientation of the chain the next vertex $v_4$ is uniquely determined as well as $v_5$ and so on. Then we have a closed chain of vertices $v_1, v_2, ..., v_m$ which uniquely determines the triangulations of $\partial P$ and $P$. Now we have a closed chain of internal triangles

$$T_1 = v_1v_2v_3,\ T_2 = v_4v_5v_6, , ..., T_k = v_{m-2-j}v_{m-1-j}v_{m-j}, \quad k = \lfloor m/3 \rfloor,\ j = m - 3k,$$

that have no common vertices and are fully labeled.

Let $M := L(v_1)L(v_2)...L(v_m)$. Then $M$ is a sequence ("word") which contains only three letters $A, B, C$. Let

$$\deg(M) := p_* - n_*,$$

where $p_*$ (respectively, $n_*$) is the number of consecutive pairs $AB$ (respectively, $BA$) in $M' = ML(v_1)$. (For instance, $\deg(ABCABCABC) = 3$ and $\deg(ABCBACABC) = 1$.) Note that if instead of $AB$ we take $BC$ or $CA$, then we obtain the same number, see [8, Lemma 2.1].

It is easy to see that the *minimum* length of $M$ of degree $n$ is $3n$. We may assume that $L(T_1) = ABC$. Then $M$ has the maximum degree $k$ if $m = 3k$ and $M = ABCABC...ABC$. In this case every triangle $T_i$ has labels $ABC$.

Suppose $\Pi$ has only one connected component. Then $H(P) = d$, i.e. $\mathrm{link}(\gamma_A, \gamma_B) = d$, where $\gamma_A = f^{-1}(A)$ and $\gamma_B = f^{-1}(B)$ are curves on the torus $\partial P$. Observe that the linking number can be computed as the *rotation number* of curves $\gamma_A$ or $\gamma_B$ on $\partial P$ [19].

The labeling $L$ divides $\mathrm{Vert}(P)$ into three groups which of them contains at least $k$ vertices. Since $m - 3k \leq 2$, we have that at least one group, say $A$, contains exactly $k$ vertices. Hence we have a chain of vertices $A_1, ..., A_k$, where $f(A_i) = A$ and $A_i$ is a vertex of $T_i$. Note that the rotation angle from $A_i$ to $A_{i+1}$ is less than $2\pi$. Therefore, the sum of rotation angles of this chain is less than $2\pi k$ and the rotation number is at most $k-1$.

Finally, we have $k \geq d+1$ and $m \geq 3d+3$. (Moreover, we can have the equality only if $M = ABCABC...ABC$.) Obviously, this inequality gives the minimum for $\Pi$ that contains only one connected component. Thus, $\mu(d) \geq 3d+3$. □

**Remark 2.1.** It is not clear, is the lower bound $\mu(d) \geq 3d+3$ sharp for $d > 2$? Madahar [5] gives a simplicial map $h_d : S^3_{6d} \to S^2_4$ of Hopf invariant $d \geq 2$ with $\mu(h_d, ABC) = 6d-3$ [5, Fig. 4]. (Note that $\mu(h_d, ABD) = \mu(h_d, ACD) = \mu(h_d, BCD) = (2d-1)(3d-2)$, i.e. grows quadratically in $d$.) However, $\mu(h_d, ABC) > \mu(d)$ for $d > 3$. Indeed, if we take for even $d$ the connected sum of $d/2$ spheres $S^3_{12}$ with labeling $h_2$ and $(d-1)/2$ spheres $S^3_{12}$ and one $\tilde{S}^3_{12}$ with labeling $h_1$ for odd $d$, then we obtain the triangulation and labeling of $S^3$ with $\mu = 9\lceil d/2 \rceil$. Hence we have $\mu(d) \leq 9\lceil d/2 \rceil$.

**Remark 2.2.** Observe that Madahar's triangulation $S^3_{12}$ in [5] *is not geometric.* Indeed, in this case $H(\Pi(h_2, ABC)) = \mathrm{link}(h_2^{-1}(A), h_2^{-1}(B)) = 2$. However, for geometric triangulations $h_2^{-1}(A)$ and $h_2^{-1}(B)$ are triangles, therefore their linking number cannot be 2.

**Lemma 2.2.** *Let $T$ be a triangulation of $D^4$. Let $L : \mathrm{Vert}(T) \to \{A, B, C, D\}$ be a labeling such that $T$ has no fully labelled 3–simplices on the boundary $\partial T \cong S^3$. If the Hopf invariant of $\partial f_L$ on $\partial T$ is $d$, then $T$ must contain at least $\mu(d)$ fully labeled 3–simplices.*

*Proof.* This lemma is a particular case of Theorem 1.3. We have $d = [\partial f_L] \in \pi_3(S^2) = \mathbb{Z}$. Then there are at least $\mu(d)$ fully labeled 3–simplices. □

It is easy to see that Lemmas 2.1 and 2.2 yield Theorem 1.1.

# 3 Framed cobordisms and homotopy group of spheres

A *framing* of an $k$–dimensional smooth submanifold $M^k \hookrightarrow X^{n+k}$ is a smooth map which for any $x \in M$ assigns a a basis of the normal vectors to $M$ in $X$ at $x$:

$$v(x) = \{v_1(x), ..., v_n(x)\},$$

where vectors $\{v_i(x)\}$ form a basis of $T_x^{\perp}(M) \subset T_x(X)$.

A *framed cobordism* between framed $k$–manifolds $M^k$ and $N^k$ in $X^{n+k}$ is a $(k+1)$–dimensional submanifold $C^{k+1}$ of $X \times [0,1]$ such that

$$\partial C = C \cap (X \times [0,1]) = (M \times \{0\}) \cup (N \times \{1\}) \tag{3.1}$$

together with a framing on $C$ that restricts to the given framings on $M \times \{0\}$ and $N \times \{1\}$. This defines an equivalence relation on the set of framed $k$–manifolds in $X$. Let $\Omega_k^{fr}(X)$ denote the set of equivalence classes.

The main result concerning $\Omega_k^{fr}(X)$ is the theorem of Pontryagin [16]: *$\Omega_k^{fr}(X^{n+k})$ with $n \geq 1$ and $k \geq 0$ corresponds bijectively to the set $[X, S^n]$ of homotopy classes of maps $X \to S^n$.* In particular,

$$\Omega_k^{fr}(S^{n+k}) \cong \pi_{n+k}(S^n).$$

Let $f : X^{n+k} \to S^n$ be a smooth map and $y \in S^n$ be a regular image of $f$. Let $v = \{v_1, ..., v_n\}$ be a positively oriented basis for the tangent space $T_y S^n$. Note that for every $x \in f^{-1}(y)$, $f$ induces the isomorphism between $T_y S^n$ and $T_x^{\perp} f^{-1}(y)$. Then $v$ induces a framing of the submanifold $M = f^{-1}(y)$ in $X$. This submanifold together with a framing is called the *Pontryagin manifold associated to $f$ at $y$.* We denote it by $\Pi(f, y)$.

Actually, the Pontryagin theorem states that

1. Under the framed cobordism $\Pi(f, y)$ does not depend on the choice of $y \in S^n$.

2. Under the framed cobordism $\Pi(f, y)$ depends only on homotopy classes of $[f]$.

3. $\Pi : [X, S^n] \to \Omega_k^{fr}(X)$ is a bijection.

Let $A^{\ell+k}$ be a submanifold of $X^{m+k}$. It is not hard to define *relative framed cobordisms* and the set of equivalence classes $\Omega_k^{fr}(X, A)$.

Let us describe the case $A = \partial X$, $\dim X = n + k + 1$, in more details. Let $M^k$ be a submanifolds of $X \setminus \partial X$ with a framing $\{v_0(x), v_1(x), ..., v_n(x)\}$. Let $N^k$ be a submanifolds of $\partial X$ with a framing $\{u_1(x), ..., u_n(x)\}$. We say that $(M, N)$ is a *framed relative pair* if there are submanifold $W$ in $X$ and $n$–framing $\omega = \{w_1(x), ..., w_n(x)\}$ of $W$ such that $\partial W = M \sqcup N$, $\omega|_M = \{v_1, ..., v_n\}$ and $\omega|_N = \{u_1, ..., u_n\}$. Then the framed cobordisms of framed relative pairs define the set of equivalence classes $\Omega_k^{fr}(X, \partial X)$.

**Theorem 3.1.** *Let $X^{n+k+1}$ with $n \geq 1$ and $k \geq 0$ be a compact orientable smooth manifold with boundary $\partial X$. Then $\Omega_k^{fr}(X, \partial X)$ corresponds bijectively to the set $[(X, \partial X), (D^{n+1}, S^n)]$ of relative homotopy classes of maps $(X, \partial X)$ to $(D^{n+1}, \partial D^{n+1})$.*

*Proof.* The proof of Pontryagin's theorem is cogently described in many textbooks, for instance, Milnor's book [7], Hirsch's and Ranicki's books [2, 17]. Actually, this theorem can be proved by very similar arguments as the Pontryagin theorem.

Let $f : (X, \partial X) \to (D^{n+1}, S^n)$ be a smooth map, $y \in S^n$ be a regular value of $\partial f$, $z \in D^{n+1} \setminus S^n$ be a regular value of $f$, $v = \{v_1, ..., v_n\}$ be a positively oriented basis for the tangent space $T_y S^n$ and $v_0$ be a vector in $\mathbb{R}^n$ such that $\{v_0, v_1, ..., v_n\}$ is its basis. Let $\gamma$ be a smooth non-singular path in $D^{n+1}$ framed with $v$, connecting $z$ and $y$ such that the tangent vector to $\gamma$ at $z$ is $v_0$. Then $\Pi(f, y, z, \gamma)$ can be defined as a framed relative pair $(f^{-1}(z), f^{-1}(y))$ with $W = f^{-1}(\gamma)$.

To prove the theorem we can use the same steps 1, 2, 3 as above. It can be shown that $\Pi : [(X, \partial X), (D^{n+1}, S^n)] \to \Omega_k^{fr}(X, \partial X)$ is well–defined and is a bijection. In the next section we consider details of this construction for simplicial maps. □

*Proof of Theorem 1.2.* Pontryagin's theorem and Theorem 3.1 yield bijective correspondences $\Omega_k^{fr}(S^{n+k}) \cong \pi_{n+k}(S^n)$ and $\Omega_k^{fr}(D^{n+k+1}, S^{n+k}) \cong \pi_{n+k+1}(D^{n+1}, S^n)$. The well–known isomorphism $\pi_{n+k+1}(D^{n+1}, S^n) \cong \pi_{n+k}(S^n)$ follows from the long exact sequence of relative homotopy groups:

$$... \to 0 = \pi_{n+k+1}(D^{n+1}) \to \pi_{n+k+1}(D^{n+1}, S^n) \to \pi_{n+k}(S^n) \to \pi_{n+k}(D^{n+1}) = 0 \to ...$$

This completes the proof. □

# 4 Quantitative Sperner–type lemma

Theorem 1.2 can be considered as a smooth version of a quantitative Sperner–type lemma. In this section we consider the bijective correspondence $\Omega_k^{fr}(D^{n+k+1}, S^{n+k}) \cong \Omega_k^{fr}(S^{n+k})$ for labelings (simplicial maps).

Let $T$ be a triangulation of a smooth manifold $X^{n+k}$. An *S–framing* of a $k$–dimensional submanifold $M^k \hookrightarrow X$ is a simplicial embedding $h : P \to T$, where $P \cong M \times D^n$ with $\text{Vert}(P) \subset \partial P$, and a labelling $L : \text{Vert}(P) \to \{1, ..., n+1\}$ such that (i) an $n$–simplex of $P$ is internal iff it is fully labeled, (ii) $M$ lies in the interior of $h(P)$ and (iii) $h^{-1}(M) \cong M$.

An *S–framed cobordism* between two $S$–framed manifolds $M^k$ and $N^k$ can be defined by the same way as the framed cobordism in (3.1). If between $M$ and $N$ there is an $S$–framed cobordism then we write $[M] = [N]$. Let $\Omega_k^{Sfr}(X)$ denote the set of equivalence classes under $S$–framed cobordisms.

Let $f : T \to Y$ be a simplicial map, where $Y$ is a triangulation of $S^n$ . For any simplex $s$ in $Y$ can be defined a simplicial complex $\Pi = \Pi(f, s)$ in $X$, see Definition 1.1. Let $s' \subset s$ be an $n$–simplex with vertices $v_1, ..., v_{n+1}$. If $\Pi$ is not empty, then it is an $(n+k)$–submanifold of $X$, all vertices of $\Pi$ lie on its boundary and $f : \text{Vert}(\Pi) \to \{v_1, ..., v_{n+1}\}$. Moreover, if $y \in \text{int}(s')$ then $M = f^{-1}(y)$ is a $k$–dimensional submanifold of $\Pi \subset X$. (*Here* $\text{int}(S)$ *denote the interior of a set* $S$.) Thus $\Pi$ is an $S$–framing of $M$.

There is a natural framing of $M$. Let $u = \{u_1, ..., u_n\}$, where $u_i$ is a vector $yv_i$. Then $u$ induces a framing of $M$ in $X$. Hence we have a correspondence between $\Pi(f, s)$ and $\Pi(f, y)$. It is not hard to see that this correspondence yield a bijection.

**Lemma 4.1.** $\Omega_k^{Sfr}(X) \cong \Omega_k^{fr}(X)$.

We observe that relative $S$–framining, relative $S$–framed cobordisms and a correspondence between relative $S$–framed and relative framed manifolds can be defined by a similar way. It can be shown that

$$\Omega_k^{Sfr}(X, \partial X) \cong \Omega_k^{fr}(X, \partial X).$$

Let us take a closer look at the bijection

$$\Omega_k^{Sfr}(D^{n+k+1}, S^{n+k}) \cong \Omega_k^{Sfr}(S^{n+k}) \cong \pi_{n+k}(S^n).$$

Let $T$ be a triangulation of $D^{n+k+1}$ and $L : \mathrm{Vert}(T) \to \{0, \dots, n+1\}$ be a labeling of $T$ such that $T$ has no fully labelled $n$–simplices on the boundary $\partial T \cong S^{n+k}$. Then we have simplicial maps:

$$f_L : T \cong D^{n+k+1} \to \Delta^{n+1} \cong D^{n+1}, \qquad \partial \mathrm{f_L} : \partial T \cong S^{n+k} \to \partial \Delta^{n+1} \cong S^n,$$

where $\Delta = \Delta^{n+1}$ denote the $(n+1)$–simplex with vertices $\{v_0, v_1, ..., v_{n+1}\}$. Hence the homotopy class $[\partial \mathrm{f_L}] \in \pi_{n+k}(S^n)$.

Let $s_0$ denote the $n$–simplex of $\Delta$ with vertices $\{v_1, ..., v_{n+1}\}$. Define

$$M_0 := f_L^{-1}(z),\ z \in \mathrm{int}(\Delta'), \quad N_0 := \partial \mathrm{f_L}^{-1}(y),\ y \in \mathrm{int}(s_0'), \quad W_0 := f_L^{-1}([z, y]).$$

**Lemma 4.2.** *We have that $(M_0, N_0)$ is an $S$–framed relative pair in $(D^{n+k+1}, S^{n+k})$ and $F([(M_0, N_0)]) = [N_0]$ defines a bijection*

$$F : \Omega_k^{Sfr}(D^{n+k+1}, S^{n+k}) \to \Omega_k^{Sfr}(S^{n+k}).$$

*Proof.* Since $z$ and $y$ are regular values of $f_L$ and $\partial \mathrm{f_L}$, we have that $M_0$ and $N_0$ are manifolds of $k$ dimensions with a cobordism $W_0$. In fact, $\Pi(f_L, \Delta)$ and $\Pi(\partial \mathrm{f_L}, s_0)$ define $S$–framings of $M_0$ and $N_0$. □

**Lemma 4.3.** *Let $C$ be a connected component of $W_0$ such that $N_C := \partial C \cap N_0 \neq \emptyset$. Then $\Pi(f_L, s_0)$ induces an $S$–framing of $M_C := \partial C \cap M_0$ in $S^{n+k}$ and $[M_C] = [N_C]$ in $\Omega_k^{Sfr}(S^n)$.*

*Proof.* Note that $\partial C = M_C \cup N_C$. Actually, $C$ is a cobordism between $M_C$ and $N_C$ in $D^{n+k+1}$. We obviously have that if $M_C$ is empty then $N_C$ is null–cobordant, i.e. $[N_C] = 0$ in $\Omega_k^{Sfr}(S^n)$.

Let $\Gamma$ be the closure of $f_L^{-1}(\mathrm{int}(\Delta))$ and $K_C := C \cap \Gamma \subset \Pi(f_L, s_0)$. Note that $\Pi(f_L, s_0)$ induces an $S$–framing of $K_C$ with $(n+1)$–labels. Let $t := [z, y)$ in $\Delta$ and $C_t := f_L^{-1}(t)$. Since $f_L$ is linear on $C_t$ we have $C_t \cong M_0 \times [0, 1)$. That induces an $S$–framing of $M_C$ with $(n+1)$–labels.

The last of the proof to show that this $S$–framing of $M_C$ is in $S^{n+k}$. We have that $S$–framing of $N_C$ is in $S^{n+k}$. It can be proved that using shelling along $C$ of fully labeled $n$-ssimplices we can contract $M_C$ to $N_C$ such that at each step the boundary lies in $S^{n+k}$. That completes the proof. □

*Proof of Theorem 1.3.* Lemma 4.1 and Pontryagin theorem yield

$$\Omega_k^{Sfr}(S^n) \cong \Omega_k^{Sr}(S^n) \cong \pi_{n+k}(S^n).$$

Let $[\partial \mathrm{f_L}] = a$ in $\pi_{n+k}(S^n)$. Then $[N_0] = a$ in $\Omega_k^{Sfr}(S^n)$. If $\{C_1, ..., C_k\}$ are connected components of $W_0$ then Lemma 4.3 yields the equality

$$[M_{C_1}] + ... + [M_{C_k}] = [N_{C_1}] + ... + [N_{C_k}] = [N_0] = a.$$

Therefore, $\Pi(f_L, \Delta)$ contains at least $\mu(a)$ $n$–simplices with labels $1, ..., n+1$. The same we have for every $(n+1)$-labeling. Since $\Pi(f_L, \Delta)$ contains all fully labeled $(n+1)$–simplices, it is not hard to see that this number is not less than $\mu(a)$. □

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