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Linear Stability Analysis with Solution Patterns due to Varying Thermal Diffusivity for a Convective Flow in a Porous Medium

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Abstract

Here we investigate the effect of the vertical rate of change in thermal diffusivity due to a hydro-thermal convective flow in a horizontal porous medium. The continuity equation, the heat equation and the momentum-Darcy equation constitute the governing system for the flow in a porous medium. Assuming a vertically varying basic state, we derive the linear system and from this linear system, we compute the critical Rayleigh and wave numbers. Using fourth-order Runge-Kutta and shooting methods, we obtain the marginal stability curves and linear solutions to analyze the solution pattern for different diffusivity parameters.

Keywords: Thermal Diffusivity; Marginal Stability; Porous Media; Critical Rayleigh Number; Hydro-thermal; Convective Flow

MSC 2010 No.: 35Q35, 76E06, 76M25, 76S99, 80A20

1. Introduction

Study of hydrodynamic stability was carried by many scientists including Helmholtz, Kelvin, Rayleigh and Reynolds in the nineteenth century because of its practical importance. Various case studies on hydrodynamic stability has been presented by Chandrasekhar (1981). The Landau equation has been derived for various cases and the dependence of Landau constant on stability has been studied by Drazin and Reid (2004). Stability analysis of natural convection in porous cavities was carried out by Alves et al. (2001) by using integral transforms. Riahi (1989) carried out nonlinear stability analysis in a porous layer with permeable boundaries. The case of a continuous finite bandwidth of convection modes in a horizontal layer was analyzed by Riahi (1996).

A porous medium is characterized by its porosity and an important property of the medium is permeability. Darcy's law for porous media, which is analogous to the Navier-Stokes equation, is taken into account in the momentum equation,. In natural porous media the distribution of voids or pores with respect to shape and size is not regular. Heat transfer through a porous medium is a very common phenomenon. Fluid tries to expand when heated causing a density inversion to follow, if the heating is strong, a circulatory motion occurs, which is called convection. This convection phenomenon in fluid layer is well-studied and occurs in various natural settings and in many industrial applications.

Theoretical and experimental treatments and research on convection in porous media have been studied by various authors through many research articles (Nield and Bejan 2017, Vafai 2005). Rubin (1981) investigated the onset of thermohaline convection in a porous medium with varying hydraulic resistivity. Vafai (1984) investigated wall effects due to variable porosity. Variable porosity and thermal dispersion effects due to natural convection in an inclined porous cavity were studied by Hsiao (1998). Kim and Vafai (1989) obtained numerical solution based on the similarity transformations and an analytical solutions using the method of matched asymptotic expansions. Riahi (1989) investigated nonlinear convection in a porous layer with permeable boundaries. Modal package convection for a porous medium with boundary imperfections was analyzed by Riahi (1996). The effect of variable permeability in porous media was analyzed analytically and numerically by Rees and Pop (2000). Hassanien and Omer (2005) studied the effect of variable permeability.

A model for an aquifer which can be treated as porous media was presented by Fowler (1997). Rubin (1982) used Galerkin method to analyze the effects of hydraulic resistivity and thermal diffusivity on stability in a nonhomogeneous aquifer. Bhatta and Riahi (2017) investigated weakly nonlinear hydro-thermal two-dimensional convective flow in a horizontal aquifer by treating it as porous medium. They investigate the effects of small linear vertical variations in the permeability and thermal conductivity. Convective flows, in a horizontal dendritic porous layer (also known as mushy layer) during alloy solidification, are known to produce undesirable effects in the final form of the alloy. Hydro-thermal convective solutions for an aquifer system heated from below was studied by Bhatta (2013). Study on oscillatory modes of nonlinear compositional convection in mushy layers was carried out by Riahi (2009). Muddamallappa et al. (2009) used a modified mushy layer model based on the standard near eutectic approximation. They used linear stability analysis and calculated critical Rayleigh number for the cases of both constant and variable permeability. However, the issues of nonlinear effects and transition effects on the chimney formation did not feature in their investigation. Linear stability analysis for convective flow in a mushy layer with non-uniform magnetic field were carried out Muddamallappa et al. (2010). An active (with varying permeability) mushy layer with permeable mush-liquid interface was analyzed by Bhatta et al. (2010) using weakly nonlinear procedure.

In this work, we analyze the linear stability and solution patterns of the governing system due to linear variation in thermal diffusivity in the vertical direction for a convective flow in a horizontal porous medium.

2. Mathematical Formulation

Equations Governing the System

The nondimensional system can be expressed as (Nield & Bejan, Riahi, Rubin, Fowler)

$$\nabla \cdot \overrightarrow{q} = 0, \tag{1}$$

$$\overrightarrow{q} - R\theta\,\hat{k} + \nabla p = 0, \tag{2}$$

$$\frac{\partial \theta}{\partial t} + \overrightarrow{q} \cdot \nabla\theta = \nabla \cdot (\beta \nabla \theta)\,. \tag{3}$$

Here $\overrightarrow{q}$, θ, R, p, β, t and $\hat{k}$, respectively, represent the velocity, temperature, Rayleigh number, pressure, thermal diffusivity, time and unit vector in the upward vertical direction. The governing system consists of continuity equation for conservation of mass, Darcy equation for conservation of momentum and heat equation for conservation of energy. For the steady state solution, we have $\frac{\partial \theta}{\partial t} = 0$ and all dependent variables ($\overrightarrow{q}$, θ, p) are functions of x, y, z.

We assume a linear variation in diffusivity β in the vertical z-direction as

$$\beta = 1 + \beta_0 z. \tag{4}$$

Here β_0 is a constant. Here β_0 represents the vertrical rate of change in thermal diffusivity.

The boundary conditions are

$$\theta = 1, q_z = 0 \text{ at } z = 0, \tag{5}$$

$$\theta = 0, q_z = 0 \text{ at } z = 1, \tag{6}$$

where q_z is the vertical z-component of $\overrightarrow{q}$.

3. Solution Procedure

We perturb the system given by (1) - (3) as follows

$$\begin{aligned} \theta(x, y, z) &= \theta_b(z) + \epsilon\Theta(x, y, z), \\ \overrightarrow{q}(x, y, z) &= \overrightarrow{q}_b + \epsilon\overrightarrow{Q}(x, y, z), \\ p(x, y, z) &= p_b(z) + \epsilon P(x, y, z), \end{aligned} \tag{7}$$

where θ_b, $\overrightarrow{q}_b$, p_b are solutions to the basic steady state system (system with no flow) and Θ, $\overrightarrow{Q}$, P are perturbed solutions. The perturbation parameter is given by $\epsilon = (R - R_c)/R_1 > 0$ i.e., $R = R + \epsilon R_1$. Here R_c is the critical Rayleigh number and R_1 is the nonlinear contribution to R beyond the critical number.

3.1. Basic State System

The basic state system is the system with no flow. Using the equations given by (7) in (1) through (3) and by comparing the coefficients of ϵ^0, we have

$$\beta \frac{d^2\theta_b}{dz^2} + \left(\frac{d\beta}{dz}\right)\left(\frac{d\theta_b}{dz}\right) = 0,$$
$$\frac{dp_b}{dz} - R_c\theta_b = 0, \tag{8}$$
$$\overrightarrow{q}_b = \overrightarrow{0},$$

with boundary conditions $\theta_b = 1,\, p_b = 0$ at $z = 0$ and $\theta_b = 0$ at $z = 1$.

Basic State Solutions

Solving the basic state system (8), we obtain the solutions as

$$\theta_b(z) = 1 - \frac{\ln(1+\beta_0 z)}{\ln(1+\beta_0)}, \tag{9}$$

$$p_b(z) = R_c\left[z - \frac{1+\beta_0 z}{\beta_0 \ln(1+\beta_0)}\{\ln(1+\beta_0 z) - 1\}\right], \tag{10}$$

$$\overrightarrow{u}_b = \overrightarrow{0}. \tag{11}$$

3.2. Perturbed System

The perturbed system can be expressed as

$$\nabla \cdot \overrightarrow{Q} = 0, \tag{12}$$

$$\overrightarrow{Q} - R_c\Theta\hat{k} - R_1\theta_b\hat{k} + \nabla P = \epsilon R_1\Theta\hat{k}, \tag{13}$$

$$\nabla^2\Theta - \overrightarrow{Q}\cdot\nabla\theta_b = \epsilon\overrightarrow{Q}\cdot\nabla\Theta, \tag{14}$$

with the boundary conditions

$$\Theta = Q_w = 0 \qquad \text{at} \quad z = 0,\ 1, \tag{15}$$

where Q_w is the vertical z-component of $\overrightarrow{Q}$.

3.2.1. *Elimination of Pressure*

To eliminate the pressure from Equation (13), we use poloidal ($\mathscr{P}$) and toroidal ($\mathscr{T}$) representations of $\overrightarrow{Q}$ (since $\nabla\cdot\overrightarrow{Q} = 0$, Chandrasekhar) which is given by

$$\overrightarrow{Q} = (Q_u,\, Q_v,\, Q_w) = \overrightarrow{\mathscr{P}}\,\phi + \overrightarrow{\mathscr{T}}\,\psi, \tag{16}$$

with

$$\overrightarrow{\mathscr{P}} = \nabla\times\nabla\times\hat{k}, \qquad \overrightarrow{\mathscr{T}} = \nabla\times\hat{k}, \tag{17}$$

Thus, we have

$$Q_w = -\triangle_2\phi, \quad \text{where} \quad \triangle_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{18}$$

After elimination of pressure by taking the vertical component of the double curl of the equation (13), the perturbed system becomes:

$$\nabla^2 Q_w - R_c\left(\triangle_2\Theta\right) = \epsilon R_1\left(\triangle_2\Theta\right), \tag{19}$$

$$\beta\nabla^2\Theta + \frac{d\beta}{dz}\left(D\Theta\right) - Q_w\,\frac{d\theta_b}{dz} = \epsilon\overrightarrow{Q}\cdot\nabla\Theta, \tag{20}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad D = \frac{\partial}{\partial z}.$$

The boundary conditions are

$$\Theta = Q_w = 0 \qquad \text{at} \quad z = 0,\ 1. \tag{21}$$

Now, expressing $\Theta(x,y,z)$ and $Q_w(x,y,z)$ as

$$\Theta(x,y,z) = \Theta_0(x,y,z) + \epsilon\Theta_1(x,y,z) + \epsilon^2\Theta_2(x,y,z) + \dots\ , \tag{22}$$

$$Q_w(x,y,z) = Q_{w0}(x,y,z) + \epsilon Q_{w1}(x,y,z) + \epsilon^2 Q_{w2}(x,y,z) + \dots\ , \tag{23}$$

and using the equations (19) and (20), we obtain different order systems, namely, linear, first order, etc.

4. Linear System

The linear system can be expressed by the following equations:

$$\nabla^2 Q_{w0} - R_c\left(\triangle_2\Theta_0\right) = 0, \tag{24}$$

$$\beta\nabla^2\Theta_0 + \frac{d\beta}{dz}\left(D\Theta_0\right) - Q_{w0}\,\frac{d\theta_b}{dz} = 0. \tag{25}$$

The boundary conditions are

$$\Theta_0 = Q_{w0} = 0 \qquad \text{at} \quad z = 0,\ 1. \tag{26}$$

4.1. 2-D Solutions: Normal Mode Approach

Using normal mode approach, we consider the following form for linear solutions for the two dimensional case,

$$h_0\left(x,z\right) = \hat{h}_0(z)\,e^{i\alpha x}, \tag{27}$$

which yields

$$\nabla^2 h_0 = e^{i\alpha x}\left(\frac{d^2}{dz^2} - \alpha^2\right)\hat{h}_0(z). \tag{28}$$

Using (27) in the linear system (24)-(25), we have

$$\left(\frac{d^2}{dz^2} - \alpha^2\right)\hat{Q}_{w0} + \alpha^2 R_c\hat{\Theta}_0 = 0, \tag{29}$$

$$\beta\left(\frac{d^2}{dz^2} - \alpha^2\right)\hat{\Theta}_0 + \beta_0\frac{d\hat{\Theta}_0}{dz} - \hat{Q}_{w0}\frac{d\theta_b}{dz} = 0. \tag{30}$$

The boundary conditions are

$$\hat{\Theta}_0 = \hat{Q}_{w0} = 0 \qquad \text{at} \quad z = 0,\ 1. \tag{31}$$

5. Numerical Results

We use fourth order Runge Kutta method together with shooing method to solve the ODE system. The critical wave and Rayleigh numbers obtained numerically for different values of β_0 are shown in Table1.

Table 1. Critical wave and Rayleigh numbers

β_0	-0.8	-0.3	0.0	0.7	1.2
α_c	3.25534	3.14729	3.14159	3.15431	3.16939
R_c	21.783967	33.382175	39.478427	52.693903	61.652823

Figure 1 presents the marginal stability curves at different values of β_0.

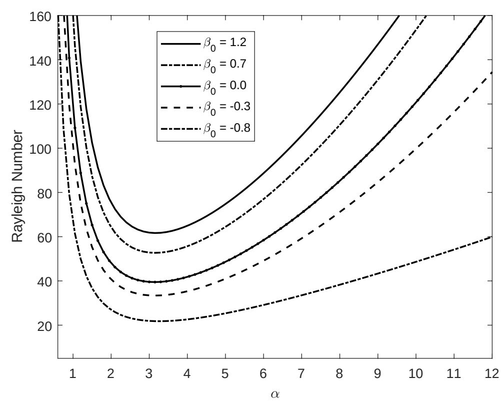

Figure 1. Marginal Stability Curves

From the marginal stability curves, it is observed that stability region is effected by the values of the vertrical rate of change in thermal diffusivity, β_0. As β_0 increases, unstable region decreases. Higher diffusivity parameter yields higher stability.

For different values of γ_0, the linear solutions for the vertical component of the velocity in the z-direction,$\hat{Q}_{w0}(z)$ are displayed in the Figure 2.

To investigate the behavior of the solution in the middle of the layer, we present two tables, Table-2 and Table-3. The linear solutions for the vertical component of the velocity, $\hat{Q}_{w0}(z)$, for $\beta_0 = 0.0,\ 1.2$ and -0.8 somewhere in the middle of layer, i.e., from $z = 0.43$ to $z = 0.62$ are presented in Table 2.

From Table 2, it is noticed that for $\beta_0 = 0.0$, the velocity component has a maximum value of 0.3183099 at $z = 0.5$, i.e., at the middle of the layer, and velocity component is symmetric about the middle of the layer. Results are not symmetric for other values of z. For $\beta_0 = 1.2$, the maximum value of the velocity component, which is 0.2719879, is attained at $z = 0.45$, i.e., it shifts downward

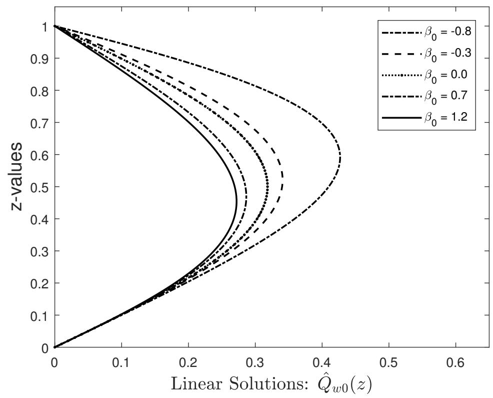

Figure 2. Linear Solutions $\hat{Q}_{w0}(z)$ for various γ_0

in the layer. For $\beta_0 = -0.8$, the maximum value of the velocity component, which is 0.4269370, is attained at $z = 0.59$, i.e., it moves upward. For different values of β_0,

Table 2. Linear Solutions $\hat{Q}_{w0}(z)$ for various β_0

z	$\beta_0 = 0.0$	$\beta_0 = 1.2$	$\beta_0 = -0.8$
0.62	0.2959570	0.2377564	0.4250248
0.61	0.2994916	0.2415681	0.4261251
0.60	0.3027306	0.2451810	0.4267589
0.59	0.3056709	0.2485894	0.4269370
0.58	0.3083095	0.2517882	0.4266706
0.57	0.3106439	0.2547718	0.4259712
0.56	0.3126717	0.2575350	0.4248501
0.55	0.3143909	0.2600726	0.4233188
0.54	0.3157999	0.2623796	0.4213891
0.53	0.3168972	0.2644509	0.4190725
0.52	0.3176817	0.2662818	0.4163808
0.51	0.3181528	0.2678675	0.4133255
0.50	0.3183099	0.2692034	0.4099183
0.49	0.3181528	0.2702851	0.4061708
0.48	0.3176817	0.2711084	0.4020943
0.47	0.3168972	0.2716692	0.3977003
0.46	0.3157999	0.2719636	0.3930001
0.45	0.3143909	0.2719879	0.3880048
0.44	0.3126717	0.2717387	0.3827254
0.43	0.3106439	0.2712128	0.3771726

Figure 3 shows the linear solutions for the temperature in the z-direction, $\hat{\Theta}_0(z)$.

Table 3 presents the linear solutions for the temperature, $\hat{\Theta}_0(z)$, for $\beta_0 = 0.0,\ 1.2$ and -0.8 somewhere in the middle of layer, i.e., from $z = 0.38$ to $z = 0.72$. From Table 3, it is noticed that for

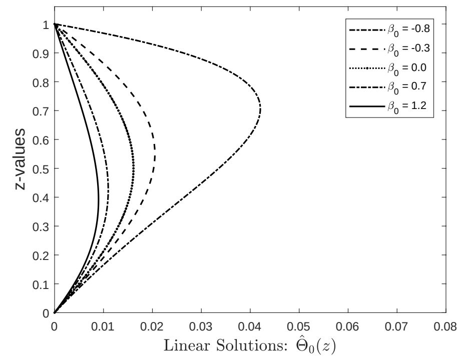

Figure 3. Linear Solutions $\hat{\Theta}_0(z)$ for various β_0

Table 3. Linear Solutions $\hat{\Theta}_0(z)$ for various β_0

z	$\beta_0 = 0.0$	$\beta_0 = 1.2$	$\beta_0 = -0.8$
0.72	0.0124251	0.0052912	0.0419984
0.71	0.0127419	0.0054713	0.0420511
0.70	0.0130460	0.0056493	0.0420435
0.69	0.0133373	0.0058252	0.0419788
0.68	0.0136154	0.0059987	0.0418604
0.67	0.0138801	0.0061696	0.0416913
0.66	0.0141311	0.0063377	0.0414745
0.65	0.0143682	0.0065029	0.0412130
0.64	0.0145910	0.0066649	0.0409095
0.63	0.0147995	0.0068235	0.0405667
0.62	0.0149934	0.0069784	0.0401871
0.61	0.0151724	0.0071296	0.0397733
0.60	0.0153365	0.0072767	0.0393275
0.59	0.0154855	0.0074194	0.0388521
0.58	0.0156192	0.0075577	0.0383491
0.57	0.0157374	0.0076912	0.0378206
0.56	0.0158402	0.0078197	0.0372686
0.55	0.0159272	0.0079429	0.0366949
0.54	0.0159986	0.0080606	0.0361013
0.53	0.0160542	0.0081725	0.0354896
0.52	0.0160940	0.0082784	0.0348613
0.51	0.0161178	0.0083780	0.0342180
0.50	0.0161258	0.0084710	0.0335611
0.49	0.0161178	0.0085573	0.0328920
0.48	0.0160940	0.0086364	0.0322120
0.47	0.0160542	0.0087082	0.0315223
0.46	0.0159986	0.0087723	0.0308242
0.45	0.0159272	0.0088285	0.0301187
0.44	0.0158402	0.0088766	0.0294069
0.43	0.0157374	0.0089162	0.0286898
0.42	0.0156192	0.0089471	0.0279683
0.41	0.0154855	0.0089689	0.0272433
0.40	0.0153365	0.0089816	0.0265156
0.39	0.0151724	0.0089846	0.0257859
0.38	0.0149934	0.0089779	0.0250551

$\beta_0 = 0.0$, the temperature has a maximum value of 0.0161258 at $z = 0.5$, i.e., at the middle of the layer, and velocity component is symmetric about the middle of the layer. Results are not symmetric for other values of z. For $\beta_0 = 1.2$, the maximum value of the temperature, which is 0.0089846, is attained at $z = 0.39$, i.e., it shifts downward in the layer. For $\beta_0 = -0.8$, the maximum value of the temperature, which is 0.0420511, is attained at $z = 0.71$, i.e., it moves upward.

Both velocity and temperature display similar behavior around the middle of the layer. These are symmetric about $z = 0.5$ for $\beta_0 = 0.0$, and the maximum values shift downward for positive values of β_0 and upward for negative values of β_0.

6. Conclusion

Here we study the effect of the vertical rate of change in diffusivity due to a hydro-thermal convective flow in a horizontal porous medium. The critical Rayleigh and wave numbers are computed from the linear system. Marginal stability curves are obtained for different values of the vertical rate of change in thermal diffusivity. The vertical variation in the dependent variables, namely, the vertical velocity component and the temperature are computed numerically using Runge-Kutta method in combination of shooting method. Results for different diffusivity parameter are presented in tabular and graphical forms. Results indicate stabilizing effect on the dependent variables for the case of positive rate of increase of diffusivity, whereas destabilizing effect for the case of negative rate of increase of diffusivity. In addition, flow is not symmetric with respect to middle of the layer for non-zero rate of resistivity.

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