2017

Using Technology to Determine Factorability or Non-factorability of Quadratic Algebraic Trinomials

John E. T. Bernard
The University of Texas Rio Grande Valley

Olga Ramirez
The University of Texas Rio Grande Valley

Cristina Villalobos
The University of Texas Rio Grande Valley

Using Technology to Determine Factorability or Non-factorability of Quadratic Algebraic Trinomials

John Bernard
University of Texas Rio Grande Valley
United States
john.bernard@utrgv.edu

Olga Ramirez
University of Texas Rio Grande Valley
United States
olga.ramirez@utrgv.edu

Cristina Villalobos
University of Texas Rio Grande Valley
United States
cristina.villalobos@utrgv.edu

Abstract: This paper is aimed for mathematics educators who teach algebra, more specifically, the factoring of quadratic algebraic expressions, and who want to enhance student learning of this topic using technology in conjunction with the Middle Term Splitting Method (Donnell, 2010; MTSM 2016a; MTSM 2016b). We will use technology-based algebra and geometry connections to help determine factorability or non-factorability of quadratic algebraic trinomials over the integers, over the real numbers, and over the complex numbers, both with clarity, certainty and with understanding by using two equations, one derived from the coefficients of the outer terms and the other from the middle term of the quadratic algebraic trinomial. Overall, the efficiency of determining factorability or non-factorability of quadratic algebraic trinomials, using Cabri Geometry II Plus (or a graphing calculator) together with the MTSM, makes this topic comprehensible in a technology-rich environment.

Keywords: Algebra, factoring quadratic algebraic trinomials, developmental mathematics, mathematics education, algebra and technology

Introduction

The National Council of Teachers of Mathematics Principles and Standards advocate that technology is not only essential in teaching and learning mathematics, but that it also influences the mathematics that is taught and enhances students' learning (NCTM, 2000). In compliance with this recommendation, the authors utilize technology to promote student learning and understanding. The method for factoring quadratic algebraic trinomials $ax^2 + bx + c$ with $a \neq 0$ that will be described in subsequent examples provides enlightened technological enhanced opportunities to make connections with algebra and geometry along with the Middle Term Splitting Method (MTSM), a systematic factoring approach for quadratic algebraic trinomials. In particular, we will show such connections with technology when solving a system of equations that play a critical role in determining factorability or non-factorability of quadratic algebraic trinomials. We will demonstrate this connection with examples of trinomials. The authors have used this method in teaching algebra to students in developmental college courses and have noted the ease by which students learn the method.

Factorability of Quadratic Algebraic Trinomials over the Integers Using Technology and the MTSM

In this section we consider the problem of factoring the quadratic algebraic trinomial $ax^2 + bx + c = 20x^2 + 104x - 33$ to show how technology can be utilized to show the factorability of quadratic algebraic trinomials over the integers. To do this, we need to solve the following set of equations for integers M and N:

let M* N = (20) (-33), where 20 = *a* and where -33 = *c,* and

let M + N = 104, where 104 = *b*.

Then the system of equations that we will solve is as follows:

$$M * N = (20)(-33) = -660 \quad \text{and} \quad M + N = 104.$$

One can think of N as representing the independent variable "X" and M representing the dependent variable "Y" in the XY-real number coordinate plane so that the system of equations becomes:

$$Y * X = (20)(-33) = -660 \qquad \text{and} \qquad Y + X = 104.$$

Solving for Y, we obtain the following two functions of X:

$$Y = -660/X \qquad \text{and} \qquad Y = 104 - X.$$

Using Cabri Geometry II Plus (Laborder, 2007) as a software tool to graph both functions (a graphing calculator can also be used), we obtain the following graphs (see **Figure** 1) where the graph of Y*X= -660 [that is; Y = -660/X] is the hyperbola and the graph of X+Y = 104 [that is; Y = 104- X] is the line. In **Figure** 1, one notes that the hyperbola and the line intersect at two points, namely (-6, 110) and (110, -6). Therefore, we have found real solutions in the XY-real number coordinate plane to the set of equations. In particular, we have obtained *integer* solutions for the set of equations. Since the solutions are of an integral form, the set of equations and its condition that X and Y be integers have been satisfied. For the first point (-6, 110), we observe that X = N = -6 and Y = M = 110. Similarly, for the second point, X = N = 110 and Y = M = -6. Thus, we can select either pair of integer values for M and N and both pairs will allow us to factor the trinomial $20x^2 + 104x - 33$ into linear factors over the integers. Now that we have obtained the integer values for M and N we can continue the factorization process over the integers for the quadratic algebraic trinomial $20x^2 + 104x - 33$. In conclusion, given any quadratic algebraic trinomial $ax^2 + bx + c$ with $a \neq 0$, notably when the graphs of the hyperbola M*N = a*c and the line M + N = b intersect at points where both the coordinates are integers, the quadratic algebraic expression is factorable over the integers.

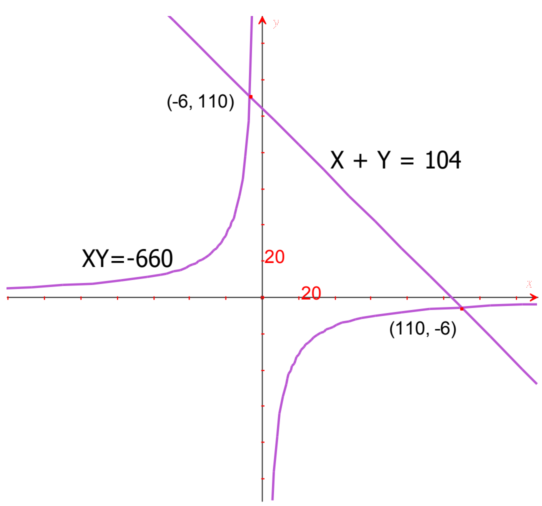

Figure 1: Example of Factorability over the integers

Now, using that M = 110 and N = -6, we show the factorization of $20x^2 + 104x - 33$, using the Middle Term Splitting Method (Donnell, 2010; MTSM 2016a; MTSM 2016b) as follows:

$$20x^2 + 104x - 33$$
$$= 20x^2 + (M+N)x - 33$$

$= 20x^2 + (110\text{-}6)x - 33$

$= 20x^2 + 110x \text{ -}6x - 33$

$= 5x\ (4x + 22) + \text{-}3(2x + 11)$

$= 10x\ (2x + 11) + \text{-}3(2x + 11)$

$= (2x + 11)\ (10x\ \text{-}3).$

Factorability of Quadratic Algebraic Trinomials over the Real Numbers Using Technology and the MTSM

This technological method can be used to determine when trinomials are not factorable over the integers. In this section, we will make the algebraic and geometric connection of non-factorability of trinomials over the integers using technology. Consider the quadratic algebraic trinomial $ax^2 + bx + c = 12x^2 + 15x - 20$. In this case, we wish to identify integers M and N that satisfy the following set of equations:

let M* N = (12) (-20), where 12 = *a* and where -20 = *c*, and

let M + N = 15, where 15 = *b*. .

Then the system of equations that we will solve is as follows:

M * N = (12)(-20) = -240 and M + N = 15.

Once again, we can think of N as representing the independent variable “X” and M representing the dependent variable “Y” in the XY-real number coordinate plane so that the system of equations becomes:

Y * X = (12)(-20) = -240 and Y + X = 15.

Solving for Y, we obtain the following two functions of X:

Y = -240/X and Y = 15 - X.

Letting Y denote the dependent variable M and X representing the independent variable N in the graphs (see **Figure** 2), we note that the line X+Y = 15 [that is; Y = 15 – X] and the hyperbola XY = -240 [that is; Y = -240/X] intersect at points with *non-integer* coordinates, namely at (⁻9.711914, 24.711914) and at (24.711914, ⁻9.711914) whose coordinates have been truncated to two decimal places on the graph below and notably are shown as (-9.72, 24.72) and (24.72, -9.72). In conclusion, given any quadratic algebraic trinomial $ax^2 + bx + c$ with $a \neq 0$, notably when the graphs of the hyperbola M*N = *a***c* and the line M + N = *b* intersect at points where both the coordinates are real numbers, the quadratic algebraic expression is factorable over the real numbers.

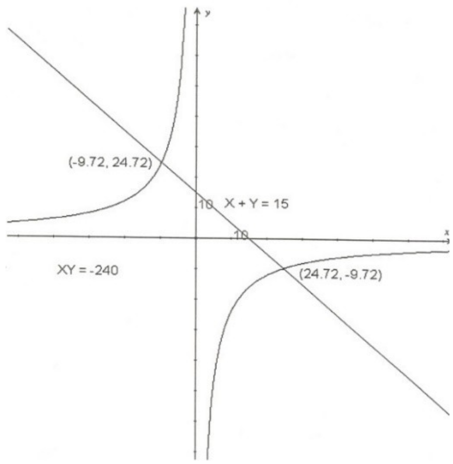

Figure 2: Example of Factorability over the real numbers

This analysis coincides with our initial observation that the two equations do not have integer solutions M and N. Thus, the trinomial $12x^2 + 15x - 20$ is *not factorable over the integers,* but it is *factorable over the real numbers*. In fact, relative to the latter, the Middle Term Splitting Method can be used here to factor the trinomial, with approximations, as follows, where M = -9.711914 and N =24.711914 we show the factorization of $12x^2 + 15x - 20$, using the Middle Term Splitting Method:

$$12x^2 + 15x - 20$$

$$= 12x^2 + (M+N)x - 20$$

$$\approx 12x^2 + ({}^{-}9.711914 + 24.711914)x - 20$$

$$\approx 12x^2 - 9.711914x + 24.711914x - 20$$

$$\approx (12x^2 - 9.711914x) + (24.711914x - 20)$$

$$\approx 12x(x - 0.809326) + 24.711914(x - 0.809326)$$

$$\approx (12x + 24.711914)\,(x - 0.809326).$$

Factorability of Quadratic Algebraic Trinomials over the Complex Numbers Using Technology and the MTSM

In this section we offer the following example to demonstrate the use of technology to show when a quadratic trinomial is not factorable over the real numbers, but will be factorable over the complex numbers. To exemplify, we begin to factor $ax^2 + bx + c = 12x^2 + 15x + 20$, perhaps at the request of it being done over the integers, which however is impossible as no integer solutions M and N can be found for satisfying the following system of equations:

let M* N = (12) (20), where 12 = a and where 20 = c and

let M + N = 15, where 15 = b. .

Then the system of equations that we will solve is as follows:

$$M * N = (12)(20) = 240 \quad \text{and } M + N = 15.$$

Once again, we can think of N as representing the independent variable "X" and M representing the dependent variable "Y" in the XY-real number coordinate plane so that the system of equations becomes:

Y * X = (12)(20) = 240 and Y + X = 15.

Solving for Y, we obtain the following two functions of X:

Y = 240/X and Y = 15 - X.

We easily notice that the graph (see **Figure** 3) of the hyperbola given by equation X*Y = 240 [that is; Y = 240/X] and the graph of the line given by X + Y = 15 [that is, Y = 15 – X] do not intersect, that is, there are no pair of integers M or N nor any pair of real numbers that satisfy the two equations above. Thus, the quadratic trinomial $12x^2 + 15x + 20$ *is not factorable over the integers nor over the real numbers but it is factorable over the complex numbers.* **Figure** 3 demonstrates our observation. In conclusion, given any quadratic algebraic trinomial $ax^2 + bx + c$ with $a \neq 0$, notably when the graphs of the hyperbola M*N = a*c and the line M + N = b do not intersect, the quadratic algebraic expression is not factorable over the real numbers, but it is factorable over the complex numbers.

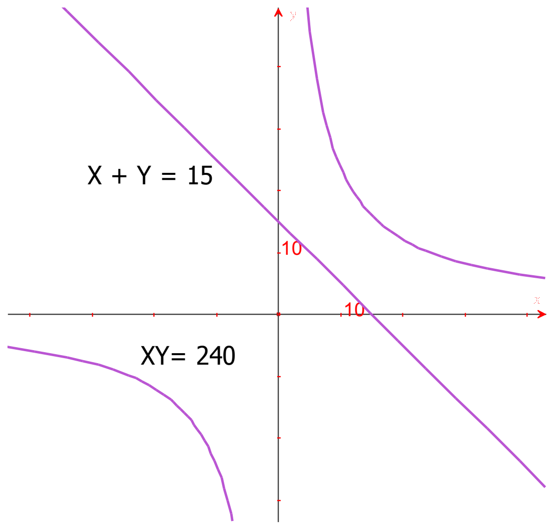

Figure 3: Example of factorability over the complex numbers

Now, given $12x^2 + 15x + 20$ and M*N = 240 and M + N =15, we find using the Quadratic Formula (Bittinger, 2012, page 786) that:

M = and N = .

In fact, relative to the latter, the Middle Term Splitting Method can be used here to factor the trinomial $12x^2 + 15x + 20$ where M = and N = . Proceeding to factor $12x^2 + 15x + 20$ over the complex numbers we obtain:

=

=

=

=

=

Conclusion

We conclude that the set of equations, M*N = $a*c$ and M + N = b for $ax^2 + bx + c$, involved in setting up the system of equations provide connections with algebra and geometry to graphically determine, with technology usage and connections to the MTSM, whether the quadratic algebraic trinomial will be factorable over the integers, over the real numbers that are not integers, or over the complex numbers that are not real numbers. This use of technology that provides insight and understanding to algebra and geometry connections is an excellent approach that simplifies notions about factorability and non-factorability of quadratic algebraic trinomials.

Acknowledgements

This work was supported by the Quality Enhancement Plan from September 2007 to May 2013 at the University of Texas Pan American, Edinburg, TX

Literature References

Bittinger M. L., Ellenbogen D. J., Johnson B. L. (2012). Elementary and Intermediate Algebra: Graphs and Models. 4th ed. Boston, MA: Pearson Education, Inc.

Donnell, W. A. (2010). Elementary theory of factoring trinomials with integer coefficients over the integers. *International Journal of Mathematical Education in Science and Technology*, *41*(8), 1114-1121.

MTSM (2016a) (Source retrieved September 27, 2016) from http://www.regentsprep.org/Regents/math/algtrig/ATV1/revFactorGrouping.htm

MTSM (2016b) (Source retrieved September 27, 2016) from http://www.ask-math.com/quadratic-factorization-using-splitting-of-middle-term.html

NCTM (2000). National Council of Teachers of Mathematics. Principles and Standards for School Mathematics: Standards for Grades 9-12 (Algebra). Reston, VA: NCTM Inc.

Laborder Jean-Marie, Bellemain Franck (2007). CABRI Geometry II Plus [CD-ROM]. Version 1.4 Instituit d' Informatique et Mathematiques Applique es de Grenoble, Universite Joseph Fourier, Grenoble, France.