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Joel Hansen
The University of Texas Rio Grande Valley

Itzamar Marquez
The University of Texas Rio Grande Valley

Mrinal Kanti Roychowdhury
The University of Texas Rio Grande Valley

Eduardo Torres
The University of Texas Rio Grande Valley

QUANTIZATION COEFFICIENTS FOR UNIFORM DISTRIBUTIONS ON THE BOUNDARIES OF REGULAR POLYGONS

[1]JOEL HANSEN, [2]ITZAMAR MARQUEZ, [3]MRINAL K. ROYCHOWDHURY, AND [4]EDUARDO TORRES

Abstract. In this paper, we give a general formula to determine the quantization coefficients for uniform distributions defined on the boundaries of different regular m-sided polygons inscribed in a circle. The result shows that the quantization coefficient for the uniform distribution on the boundary of a regular m-sided polygon inscribed in a circle is an increasing function of m, and approaches to the quantization coefficient for the uniform distribution on the circle as m tends to infinity.

1. Introduction

Let $\mathbb{R}^d$ denote the d-dimensional Euclidean space, $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^d$ for any $d \geq 1$, and $n \in \mathbb{N}$. For a finite set $\alpha \subset \mathbb{R}^d$, the *cost* or *distortion error* for P with respect to the set α, denoted by $V(P;\alpha)$, is defined by

$$V(P;\alpha) := \int \min_{a\in\alpha} \|x-a\|^2 dP(x).$$

Then, the nth quantization error for P, denoted by $V_n := V_n(P)$, is defined by

$$V_n := V_n(P) = \inf \Big\{ V(P;\alpha) : \alpha \subset \mathbb{R}^d, 1 \leq \text{ card}(\alpha) \leq n \Big\}.$$

A set α for which the infimum is achieved and contains no more than n points is called an *optimal set of n-means*. It is well-known that for a continuous probability measure an optimal set of n-means always contains exactly n elements. If P is the probability distribution, then an optimal set of n-means is denoted by $\alpha_n := \alpha_n(P)$. Optimal sets of n-means for different probability distributions were determined by several authors, for example, see [CR1, CR2, DR1, DR2, GL2, L, R1, R2, R3, R4, R5, R6, RR1, RS]. It has broad applications in engineering and technology (see [GG, GN, Z]). For any $s \in (0, +\infty)$, the number

$$\lim_{n\to\infty} n^{\frac{2}{s}} V_n(P),$$

if it exists, is called the s-dimensional *quantization coefficient* for P. Bucklew and Wise (see [BW]) showed that for a Borel probability measure P with non-vanishing absolutely continuous part the quantization coefficient exists as a finite positive number. For some more details interested readers can also see [GL1, P]. Let $E(X)$ represent the expected value of a random variable X associated with a probability distribution P. Let α be an optimal set of n-means for P, and $a \in \alpha$. Then, it is well-known that $a = E(X : X \in M(a|\alpha))$, where $M(a|\alpha)$ is the Voronoi region of $a \in \alpha$, i.e., $M(a|\alpha)$ is the set of all elements x in $\mathbb{R}^d$ which are closest to a among all the elements in α (see [GG, GL1]).

From the work of Rosenblatt and Roychowdhury (see [RR2]), it is known that the quantization coefficient for the uniform distribution on a unit circle is $\frac{\pi^2}{3}$; on the other hand, from the work of Pena et al. (see [PRRSS]), it is known that the quantization coefficient for the uniform distribution on the boundary of a regular hexagon inscribed in a unit circle is 3. Notice that a regular m-sided polygon inscribed in a circle tends to the circle as m tends to infinity. Pena et

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al. conjectured that the quantization coefficient for the uniform distribution on the boundary of a regular m-sided polygon inscribed in a circle is an increasing function of m (see [PRRSS]), and approaches to the quantization coefficient for the uniform distribution on the circle as m tends to infinity. In this paper, we prove that the conjecture is true.

The arrangement of the paper is as follows: First, we prove a theorem Theorem 2.1, which gives a technique how to calculate the optimal sets of n-means and the nth quantization errors for all positive integers n for a uniform distribution defined on any line segment. Next, let P be the uniform distribution defined on the boundary of a regular m-sided polygon inscribed in a unit circle. In Proposition 2.3, for $k \geq 2$, we determine the optimal set of mk-means and the mkth quantization error for the probability distribution P. Then, with the help of the proposition, in Theorem 2.4, we have shown that the quantization coefficient for P exists, and equals $\frac{1}{3}m^2 \sin^2\left(\frac{\pi}{m}\right)$, i.e.,

$$\lim_{n\to\infty} n^2 V_n(P) = \frac{1}{3}m^2 \sin^2\left(\frac{\pi}{m}\right).$$

Notice that $\frac{1}{3}m^2 \sin^2\left(\frac{\pi}{m}\right)$ is an increasing function of m, and $\lim\limits_{m\to\infty} \frac{1}{3}m^2 \sin^2\left(\frac{\pi}{m}\right) = \frac{\pi^2}{3}$, which is the quantization coefficient for a uniform distribution on the unit circle (see [RR2]). Thus, the result in this paper, shows that the conjecture given by Pena et al. in [PRRSS] is true.

2. Main result

In this section, first we give some basic definitions.

Let i and j be the unit vectors in the positive directions of the x_1- and x_2-axes, respectively. By the position vector a of a point A, it is meant that $\overrightarrow{OA} = a$. We will identify the position vector of a point (a_1, a_2) by $(a_1, a_2) := a_1 i + a_2 j$, and apologize for any abuse in notation. For any two position vectors $a := (a_1, a_2)$ and $b := (b_1, b_2)$, we write $\rho(a, b) := \|(a_1, b_1) - (a_2, b_2)\|^2 = (a_1 - a_2)^2 + (b_1 - b_2)^2$, which gives the squared Euclidean distance between the two points (a_1, a_2) and (b_1, b_2). Let P and Q belong to an optimal set of n-means for some positive integer n, and let D be a point on the boundary of the Voronoi regions of the points P and Q. Since the boundary of the Voronoi regions of any two points is the perpendicular bisector of the line segment joining the points, we have $|\overrightarrow{DP}| = |\overrightarrow{DQ}|$, i.e., $(\overrightarrow{DP})^2 = (\overrightarrow{DQ})^2$ implying $(p-d)^2 = (q-d)^2$, i.e., $\rho(d, p) - \rho(d, q) = 0$, where p, q, d are, respectively, the position vectors of the points P, Q, D. We call such an equation a *canonical equation.*

Let us now give the following theorem.

Theorem 2.1. *Let AB be a line segment joining the two points A and B given by the position vectors $a := (a_1, b_1)$ and $b := (a_2, b_2)$, respectively. Let μ be a uniform distribution on AB. Let $M(t)$ be the parametric representation of AB for $0 \leq t \leq 1$ such that $M(0) = a$, and $M(1) = b$. Let D_1 and D_2 be two points on the segment AB at distances r_1 and r_2 from A and B, respectively (see Figure 1). Then, the optimal set of n-means for μ on the segment D_1D_2, is given by*

$$\alpha_n(\mu, D_1D_2) := \Big\{M\big(\frac{r_1}{\ell} + \frac{2j-1}{2n}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell})\big) : 1 \leq j \leq n\Big\},$$

with the nth quantization error for μ on the segment D_1D_2,

$$V_n(\mu, D_1D_2) := n\int_{\frac{r_1}{\ell}}^{\frac{r_1}{\ell}+\frac{1}{n}(1-\frac{r_2}{\ell}-\frac{r_1}{\ell})} \rho\Big(M(t), M\big(\frac{r_1}{\ell} + \frac{1}{2n}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell})\big)\Big) d\mu,$$

where ℓ is the length of the line segment AB.

Proof. Since ℓ is the length of the line segment AB, the probability density function (pdf) f of the uniform distribution μ on AB is given by $f(x_1, x_2) = \frac{1}{\ell}$ for all $(x_1, x_2) \in AB$, and zero, otherwise. Let s represent the distance of any point on AB from the point A. Then, we have $d\mu = d\mu(s) = \mu(ds) = f(x_1, x_2)ds = \frac{1}{\ell}\, ds$. Notice that $ds = \sqrt{(\frac{dx_1}{dt})^2 + (\frac{dx_2}{dt})^2}\, dt = \ell\, dt$ yielding

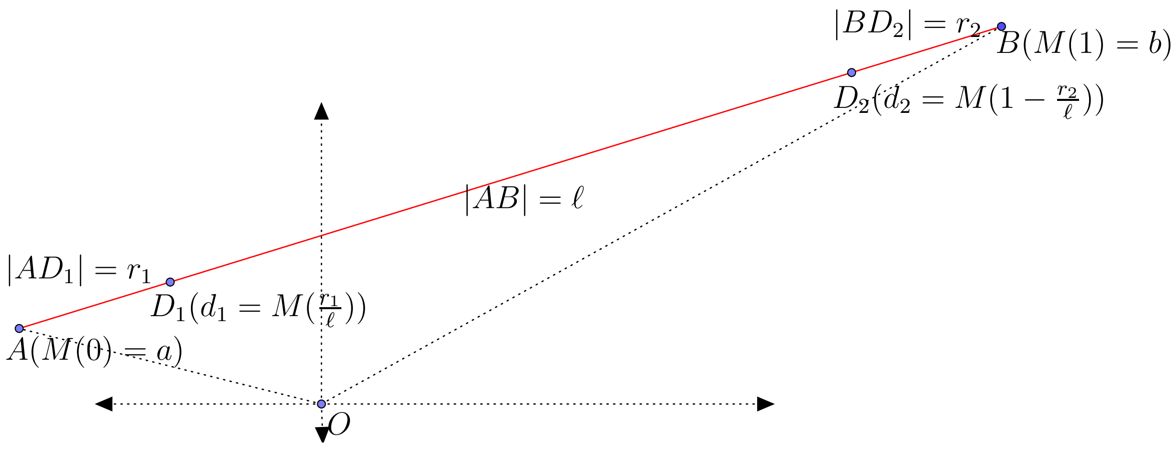

FIGURE 1.

$d\mu = dt$. Given, the parametric representation of the line segment AB is $M(t)$ for $0 \le t \le 1$ with $M(0) = a$ and $M(1) = b$. Hence, the parameters for the points D_1 and D_2, which are at distances r_1 and r_2 from A and B are, respectively, given by $t = \frac{r_1}{\ell}$ and $t = 1 - \frac{r_2}{\ell}$, i.e., if d_1 and d_2 are the position vectors of the points D_1 and D_2 (see Figure 1), then we have

$$d_1 = M(\frac{r_1}{\ell}), \text{ and } d_2 = M(1 - \frac{r_2}{\ell}).$$

In fact, we can identify the line segment D_1D_2 by its parameters in the closed interval $[\frac{r_1}{\ell}, 1-\frac{r_2}{\ell}]$. By [RR2], we know that the optimal set of n-means with respect to an uniform distribution in the closed interval $[\frac{r_1}{\ell}, 1 - \frac{r_2}{\ell}]$ is given by the set

$$\Big\{\frac{r_1}{\ell} + \frac{2j-1}{n}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell}) : 1 \le j \le n\Big\}.$$

Hence, the optimal set of n-means for μ on the segment D_1D_2, is given by

$$\alpha_n(\mu,\, D_1D_2) := \Big\{M(\frac{r_1}{\ell} + \frac{2j-1}{2n}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell})) : 1 \le j \le n\Big\}.$$

If $V_n(\mu,\, D_1D_2)$ is the corresponding quantization error, we have

$$V_n(\mu,\, D_1D_2) = n\Big(\text{Quantization error due to the point } M(\frac{r_1}{\ell} + \frac{1}{2n}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell}))\Big).$$

Again, notice that any point on the line segment D_1D_2 is given by $M(t)$ for $\frac{r_1}{\ell} \le t \le 1 - \frac{r_2}{\ell}$, and the parameters for the points at which the boundary of the Voronoi region of $M(\frac{r_1}{\ell} + \frac{1}{2n}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell}))$ cuts the segment D_1D_2 are given by $t = \frac{r_1}{\ell}$, and $t = \frac{r_1}{\ell} + \frac{1}{n}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell})$. Hence, we have

$$V_n(\mu,\, D_1D_2) = n\int_{\frac{r_1}{\ell}}^{\frac{r_1}{\ell}+\frac{1}{n}(1-\frac{r_2}{\ell}-\frac{r_1}{\ell})} \rho\Big(M(t), M(\frac{r_1}{\ell} + \frac{1}{2n}(1 - \frac{r_2}{\ell} - \frac{r_1}{\ell}))\Big) d\mu.$$

Thus, the proof of the theorem is complete. □

Let the equation of the unit circle be $x_1^2 + x_2^2 = 1$. Let $A_1A_2A_3 \cdots A_m$ be a regular m-sided polygon for some $m \ge 3$ inscribed in the circle. Without any loss of generality due to rotational symmetry, we can always assume that the vertex A_1 lies on the x_1-axis, i.e., the vertex A_1 is the point where the circle intersects the positive direction of the x_1-axis. Again, notice that each side of the regular m-sided polygon subtends a central angle of radian $\frac{2\pi}{m}$. Thus, the position vectors $\tilde{a}_j$ of the vertices A_j are given by $\tilde{a}_j = (\cos\frac{2\pi}{m}(j-1), \sin\frac{2\pi}{m}(j-1))$ for $1 \le j \le m$. Let ℓ be the length of each side of the polygon, then we have

$$\ell = \|\tilde{a}_m - \tilde{a}_{m-1}\| = \|\tilde{a}_{m-1} - \tilde{a}_{m-2}\| = \cdots = \|\tilde{a}_2 - \tilde{a}_1\| = 2\sin\frac{\pi}{m}. \tag{1}$$

Let L be the boundary of the polygon. Then, we can write

$$L = \bigcup_{j=1}^{m} L_j,$$

where L_j is the side A_jA_{j+1}, and A_{m+1} is identified as the vertex A_1. Then, for $1 \leq j \leq m$, we can write

$$L_j := A_jA_{j+1} = \{M_j : 0 \leq t \leq 1\}, \text{ where } M_j = \tilde{a}_{j+1}t + (1-t)\tilde{a}_j.$$

Notice that M_j is a function of t, and any point on the side A_jA_{j+1} can be represented by $M_j := M_j(t)$ for $0 \leq t \leq 1$. Thus, we see that $M_j(0) = \tilde{a}_j$, and $M_j(1) = \tilde{a}_{j+1}$ for $1 \leq j \leq m$. Let P be the uniform distribution defined on the boundary L of the polygon. Then, the probability density function (pdf) f of the uniform distribution P is given by $f(x_1, x_2) = \frac{1}{m\ell}$ for all $(x_1, x_2) \in L$, and zero, otherwise. Let s represent the distance of any point on L from the vertex A_1 tracing along the boundary L in the counterclockwise direction. Then, we have $dP = dP(s) = P(ds) = f(x_1, x_2)ds = \frac{1}{m\ell}ds$. For $1 \leq j \leq m$, on each L_j, we have $ds = \sqrt{(\frac{dx_1}{dt})^2 + (\frac{dx_2}{dt})^2}|dt| = \ell|dt|$ yielding $dP(s) = \frac{\ell}{m\ell}|dt| = \frac{1}{m}|dt|$.

Remark 2.2. Since P is uniform, and a regular m-sided polygon has symmetry of order m, it is not difficult to show that an optimal set α_m will contain m points, each from the interior of the m angles of the regular m-sided polygon; and for any positive integer $k \geq 2$, α_{mk} will contain m points, each from the interior of the m angles, and $(k-1)$ points from each side of the regular m-sided polygon. Moreover, the following is true: Let A be one of the vertices of the regular m-sided polygon, and for $k \geq 2$, let a be an element of an optimal set of mk-means that lies in the interior of $\angle A$. Further, let AA_1 and AA_2 be the two adjacent sides of the vertex A. Then, the boundary of the Voronoi region of a will cut AA_1 and AA_2 at two points D_1 and D_2 such that $|AD_1| = |AD_2| = r$ for some real r such that $0 < r \leq \frac{\ell}{2}$, where ℓ is the length of the sides of the polygon.

Proposition 2.3. *Let α_n be an optimal set of n-means such that $n = mk$, where $k \in \mathbb{N}$, and $k \geq 2$. Let a_j be the points that α_n contains from the interior of the angles A_j of the regular m-sided polygon, $1 \leq j \leq m$. Then,*

$$\alpha_n = \{a_j : 1 \leq j \leq m\} \cup \bigcup_{j=1}^{m} \alpha_{j,k-1},$$

where

$$a_1 = (1 - \frac{1}{2}r\sin(\frac{\pi}{m}), 0),$$

$$a_j = (\frac{1}{4}\cos\frac{2\pi(j-1)}{m}(r(\cos(\frac{2\pi}{m}) - 1)\csc(\frac{\pi}{m}) + 4), \sin\frac{2\pi(j-1)}{m}(\frac{1}{4}r(\cos(\frac{2\pi}{m}) - 1)\csc(\frac{\pi}{m}) + 1))$$

for $2 \leq j \leq m$, and $\alpha_{j,k-1} := \{M_j(\frac{r}{\ell} + \frac{2i-1}{2(k-1)}(1 - \frac{2r}{\ell})) : 1 \leq i \leq k-1\}$ for $1 \leq j \leq m$, and

$$r = \frac{4\sin(\frac{\pi}{m})}{2(k-1)\sqrt{3\cos^2(\frac{\pi}{m}) + 1} + 4}.$$

Moreover, the quantization error for n-means is given by

$$V_n = \frac{2\sin^2(\frac{\pi}{m})(3\cos(\frac{2\pi}{m}) + 5)}{3\Big(k\sqrt{6\cos(\frac{2\pi}{m}) + 10} - \sqrt{6\cos(\frac{2\pi}{m}) + 10} + 4\Big)^2}.$$

Proof. Let α_n be an optimal set of n-means, where $n = mk$ for some positive integer $k \geq 2$. Since a_j are the points that α_n contains from the interior of the angles A_j, by Remark 2.2, due to uniform distribution and symmetry, we can say that there exists a real number r, where $0 < r \leq \frac{\ell}{2}$, such that the boundary of the Voronoi region of each a_j will cut the the two adjacent sides at distances r from the vertex A_j. Notice that the two adjacent sides of the vertex A_1 are A_mA_1 and A_1A_2 in the polygon. Again, by the hypothesis a_1 is the point that α_n contains from $\angle A_1$. If the boundary of the Voronoi region of a_1 cuts A_mA_1 and A_1A_2 at D_1 and D_2, respectively, we have

$$a_1 = E(X : X \in D_1A_1 \cup A_1D_2) = \frac{\int_{D_1A_1}(x_1,x_2)dP + \int_{A_1D_2}(x_1,x_2)dP}{\int_{D_1A_1} 1dP + \int_{A_1D_2} 1dP},$$

which implies

$$a_1 = \frac{\int_{1-\frac{r}{L}}^1 M_n(t)\,dt + \int_0^{\frac{r}{L}} M_1(t)\,dt}{\int_0^{\frac{r}{L}} 1\,dt + \int_{1-\frac{r}{L}}^1 1\,dt} = (1 - \frac{1}{2}r\sin\left(\frac{\pi}{m}\right), 0).$$

Similarly, for $2 \leq j \leq m$, we obtain

$$a_j = \frac{\int_{1-\frac{r}{L}}^1 M_{j-1}(t)\,dt + \int_0^{\frac{r}{L}} M_j(t)\,dt}{\int_0^{\frac{r}{L}} 1\,dt + \int_{1-\frac{r}{L}}^1 1\,dt}$$

yielding

$$a_j = (\frac{1}{4}\cos\frac{2\pi(j-1)}{m}(r(\cos(\frac{2\pi}{m}) - 1)\csc(\frac{\pi}{m}) + 4), \sin\frac{2\pi(j-1)}{m}(\frac{1}{4}r(\cos(\frac{2\pi}{m}) - 1)\csc(\frac{\pi}{m}) + 1)).$$

Again, by Remark 2.2, α_n contains $(k-1)$ points from each side of the regular m-sided polygon. For $1 \leq j \leq m$, let $\alpha_{j,k-1}$ be the optimal set of $(k-1)$-means that α_n contains from the side A_jA_{j+1}. Recall that the parametric representation of the side A_jA_{j+1} is $M_j(t)$, and the $(k-1)$ means from each side occur due to an uniform distribution on the segment bounded by the two points represented by the parameters $t = \frac{r}{\ell}$ and $t = 1 - \frac{r}{\ell}$. Hence, by Theorem 2.1, we have

$$\alpha_{j,k-1} := \Big\{M_j(\frac{r}{\ell} + \frac{2i-1}{2(k-1)}(1 - \frac{2r}{\ell})) : 1 \leq i \leq k-1\Big\}.$$

To calculate the quantization error, we proceed as follows: By symmetry, the quantization error contributed by all the points a_j for $1 \leq j \leq m$ is given by

$$m\int_{D_1A_1\cup A_1D_2} \rho((x_1,x_2),a_1)dP = 2m\int_{A_1D_2} \rho((x_1,x_2),a_1)dP = 2\int_0^{\frac{r}{\ell}} \rho(M_1(t),a_1))dt,$$

implying

$$m\int_{D_1A_1\cup A_1D_2} \rho((x_1,x_2),a_1)dP = \frac{1}{24}r^3(3\cos(\frac{2\pi}{m}) + 5)\csc(\frac{\pi}{m}). \tag{2}$$

Again, by Theorem 2.1, the quantization error contributed by all the sets $\alpha_{j,k-1}$ for $1 \leq j \leq m$ is given by

$$mV_n(\mu, \alpha_{j,k-1}) := (k-1)\int_{\frac{r}{\ell}}^{\frac{r}{\ell}+\frac{1}{k-1}(1-\frac{2r}{\ell})} \rho\Big(M(t), M(\frac{r}{\ell} + \frac{1}{2(k-1)}(1 - \frac{2r}{\ell}))\Big)dt.$$

implying

$$mV_n(\mu, \alpha_{j,k-1}) = \frac{1}{3(k-1)^2}\csc(\frac{\pi}{m})(\sin(\frac{\pi}{m}) - r)^3. \tag{3}$$

Hence, by (2) and (3), the quantization error for n-means is given by

$$V_n = \frac{1}{24}\csc(\frac{\pi}{m})\Big(r^3(3\cos(\frac{2\pi}{m}) + 5) + \frac{8}{(k-1)^2}(\sin(\frac{\pi}{m}) - r)^3\Big).$$

Notice that for a given k, the quantization error V_n is a function of r. Solving $\frac{\partial V_n}{\partial r}=0$, we have $r=\frac{4\sin(\frac{\pi}{m})}{2(k-1)\sqrt{3\cos^2(\frac{\pi}{m})+1}+4}$. Putting $r=\frac{4\sin(\frac{\pi}{m})}{2(k-1)\sqrt{3\cos^2(\frac{\pi}{m})+1}+4}$, we have

$$V_n=\frac{2\sin^2(\frac{\pi}{m})(3\cos(\frac{2\pi}{m})+5)}{3\Big(k\sqrt{6\cos(\frac{2\pi}{m})+10}-\sqrt{6\cos(\frac{2\pi}{m})+10}+4\Big)^2}.$$

Thus, the proof the proposition is complete. ☐

Let us now prove the following theorem.

Theorem 2.4. *Let P be the uniform distribution on the boundary of a regular m-sided polygon inscribed in a unit circle. Then, the quantization coefficient for P exists as a finite positive number which equals $\frac{1}{3}m^2\sin^2(\frac{\pi}{m})$, i.e., $\lim\limits_{n\to\infty} n^2V_n=\frac{1}{3}m^2\sin^2(\frac{\pi}{m})$.*

Proof. Let $n\in\mathbb{N}$ be such that $n\geq 2m$. Then, there exists a unique positive integer $\ell(n)\geq 2$ such that $m\ell(n)\leq n<m(\ell(n)+1)$. Then,

$$(m\ell(n))^2V_{m(\ell(n)+1)}<n^2V_n<(m(\ell(n)+1))^2V_{m\ell(n)}. \tag{4}$$

We have

$$\begin{aligned}&\lim_{n\to\infty}(m\ell(n))^2V_{m(\ell(n)+1)}\\ &=\lim_{\ell(n)\to\infty}(m\ell(n))^2\frac{2\sin^2(\frac{\pi}{m})(3\cos(\frac{2\pi}{m})+5)}{3\Big((\ell(n)+1)\sqrt{6\cos(\frac{2\pi}{m})+10}-\sqrt{6\cos(\frac{2\pi}{m})+10}+4\Big)^2}=\frac{1}{3}m^2\sin^2(\frac{\pi}{m}),\end{aligned}$$

and

$$\begin{aligned}&\lim_{n\to\infty}(m(\ell(n)+1))^2V_{m\ell(n)}\\ &=\lim_{\ell(n)\to\infty}(m(\ell(n)+1))^2\frac{2\sin^2(\frac{\pi}{m})(3\cos(\frac{2\pi}{m})+5)}{3\Big(\ell(n)\sqrt{6\cos(\frac{2\pi}{m})+10}-\sqrt{6\cos(\frac{2\pi}{m})+10}+4\Big)^2}=\frac{1}{3}m^2\sin^2(\frac{\pi}{m}),\end{aligned}$$

and hence, by (4) we have $\lim\limits_{n\to\infty} n^2V_n=\frac{1}{3}m^2\sin^2(\frac{\pi}{m})$, i.e., the quantization coefficient exists as a finite positive number which equals $\frac{1}{3}m^2\sin^2(\frac{\pi}{m})$. Thus, the proof of the theorem is complete. ☐

Remark 2.5. Since $\lim\limits_{m\to\infty} 2\sin\frac{\pi}{m}=0$, by (1), we can conclude that when m tends to ∞, then the length of each side of the regular m-sided polygon becomes zero, i.e., the regular m-sided polygon coincides with the circle. Moreover, for $m\geq 3$, we have

$$\frac{d}{dm}(\frac{1}{3}m^2\sin^2(\frac{\pi}{m}))=\frac{2}{3}\sin(\frac{\pi}{m})(m\sin(\frac{\pi}{m})-\pi\cos(\frac{\pi}{m}))>0$$

yielding the fact that the quantization coefficient $\frac{1}{3}m^2\sin^2(\frac{\pi}{m})$ for the uniform distribution on the boundary of the regular m-sided polygon is an increasing function of m. Again,

$$\lim_{m\to\infty}\frac{1}{3}m^2\sin^2(\frac{\pi}{m})=\frac{\pi^2}{3},$$

i.e., when m tends to infinity, then the quantization coefficient of the regular m-sided polygon equals $\frac{\pi^2}{3}$. Recall that $\frac{\pi^2}{3}$ is the quantization coefficient for the uniform distribution on the unit circle. Thus, the result in this paper, proves the conjecture given by Pena et al. in the paper [PRRSS].

Remark 2.6. If $m=6$, we see that $\lim\limits_{n\to\infty} n^2V_n=3$, which is the quantization coefficient for the uniform distribution on the boundary of a hexagon inscribed in a unit circle. Thus, the result in this paper, also generalizes a result given by Pena et al. in the paper [PRRSS].

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School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, 1201 West University Drive, Edinburg, TX 78539-2999, USA.

E-mail address: {[1]`joel.hansen01`, [2]`itzamar.marquez01`, [3]`mrinal.roychowdhury`, [4]`eduardo.torres05`} `@utrgv.edu`