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Abstract

In this paper, we study the orbital stability of peakons and periodic peakons for a nonlinear quartic Camassa-Holm equation (QCHE).We first verify that the QCHE has global peakon and periodic peakon solutions. Then by the invariants of the equation and controlling the extrema of the solution, we prove that the shapes of the peakons and periodic peakons are stable under small perturbations in the energy space.

Key words: Camassa-Holm equation, peakon, patched peakon, orbital stability

1. Introduction

The Camassa-Holm (CH) equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.1}$$

was proposed as a model for describing the unidirectional propagation of the shallow water waves over a flat bottom [2, 3], with $u(x,t)$ representing the water's free surface in non-dimensional variables. It may also be found using the method of recursion operators as an example of bi-Hamiltonian equation with

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an infinite number of conserved quantities by Fokas and Fuchssteiner [11]. The CH equation has attracted much attention in the last two decades because of its interesting properties: complete integrability [2], the presence of breaking waves [5–7] (i.e. a wave profile remains bounded while its slope becomes unbounded in finite time), and algebro-geometric formulations [20]. Among those properties, remarkable is that the CH equation admits the peakons solutions in the following form

$$u(x,t) = c\varphi(x-ct) = ce^{-|x-ct|}. \tag{1.2}$$

The peakons were proved orbital stable by Constantin and Strauss in [8]. A variational approach for proving the orbital stability of the peakons was introduced by Constantin and Molient [9]. Orbital stability of multi-peakon solutions was discussed by Dika and Molient in [10]. Liu, Liu and Qu [17] considered the modified Camassa-Holm equation with cubic nonlinearity, which is integrable and admits the single peakons and mult-peakons. Using energy argument and combining the method of the orbital stability of a single peakon with monotonicity of the local energy norm, they proved that the sum of N sufficiently decoupled peakons is orbitally stable in the energy space. Moreover, the orbital stability of the single peakons for the DP equation was proved by Lin and Liu [15]. They developed the approach due to Constantin and Strauss [8] in a delicate way. The approach in [8] was extended in [18] to prove the orbital stability of the peakons for the Novikov equation. Liu et al. [16] investigated the orbital stability of the peaked solitary-wave solutions for a generalization of the modified Camassa-Holm equation with both cubic and quadratic nonlinearities. Very recently, Guo, Liu, Liu and Qu [13] studied the orbital stability of peakons for a generalized modified Camassa-Holm (gmCH) equation. It is worth to point out that the proof for orbital stability of peakons could be utilized to periodic peakons as well. Orbital stability of the periodic peakons for the CH equation was studied by Lenells in [14]. Wang and Tian [22] extended Lenell's approach to discuss the orbital stability of the periodic peakons for the Novikov equation. Chen, Lenells and Liu [4] showed that the periodic peakons of the μCH equation are orbitally stable. Liu, Qu and Zhang [19] further proved that the periodic peakons of the modified μCH equation are orbitally stable.

In this paper, we shall discuss the stability issue of peakons and periodic peakons for the following nonlinear quartic Camassa-Holm equation (QCHE)

$$m_t + \left(\frac{1}{4}(u^2-u_x^2)^2 + u(u^2-u_x^2)m\right)_x = 0, \quad m = u - u_{xx}, \tag{1.3}$$

which was proposed by Anco and Recio in [1]. Eq. (1.3) has bi-Hamiltonian structures and peakon solutions in the form of

$$\varphi(x,t) = \sqrt[3]{\frac{3c}{2}}e^{-|x-ct|}, \tag{1.4}$$

and the periodic peakon solution in the form of

$$\varphi(x,t) = \sqrt[3]{\frac{3c}{\left(2\cosh^2\left(\frac{1}{2}\right)+1\right)\cosh\left(\frac{1}{2}\right)}}\cosh\left(\frac{1}{2}-(x-ct)+[x-ct]\right). \tag{1.5}$$

Recently, Gao, Li and Liu [12] provide a method, called patching technic, to truncate traveling wave solutions and patch different segments to obtain patched bounded single-valued peakon weak solutions which satisfy jump conditions at peakons. Here peakon (1.4) and periodic peakon solution (1.5) are also patched peakon weak solutions of the QCHE. More recently, Qu and Fu [21] proved the local well-posedness to the Cauchy problem (1.3) in Besov spaces, and established a few criteria for the blow-up of solutions in Sobolev spaces, then they derived several types of conditions on initial data which could lead to finite time curvature blow-up. In the present work, we shall prove the stability for both peakons (1.4) and periodic peakons (1.5). The proof is inspired by [8] where the case of peakons of the Camassa-Holm equation is considered. The approach taken here is similar but there are differences. It is found that the conservation law $H_2[u]$ (see equation 3.1) of the nonlinear quartic Camassa-Holm equation is much more complicated than $F(u)$ of the CH equation (see equation (2.1) in [8]). Therefore, the stability issue of the peaked solitons of the QCHE is more subtle in both the periodic and non-periodic cases. Let us first state our main results below, and then the proof is followed in the remaining sections.

Theorem 1.1. *Let $\mathbb{X} = \mathbb{R}$ or $\mathbb{S}$, where $\mathbb{R}$ and $\mathbb{S}$ are referred to the real field and the unit circle. For all $\varepsilon > 0$, there is a $\delta > 0$ such that if $u \in C([0,T); H^1(\mathbb{X}))$ is a solution to (1.3) with*

$$\|u(\cdot, 0) - \varphi\|_{H^1(\mathbb{X})} < \delta, \tag{1.6}$$

then

$$\|u(\cdot, t) - \varphi(\cdot - \xi(t))\|^2_{H^1(\mathbb{X})} < \varepsilon, \tag{1.7}$$

where $t \in (0,T)$ and $\xi(t) \in \mathbb{X}$ is an extreme point where the function $u(\cdot, t)$ attains its maximum. Therefore, the peakons (or periodic peakons) are orbitally stable.

2. Peakons and periodic peakons

In this section, we will consider the Cauchy problem for the QCHE (1.3) on both the line and the unit circle:

$$\begin{cases} m_t + \left(\frac{1}{4}(u^2 - u_x^2)^2 + u(u^2 - u_x^2)m\right)_x = 0, \quad t > 0, \quad x \in \mathbb{X} = \mathbb{R} \text{ or } \mathbb{S}, \\ u(0,x) = u_0(x), \quad m = u - u_{xx}, \quad x \in \mathbb{X}. \end{cases} \tag{2.1}$$

Substituting the formula $m = u - u_{xx}$ into equation (2.1) produces the following integral-differential equation:

$$\begin{aligned} &u_t + \left(u^2 - \frac{1}{3}u_x^2\right) u u_x + (1 - \partial_x^2)^{-1} \left(\frac{1}{3} u u_x^3\right) \\ &+ \partial_x (1 - \partial_x^2)^{-1} \left(u^4 + \frac{3}{2} u^2 u_x^2 - \frac{1}{12} u_x^4\right) = 0. \end{aligned} \tag{2.2}$$

Due to $m = u - u_{xx}$, u is able to be rewritten as

$$u = (1 - \partial_x^2)^{-1} m = p * m \text{ or } G * m, \tag{2.3}$$

where $p(x) = \frac{1}{2}\, e^{-|x|}$ is for the regular peakon case, while $G(x) = \frac{\cosh\left(\frac{1}{2} - x + [x]\right)}{2\sinh\left(\frac{1}{2}\right)}$ for the periodic case, and $*$ stands for the convolution product on $\mathbb{X}$,

$$(f * g)(x) = \int_{\mathbb{X}} f(y) g(x - y)\ dy.$$

This formulation allows us to define the weak solution as follows.

Definition 2.1. *Given the initial data $u_0 \in W^{1,3}(\mathbb{X})$ and the function $u \in L^\infty([0,T],\ W^{1,3}(\mathbb{X}))$ is said to be a weak solution to the initial-value problem (2.1) if it satisfies the following identity:*

$$\begin{aligned} \int_0^T \int_{\mathbb{X}} \Big[u\psi_t + \frac{1}{4} u^4 \psi_x + \frac{1}{3} u u_x^3 \psi + p * \Big(u^4 + \frac{3}{2} u^2 u_x^2 - \frac{1}{12} u_x^4 \Big) \partial_x \psi \\ - \frac{1}{3} (p * u u_x^3) \psi \Big]\ dxdt + \int_{\mathbb{X}} u_0(x) \psi(x, 0)\ dx = 0, \end{aligned} \tag{2.4}$$

for any smooth test function $\psi(x,t) \in C_c^\infty([0,T] \times \mathbb{X})$. If u is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global weak solution.

In the following two subsections, we will prove that (1.4) and (1.5) are weak solutions of Eq. (1.3) in the case of $\mathbb{X} = \mathbb{R}$ and $\mathbb{X} = \mathbb{S}$, respectively.

2.1. Peakon solutions

In the subsection, we just verify that (1.4) is a weak solution to Eq. (2.4) for $\mathbb{X} = \mathbb{R}$.

Theorem 2.1. *For any $a \neq 0$, the peaked functions of the form*

$$u(x,t) = a\ e^{-|x - ct|}, \quad \text{where} \quad c = \frac{2}{3}\ a^3, \tag{2.5}$$

is a global weak solution to (2.1) in the sense of Definition 2.1.

Proof. Apparently, for all $t \in \mathbb{R}^+$, the following formulation

$$u_x(x,t) = -\text{sign}(x - ct) u(x,t), \tag{2.6}$$

is true in the sense of distribution $\mathcal{S}'(\mathbb{R})$.

Let us define $u_{o,c}(x) := u(0,x)$ for $x \in \mathbb{R}$. Then

$$\lim_{t \to 0^+} \|u(\cdot, t) - u_{0,c}(\cdot)\|_{W^{1,\infty}} = 0. \tag{2.7}$$

As shown in (2.6), we have

$$u_t(x,t) = c \ \mathrm{sign}(x-ct)u(x,t) \ \in L^\infty(\mathbb{R}) \ \ for\ all\ t \geq 0. \tag{2.8}$$

Hence, using (2.6), (2.7), (2.8), and integration by parts, we are able to arrive at the following result

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}} \left(u\psi_t + \frac{1}{4}u^4\psi_x + \frac{1}{3}uu_x^3\psi \right) \ dxdt + \int_{\mathbb{R}} u(x,0)\psi(x,0) \ dx \\
&\quad = -\int_0^{+\infty}\int_{\mathbb{R}} \left(u_t + u^3u_x - \frac{1}{3}uu_x^3 \right) \psi \ dxdt \\
&\quad = -\int_0^{+\infty}\int_{\mathbb{R}} \mathrm{sign}(x-ct)u\left(c - \frac{2}{3}u^3 \right) \psi \ dxdt,
\end{aligned} \tag{2.9}$$

where $\psi(x,t) \in C_c^\infty([0,+\infty)\times\mathbb{R})$ is an arbitrary test function.

On the other hand, casting Eq. (2.3) into

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}} \left[(1-\partial_x^2)^{-1}\left(u^4 + \frac{3}{2}u^2u_x^2 - \frac{1}{12}u_x^4 \right)\psi_x - \frac{1}{3}(1-\partial_x^2)^{-1}(uu_x^3)\psi \right] dxdt \\
&= -\int_0^{+\infty}\int_{\mathbb{R}} \left[\partial_x p * \left(\frac{3}{2}u^2u_x^2 - \frac{1}{12}u_x^4 \right) + p * \left(\frac{1}{3}uu_x^3 + 4u^3u_x \right) \right] \psi \ dxdt,
\end{aligned} \tag{2.10}$$

yields

$$\frac{1}{3}uu_x^3 + 4u^3u_x = -\frac{1}{3} \ \mathrm{sign}^3(x-ct)u^4 - 4 \ \mathrm{sign}(x-ct)u^4 = \frac{13}{12}(u^4)_x,$$

and

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}} \left[(1-\partial_x^2)^{-1}\left(u^4 + \frac{3}{2}u^2u_x^2 - \frac{1}{12}u_x^4 \right)\psi_x - \frac{1}{3}(1-\partial_x^2)^{-1}(uu_x^3)\psi \right] dxdt \\
&= -\int_0^{+\infty}\int_{\mathbb{R}} \left[\partial_x p * \left(\frac{3}{2}u^2u_x^2 - \frac{1}{12}u_x^4 + \frac{13}{12}u^4 \right) \right] \psi \ dxdt.
\end{aligned} \tag{2.11}$$

Furthermore, Eq. (2.6) implies $\partial_x p(x) = -\frac{1}{2} \ \mathrm{sign}(x)e^{-|x|}$ for $x \in \mathbb{R}$. Therefore, the kernel function in Eq. (2.11) can explicitly be computed and split into the following three parts:

$$\begin{aligned}
&\partial_x p * \left(\frac{3}{2}u^2u_x^2 - \frac{1}{12}u_x^4 + \frac{13}{12}u^4\right)(x,t)\\
&= -\frac{1}{2}\left(\frac{3\ c}{2}\right)^{4/3}\int_{-\infty}^{+\infty} \operatorname{sign}(x-y)e^{-|x-y|}\\
&\quad\times\left(\frac{3}{2}\ \operatorname{sign}^2(y-ct) - \frac{1}{12}\ \operatorname{sign}^4(y-ct) + \frac{13}{12}\right)e^{-4|y-ct|}\ dy,\\
&= -\frac{1}{2}\left(\frac{3\ c}{2}\right)^{4/3}\left(\int_{-\infty}^{ct} + \int_{ct}^{x} + \int_{x}^{+\infty}\right)\operatorname{sign}(x-y)e^{-|x-y|}\\
&\quad\times\left(\frac{3}{2}\ \operatorname{sign}^2(y-ct) - \frac{1}{12}\ \operatorname{sign}^4(y-ct) + \frac{13}{12}\right)e^{-4|y-ct|}\ dy\\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{2.12}$$

Some lengthy computations lead I_1, I_2, I_3 to the following results.

$$\begin{aligned}
I_1 &= -\frac{1}{2}\left(\frac{3\ c}{2}\right)^{4/3}\int_{-\infty}^{ct}\left(\frac{5}{2}\ e^{-(x-y)}\ e^{4(y-ct)}\right)\ dy\\
&= -\frac{5}{4}\left(\frac{3\ c}{2}\right)^{4/3} e^{-(x+4ct)}\int_{-\infty}^{ct} e^{5y}\ dy = -\frac{1}{4}\left(\frac{3\ c}{2}\right)^{4/3} e^{(ct-x)}.
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
I_2 &= -\frac{1}{2}\left(\frac{3\ c}{2}\right)^{4/3}\int_{ct}^{x}\left(\frac{5}{2}\ e^{-(x-y)}\ e^{-4(y-ct)}\right)\ dy\\
&= -\frac{5}{4}\left(\frac{3\ c}{2}\right)^{4/3} e^{-(x-4ct)}\int_{-\infty}^{ct} e^{-3y}\ dy\\
&= -\frac{5}{12}\left(\frac{3\ c}{2}\right)^{4/3}\left(e^{(ct-x)} - e^{4(ct-x)}\right),
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
I_3 &= -\frac{1}{2}\left(\frac{3\ c}{2}\right)^{4/3}\int_{x}^{+\infty}\left(\frac{5}{2}\ e^{(x-y)}\ e^{-4(y-ct)}\right)\ dy\\
&= -\frac{5}{4}\left(\frac{3\ c}{2}\right)^{4/3} e^{(x+4ct)}\int_{x}^{+\infty} e^{-5y}\ dy = \frac{1}{4}\left(\frac{3\ c}{2}\right)^{4/3} e^{4(ct-x)}.
\end{aligned} \tag{2.15}$$

Plugging (2.13)-(2.15) into (2.12) generates

$$\partial_x p * \left(\frac{3}{2}u^2u_x^2 - \frac{1}{12}u_x^4 + \frac{13}{12}u^4\right)(x,t) = \frac{2}{3}\left(\frac{3\ c}{2}\right)^{4/3}\left(e^{4(ct-x)} - e^{(ct-x)}\right), x > ct. \tag{2.16}$$

When $x \le ct$, we split the right hand side of (2.12) into the following three parts

$$\begin{aligned}
&\partial_x p * \left(\frac{3}{2}u^2u_x^2 - \frac{1}{12}u_x^4 + \frac{13}{12}u^4\right)(x,t) \\
&= -\frac{1}{2}\left(\frac{3\ c}{2}\right)^{4/3}\left(\int_{-\infty}^{x} + \int_{x}^{ct} + \int_{ct}^{+\infty}\right)\operatorname{sign}(x-y)e^{-|x-y|} \\
&\quad\times\left(\frac{3}{2}\ \operatorname{sign}^2(y-ct) - \frac{1}{12}\ \operatorname{sign}^4(y-ct) + \frac{13}{12}\right)e^{-4|y-ct|}\ dy \\
&=: J_1 + J_2 + J_3,
\end{aligned} \tag{2.17}$$

where J_1, J_2, J_3 can be worked out in the following formulas through some computations similar to the case of $x > ct$:

$$\begin{aligned}
J_1 &= -\frac{1}{2}\left(\frac{3\ c}{2}\right)^{4/3}\int_{-\infty}^{x}\left(\frac{5}{2}\ e^{-(x-y)}\ e^{4(y-ct)}\right)\ dy \\
&= -\frac{5}{4}\left(\frac{3\ c}{2}\right)^{4/3}e^{-(x+4ct)}\int_{-\infty}^{x} e^{5y}\ dy = -\frac{a}{4}\left(\frac{3\ c}{2}\right)^{4/3}e^{4(x-ct)},
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
J_2 &= \frac{1}{2}\left(\frac{3\ c}{2}\right)^{4/3}\int_{x}^{ct}\left(\frac{5}{2}\ e^{(x-y)}\ e^{4(y-ct)}\right)\ dy \\
&= \frac{5}{4}\left(\frac{3\ c}{2}\right)^{4/3}e^{(x-4ct)}\int_{-\infty}^{ct} e^{3y}\ dy \\
&= \frac{5}{12}\left(\frac{3\ c}{2}\right)^{4/3}\left(e^{(x-ct)} - e^{4(x-ct)}\right).
\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
J_3 &= \frac{1}{2}\left(\frac{3\ c}{2}\right)^{4/3}\int_{ct}^{+\infty}\left(\frac{5}{2}\ e^{(x-y)}\ e^{-4(y-ct)}\right)\ dy \\
&= \frac{5}{4}\left(\frac{3\ c}{2}\right)^{4/3}e^{(x+4ct)}\int_{ct}^{+\infty} e^{-5y}\ dy = \frac{1}{4}\left(\frac{3\ c}{2}\right)^{4/3}e^{(x-ct)}.
\end{aligned} \tag{2.20}$$

Therefore, we arrive at

$$\partial_x p*\left(\frac{3}{2}u^2u_x^2 - \frac{1}{12}u_x^4 + \frac{13}{12}u^4\right)(x,t) = -\frac{2}{3}\left(\frac{3\ c}{2}\right)^{4/3}\left(e^{4(x-ct)} - e^{(x-ct)}\right), x \le ct. \tag{2.21}$$

On the other hand, we have

$$\operatorname{sign}(x-ct)u\left(c - \frac{2}{3}u^3\right) = \begin{cases} -\frac{2}{3}\left(\frac{3\ c}{2}\right)^{4/3}\left(e^{4(ct-x)} - e^{(ct-x)}\right), & x > ct, \\[2ex] \frac{2}{3}\left(\frac{3\ c}{2}\right)^{4/3}\left(e^{4(x-ct)} - e^{(x-ct)}\right), & x \le ct. \end{cases}$$

This along with (2.16) and (2.21) yields

$$\partial_x p \ * \left(\frac{3}{2}u^2u_x^2 - \frac{1}{12}u_x^4 + \frac{13}{12}u^4\right)(x,t) + \operatorname{sign}(x-ct)u\left(c - \frac{2}{3}u^3\right)(x,t) = 0, \tag{2.22}$$

which completes the proof of Theorem 2.1. □

2.2. Periodic peakon solutions

In this subsection, let us verify that (1.5) is a solution to Eq. (2.4) for $\mathbb{X} = \mathbb{S}$.

Theorem 2.2. *For any $b \neq 0$, the peaked functions in the form of*

$$u(x,t) = b\cosh(\zeta), \quad \zeta = \frac{1}{2} - (x-ct) + [x-ct], \tag{2.23}$$

is a global weak solution to (2.1) in the sense of Definition 2.1, where b satisfies $b^3 = \frac{3c}{\left(2\cosh^2\left(\frac{1}{2}\right)+1\right)\cosh\left(\frac{1}{2}\right)}$.

Proof. Obviously, for all $t \in \mathbb{S}^+$,

$$u_x(x,t) = -b\sinh(\zeta), \;\; u_t(x,t) = ac\sinh(\zeta), \tag{2.24}$$

hold in the distribution sense. Define $u_{o,c}(x) := u(0,x)$ for $x \in \mathbb{S}$. Then, we have

$$\lim_{t\to 0^+} \|u(\cdot,t) - u_{0,c}(\cdot)\|_{W^{1,\infty}} = 0, \tag{2.25}$$

and

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{S}} \left(u\psi_t + \frac{1}{4}u^4\psi_x + \frac{1}{3}uu_x^3\psi\right)\, dxdt + \int_{\mathbb{S}} u(x,0)\psi(x,0)\, dx \\
&= -\int_0^{+\infty}\int_{\mathbb{S}} \left(u_t + u^3u_x - \frac{1}{3}uu_x^3\right)\psi\, dxdt \\
&= -\int_0^{+\infty}\int_{\mathbb{S}} \left[bc\sinh(\zeta) - b^4\cosh(\zeta)\sinh(\zeta) - \frac{2}{3}b^4\cosh(\zeta)\sinh^3(\zeta)\right]\psi\, dxdt.
\end{aligned} \tag{2.26}$$

where $\psi(x,t) \in C_c^\infty([0,+\infty)\times\mathbb{S})$ is an arbitrary test function. On the other hand, employing integration by parts leads to the following formulation:

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{S}} \left[G * \left(u^4 + \frac{3}{2}u^2u_x^2 - \frac{1}{12}u_x^4\right)\psi_x - \frac{1}{3}G*(uu_x^3)\psi\right] dxdt \\
&= -\int_0^{+\infty}\int_{\mathbb{S}} \left[\partial_x G * \left(\frac{3}{2}u^2u_x^2 - \frac{1}{12}u_x^4\right) + G * \left(\frac{1}{3}uu_x^3 + 4u^3u_x\right)\right]\psi\, dxdt \\
&= \int_0^{+\infty}\int_{\mathbb{S}} \left[G(x) * \left(\frac{1}{3}b^4\cosh(\zeta)\sinh^3(\zeta) + 4b^4\cosh^3(\zeta)\sinh(\zeta)\right)\right]\psi\, dxdt. \\
&\quad -\int_0^{+\infty}\int_{\mathbb{S}} \left[G_x(x) * \left(\frac{3}{2}b^4\sinh^2(\zeta) + \frac{17}{12}b^4\sinh^4(\zeta)\right)\right]\psi\, dxdt.
\end{aligned} \tag{2.27}$$

Noticing for the periodic case, we have the following three identities:

$$\partial_x G(x) = -\frac{\sinh\left(\frac{1}{2} - x + [x]\right)}{2\sinh\left(\frac{1}{2}\right)}, \quad x \in \mathbb{S},$$

$$\begin{aligned}
&G(x) * \left(\frac{1}{3}b^4 \cosh(\zeta)\sinh^3(\zeta) + 4b^4\cosh^3(\zeta)\sinh(\zeta)\right)(x,t) \\
&= -\frac{b^4}{2\sinh\left(\frac{1}{2}\right)} \int_{\mathbb{S}} \cosh\left(\frac{1}{2} - (x-y) + [x-y]\right) \\
&\quad \left[\frac{1}{3}\cosh\left(\frac{1}{2} - (y-ct) + [y-ct]\right)\sinh^3\left(\frac{1}{2} - (y-ct) + [y-ct]\right)\right. \\
&\quad \left. + 4\cosh^3\left(\frac{1}{2} - (y-ct) + [y-ct]\right)\sinh\left(\frac{1}{2} - (y-ct) + [y-ct]\right)\right] dy,
\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}
&G_x(x) * \left(\frac{3}{2}b^4\sinh^2(\zeta) + \frac{17}{12}b^4\sinh^4(\zeta)\right)(x,t) \\
&= -\frac{b^4}{2\sinh\left(\frac{1}{2}\right)} \int_{\mathbb{S}} \sinh\left(\frac{1}{2} - (x-y) + [x-y]\right)\left[\frac{3}{2}\sinh^2\left(\frac{1}{2} - (y-ct) + [y-ct]\right)\right. \\
&\quad \left. + \frac{17}{12}\sin^4\left(\frac{1}{2} - (y-ct) + [y-ct]\right)\right] dy.
\end{aligned} \tag{2.29}$$

When $x > ct$, the right hand side of (2.28) can be split into the following three parts:

$$\begin{aligned}
&G(x) * \left(\frac{1}{3}b^4 \cosh(\zeta)\sinh^3(\zeta) + 4b^4\cosh^3(\zeta)\sinh(\zeta)\right)(x,t) \\
&= \frac{b^4}{2\sinh\left(\frac{1}{2}\right)} \left(\int_0^{ct} + \int_{ct}^{x} + \int_x^1\right) \cosh\left(\frac{1}{2} - (x-y) + [x-y]\right) \\
&\quad \left[\frac{1}{3}\cosh\left(\frac{1}{2} - (y-ct) + [y-ct]\right)\sinh^3\left(\frac{1}{2} - (y-ct) + [y-ct]\right)\right. \\
&\quad \left. + 4\cosh^3\left(\frac{1}{2} - (y-ct) + [y-ct]\right)\sinh\left(\frac{1}{2} - (y-ct) + [y-ct]\right)\right] dy \\
&=: (III_1 + III_2 + III_3)\frac{b^4}{2\sinh\left(\frac{1}{2}\right)},
\end{aligned} \tag{2.30}$$

where III_1, III_2 and III_3 are calculated in details in Appendix A. So, we arrive at

$$
\begin{aligned}
&G(x) * \left(\frac{1}{3} b^4 \cosh(\zeta) \sinh^3(\zeta) + 4 b^4 \cosh^3(\zeta) \sinh(\zeta)\right)(x,t) \\
&= \frac{b^4}{2\sinh\left(\frac{1}{2}\right)} \left[\frac{13}{360} \cosh\left(\frac{3}{2} - 4x + 4ct\right) - \frac{13}{360} \cosh\left(\frac{5}{2} - 4x + 4(t)\right) \right. \\
&\quad + \frac{11}{36} \cosh\left(\frac{1}{2} - 2x + 2ct\right) - \frac{11}{36} \cosh\left(\frac{3}{2} - 2x + 2ct\right) \\
&\quad + \frac{13}{360} \cosh\left(\frac{5}{2} - x + ct\right) - \frac{13}{360} \cosh\left(-\frac{3}{2} - x + ct\right) \\
&\quad \left. + \frac{11}{36} \cosh\left(\frac{3}{2} - x + ct\right) - \frac{11}{36} \cosh\left(-\frac{1}{2} - x + ct\right) \right]. \qquad (2.31)
\end{aligned}
$$

A similar way sends Eq. (2.29) to the following form

$$
\begin{aligned}
&G_x(x) * \left(\frac{3}{2} b^4 \sinh^2(\zeta) + \frac{17}{12} b^4 \sinh^4(\zeta)\right)(x,t) \\
&= -\frac{b^4}{2\sinh\left(\frac{1}{2}\right)} \left(\int_0^{ct} + \int_{ct}^{x} + \int_x^1\right) \sinh\left(\frac{1}{2} - (x-y) + [x-y]\right) \\
&\quad \times \left[\frac{3}{2} \sinh^2\left(\frac{1}{2} - (y-ct) + [y-ct]\right) + \frac{17}{12} \sinh^4\left(\frac{1}{2} - (y-ct) + [y-ct]\right)\right] dy \\
&=: (IV_1 + IV_2 + IV_3) \frac{b^4}{2\sinh\left(\frac{1}{2}\right)}, \qquad (2.32)
\end{aligned}
$$

where $(IV_1 + IV_2 + IV_3)$ are computed in details in Appendix B. Thus, we have

$$
\begin{aligned}
&G_x(x) * \left(\frac{3}{2} b^4 \sinh^2(\zeta) + \frac{17}{12} b^4 \sinh^4(\zeta)\right)(x,t) \\
&= \frac{b^4}{2\sinh\left(\frac{1}{2}\right)} \left[\frac{17}{360} \cosh\left(\frac{5}{2} - 4x + 4ct\right) - \frac{17}{360} \cosh\left(\frac{3}{2} - 4x + 4ct\right) \right. \\
&\quad + \frac{1}{36} \cosh\left(\frac{3}{2} - 2x + 2ct\right) - \frac{1}{36} \cosh\left(\frac{1}{2} - 2x + 2ct\right) \\
&\quad + \frac{17}{360} \cosh\left(-\frac{3}{2} - x + ct\right) - \frac{17}{360} \cosh\left(\frac{5}{2} - x + ct\right) \\
&\quad \left. + \frac{1}{36} \cosh\left(-\frac{1}{2} - x + ct\right) - \frac{1}{36} \cosh\left(\frac{3}{2} - x + ct\right) \right]. \qquad (2.33)
\end{aligned}
$$

Combining Eq. (2.31) with Eq. (2.33) yields

$$\int_0^{+\infty}\int_{ct}^1\left[G*\left(u^4+\frac{3}{2}u^2u_x^2-\frac{1}{12}u_x^4\right)\psi_x-\frac{1}{3}G*(uu_x^3)\psi\right]dxdt$$
$$=\int_0^{+\infty}\int_{ct}^1\left[\frac{2}{3}b^4\sinh^2\left(\frac{1}{2}\right)\cosh\left(\frac{1}{2}\right)\sinh\left(\frac{1}{2}-x+ct\right)\right.$$
$$+b^4\cosh\left(\frac{1}{2}\right)\sinh\left(\frac{1}{2}-x+ct\right)-\frac{2}{3}b^4\sinh^3\left(\frac{1}{2}-x+ct\right)\cosh\left(\frac{1}{2}-x+ct\right)$$
$$\left.-b^4\sinh\left(\frac{1}{2}-x+ct\right)\cosh\left(\frac{1}{2}-x+ct\right)\right]\psi dxdt \tag{2.34}$$

On the other hand, when $x<ct$, the right-hand side of (2.28) can be split into the following three parts:

$$G(x)*\left(\frac{1}{3}b^4\cosh(\zeta)\sinh^3(\zeta)+4b^4\cosh^3(\zeta)\sinh(\zeta)\right)(x,t)$$
$$=\frac{b^4}{2\sinh\left(\frac{1}{2}\right)}\left(\int_0^x+\int_x^{ct}+\int_{ct}^1\right)\cosh\left(\frac{1}{2}-(x-y)+[x-y]\right)$$
$$\times\left[\frac{1}{3}\cosh\left(\frac{1}{2}-(y-ct)+[y-ct]\right)\sinh^3\left(\frac{1}{2}-(y-ct)+[y-ct]\right)\right.$$
$$\left.+4\cosh^3\left(\frac{1}{2}-(y-ct)+[y-ct]\right)\sinh\left(\frac{1}{2}-(y-ct)+[y-ct]\right)\right]dy$$
$$=:(V_1+V_2+V_3)\frac{b^4}{2\sinh\left(\frac{1}{2}\right)} \tag{2.35}$$

and the right-hand side of (2.29) can be split into the following three parts:

$$G_x(x)*\left(\frac{3}{2}b^4\sinh^2(\zeta)+\frac{17}{12}b^4\sinh^4(\zeta)\right)(x,t)$$
$$=-\frac{b^4}{2\sinh\left(\frac{1}{2}\right)}\left(\int_0^x+\int_x^{ct}+\int_{ct}^1\right)\sinh\left(\frac{1}{2}-(x-y)+[x-y]\right)$$
$$\times\left[\frac{3}{2}\sinh^2\left(\frac{1}{2}-(y-ct)+[y-ct]\right)+\frac{17}{12}\sin^4\left(\frac{1}{2}-(y-ct)+[y-ct]\right)\right]dy$$
$$=:(VI_1+VI_2+VI_3)\frac{b^4}{2\sinh\left(\frac{1}{2}\right)}. \tag{2.36}$$

According to the computations in Appendix C and Appendix D, we arrive at

$$
\begin{aligned}
&G(x) * \left(\frac{1}{3} b^4 \cosh(\zeta) \sinh^3(\zeta) + 4 b^4 \cosh^3(\zeta) \sinh(\zeta)\right)(x,t) \\
&= \frac{b^4}{2\sinh\left(\frac{1}{2}\right)} \left[\frac{13}{360} \cosh\left(-\frac{5}{2} - 4x + 4ct\right) - \frac{13}{360} \cosh\left(-\frac{1}{2} - 4x + 4(t)\right) \right. \\
&\quad + \frac{11}{36} \cosh\left(-\frac{3}{2} - 2x + 2ct\right) - \frac{13}{360} \cosh\left(-\frac{3}{2} - 4x + 4ct\right) \\
&\quad + \frac{13}{360} \cosh\left(\frac{5}{2} - x + ct\right) - \frac{11}{36} \cosh\left(-\frac{3}{2} - x + ct\right) \\
&\quad \left. + \frac{11}{36} \cosh\left(\frac{1}{2} - x + ct\right) - \frac{13}{360} \cosh\left(-\frac{5}{2} - x + ct\right) \right]
\end{aligned} \tag{2.37}
$$

and

$$
\begin{aligned}
&G_x(x) * \left(\frac{3}{2} b^4 \sinh^2(\zeta) + \frac{17}{12} b^4 \sinh^4(\zeta)\right)(x,t) \\
&= \frac{b^4}{2\sinh\left(\frac{1}{2}\right)} \left[-\frac{17}{360} \cosh\left(\frac{3}{2} - x + ct\right) + \frac{17}{360} \cosh\left(-\frac{5}{2} - x + ct\right) \right. \\
&\quad - \frac{1}{36} \cosh\left(\frac{1}{2} - x + ct\right) + \frac{1}{36} \cosh\left(-\frac{3}{2} - x + ct\right) \\
&\quad + \frac{1}{36} \cosh\left(-\frac{1}{2} - 2x + 2ct\right) - \frac{17}{360} \cosh\left(-\frac{5}{2} - 4x + 4ct\right) \\
&\quad \left. + \frac{17}{360} \cosh\left(-\frac{3}{2} - 4x + 4ct\right) - \frac{1}{36} \cosh\left(-\frac{3}{2} - 2x + 2ct\right) \right].
\end{aligned} \tag{2.38}
$$

Combining (2.37) with (2.38) yields

$$
\begin{aligned}
&\int_0^{+\infty} \int_0^{ct} \left[G * \left(u^4 + \frac{3}{2} u^2 u_x^2 - \frac{1}{12} u_x^4\right) \psi_x - \frac{1}{3} G * (u u_x^3) \psi \right] dx dt \\
&= -\int_0^{+\infty} \int_0^{ct} \left[\frac{2}{3} b^4 \sinh^2\left(\frac{1}{2}\right) \cosh\left(\frac{1}{2}\right) \sinh\left(\frac{1}{2} + x - ct\right) \right. \\
&\quad + b^4 \cosh\left(\frac{1}{2}\right) \sinh\left(\frac{1}{2} + x - ct\right) - \frac{2}{3} b^4 \sinh^3\left(\frac{1}{2} + x - ct\right) \cosh\left(\frac{1}{2} + x - ct\right) \\
&\quad \left. - b^4 \sinh\left(\frac{1}{2} + x - ct\right) \cosh\left(\frac{1}{2} + x - ct\right) \right] \psi dx dt.
\end{aligned} \tag{2.39}
$$

Meanwhile, combining (2.34) with (2.39) leads to

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{S}}\left[G*\left(u^4+\frac{3}{2}u^2u_x^2-\frac{1}{12}u_x^4\right)\psi_x-\frac{1}{3}G*(uu_x^3)\psi\right]dxdt\\
&=\int_0^{+\infty}\int_{\mathbb{S}}\left[\frac{2}{3}b^4\sinh^2\left(\frac{1}{2}\right)\cosh\left(\frac{1}{2}\right)\sinh(\zeta)+b^4\cosh\left(\frac{1}{2}\right)\sinh(\zeta)\right.\\
&\quad\left.-\frac{2}{3}b^4\sinh^3(\zeta)\cosh(\zeta)-b^4\sinh(\zeta)\cosh(\zeta)\right]\psi dxdt. \qquad (2.40)
\end{aligned}$$

So, by (2.26) and (2.40), we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{S}}\left[u\psi_t+\frac{1}{4}u^4\psi_x+\frac{1}{3}uu_x^3\psi+G*\left(u^4+\frac{3}{2}u^2u_x^2-\frac{1}{12}u_x^4\right)\partial_x\psi\right.\\
&\left.-\frac{1}{3}G*(uu_x^3)\psi\right]dxdt+\int_{\mathbb{S}}u_0(x)\psi(x,0)\,dx\\
&=\left(-bc+\frac{1+2\cosh^2\left(\frac{1}{2}\right)}{3}b^4\cosh\left(\frac{1}{2}\right)\right)\sinh(\zeta)\\
&=0, \qquad (2.41)
\end{aligned}$$

which completes the proof of the theorem. □

3. Stability

3.1. Stability of peakons

In the subsection, we first prove the orbital stability of peakons. Eq. (1.3) has the following three conservation laws

$$\begin{aligned}
H_0[u]&=\int_{\mathbb{X}}udx, \quad H_1[u]=\int_{\mathbb{X}}\left(u^2+u_x^2\right)dx,\\
H_2[u]&=\int_{\mathbb{X}}\left(u^5+2u^3u_x^2-\frac{1}{3}uu_x^4\right)dx,
\end{aligned} \qquad (3.1)$$

which will play a key role in proving the orbital stability of the peakon solutions and the periodic peakon solutions. Replacing u by $\varphi(x)=\sqrt[3]{\frac{3c}{2}}e^{-|x|}$ leads $H_1[u]$ and $H_2[u]$ to

$$\begin{aligned}
H_1[\varphi]&=\int_{\mathbb{R}}(\varphi^2+\varphi_x^2)\,dx=2\sqrt[3]{\frac{9c^2}{4}},\\
H_2[\varphi]&=\int_{\mathbb{R}}\left(\varphi^5+2\varphi^3\varphi_x^2-\frac{1}{3}\varphi\varphi_x^4\right)dx=\frac{8c}{5}\sqrt[3]{\frac{9c^2}{4}}.
\end{aligned} \qquad (3.2)$$

Next, let us consider the expansion of the conservation law H_1 around the peakon φ in the $H^1(\mathbb{R})$-norm.

Lemma 3.1. *For all $u \in H^1(\mathbb{R})$ and $\xi \in \mathbb{R}$,*

$$H_1[u] - H_1[\varphi] = \|u - \varphi(\cdot - \xi)\|^2_{H^1(\mathbb{R})} + 4\sqrt[3]{\frac{3c}{2}}\left(u(\xi) - \sqrt[3]{\frac{3c}{2}}\right). \qquad (3.3)$$

Proof. Apparently, we have

$$\begin{aligned}
&\|u - \varphi(\cdot - \xi)\|^2_{H^1(\mathbb{R})} \\
&= H_1[u] + H_1[\varphi] - 2\int_{\mathbb{R}} u_x(x)\varphi_x(x-\xi)\ dx - 2\int_{\mathbb{R}} u(x)\varphi(x-\xi)\ dx \\
&= H_1[u] + H_1[\varphi] - 2\int_{-\infty}^{\xi} u_x(x)\varphi_x(x-\xi)dx - 2\int_{\xi}^{+\infty} u_x(x)\varphi_x(x-\xi)dx \\
&\quad - 2\int_{\mathbb{R}} u(x)\varphi(x-\xi)dx. \qquad (3.4)
\end{aligned}$$

Due to

$$\int_{-\infty}^{\xi} u_x(x)\varphi_x(x-\xi)dx = \sqrt[3]{\frac{3c}{2}}u(\xi) - \int_{-\infty}^{\xi} u(x)\varphi(x-\xi)dx, \qquad (3.5)$$

and

$$\int_{\xi}^{+\infty} u_x(x)\varphi_x(x-\xi)dx = \sqrt[3]{\frac{3c}{2}}u(\xi) - \int_{\xi}^{+\infty} u(x)\varphi(x-\xi)dx, \qquad (3.6)$$

we obtain

$$\|u - \varphi(\cdot - \xi)\|^2_{H^1(\mathbb{R})} = H_1[u] - H_1[\varphi] + 4\sqrt[3]{\frac{3c}{2}}\left(\sqrt[3]{\frac{3c}{2}} - u(\xi)\right), \qquad (3.7)$$

which completes the proof of the lemma. □

Lemma 3.2. *For $0 < u(x) \in H^s(\mathbb{R})$, $s > \frac{5}{2}$, let $M = \max_{x\in\mathbb{R}}\{u(x)\}$. Then*

$$H_2[u] \le \frac{4}{3}M^3 H_1[u] - \frac{8}{5}M^5. \qquad (3.8)$$

Proof. Assume $u(x)$ attains the maximum at $\xi \in \mathbb{R}$. Then $M = u(\xi)$. Define

$$g(x) = \begin{cases} u(x) - u_x(x), & x < \xi, \\ u(x) + u_x(x), & x > \xi, \end{cases} \qquad (3.9)$$

Then, we have

$$\begin{aligned}
\int_{\mathbb{R}} g^2(x)dx &= \int_{-\infty}^{\xi}[u(x) - u_x(x)]^2dx + \int_{\xi}^{+\infty}[u(x) + u_x(x)]^2dx \\
&= \int_{\mathbb{R}}[u^2(x) + u_x^2(x)]dx - u^2(x)\Big|_{-\infty}^{\xi} + u^2(x)\Big|_{\xi}^{+\infty} \\
&= H_1[u] - 2M^2.
\end{aligned}$$

Next, let us define

$$h(x)=\begin{cases} u^3(x)-\frac{2}{3}u^2(x)u_x(x)-\frac{1}{3}u(x)u_x^2(x), & x<\xi,\\ \\ u^3(x)+\frac{2}{3}u^2(x)u_x(x)-\frac{1}{3}u(x)u_x^2(x), & x>\xi. \end{cases}$$

Then, we have

$$\begin{aligned}
&\int_{\mathbb{R}} h(x)g^2(x)dx\\
&=\int_{-\infty}^{\xi}\left(u^3-\frac{2}{3}u^2u_x-\frac{1}{3}uu_x^2\right)(u-u_x)^2\,dx\\
&\quad+\int_{\xi}^{+\infty}\left(u^3+\frac{2}{3}u^2u_x-\frac{1}{3}uu_x^2\right)(u+u_x)^2\,dx\\
&=\int_{\mathbb{R}}\left(u^5+2u^3u_x^2-\frac{1}{3}uu_x^4\right)dx-\frac{8}{3}\int_{-\infty}^{\xi}u^4u_xdx+\frac{8}{3}\int_{\xi}^{+\infty}u^4u_xdx\\
&=H_2[u]-\frac{16}{15}M^5. && (3.10)
\end{aligned}$$

Employing the Young's inequality leads to

$$h(x)=u^3(x)\pm\frac{2}{3}u^2(x)u_x(x)-\frac{1}{3}uu_x^2(x)\le u^3(x)+\frac{1}{3}u^3(x)\le\frac{4}{3}M^3.$$

Therefore, we obtain

$$\int_{\mathbb{R}} h(x)g^2(x)dx\le\frac{4}{3}M^3\int_{\mathbb{R}} g^2(x)dx. \tag{3.11}$$

Combining the above three relations (3.10) and (3.11) reveals

$$H_2[u]\le\frac{4}{3}M^3H_1[u]-\frac{8}{5}M^5,$$

which is desired in Lemma 3.4. □

Lemma 3.3. *For all $u\in H^s(\mathbb{R})$, $s>\frac{5}{2}$, if $\|u-\varphi\|_{H^1}<\delta$ with $\delta\in(0,1)$, then*

$$|H_1[u]-H_1[\varphi]|\le\left(1+4\sqrt[3]{c}\right)\delta, \tag{3.12}$$

and

$$|H_2[u]-H_2[\varphi]|\le B(c)\delta, \tag{3.13}$$

where

$$\begin{aligned}
B(c)=&\left[\frac{\sqrt{2}}{2}\left(1+4\sqrt[3]{c}+12\sqrt[3]{c^2}\right)\left(1+4\sqrt[3]{c}+\sqrt[3]{18c^2}\right)+3c\left(1+4\sqrt[3]{c}\right)\right.\\
&\left.+\frac{\sqrt{2}}{12}\left(\frac{3c}{2}\right)^{\frac{4}{3}}+\frac{\sqrt{3}}{3}\left(1+2\sqrt[3]{c}\right)\sqrt{A+c^2}\right],
\end{aligned}$$

and $A > 0$ is a constant depending only on the norm $\|u\|_{H^s(\mathbb{R})}$.

Proof. See Appendix E.
Lemma 3.4. *For all $u \in H^s(\mathbb{R})$, $s > \frac{5}{2}$, let $M = \max_{x\in\mathbb{R}}\{u(x)\}$. If*

$$|H_1[u] - H_1[\varphi]| \leq \left(1 + 4\sqrt[3]{c}\right)\delta$$

and

$$|H_2[u] - H_2[\varphi]| \leq B(c)\delta$$

for some $\delta \in (0,1)$, then

$$|M - \sqrt[3]{\frac{3c}{2}}| \leq \delta^{\frac{1}{2}} \sqrt{\frac{5}{6c}\left(\frac{1}{2} + 2\sqrt[3]{c} + \frac{\sqrt[3]{18c^2}}{2}\right)^{\frac{3}{2}} \left(1 + 4\sqrt[3]{c}\right) + B(c)}. \tag{3.14}$$

Proof. See Appendix F.

Now, let us come to the **Proof of Theorem 1.1 for $\mathbb{X} = \mathbb{R}$.** Since $H_1[u]$, $H_2[u]$ are both conserved by Eq. (1.3), we have

$$H_1[u(\cdot,t)] = H_1[u_0],\ H_2[u(\cdot,t)] = H_2[u_0],\ t \in (0,T). \tag{3.15}$$

Applying Lemma 3.3 to u_0 with $0 < \delta < 1$ leads the hypotheses of Lemma 3.4 to hold for $u(,t)$ due to (3.15). Then, we obtain

$$\left|u(\xi(t),t) - \sqrt[3]{\frac{3c}{2}}\right| \leq \delta^{\frac{1}{2}} \sqrt{\frac{5}{6c}\left(\frac{1}{2} + 2\sqrt[3]{c} + \frac{\sqrt[3]{18c^2}}{2}\right)^{\frac{3}{2}} \left(1 + 4\sqrt[3]{c}\right) + B(c)}. \tag{3.16}$$

Combining (3.3) with Lemma 3.3 yields

$$\begin{aligned}
&\|u(\cdot,t) - \varphi(\cdot - \xi(t))\|^2_{H^1(\mathbb{R})} \\
&= H_1[u] - H_1[\varphi] + 4\sqrt[3]{\frac{3c}{2}}\left(\sqrt[3]{\frac{3c}{2}} - u(\xi,t)\right) \\
&\leq |H_1[u] - H_1[\varphi]| + 4\sqrt[3]{\frac{3c}{2}}\left|\sqrt[3]{\frac{3c}{2}} - u(\xi,t)\right| \\
&\leq \left(1 + 4\sqrt[3]{c}\right)\delta + \delta^{\frac{1}{2}}\sqrt{\frac{5}{6c}\left(\frac{1}{2} + 2\sqrt[3]{c} + \frac{\sqrt[3]{18c^2}}{2}\right)^{\frac{3}{2}} \left(1 + 4\sqrt[3]{c}\right) + B(c)} \\
&\leq \delta^{\frac{1}{2}}\left[\left(1 + 4\sqrt[3]{c}\right) + \sqrt{\frac{5}{6c}\left(\frac{1}{2} + 2\sqrt[3]{c} + \frac{\sqrt[3]{18c^2}}{2}\right)^{\frac{3}{2}} \left(1 + 4\sqrt[3]{c}\right) + B(c)}\right].
\end{aligned}$$

Therefore, for any $\epsilon > 0$, let us choose

$$\delta = \epsilon^2 \left[(1 + 4\sqrt[3]{c}) + \sqrt{\frac{5}{6c}\left(\frac{1}{2} + 2\sqrt[3]{c} + \frac{\sqrt[3]{18c^2}}{2}\right)^{\frac{3}{2}} (1 + 4\sqrt[3]{c}) + B(c)} \right]^{-2},$$

then $\|u(\cdot,t) - \varphi(\cdot - \xi(t))\|^2_{H^1(\mathbb{R})} < \epsilon$. Thus, we complete the proof of Theorem 1.1 for $\mathbb{X} = \mathbb{R}$. $\square$

3.2. Stability of periodic peakons

This subsection is devoted to proving the stability of periodic peakons for Eq. (1.3). Let us first give some basic properties of periodic peakons. It is obvious that the periodic peaked function

$$u(x,t) = \varphi(x - ct),$$

can be extended to the whole line, where $\varphi(x)$ is given for $x \in [0,1]$ by

$$\varphi(x) = b\cosh\left(\frac{1}{2} - x\right), \quad b = \sqrt[3]{\frac{3c}{\left(2\cosh^2\left(\frac{1}{2}\right) + 1\right)\cosh\left(\frac{1}{2}\right)}}\cosh\left(\frac{1}{2} - x\right).$$

Let us still use $\mathbb{S}$ with the interval $[0,T)$ and treat all functions on $\mathbb{S}$ as periodic functions with the period T on the entire line.

As we know, Eq. (1.3) has the following three conservation laws

$$H_0[u] = \int_{\mathbb{S}} u dx,\ H_1[u] = \int_{\mathbb{S}} \left(u^2 + u_x^2\right) dx,\ H_2[u] = \int_{\mathbb{S}} \left(u^5 + 2u^3u_x^2 - \frac{1}{3}uu_x^4\right) dx. \tag{3.17}$$

For an integer $n \geq 1$, let $H^n(\mathbb{S})$ be the Sobolev space of all square integrable functions $f \in L^2(\mathbb{S})$ with distributional derivatives $\partial_x^i f \in L^2(\mathbb{S})$ for $i = 1, ..., n$. These Hilbert spaces are endowed with the inner product

$$\langle f, g\rangle_{H^n(\mathbb{S})} = \sum_{i=0}^{n} \int_{\mathbb{S}} (\partial_x^i f)(x)(\partial_x^i g)(x)\ dx. \tag{3.18}$$

A function $u \in C([0,T]; H^1(\mathbb{S}))$ is referred to a solution to the QCHE (1.3) on $[0,T]$ with the period $T > 0$ if the equation holds in the distribution sense. Apparently, the functionals $H_i[u], i = 0,1,2$, defined in (3.17) are independent of $t \in [0,T]$.

Note that φ is continuous on $\mathbb{S}$ with a peak at $x = 0$. A simple calculation yields

$$M_\varphi = \varphi(0) = b\cosh\left(\frac{1}{2}\right), \quad m_\varphi = \varphi\left(\frac{1}{2}\right) = b. \tag{3.19}$$

Moreover, φ is smooth on $(0,T)$, where $T = 1$. This reveals, as $\varphi_{xx}(x) = \varphi(x)$ on $(0,1)$, that the integration-by-parts formula $\int \varphi_{xx} f dx = -\int \varphi_x f_x dx$, holds with $\varphi_{xx}(x) = \varphi(x) - 2b\sinh(\frac{1}{2})\delta(x)$. Here, δ denotes the Dirac delta distribution. For simplicity, let us adopt the notation by writing integrals, and then we have

$$H_0[\varphi] = \int_0^1 b\cosh\left(\frac{1}{2} - x\right)\ dx = 2b\sinh\left(\frac{1}{2}\right), \tag{3.20}$$

$$\begin{aligned} H_1[\varphi] &= \int_{\mathbb{S}} (\varphi^2 + \varphi_x^2)dx = \int_{\mathbb{S}} (\varphi^2 - \varphi\varphi_{xx})dx \\ &= \int_{\mathbb{S}} \left(\varphi^2 - \varphi\left(\varphi - 2b\sinh\left(\frac{1}{2}\right)\delta\right)\right)dx \\ &= 2b\sinh\left(\frac{1}{2}\right)M_\varphi, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} H_2[\varphi] &= \int_{\mathbb{S}} \left(\varphi^5 + 2\varphi^3\varphi_x^2 - \frac{1}{3}\varphi\varphi_x^4\right)\ dx \\ &= b^5 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\cosh^5(x) + 2\cosh^3(x)\sinh^2(x) - \frac{1}{3}\cosh(x) sinh^4(x)\right)dx \\ &= b^5 \left[\frac{6}{5}\cosh^4\left(\frac{1}{2}\right)\sinh\left(\frac{1}{2}\right) + \frac{4}{15}\sinh\left(\frac{1}{2}\right)\cosh^2\left(\frac{1}{2}\right)\right. \\ &\quad \left. + \frac{8}{15}\sinh\left(\frac{1}{2}\right) - \frac{2}{15}\sinh^5\left(\frac{1}{2}\right)\right]. \end{aligned} \tag{3.22}$$

Lemma 3.5. *For all $u \in H^1(\mathbb{S})$ and $\xi \in \mathbb{S}$,*

$$H_1[u] - H_1[\varphi] = \|u - \varphi(\cdot - \xi)\|^2_{H^1(\mathbb{S})} + 4b\sinh\left(\frac{1}{2}\right)(u(\xi) - M_\varphi). \tag{3.23}$$

Proof. Due to $\varphi_{xx}(x) = \varphi(x) - 2b\sinh(\frac{1}{2})\delta(x)$, we have

$$\begin{aligned} &\|u - \varphi(\cdot - \xi)\|^2_{H^1(\mathbb{S})} \\ &= H_1[u] + H_1[\varphi] - 2\int_{\mathbb{S}} u_x(x)\varphi_x(x-\xi)\ dx - 2\int_{\mathbb{S}} u(x)\varphi(x-\xi)\ dx \\ &= H_1[u] + H_1[\varphi] + 2\int_{\mathbb{S}} u(x+\xi)\varphi_{xx}(x)dx - 2\int_{\mathbb{S}} u(x+\xi)\varphi(x)dx \\ &= H_1[u] + H_1[\varphi] - 4b\sinh\left(\frac{1}{2}\right)u(\xi) \\ &= H_1[u] - H_1[\varphi] + 4b\sinh\left(\frac{1}{2}\right)(M_\varphi - u(\xi)). \end{aligned}$$

which completes the proof of this lemma. □

Lemma 3.6 *For any positive $u \in H^1(\mathbb{S})$, let*

$$F_u : \{(M,m) \in \mathbb{R}^2 : M \geq m > 0\} \to \mathbb{R} \tag{3.24}$$

be the function defined by

$$\begin{aligned} &F_u(M,m) \\ &= \left(\frac{4}{3}M^3 + \frac{2}{3}Mm^2\right)\left[H_1[u] + 2m^2 \ln\left(\frac{M+\sqrt{M^2-m^2}}{m}\right)\right. \\ &\left. - 2M\sqrt{M^2-m^2} - m^2\right] + \left(\frac{16}{15}M^2 + \frac{8}{5}m^2\right)\left(M^2-m^2\right)\sqrt{M^2-m^2} \\ &+ m^4 H_0[u] - H_2[u]. \end{aligned}$$

Then, we have

$$F_u(M_u, m_u) \geq 0,$$

where $M_u = \max_{x\in\mathbb{S}}\{u(x)\}$ and $m_u = \min_{x\in\mathbb{S}}\{u(x)\}$.

Proof. Note that the periodic peakon φ satisfies the following differential relations

$$\varphi_x = \begin{cases} -\sqrt{\varphi^2 - m_\varphi^2}, & 0 < x \leq \frac{1}{2}, \\ \sqrt{\varphi^2 - m_\varphi^2}, & \frac{1}{2} < x < 1. \end{cases} \tag{3.25}$$

Let $u \in H^1(\mathbb{S}) \subset C(\mathbb{S})$ be a positive function and write $M = M_u = \max_{x\in\mathbb{S}}\{u(x)\}$ and $m = m_u = \min_{x\in\mathbb{S}}\{u(x)\}$. Let ξ and η be two extreme points such that $u(\xi) = M$ and $u(\eta) = m$. Define the real function $g(x)$ as follows

$$g(x) = \begin{cases} u_x + \sqrt{u^2-m^2}, & \xi < x \leq \eta, \\ u_x - \sqrt{u^2-m^2}, & \eta < x < \xi + 1, \end{cases} \tag{3.26}$$

and extend it periodically to the whole real line. Then, we have

$$\begin{aligned} \int_S g^2(x)dx &= \int_\xi^\eta \left(u_x + \sqrt{u^2-m^2}\right)^2 dx + \int_\eta^{\xi+1}\left(u_x - \sqrt{u^2-m^2}\right)^2 dx \\ &= 2m^2 \ln\left(\frac{M+\sqrt{M^2-m^2}}{m}\right) - 2M\sqrt{M^2-m^2} - m^2 + H_1[u] \end{aligned}$$

Next, define the function $h(x)$ as follows

$$h(x) = \begin{cases} u^3 + \frac{2}{3}uu_x\sqrt{u^2-m^2} - \frac{1}{3}uu_x^2 + um^2, & \xi < x \leq \eta, \\ u^3 - \frac{2}{3}uu_x\sqrt{u^2-m^2} - \frac{1}{3}uu_x^2 + um^2, & \eta \leq x < \xi + 1. \end{cases} \tag{3.27}$$

and extend it periodically to the whole real line as well. Then, we obtain

$$
\begin{aligned}
&\int_{\mathbb{S}} h(x)g^2(x)dx \\
&= \int_{\xi}^{\eta} \left(u^3 + \frac{2}{3}uu_x\sqrt{u^2-m^2} - \frac{1}{3}uu_x^2 + um^2\right)\left(u_x + \sqrt{u^2-m^2}\right)^2 dx \\
&\quad + \int_{\eta}^{\xi+1} \left(u^3 - \frac{2}{3}uu_x\sqrt{u^2-m^2} - \frac{1}{3}uu_x^2 + um^2\right)\left(u_x - \sqrt{u^2-m^2}\right)^2 dx.
\end{aligned}
$$

A direct calculation leads to

$$
\begin{aligned}
&\int_{\xi}^{\eta} \left(u^3 + \frac{2}{3}uu_x\sqrt{u^2-m^2} - \frac{1}{3}uu_x^2 + um^2\right)\left(u_x^2 + 2u_x\sqrt{u^2-m^2} + u^2 - m^2\right) dx \\
&= \int_{\xi}^{\eta} \left(u^5 + 2u^3u_x^2 - \frac{1}{3}uu_x^4\right) dx + \frac{8}{3}\int_{\xi}^{\eta} u^3u_x\sqrt{u^2-m^2}dx \\
&\quad + \frac{4}{3}m^2\int_{\xi}^{\eta} uu_x\sqrt{u^2-m^2}dx - m^4\int_{\xi}^{\eta} udx
\end{aligned} \tag{3.28}
$$

and

$$
\begin{aligned}
&\int_{\eta}^{\xi+1} \left(u^3 + \frac{2}{3}uu_x\sqrt{u^2-m^2} - \frac{1}{3}uu_x^2 + um^2\right)\left(u_x^2 + 2u_x\sqrt{u^2-m^2} + u^2 - m^2\right) dx \\
&= \int_{\eta}^{\xi+1} \left(u^5 + 2u^3u_x^2 - \frac{1}{3}uu_x^4\right) dx - \frac{8}{3}\int_{\eta}^{\xi+1} u^3u_x\sqrt{u^2-m^2}dx \\
&\quad - \frac{4}{3}m^2\int_{\eta}^{\xi+1} uu_x\sqrt{u^2-m^2}dx - m^4\int_{\eta}^{\xi+1} udx.
\end{aligned} \tag{3.29}
$$

Since

$$
\frac{d}{dx}\left(-\frac{1}{15}(M-m)(M+m)\left(3M^2+2m^2\right)\sqrt{M^2-m^2}\right) = u^3u_x\sqrt{M^2-m^2} \tag{3.30}
$$

and

$$
\frac{d}{dx}\left(-\frac{1}{3}(M-m)(M+m)\sqrt{M^2-m^2}\right) = uu_x\sqrt{M^2-m^2}, \tag{3.31}
$$

we arrive at

$$
\begin{aligned}
\int_{\mathbb{S}} h(x)g^2(x)dx =& H_2[u] - \left(\frac{16}{15}M^2 + \frac{8}{5}m^2\right)\left(M^2-m^2\right)\sqrt{M^2-m^2} \\
&- m^4H_0[u].
\end{aligned} \tag{3.32}
$$

Adopting the Young's inequality generates

$$
\begin{aligned}
h(x) &= u^3(x) \pm \frac{2}{3}uu_x(x)\sqrt{u^2(x)-m^2} - \frac{1}{3}uu_x^2(x) + um^2 \\
&\le u^3(x) + \frac{1}{3}\left(u^3(x) - um^2\right) + um^2 \\
&\le \frac{4}{3}M^3 + \frac{2}{3}Mm^2.
\end{aligned} \tag{3.33}
$$

Since

$$\int_{\mathbb{S}} h(x)g^2(x)dx \le \left(\frac{4}{3}M^3 + \frac{2}{3}Mm^2\right) \int_{\mathbb{S}} g^2(x)dx,$$

we arrive at

$$\begin{aligned}
0 \le &\left(\frac{4}{3}M^3 + \frac{2}{3}Mm^2\right) \Bigg[H_1[u] + 2m^2 \ln\left(\frac{M + \sqrt{M^2 - m^2}}{m}\right) \\
&- 2M\sqrt{M^2 - m^2} - m^2\Bigg] + \left(\frac{16}{15}M^2 + \frac{8}{5}m^2\right)\left(M^2 - m^2\right)\sqrt{M^2 - m^2} \\
&+ m^4 H_0[u] - H_2[u],
\end{aligned}$$

which completes the proof of lemma. □

Lemma 3.7. *The peaked function φ satisfies the following relations:*

$$\begin{aligned}
&F_\varphi(M_\varphi, m_\varphi) = 0, \quad \frac{\partial F_\varphi}{\partial M}(M_\varphi, m_\varphi) = 0, \\
&\frac{\partial F_\varphi}{\partial m}(M_\varphi, m_\varphi) = 0, \quad \frac{\partial^2 F_\varphi}{\partial M \partial m}(M_\varphi, m_\varphi) = 0, \\
&\frac{\partial^2 F_\varphi}{\partial M^2}(M_\varphi, m_\varphi) = -\frac{272}{15} b^3 \sinh^3\left(\frac{1}{2}\right) - \frac{344}{15} b^3 \sinh\left(\frac{1}{2}\right) - \frac{32}{15\sinh\left(\frac{1}{2}\right)} b^3, \\
&\frac{\partial^2 F_\varphi}{\partial m^2}(M_\varphi, m_\varphi) = -\frac{16}{3} b^3 \cosh^2\left(\frac{1}{2}\right) \sinh\left(\frac{1}{2}\right) - 8b^3 \sinh\left(\frac{1}{2}\right).
\end{aligned}$$

Proof. Verification with a lengthy computation for each equation is done in Appendix G.

Lemma 3.8. *Suppose $f \in H^1(\mathbb{S})$, then*

$$\max_{x \in \mathbb{S}} |f(x)| \le \sqrt{\frac{\cosh\left(\frac{1}{2}\right)}{2\sinh\left(\frac{1}{2}\right)}} \|f\|_{H^1(\mathbb{S})}. \tag{3.34}$$

Here, "equal" holds if and only if $f = \varphi(\cdot - \xi)$ for some $\xi \in \mathbb{R}$, that is, f is a peakon.

Proof. For $x \in \mathbb{S}$, we have

$$\begin{aligned}
\langle \varphi(\cdot - x), f \rangle_{H^1(\mathbb{S})} &= \int_{\mathbb{S}} \left(\varphi(y - x)f(y) + \varphi'(y - x)f'(y)\right) \, dy \\
&= \int_{\mathbb{S}} (\varphi - \varphi'')(y - x)f(y) \, dy \\
&= \int_{\mathbb{S}} 2b\sinh\left(\frac{1}{2}\right) \delta(y - x)f(y)dy = 2b\sinh\left(\frac{1}{2}\right) f(x).
\end{aligned}$$

Since

$$H_1[\varphi] = \|\varphi\|^2_{\mathrm{H}^1(\mathbb{S})} = 2b^2 \cosh\left(\frac{1}{2}\right) \sinh\left(\frac{1}{2}\right),$$

we have

$$\begin{aligned} f(x) = \frac{1}{2b\sinh\left(\frac{1}{2}\right)} \langle \varphi(\cdot - x), f\rangle_{\mathrm{H}^1(S)} &\leq \frac{1}{2b\sinh\left(\frac{1}{2}\right)} \|\varphi\|_{\mathrm{H}^1(\mathbb{S})} \|f\|_{\mathrm{H}^1(S)} \\ &= \sqrt{\frac{\cosh\left(\frac{1}{2}\right)}{2\sinh\left(\frac{1}{2}\right)}} \|f\|_{H^1(\mathbb{S})}, \end{aligned} \tag{3.35}$$

where "equal" holds true if and only if f and $\varphi(\cdot - x)$ are proportional. Taking the maximum of (3.35) over $\mathbb{S}$ completes the proof of the lemma. □

Lemma 3.9. *If $u \in C([0,T); H^1(\mathbb{S}))$, then $M_{u(t)} = \max_{x\in S} u(x,t)$ and $m_{u(t)} = \min_{x\in S} u(x,t)$ are continuous functions of $t \in [0,T)$.*

Proof. By Lemma 3.8, for $t, s \in [0,T)$, we have

$$\begin{aligned} |M_{u(t)} - M_{u(s)}| &= |\max_{x\in\mathbb{S}} u(x,t) - \max_{x\in\mathbb{S}} u(x,s)| \\ &\leq \max_{x\in\mathbb{S}} |u(x,t) - u(x,s)| \\ &\leq \sqrt{\frac{\cosh\left(\frac{1}{2}\right)}{2\sinh\left(\frac{1}{2}\right)}} \|u(x,t) - u(x,s)\|_{H^1(\mathbb{S})}, \end{aligned}$$

which implies that $M_{u(t)}$ is continuous. The continuity of $m_{u(t)}$ is evident since $m_{u(t)} = -M_{-u(t)}$. □

Lemma 3.10. *Let $u \in C([0,T); H^1(\mathbb{S}))$ be a solution of (1.3). Given a small neighborhood $\mathcal{U}$ of (M_φ, m_φ) in $\mathbb{R}^2$, there is a $\delta > 0$ such that*

$$(M_{u(t)}, m_{u(t)}) \in \mathcal{U} \quad for\ t \in [0,T)\ if\ \|u(\cdot,0) - \varphi\|_{H^1(\mathbb{S})} < \delta. \tag{3.36}$$

Proof. Suppose $H_i[u] = H_i[\varphi] + \varepsilon_i$, $i = 0,1,2$. Then, we have

$$F_u(M,m) = F_\varphi(M,m) + \left(\frac{4}{3}M^3 + \frac{2}{3}Mm^2\right)\varepsilon_1 + m^4\varepsilon_0 - \varepsilon_2. \tag{3.37}$$

So, F_u is a small perturbation of F_φ. The effect of the perturbation near the point (M_φ, m_φ) can be made arbitrarily small by choosing a small ε'. Lemma 3.7 says that $F_\varphi(M_\varphi, m_\varphi) = 0$ and that F_φ has a critical point with negative definite second derivative at (M_φ, m_φ). By continuity of the second derivative, there is a neighborhood around (M_φ, m_φ), where F_φ is concave with curvature bounded away from zero. Therefore, after a small perturbation, the set where $F_u \geq 0$ near (M_φ, m_φ) will be contained in a neighbor-hood of (M_φ, m_φ).

Let $\mathcal{U}$ be the neighborhood given as in the statement. Shrinking $\mathcal{U}$ if necessary, we infer the existence of a $\delta' > 0$ such that for u

$$H_i[u] - H_i[\varphi] < \delta', \quad i = 0, 1, 2, \tag{3.38}$$

which reveals that the set where $F_u \geq 0$ near (M_φ, m_φ) is contained in $\mathcal{U}$, and $\mathcal{U}$ is surrounded by a set where $F_u < 0$. Lemmas 3.9 and 3.6 say that $M_{u(t)}$ and $m_{u(t)}$ are continuous functions of $t \in [0, T)$, and $F_u(M_{u(t)}, m_{u(t)}) \geq 0$ for $t \in [0, T)$. Thus, for the u satisfying (3.38), we have

$$(M_{u(t)}, m_{u(t)}) \in \mathcal{U} \;\; for \;\; t \in [0, T) \; if \; (M_{u(0)}, m_{u(0)}) \in \mathcal{U}. \tag{3.39}$$

However, the continuity of the conserved functionals $H_i : H^1(\mathbb{S}) \to \mathbb{R},\ i = 0, 1, 2$, shows that there is a $\delta > 0$ such that (3.38) holds for all u with

$$\|u(\cdot, 0) - \varphi\|_{H^1(\mathbb{S})} < \delta. \tag{3.40}$$

Moreover, by Lemma 3.8, taking a smaller δ, we may cast

$$(M_{u(0)}, m_{u(0)}) \in \mathcal{U} \;\; if \; \|u(\cdot, 0) - \varphi\|_{H^1(\mathbb{S})} < \delta,$$

which completes the lemma. □

Now, let us come to the **Proof of Theorem 1.1 for $\mathbb{X} = \mathbb{S}$.** Let $u \in C([0, T); H^1(\mathbb{S}))$ be a solution of (1.3) and suppose we are given an $\varepsilon > 0$. Pick a neighborhood $\mathcal{U}$ of (M_φ, m_φ) small enough such that $|M - M_\varphi| < \frac{1}{8b\sinh\left(\frac{1}{2}\right)}\varepsilon$ if $(M, m) \in \mathcal{U}$. Choose a $\delta > 0$ as in Lemma 3.10 so that (3.36) holds. Taking a smaller δ if necessary, we may place

$$|H_1[u] - H_1[\varphi]| < \frac{1}{2}\varepsilon,$$

if $|u(\cdot, 0) - \varphi\|_{H^1(\mathbb{S})} < \delta$.

From Lemma 3.7, we conclude that

$$\|u - \varphi(\cdot - \xi(t))\|^2_{H^1(\mathbb{S})} = H_1[u] - H_1[\varphi] + 4b\sinh\left(\frac{1}{2}\right)\left(M_\varphi - M_{u(t)}\right) \leq \varepsilon \tag{3.41}$$

where $\xi(t) \in \mathbb{R}$ is any point where $u(\xi(t), t) = M_{u(t)}$. This completes the proof of the theorem for $\mathbb{X} = \mathbb{S}$. □

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Appendix A

In this Appendix, we will show some basic properties of the hyperbolic function. We list some formulas for hyperbolic functions.

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

$$\cosh(2x) = 2\sinh^2(x) + 1,$$
$$\sinh(3x) = 3\sinh(x) + 4\sinh^3(x),$$
$$\cosh(3x) = 4\cosh^3(x) - 3\cosh(x),$$
$$\cosh(4x) = 8\sinh^3(x)\cosh(x) + 4\sinh(x)\cosh(x)$$

and

$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

From the above formula, we can get the detailed calculations of III_1, III_2 and III_3.

$$\begin{aligned}
III_1 =& \int_0^{ct} \cosh\left(\frac{1}{2} - x + y\right) \left[\frac{1}{3}\cosh\left(-\frac{1}{2} - y + ct\right)\sinh^3\left(-\frac{1}{2} - y + ct\right)\right.\\
&\left. + 4\cosh^3\left(-\frac{1}{2} - y + ct\right)\sinh\left(-\frac{1}{2} - y + ct\right)\right] dy\\
=& \frac{13}{240}\cosh\left(-\frac{5}{2} - 4x + 4ct\right) + \frac{13}{144}\cosh\left(-\frac{3}{2} - 4x + 4ct\right)\\
&+ \frac{11}{24}\cosh\left(-\frac{1}{2} - x + 2ct\right) + \frac{11}{72}\cosh\left(-\frac{3}{2} + x + 2ct\right)\\
&- \frac{13}{240}\cosh\left(\frac{5}{2} - x + ct\right) - \frac{11}{24}\cosh\left(-\frac{1}{2} - x + ct\right)\\
&- \frac{11}{72}\cosh\left(\frac{3}{2} - x + ct\right) - \frac{13}{144}\cosh\left(-\frac{3}{2} - x + ct\right),
\end{aligned} \tag{3.42}$$

$$\begin{aligned}
III_2 =& \int_{ct}^{x} \cosh\left(\frac{1}{2} - x + y\right) \left[\frac{1}{3}\cosh\left(\frac{1}{2} - y + ct\right)\sinh^3\left(\frac{1}{2} - y + ct\right)\right.\\
&\left. + 4\cosh^3\left(\frac{1}{2} - y + ct\right)\sinh\left(\frac{1}{2} - y + ct\right)\right] dy\\
=& -\frac{13}{240}\cosh\left(\frac{3}{2} - 4x + 4ct\right) - \frac{13}{144}\cosh\left(\frac{5}{2} - 4x + 4ct\right)\\
&- \frac{11}{24}\cosh\left(\frac{3}{2} - 2x + 2ct\right) - \frac{11}{72}\cosh\left(\frac{1}{2} - 2x + 2ct\right)\\
&+ \frac{13}{240}\cosh\left(-\frac{3}{2} - x + ct\right) + \frac{11}{24}\cosh\left(\frac{3}{2} - x + ct\right)\\
&+ \frac{11}{72}\cosh\left(-\frac{1}{2} - x + ct\right) + \frac{13}{144}\cosh\left(\frac{5}{2} - x + ct\right)
\end{aligned} \tag{3.43}$$

and

$$
\begin{aligned}
III_3 = & \int_x^1 \cosh\left(-\frac{1}{2} - x + y\right) \left[\frac{1}{3}\cosh\left(\frac{1}{2} - y + ct\right)\sinh^3\left(\frac{1}{2} - y + ct\right)\right. \\
& \left. + 4\cosh^3\left(\frac{1}{2} - y + ct\right)\sinh\left(\frac{1}{2} - y + ct\right)\right] dy \\
= & \frac{13}{240}\cosh\left(\frac{5}{2} - 4x + 4ct\right) - \frac{13}{240}\cosh\left(-\frac{5}{2} + x + 4ct\right) \\
& + \frac{13}{144}\cosh\left(\frac{3}{2} - 4x + 4ct\right) - \frac{13}{144}\cosh\left(-\frac{3}{2} - x + 4ct\right) \\
& + \frac{11}{24}\cosh\left(\frac{1}{2} - 2x + 2ct\right) + \frac{11}{72}\cosh\left(\frac{3}{2} - 2x + 2ct\right) \\
& - \frac{11}{24}\cosh\left(-\frac{1}{2} - x + 2ct\right) - \frac{11}{72}\cosh\left(-\frac{3}{2} + x + 2ct\right). \qquad (3.44)
\end{aligned}
$$

Appendix B

In this Appendix, we can get detailed calculations for IV_1, IV_2 and IV_3 similarly.

$$
\begin{aligned}
IV_1 = & -\int_0^{ct} \sinh\left(\frac{1}{2} - x + y\right)\left[\frac{3}{2}\sinh^2\left(-\frac{1}{2} - y + ct\right) + \frac{17}{12}\sinh^4\left(-\frac{1}{2} - y + ct\right)\right] dy \\
= & \frac{17}{960}\cosh\left(-\frac{5}{2} + x + 4ct\right) - \frac{17}{576}\cosh\left(-\frac{3}{2} - x + 4ct\right) - \frac{7}{32}\cosh\left(-\frac{1}{2} + x\right) \\
& + \frac{1}{144}\cosh\left(-\frac{3}{2} + x + 2ct\right) - \frac{1}{48}\cosh\left(-\frac{1}{2} - x + 2ct\right) + \frac{7}{32}\cosh\left(\frac{1}{2} - x + ct\right) \\
& + \frac{17}{576}\cosh\left(-\frac{3}{2} - x + ct\right) - \frac{1}{144}\cosh\left(\frac{3}{2} - x + ct\right) \\
& - \frac{17}{960}\cosh\left(\frac{5}{2} - x + ct\right) + \frac{1}{48}\cosh\left(-\frac{1}{2} - x + ct\right), \qquad (3.45)
\end{aligned}
$$

$$
\begin{aligned}
IV_2 = & -\int_{ct}^{x} \sinh\left(\frac{1}{2} - x + y\right)\left[\frac{3}{2}\sinh^2\left(\frac{1}{2} - y + ct\right) + \frac{17}{12}\sinh^4\left(\frac{1}{2} - y + ct\right)\right] dy \\
= & \frac{17}{576}\cosh\left(\frac{5}{2} - 4x + 4ct\right) - \frac{17}{960}\cosh\left(\frac{3}{2} - 4x + 4ct\right) + \frac{7}{32}\cosh\left(\frac{1}{2}\right) \\
& - \frac{1}{144}\cosh\left(\frac{1}{2} - 2x + 2ct\right) + \frac{1}{48}\cosh\left(\frac{3}{2} - 2x + 2ct\right) - \frac{7}{32}\cosh\left(\frac{1}{2} - x + ct\right) \\
& - \frac{17}{576}\cosh\left(\frac{5}{2} - x + ct\right) + \frac{1}{144}\cosh\left(-\frac{1}{2} - x + ct\right) \\
& + \frac{17}{960}\cosh\left(-\frac{3}{2} - x + ct\right) - \frac{1}{48}\cosh\left(\frac{3}{2} - x + ct\right) \qquad (3.46)
\end{aligned}
$$

and

$$
\begin{aligned}
IV_3 = & -\int_x^1 \sinh\left(-\frac{1}{2} - x + y\right)\left[\frac{3}{2}\sinh^2\left(\frac{1}{2} - y + ct\right) + \frac{17}{12}\sinh^4\left(\frac{1}{2} - y + ct\right)\right] dy \\
= & \frac{17}{960}\cosh\left(\frac{5}{2} - 4x + 4ct\right) - \frac{17}{576}\cosh\left(\frac{3}{2} - 4x + 4ct\right) + \frac{7}{32}\cosh\left(-\frac{1}{2} + x\right) \\
& - \frac{1}{48}\cosh\left(\frac{1}{2} - 2x + 2ct\right) - \frac{7}{32}\cosh\left(-\frac{1}{2}\right) - \frac{17}{960}\cosh\left(-\frac{5}{2} + x + 4ct\right) \\
& + \frac{17}{576}\cosh\left(-\frac{3}{2} - x + 4ct\right) - \frac{1}{144}\cosh\left(-\frac{3}{2} + x + 2ct\right) \\
& + \frac{1}{48}\cosh\left(-\frac{1}{2} - x + 2ct\right) + \frac{1}{144}\cosh\left(\frac{3}{2} - 2x + 2ct\right). \qquad (3.47)
\end{aligned}
$$

Appendix C

In this Appendix, we calculate V_1, V_2 and V_3 in detail.

$$
\begin{aligned}
V_1 = & \int_0^x \cosh\left(\frac{1}{2} - x + y\right)\left[\frac{1}{3}\cosh\left(-\frac{1}{2} - y + ct\right)\sinh^3\left(-\frac{1}{2} - y + ct\right)\right. \\
& \left. + 4\cosh^3\left(-\frac{1}{2} - y + ct\right)\sinh\left(-\frac{1}{2} - y + ct\right)\right] dy \\
= & \frac{13}{240}\cosh\left(-\frac{5}{2} - 4x + 4ct\right) + \frac{13}{144}\cosh\left(-\frac{3}{2} - 4x + 4ct\right) \\
& + \frac{11}{24}\cosh\left(-\frac{1}{2} - x + 2ct\right) + \frac{11}{72}\cosh\left(-\frac{3}{2} + x + 2ct\right) \\
& - \frac{13}{240}\cosh\left(\frac{5}{2} - x + ct\right) - \frac{11}{24}\cosh\left(-\frac{1}{2} - x + ct\right) \\
& - \frac{11}{72}\cosh\left(\frac{3}{2} - x + ct\right) - \frac{13}{144}\cosh\left(-\frac{3}{2} - x + ct\right), \qquad (3.48)
\end{aligned}
$$

$$
\begin{aligned}
V_2 = \int_x^{ct} & \cosh\left(\frac{1}{2} - x + y\right) \left[\frac{1}{3}\cosh\left(\frac{1}{2} - y + ct\right)\sinh^3\left(\frac{1}{2} - y + ct\right)\right. \\
& \left. + 4\cosh^3\left(\frac{1}{2} - y + ct\right)\sinh\left(\frac{1}{2} - y + ct\right)\right] dy \\
= & -\frac{13}{240}\cosh\left(\frac{3}{2} - 4x + 4ct\right) - \frac{13}{144}\cosh\left(\frac{5}{2} - 4x + 4ct\right) \\
& - \frac{11}{24}\cosh\left(\frac{3}{2} - 2x + 2ct\right) - \frac{11}{72}\cosh\left(\frac{1}{2} - 2x + 2ct\right) \\
& + \frac{13}{240}\cosh\left(-\frac{3}{2} - x + ct\right) + \frac{11}{24}\cosh\left(\frac{3}{2} - x + ct\right) \\
& + \frac{11}{72}\cosh\left(-\frac{1}{2} - x + ct\right) + \frac{13}{144}\cosh\left(\frac{5}{2} - x + ct\right), \qquad (3.49)
\end{aligned}
$$

and

$$
\begin{aligned}
V_3 = \int_{ct}^{1} & \cosh\left(-\frac{1}{2} - x + y\right) \left[\frac{1}{3}\cosh\left(\frac{1}{2} - y + ct\right)\sinh^3\left(\frac{1}{2} - y + ct\right)\right. \\
& \left. + 4\cosh^3\left(\frac{1}{2} - y + ct\right)\sinh\left(\frac{1}{2} - y + ct\right)\right] dy \\
= & \frac{13}{240}\cosh\left(\frac{5}{2} - 4x + 4ct\right) - \frac{13}{240}\cosh\left(-\frac{5}{2} + x + 4ct\right) \\
& + \frac{13}{144}\cosh\left(\frac{3}{2} - 4x + 4ct\right) - \frac{13}{144}\cosh\left(-\frac{3}{2} - x + 4ct\right) \\
& + \frac{11}{24}\cosh\left(\frac{1}{2} - 2x + 2ct\right) + \frac{11}{72}\cosh\left(\frac{3}{2} - 2x + 2ct\right) \\
& - \frac{11}{24}\cosh\left(-\frac{1}{2} - x + 2ct\right) - \frac{11}{72}\cosh\left(-\frac{3}{2} + x + 2ct\right), \qquad (3.50)
\end{aligned}
$$

Appendix D

In this Appendix, we calculate similarly VI_1, VI_2 and VI_3.

$$\begin{aligned}
VI_1 &= -\int_0^x \sinh\left(\frac{1}{2}-x+y\right)\left[\frac{3}{2}\sinh^2\left(-\frac{1}{2}-y+ct\right)+\frac{17}{12}\sinh^4\left(-\frac{1}{2}-y+ct\right)\right]dy \\
&= \frac{17}{960}\cosh\left(-\frac{5}{2}+x+4ct\right)-\frac{17}{576}\cosh\left(-\frac{3}{2}-x+4ct\right)+\frac{7}{32}\cosh\left(\frac{1}{2}-x+ct\right) \\
&\quad -\frac{1}{48}\cosh\left(-\frac{1}{2}-x+2ct\right)-\frac{7}{32}\cosh\left(-\frac{1}{2}+x\right)-\frac{17}{960}\cosh\left(\frac{5}{2}-x+ct\right) \\
&\quad +\frac{17}{576}\cosh\left(-\frac{3}{2}-x+ct\right)-\frac{1}{144}\cosh\left(\frac{3}{2}-x+ct\right)+\frac{1}{48}\cosh\left(-\frac{1}{2}-x+ct\right) \\
&\quad +\frac{1}{144}\cosh\left(-\frac{3}{2}+x+2ct\right), \qquad (3.51)
\end{aligned}$$

$$\begin{aligned}
VI_2 &= -\int_x^{ct} \sinh\left(\frac{1}{2}-x+y\right)\left[\frac{3}{2}\sinh^2\left(\frac{1}{2}-y+ct\right)+\frac{17}{12}\sinh^4\left(\frac{1}{2}-y+ct\right)\right]dy \\
&= \frac{17}{576}\cosh\left(\frac{5}{2}-4x+4ct\right)-\frac{17}{960}\cosh\left(\frac{3}{2}-4x+4ct\right)+\frac{7}{32}\cosh\left(\frac{1}{2}\right) \\
&\quad -\frac{1}{144}\cosh\left(\frac{1}{2}-2x+2ct\right)+\frac{17}{960}\cosh\left(-\frac{3}{2}-x+ct\right)-\frac{7}{32}\cosh\left(\frac{1}{2}-x+ct\right) \\
&\quad -\frac{17}{576}\cosh\left(\frac{5}{2}-x+ct\right)+\frac{1}{144}\cosh\left(-\frac{1}{2}-x+ct\right)-\frac{1}{48}\cosh\left(\frac{3}{2}-x+ct\right) \\
&\quad +\frac{1}{48}\cosh\left(\frac{3}{2}-2x+2ct\right) \qquad (3.52)
\end{aligned}$$

and

$$\begin{aligned}
VI_3 &= -\int_{ct}^{1} \sinh\left(-\frac{1}{2}-x+y\right)\left[\frac{3}{2}\sinh^2\left(\frac{1}{2}-y+ct\right)+\frac{17}{12}\sinh^4\left(\frac{1}{2}-y+ct\right)\right]dy \\
&= \frac{17}{960}\cosh\left(\frac{5}{2}-4x+4ct\right)-\frac{17}{576}\cosh\left(\frac{3}{2}-4x+4ct\right)-\frac{7}{32}\cosh\left(-\frac{1}{2}\right) \\
&\quad +\frac{1}{144}\cosh\left(\frac{3}{2}-2x+2ct\right)-\frac{1}{48}\cosh\left(\frac{1}{2}-2x+2ct\right)+\frac{7}{32}\cosh\left(-\frac{1}{2}+x\right) \\
&\quad -\frac{17}{960}\cosh\left(-\frac{5}{2}+x+4ct\right)+\frac{17}{576}\cosh\left(-\frac{3}{2}-x+4ct\right) \\
&\quad -\frac{1}{144}\cosh\left(-\frac{3}{2}+x+2ct\right)+\frac{1}{48}\cosh\left(-\frac{1}{2}-x+2ct\right). \qquad (3.53)
\end{aligned}$$

Appendix E

The proof of Lemma 3.3 is presented below.

Identity (3.10) shows that for all $v \in H^1(\mathbb{R})$,

$$\sup_{x\in\mathbb{R}} |v(x)| \leq \sqrt{\frac{1}{2}H_1[v]} = \frac{\sqrt{2}}{2}\|v\|_{H^1}. \tag{3.54}$$

Equality holds if and only if v is proportional to atranslate of φ. Note that

$$\begin{aligned} |H_1[u] - H_1[\varphi]| &= |(\|u\|_{H^1} + \|\varphi\|_{H^1})(\|u\|_{H^1} - \|\varphi\|_{H^1})| \\ &\leq (\|u-\varphi\|_{H^1} + 2\|\varphi\|_{H^1})\|u-\varphi\|_{H^1} \\ &\leq \delta\left(\delta + 2\sqrt{2}\sqrt[3]{\frac{3c}{2}}\right). \end{aligned} \tag{3.55}$$

Since $\delta \in (0,1)$, then

$$\left(\delta + 2\sqrt{2}\sqrt[3]{\frac{3c}{2}}\right) < \left(1 + 4\sqrt[3]{c}\right). \tag{3.56}$$

Hence (3.10) proved to be successful. Similarly,

$$\begin{aligned} &|H_2[u] - H_2[\varphi]| \\ &= \left|\int_{\mathbb{R}}\left(u^5 + 2u^3u_x^2 - \frac{1}{3}uu_x^4\right)dx - \int_{\mathbb{R}}\left(\varphi^5 + 2\varphi^3\varphi_x^2 - \frac{1}{3}\varphi_x^4\right)dx\right| \\ &\leq \left|\int_{\mathbb{R}}\left(u^3 - \varphi^3\right)\left(u^2 + 2u_x^2\right)dx\right| + \left|\int_{\mathbb{R}}\varphi^3\left(u^2 + 2u_x^2 - \varphi^2 - 2\varphi_x^2\right)dx\right| \\ &\quad + \frac{1}{3}\left|\int_{\mathbb{R}}\varphi_x^4\left(u-\varphi\right)dx\right| + \frac{1}{3}\left|\int_{\mathbb{R}}u\left(u_x^4 - \varphi_x^4\right)dx\right| \\ &=: J_1 + J_2 + J_3 + J_4. \end{aligned}$$

For the term J_1 and term J_2, we obtain

$$\begin{aligned} J_1 &\leq 2\int_{\mathbb{R}}\left|u^2 + u\varphi + \varphi^2\right||u-\varphi|\left(u^2 + u_x^2\right)dx \\ &\leq 2\left(\|u-\varphi\|_{L^\infty}^2 + 3\|u-\varphi\|_{L^\infty}\|\varphi\|_{L^\infty} + 3\|\varphi\|_{L^\infty}^2\right)\|u-\varphi\|_{L^\infty}\int_{\mathbb{R}}\left(u^2 + u_x^2\right)dx \\ &\leq \frac{\sqrt{2}}{2}\left(\|u-\varphi\|_{H^1}^2 + 3\sqrt{2}\sqrt[3]{\frac{3c}{2}}\|u-\varphi\|_{H^1} + 6\left(\frac{3c}{2}\right)^{\frac{2}{3}}\right)\|u-\varphi\|_{H^1}H_1[u] \\ &\leq \frac{\sqrt{2}}{2}\delta\left(\delta^2 + 3\sqrt{2}\sqrt[3]{\frac{3c}{2}}\delta + 6\left(\frac{3c}{2}\right)^{\frac{2}{3}}\right)\left(\left(1 + 4\sqrt[3]{c}\right)\delta + \sqrt[3]{18c^2}\right) \end{aligned}$$

and

$$\begin{aligned} J_2 &\leq \frac{3c}{2}\left|\int_{\mathbb{R}}\left((u-\varphi)^2 + 2\left(u_x - \varphi_x\right)^2 + 2\varphi\left(u-\varphi\right) + 4\varphi_x\left(u_x - \varphi_x\right)\right)dx\right| \\ &\leq 3c\left(\|u-\varphi\|_{H^1}^2 + 2\|\varphi\|_{H^1}\|u-\varphi\|_{H^1}\right) \\ &\leq 3c\delta\left(\delta + 2\sqrt{2}\sqrt[3]{\frac{3c}{2}}\right). \end{aligned}$$

For the sake of simplicity, we make the following estimates:

$$\delta^2 + 3\sqrt{2}\sqrt[3]{\frac{3c}{2}}\delta + 6\left(\frac{3c}{2}\right)^{\frac{2}{3}} < 1 + 4\sqrt[3]{c} + 12\sqrt[3]{c^2}. \tag{3.57}$$

Using (3.10) and (3.57), we get

$$J_1 + J_2 < \delta\left[\frac{\sqrt{2}}{2}\left(1 + 4\sqrt[3]{c} + 12\sqrt[3]{c^2}\right)\left(1 + 4\sqrt[3]{c} + \sqrt[3]{18c^2}\right) + 3c\left(1 + 4\sqrt[3]{c}\right)\right]$$

On the other hand, for the term J_3, we have

$$J_3 = \frac{1}{3}\left|\int_{\mathbb{R}} \varphi_x^4 (u - \varphi)\, dx\right| \leq \frac{1}{3}\|u - \varphi\|_{L^\infty}\int_{\mathbb{R}} \varphi_x^4 dx \leq \frac{\sqrt{2}}{12}\left(\frac{3c}{2}\right)^{\frac{4}{3}}\delta,$$

where $\int_{\mathbb{R}} \varphi_x^4 dx = \frac{1}{2}\left(\frac{3c}{2}\right)^{\frac{4}{3}}$. By Hölder inequality, we can get

$$\begin{aligned}
J_4 &= \left|\frac{1}{3}\int_{\mathbb{R}} u\left(u_x^2 + \varphi_x^2\right)(u_x + \varphi_x)(u_x - \varphi_x)\, dx\right| \\
&\leq \frac{1}{3}\|u\|_{L^\infty}\left(\int_{\mathbb{R}}\left(u_x^2 + \varphi_x^2\right)^2 (u_x + \varphi_x)^2\, dx\right)^{1/2}\left(\int_{\mathbb{R}} (u_x - \varphi_x)^2\, dx\right)^{1/2} \\
&\leq \frac{\sqrt{2}}{6}\left(\|u - \varphi\|_{H^1} + \|\varphi\|_{H^1}\right)\left(\int_{\mathbb{R}}\left(u_x^2 + \varphi_x^2\right)^2 (u_x + \varphi_x)^2\, dx\right)^{1/2}\left(\int_{\mathbb{R}} (u_x - \varphi_x)^2\, dx\right)^{1/2}.
\end{aligned}$$

For convenience, we denote

$$\begin{aligned}
K &= \int_{\mathbb{R}}\left(u_x^2 + \varphi_x^2\right)^2 (u_x + \varphi_x)^2\, dx \\
&= \int_{\mathbb{R}}\left(u_x^6 + 2u_x^5\varphi_x + 3u_x^4\varphi_x^2 + 4u_x^3\varphi_x^3\right) + 3u_x^2\varphi_x^4 + 2u_x\varphi_x^5 + \varphi_x^6)dx.
\end{aligned}$$

It is inferred from Young's inequality with exact exponents respectively that K can be estimated as follows:

$$K \leq 8\left(\int_{\mathbb{R}} u_x^6 dx + \int_{\mathbb{R}} \varphi_x^6 dx\right).$$

Since $u \in H^s(\mathbb{R}) \subset H^2(\mathbb{R})$, $s > \frac{5}{2}$, $\|u_x\|_{L^6(\mathbb{R})}$ is bounded by $\|u\|_{H(\mathbb{R})}$ due to the following Gagliardo-Nirenberg inequality:

$$\|u_x\|_{L^6} \leq C\|u_{xx}\|_{L^2}^{2/3}\|u\|_{L^2}^{1/3}$$

with $C > 0$ independent of u. Hence, it follows from $\|\varphi_x\|_{L^6}^6 = \frac{3}{4}c^2$ that

$$K \leq 6\left(A + c^2\right),$$

where the constant $A > 0$ depends only on the norm $\|u\|_{H^s}$. Therefore, we obtain

$$\begin{aligned}
J_4 &\leq \frac{\sqrt{2}}{6} \left(\|u - \varphi\|_{H^1} + \|\varphi\|_{H^1}\right) K^{1/2} \|u - \varphi\|_{H^1} \\
&\leq \frac{\sqrt{3}}{3} \delta \left(\delta + \sqrt{2} \sqrt[3]{\frac{3c}{2}} \right) \sqrt{A + c^2} \\
&\leq \frac{\sqrt{3}}{3} \delta \left(1 + 2\sqrt[3]{c}\right) \sqrt{A + c^2}.
\end{aligned}$$

In view of , we conclude that

$$|H_2[u] - H_2[\varphi_c]| \leq J_1 + J_2 + J_3 + J_4 \leq B(c)\delta.$$

Hence, we end the proof of the lemma. □

Appendix F

The proof of Lemma 3.4 is given below.

In view of (3.8) in Lemma 3.4, the following inequality holds:

$$H_2[u] - \frac{4}{3} M^3 H_1[u] + \frac{8}{5} M^5 \leq 0. \tag{3.58}$$

Define the polynomial P by

$$P(y) = H_2[u] - \frac{4}{3} y^3 H_1[u] + \frac{8}{5} y^5. \tag{3.59}$$

Using (3.2), $P(y)$ takes the form

$$\begin{aligned}
P_0(y) &= H_2[\varphi] - \frac{4}{3} y^3 H_1[\varphi] + \frac{8}{5} y^5 \\
&= \frac{8}{15} \left(3y \left(y + \sqrt[3]{\frac{3c}{2}} \right)^2 + \left(\frac{3c}{2} \right)^{\frac{2}{3}} y + 3c \right) \left(y - \sqrt[3]{\frac{3c}{2}} \right)^2.
\end{aligned} \tag{3.60}$$

We calculate from (3.59) and (3.60) that

$$P_0(M) = P(M) + \frac{4}{3} M^3 \left(H_1[u] - H_1[\varphi]\right) - \left(H_2[u] - H_2[\varphi]\right), \tag{3.61}$$

which along with (3.55) and (3.57) yields

$$\frac{8}{5} c \left(M - \sqrt[3]{\frac{3c}{2}} \right)^2 \leq \frac{4}{3} M^3 \left(H_1[u] - H_1[\varphi]\right) - \left(H_2[u] - H_2[\varphi]\right). \tag{3.62}$$

By (3.62) and the relation

$$0 \le M^2 \le \frac{H_1[u]}{2} \le \frac{1}{2} + 2\sqrt[3]{c} + \frac{\sqrt[3]{18c^2}}{2}, \tag{3.63}$$

we obtain

$$\begin{aligned}\left|M - \sqrt[3]{\frac{3c}{2}}\right| &\le \sqrt{\frac{5}{6c}M^3\,|H_1[u]-H_1[\varphi]| + |H_2[u]-H_2[\varphi]|}\\ &\le \delta^{\frac{1}{2}}\sqrt{\frac{5}{6c}\left(\frac{1}{2}+2\sqrt[3]{c}+\frac{\sqrt[3]{18c^2}}{2}\right)^{\frac{3}{2}}\left(1+4\sqrt[3]{c}\right)+B(c)}.\end{aligned}$$

Hence, we end the proof of Lemma 3.4. □

Appendix G

This Appendix is devoted to proving Lemma 3.7.
Since

$$\sqrt{M_\varphi^2 - m_\varphi^2} = a\sinh\left(\frac{1}{2}\right),\quad \ln\left(\frac{M_\varphi + \sqrt{M_\varphi^2 - m_\varphi^2}}{m_\varphi}\right) = \frac{1}{2}, \tag{3.64}$$

we have

$$\begin{aligned}&F_\varphi(M_\varphi, m_\varphi)\\ &= \left(\frac{4}{3}M_\varphi^3 + \frac{2}{3}M_\varphi m_\varphi^2\right)\left[H_1[\varphi] + 2m_\varphi^2\ln\left(\frac{M_\varphi + \sqrt{M_\varphi^2 - m_\varphi^2}}{m_\varphi}\right)\right.\\ &\quad\left. - 2M_\varphi\sqrt{M_\varphi^2 - m_\varphi^2} - m_\varphi^2\right] + \left(\frac{16}{15}M_\varphi^2 + \frac{8}{5}m_\varphi^2\right)\left(M_\varphi^2 - m_\varphi^2\right)\sqrt{M_\varphi^2 - m_\varphi^2}\\ &\quad + m_\varphi^4 H_0[\varphi] - H_2[\varphi] = 0.\end{aligned}$$

On the other hand, differentiation gives

$$\begin{aligned}\frac{\partial F_u}{\partial M} =& \left(4M^2 + \frac{2}{3}m^2\right)\left(H_1[u] + 2m^2\ln\left(\frac{M+\sqrt{M^2-m^2}}{m}\right) - 2M\sqrt{M^2-m^2} - m^2\right)\\ &+ \left(\frac{4}{3}M^3 + \frac{2}{3}Mm^2\right)\left(\frac{2m^2\left(1+\frac{M}{\sqrt{M^2-m^2}}\right)}{M+\sqrt{M^2-m^2}} - 2\sqrt{M^2-m^2} - \frac{2M^2}{\sqrt{M^2-m^2}}\right)\\ &+ \frac{32}{15}M\left(M^2-m^2\right)^{\frac{3}{2}} + \left(\frac{16}{5}M^3 + \frac{24}{5}Mm^2\right)\sqrt{M^2-m^2}.\end{aligned}$$

Since

$$\left(H_1[\varphi] + 2m_\varphi^2 \ln\left(\frac{M_\varphi + \sqrt{M_\varphi^2 - m_\varphi^2}}{m_\varphi}\right) - 2M_\varphi\sqrt{M_\varphi^2 - m_\varphi^2} - m_\varphi^2\right) = 0$$

and

$$\left(\frac{2m_\varphi^2\left(1 + \frac{M_\varphi}{\sqrt{M_\varphi^2 - m_\varphi^2}}\right)}{M_\varphi + \sqrt{M_\varphi^2 - m_\varphi^2}} - 2\sqrt{M_\varphi^2 - m_\varphi^2} - \frac{2M_\varphi^2}{\sqrt{M_\varphi^2 - m_\varphi^2}}\right) = -4a\sinh\left(\frac{1}{2}\right),$$

we get

$$\frac{\partial F_\varphi}{\partial M}(M_\varphi, m_\varphi) = 0.$$

Similarly, we have

$$\begin{aligned}\frac{\partial F_u}{\partial m} =& \frac{4}{3}Mm\left(H_1[u] + 2m^2\ln\left(\frac{M + \sqrt{M^2 - m^2}}{m}\right) - 2M\sqrt{M^2 - m^2} - m^2\right)\\ &+ \left(\frac{4}{3}M^3 + \frac{2}{3}Mm^2\right)\left(4m\ln\left(\frac{M + \sqrt{M^2 - m^2}}{m}\right) - 2m\right)\\ &+ \frac{16}{5}m\left(M^2 - m^2\right)^{3/2} - \left(\frac{16}{5}M^2m + \frac{24}{5}m^3\right)\sqrt{M^2 - m^2} + 4m^3H_0[u].\end{aligned}$$

Since (3.64) and

$$\frac{16}{5}m_\varphi\left(M_\varphi^2 - m_\varphi^2\right)^{3/2} - \left(\frac{16}{5}M_\varphi^2 m_\varphi + \frac{24}{5}m_\varphi^3\right)\sqrt{M_\varphi^2 - m_\varphi^2} + 4m_\varphi^3H_0[\varphi] = 0,$$

we obtain

$$\frac{\partial F_\varphi}{\partial m}(M_\varphi, m_\varphi) = 0.$$

In a similar manner, we obtain

$$\begin{aligned}\frac{\partial^2 F_u}{\partial M\partial m} =& \frac{4}{3}m\left(H_1[u] + 2m^2\ln\left(\frac{M + \sqrt{M^2 - m^2}}{m}\right) - 2M\sqrt{M^2 - m^2} - m^2\right)\\ &+ \left(4M^2 + \frac{2}{3}m^2\right)\left(4m\ln\left(\frac{M + \sqrt{M^2 - m^2}}{m}\right) - 2m\right)\\ &- \frac{16}{3}Mm\sqrt{M^2 - m^2} + \frac{4m}{\sqrt{M^2 - m^2}}\left(\frac{4}{3}M^3 + \frac{2}{3}Mm^2\right) - \frac{8Mm^3}{\sqrt{M^2 - m^2}}.\end{aligned}$$

By (3.64) and

$$
\begin{aligned}
&-\frac{16}{3}M_\varphi m_\varphi\sqrt{M_\varphi^2-m_\varphi^2}+\frac{4m_\varphi}{\sqrt{M_\varphi^2-m_\varphi^2}}\left(\frac{4}{3}M_\varphi^3+\frac{2}{3}M_\varphi m_\varphi^2\right)-\frac{8M_\varphi m_\varphi^3}{\sqrt{M_\varphi^2-m_\varphi^2}}\\
&=\frac{1}{\sinh\left(\frac{1}{2}\right)}\Bigg[-\frac{16}{3}a^3\cosh\left(\frac{1}{2}\right)\sinh^2\left(\frac{1}{2}\right)+\frac{16}{3}a^3\cosh^3\left(\frac{1}{2}\right)\\
&\quad+\frac{8}{3}a^3\cosh\left(\frac{1}{2}\right)-8a^3\cosh\left(\frac{1}{2}\right)\Bigg]=0,
\end{aligned}
\tag{3.65}
$$

we have

$$
\frac{\partial^2 F_\varphi}{\partial M\partial m}(M_\varphi,m_\varphi)=0.
$$

In the same way, we get

$$
\begin{aligned}
\frac{\partial^2 F_u}{\partial m^2}=&\frac{4}{3}M\left(H_1[u]+2m^2\ln\left(\frac{M+\sqrt{M^2-m^2}}{m}\right)-2M\sqrt{M^2-m^2}-m^2\right)\\
&+\frac{8}{3}Mm\left(4m\ln\left(\frac{M+\sqrt{M^2-m^2}}{m}\right)-2m\right)\\
&+\left(\frac{4}{3}M^3+\frac{2}{3}Mm^2\right)\Bigg[4\ln\left(\frac{M+\sqrt{M^2-m^2}}{m}\right)-\frac{8M}{\sqrt{M^2-m^2}}\\
&+\frac{2M^2m^2-4m^4}{(M^2-m^2)^{\frac{3}{2}}(M+\sqrt{M^2-m^2})}-\frac{2Mm^2}{(M^2-m^2)(M+\sqrt{M^2-m^2})}\\
&+\frac{2Mm^2}{(M^2-m^2)^{\frac{3}{2}}}+2\Bigg]+\frac{16}{5}(M^2-m^2)^{\frac{3}{2}}-\frac{96}{5}m^2\sqrt{M^2-m^2}\\
&+\frac{\frac{16}{5}M^2m^2+\frac{24}{5}m^4}{\sqrt{M^2-m^2}}-\left(\frac{16}{5}M^2+\frac{24}{5}m^2\right)(\sqrt{M^2-m^2})+12m^2H_0[u]
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 F_u}{\partial M^2}=&8M\left(H_1[u]+2m^2\ln\left(\frac{M+\sqrt{M^2-m^2}}{m}\right)-2M\sqrt{M^2-m^2}-m^2\right)\\
&+\left(8M^2+\frac{2}{3}m^2\right)\left(\frac{2m^2\left(1+\frac{M}{\sqrt{M^2-m^2}}\right)}{M+\sqrt{M^2-m^2}}-2\sqrt{M^2-m^2}-\frac{2M^2}{\sqrt{M^2-m^2}}\right)\\
&+\left(\frac{4}{3}M^3+\frac{2}{3}Mm^2\right)\Bigg[-\frac{2m^4}{(M^2-m^2)^{\frac{3}{2}}(M+\sqrt{M^2-m^2})}-\frac{2m^2}{(M^2-m^2)}\\
&-\frac{6M}{\sqrt{M^2-m^2}}+\frac{2M^3}{(M^2-m^2)^{\frac{3}{2}}}\Bigg]+\frac{32}{15}(M^2-m^2)^{\frac{3}{2}}+\frac{64}{5}M^2\sqrt{M^2-m^2}\\
&+\frac{\frac{16}{5}M^4+\frac{24}{5}M^2m^2}{\sqrt{M^2-m^2}}+\left(\frac{16}{5}M^2+\frac{24}{5}m^2\right)\sqrt{M^2-m^2}
\end{aligned}
$$

By (3.61), (3.65) and

$$
\begin{aligned}
&\frac{2M^2m^2 - 4m^4}{(M^2 - m^2)^{\frac{3}{2}}(M + \sqrt{M^2 - m^2})} - \frac{2Mm^2}{(M^2 - m^2)(M + \sqrt{M^2 - m^2})} + \frac{2Mm^2}{(M^2 - m^2)^{\frac{3}{2}}} \\
&= \frac{4m^2}{\sqrt{M^2 - m^2}(M + \sqrt{M^2 - m^2})},
\end{aligned}
$$

we have

$$
\frac{\partial^2 F_\varphi}{\partial m^2}(M_\varphi, m_\varphi) = -\frac{16}{3} b^3 \cosh^2\left(\frac{1}{2}\right) \sinh\left(\frac{1}{2}\right) - 8b^3 \sinh\left(\frac{1}{2}\right).
$$

Since

$$
\begin{aligned}
&- \frac{2m^4}{(M^2 - m^2)^{\frac{3}{2}}(M + \sqrt{M^2 - m^2})} - \frac{2m^2}{(M^2 - m^2)} - \frac{6M}{\sqrt{M^2 - m^2}} + \frac{2M^3}{(M^2 - m^2)^{\frac{3}{2}}} \\
&= \frac{-4M}{\sqrt{M^2 - m^2}},
\end{aligned}
$$

we get

$$
\frac{\partial^2 F_\varphi}{\partial M^2}(M_\varphi, m_\varphi) = -\frac{272}{15} b^3 \sinh^3\left(\frac{1}{2}\right) - \frac{344}{15} b^3 \sinh\left(\frac{1}{2}\right) - \frac{32}{15 \sinh\left(\frac{1}{2}\right)} b^3.
$$

Consequently, we have established the lemma. $\square$