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Anahit Galstian

The University of Texas Rio Grande Valley

Karen Yagdjian

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Finite lifespan of solutions of the semilinear wave equation in the Einstein-de Sitter spacetime

Anahit Galstian and Karen Yagdjian

School of Mathematical and Statistical Sciences,
University of Texas RGV, Edinburg, TX 78539, U.S.A.

Abstract

We examine the solutions of the semilinear wave equation, and, in particular, of the φ^q model of quantum field theory in the curved space-time. More exactly, for $1 < q < 4$ we prove that the solution of the massless self-interacting scalar field equation in the Einstein-de Sitter universe has finite lifespan.

Keywords: Generalized Tricomi Equation; Einstein-de Sitter spacetime; Blowup of solution

Mathematics Subject Classification 2010: 35C15; 35Q75; 35Q05; 83F05; 76H05

1 Introduction

The equation for a self-interacting massless scalar field in the quantum field theory is the semilinear covariant wave equation

$$\square_g \psi = \lambda |\psi|^{p-1} \psi, \quad (1.1)$$

where \square_g is a covariant d'Alembert's operator (the Laplace-Beltrami operator) in the spacetime with the metric tensor g . The exponent $p > 1$ and the self-coupling constant λ show the intensity of self-interaction. The metric of the Einstein & de Sitter universe (EdeS universe, see, e.g., [5, p.123], [11, Sec. 5.3]) is a particular member of the Friedmann-Robertson-Walker metrics

$$ds^2 = -dt^2 + a_{sc}^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right],$$

where $K = -1, 0$, or $+1$, for a hyperbolic, flat, or spherical spatial geometry, respectively. For the EdeS universe the scale factor is $a_{sc}(t) = t^{2/3}$. The covariant d'Alambert's operator,

$$\square_g \psi = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ik} \frac{\partial \psi}{\partial x^k} \right),$$

in the EdeS spacetime is

$$\square_{EdeS} \psi = -\partial_t^2 \psi - 2t^{-1} \partial_t \psi + t^{-\frac{4}{3}} A(x, \partial_x) \psi,$$

where $A(x, \partial_x)$ is a second order partial differential operator.

Thus, the equation for the self-interacting massless field in the Einstein-de Sitter spacetime is the semilinear covariant wave equation (1.1) which has singular coefficients at $t = 0$. The covariant

d'Alembert's operator in the Einstein-de Sitter spacetime belongs to the family of the non-Fuchsian partial differential operators. The initial value problem for the equation (1.1) with the Cauchy data on hyperplane $t = 0$ must be defined properly. In [7] Galstian, Kinoshita and Yagdjian suggested such setting for the wave propagating in the EdeS spacetime when $A(x, \partial_x)$ is the Laplace operator on \mathbb{R}^n . In [7] the authors introduced the weighted initial value problem for the covariant (if $n = 3$) wave equation and gave explicit representation formulas for the solutions. We generalize that setting and set the problem for the semilinear equation as follows

$$\begin{cases} \psi_{tt} - t^{-4/3} A(x, D_x) \psi + 2t^{-1} \psi_t = F(\psi), & t > 0, x \in \mathbb{R}^n, \\ \lim_{t \rightarrow 0^+} t \psi(x, t) = \varphi_0(x), & x \in \mathbb{R}^n, \\ \lim_{t \rightarrow 0^+} \left(t \psi_t(x, t) + \psi(x, t) + 3t^{-1/3} A(x, D_x) \varphi_0(x) \right) = \varphi_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.2)$$

where $A(x, D_x)$ is an elliptic partial differential operator $A(x, D_x) = \sum_{|\alpha| \leq 2} a_\alpha(x) \partial_x^\alpha$ with smooth real-valued coefficients $a_\alpha(x) \in C^\infty(\mathbb{R}^n)$, which are constant outside of some compact. The two limits of (1.2) are taken in the sense of $H^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, respectively.

We define $p_0(n)$ as a positive root of the equation

$$(n+3)p^2 - (n+13)p - 2 = 0 \quad (1.3)$$

and denote

$$p_{cr}(n) := \max \left\{ p_0(n), 1 + \frac{6}{n} \right\}.$$

Consider the operator

$$A(x, D_x)u = \frac{1}{a(x)} \sum_{k,j=1,\dots,n} \frac{\partial}{\partial x_k} \left(a_{kj}(x) \frac{\partial}{\partial x_j} u \right), \quad (1.4)$$

where $a(x), a_{kj}(x) \in C^\infty(\mathbb{R}^n)$ and

$$a(x) \geq a_0 > 0, \quad a_{kj}(x) = a_{jk}(x) \quad \text{for all } x \in \mathbb{R}^n, k, j = 1, 2, \dots, n, \quad (1.5)$$

with some number a_0 . Assume that the coefficients $a(x), a_{kj}(x)$ are constant outside of some ball $B_{R_A}(0)$:

$$a_{kj}(x) = c \delta_{jk}, \quad a(x) = 1 \quad \text{for all } x \in \mathbb{R}^n, |x| \geq R_A > 0, \quad c > 0, \quad (1.6)$$

where δ_{jk} is the Kronecker delta.

We say that the solution $\psi \in C^2((0, T]; \mathcal{D}'(\mathbb{R}^n))$ of the problem (1.2) obeys the finite propagation speed property if for every point (x_0, t_0) with $t_0 > 0$ and an open ball $B_R(x_0) = \{x \in \mathbb{R}^n; |x - x_0| < R\}$, the property

$$\varphi_0(x) = \varphi_1(x) = 0 \quad \text{on } B_{R+3t_0^{1/3}s_A}(x_0),$$

implies

$$\psi(x, t_0) = 0 \quad \text{on } B_R(x_0).$$

Here

$$s_A = \max_{x \in \mathbb{R}^n, \xi \in \mathbb{R}^n, |\xi|=1} \frac{1}{a(x)} \sum_{|\alpha|=2} a_\alpha(x) \xi^\alpha.$$

Although in quantum field theory the nonlinear term typically has a gauge invariant form $F(\psi) = |\psi|^{p-1} \psi$, we will focus on semilinear equations, which are commonly used models for general nonlinear problems (see [18, 20] and the bibliography therein). Our first main result is the following theorem.

Theorem 1.1 Consider the problem (1.2) with $F(\psi) = |\psi|^p$ and $A(x, D_x)$ being an elliptic operator with the properties (1.4),(1.5),(1.6). If $p > 1$ and

$$p < p_{cr}(n),$$

then for every arbitrary small number $\varepsilon > 0$ and an arbitrary number s there exist functions $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$ with norms satisfying inequality

$$\|\varphi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^n)} < \varepsilon$$

such that the solution $\psi \in C((0, T); H^1(\mathbb{R}^n)) \cap C((0, T); L^2(\mathbb{R}^n))$ of the problem (1.2), which obeys the finite propagation speed property, blows up in finite time. More precisely, there is $T < \infty$ such that

$$\lim_{t \nearrow T} \int_{\mathbb{R}^n} a(x)\psi(x, t) dx = \infty.$$

Note, for $n = 3$ we have $p_{cr}(3) = 3$ that is the exponent of the φ^4 model of quantum field theory. The next corollary indicates that the equations (1.2) possesses global in time sign preserving solution only if $p \geq 3$.

Corollary 1.2 Assume that $F(\psi) = |\psi|^{p-1}\psi$, $1 < p < 3$. For every arbitrary small number $\varepsilon > 0$ and an arbitrary number s there exist functions $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$, $\|\varphi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^n)} < \varepsilon$, such that the positive solution $\psi \in C((0, T); H^1(\mathbb{R}^n)) \cap C((0, T); L^2(\mathbb{R}^n))$ of the problem (1.2) has a finite life-span.

Note that, for the semilinear Klein-Gordon equation a global in time solvability is proved in [9] for the problem with small initial data prescribed on the hyper-surface $t = t_0 > 0$.

In Section 5 we prove the finite propagation speed property for a subclass of operators of type (1.2). The next theorem shows that the blow up phenomenon is still present even if we remove the singularity at $t = 0$ by shifting the initial hyperplane; the blow up is caused by the semilinear term. Consider the following Cauchy problem

$$\begin{cases} \psi_{tt} - t^{-2k}A(x, D_x)\psi + 2t^{-1}\psi_t = |\psi|^p, & t > 1, x \in \mathbb{R}^n, \\ \psi(x, 1) = \varphi_0(x), \quad \psi_t(x, 1) = \varphi_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.7)$$

where $k \in (0, 1)$ and $A(x, D_x)$ is an elliptic partial differential operator with the properties (1.4), (1.5), (1.6). Let $p_0(n, k)$ be a positive root of the equation

$$p^2(n + 1 - kn) - p(2k + n + 3 - kn) - 2(1 - k) = 0. \quad (1.8)$$

The numbers $p_0(k)$ and $p_0(n, k)$ can be regarded as an analog of the Strauss exponent that was defined for the semilinear wave equation in the Minkowski spacetime. (See, e.g., [18, 20].)

The equation of (1.7) is strictly hyperbolic for every bounded interval of time and it has smooth coefficients. Consequently, for every smooth initial functions φ_0 and φ_1 the problem (1.7) has the local solution. According to the next theorem a local in time solution, in general, cannot be prolonged to the global solution.

Theorem 1.3 Assume that $p > 1$ and

$$\text{either } 1 < p < 1 + \frac{2}{n(1-k)} \quad \text{or} \quad 1 < p \leq 2 + \frac{2k}{n+1-kn} \quad \text{and} \quad p < p_0(n, k). \quad (1.9)$$

Then for every arbitrary small number $\varepsilon > 0$ and an arbitrary number s there exist functions $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$ with norms satisfying inequality

$$\|\varphi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^n)} < \varepsilon \quad (1.10)$$

such that the solution $\psi \in C([1, T]; H^1(\mathbb{R}^n)) \cap C([1, T]; L^2(\mathbb{R}^n))$ of the problem (1.7) that obeys the finite propagation speed property blows up in finite time. More precisely, there is $T < \infty$ such that

$$\lim_{t \nearrow T} \int_{\mathbb{R}^n} a(x)\psi(x, t) dx = \infty.$$

Thus, according to this theorem for $n = 3$ and $k = 2/3$ the blow-up occurs if $1 < p < 3$.

Corollary 1.4 Assume that $F(\psi) = |\psi|^{p-1}\psi$ and p satisfies (1.9). Then for every arbitrary small number $\varepsilon > 0$ and an arbitrary number s there exist functions $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$, $\|\varphi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^n)} < \varepsilon$, such that the positive solution $\psi \in C([1, T]; H^1(\mathbb{R}^n)) \cap C([1, T]; L^2(\mathbb{R}^n))$ of the problem (1.7) that obeys the finite propagation speed property blows up in finite time.

In order to illustrate results of the theorems above we discuss below several examples which include, in particular, the Einstein-de Sitter spacetime of the *matter dominated universe*.

Example 1. Consider the covariant equation

$$\psi_{tt} - t^{-4/3}\Delta\psi + 2t^{-1}\psi_t = |\psi|^p, \quad t > 0, x \in \mathbb{R}^3, \quad (1.11)$$

for the self-interacting waves propagating in the Einstein-de Sitter spacetime. Here Δ is the Laplace operator in \mathbb{R}^3 . According to Theorem 1.1 and Theorem 1.3 ($k = 2/3$) if $1 < p < 3$, then for every arbitrary small number $\varepsilon > 0$ and an arbitrary number s there exist functions $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^3)$ with norms satisfying (1.10) such that the solution ψ of the problem (1.2) or (1.7), respectively, for the equation (1.11), which obeys the finite propagation speed property, blows up in finite time. Note that $p = 3$ is the exponent of the φ^4 model of quantum field theory.

The coefficients of the operator in the next examples depend on the spatial variables as well.

Example 2. Consider the the Einstein-de Sitter spacetime with the metric defined by

$$ds^2 = -dt^2 + t^{4/3} \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right],$$

where $K = -1, 0$, or $+1$. In the Cartesian coordinates $x = (x_1, x_2, x_3)$ this metric tensor is

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & t^{4/3} \frac{1-K(x_2^2+x_3^2)}{1-K|x|^2} & t^{4/3} \frac{Kx_1x_2}{1-K|x|^2} & t^{4/3} \frac{Kx_1x_3}{1-K|x|^2} \\ 0 & t^{4/3} \frac{Kx_1x_2}{1-K|x|^2} & t^{4/3} \frac{1-K(x_1^2+x_3^2)}{1-K|x|^2} & t^{4/3} \frac{Kx_2x_3}{1-K|x|^2} \\ 0 & t^{4/3} \frac{Kx_1x_3}{1-K|x|^2} & t^{4/3} \frac{Kx_2x_3}{1-K|x|^2} & t^{4/3} \frac{1-K(x_1^2+x_2^2)}{1-K|x|^2} \end{pmatrix}$$

and the semilinear covariant wave equation in this metric reads

$$\psi_{tt} - t^{-4/3}\mathcal{A}(x, \partial_x)\psi + 2t^{-1}\partial_t\psi = |\psi|^p, \quad (1.12)$$

where

$$\begin{aligned} \mathcal{A}(x, \partial_x) = & (1 - Kx_1^2)\partial_{x_1}^2 + (1 - Kx_2^2)\partial_{x_2}^2 + (1 - Kx_3^2)\partial_{x_3}^2 - 2Kx_1x_2\partial_{x_1}\partial_{x_2} \\ & - 2Kx_1x_3\partial_{x_1}\partial_{x_3} - 2Kx_2x_3\partial_{x_2}\partial_{x_3} - 3Kx_1\partial_{x_1} - 3Kx_2\partial_{x_2} - 3Kx_3\partial_{x_3}. \end{aligned} \quad (1.13)$$

Thus, in the notation of (1.4) we have

$$a_{kj}(x) = \frac{\delta_{kj} - Kx_kx_j}{\sqrt{1 - K|x|^2}}, \quad k, j = 1, 2, 3, \quad a(x) = \frac{1}{\sqrt{1 - K|x|^2}}.$$

We consider equation of (1.2) that coincides with (1.12) inside of the ball $B_R(0) \subset \mathbb{R}^3$ and with (1.11) outside of the ball $B_{2R}(0)$. The curvature \mathcal{R} of such spacetime is

$$\mathcal{R} = \frac{4}{3t^2} + \frac{6K}{t^{4/3}} \quad \text{in } B_R(0) \quad \text{while} \quad \mathcal{R} = \frac{4}{3t^2} \quad \text{in } (\mathbb{R}^3 \setminus B_{2R}(0)).$$

In order to make coefficients of this operator more explicit in $B_{2R}(0) \setminus B_R(0)$ one can use the standard cut-off function $\chi = \chi(x) \geq 0$ and attach to K the factor $\varepsilon\chi(x)$. For sufficiently small $\varepsilon > 0$ the conditions (1.4),(1.5),(1.6) are fulfilled. Another equation satisfying all conditions is the following one

$$\psi_{tt} - t^{-4/3} \frac{1}{1 + a(x)\chi(x)} \mathcal{A}(x, \partial_x)\psi + 2t^{-1}\partial_t\psi = |\psi|^p,$$

where $a(x)$ is any smooth non-negative function and the operator $\mathcal{A}(x, \partial_x)$ is given by (1.13) inside of $B_R(0)$ and is Δ outside of $B_{2R}(0)$. Then all conclusions of Example 1 are valid also for these equations.

Example 3. Consider now problem (1.7) with $A(x, D_x) = \Delta$. For the radiation dominated universe $k = 1/2$ and $n = 3$. The first case of (1.9) in Theorem 1.3 reads $1 < p < \frac{7}{3}$. We obtain $1 < p \leq 2$ from the second one. Thus, for the equation (1.7) there is a blowing up small data solution if $1 < p < \frac{7}{3}$. Another example can be obtained by replacing Δ with $\mathcal{A}(x, \partial_x)$ (1.13) inside of some ball in \mathbb{R}^3 and with the modification similar to the one has used in Example 2.

Next two examples have spacetimes with non flat spatial slices.

Example 4. Let a spacetime be defined by the following metric

$$ds^2 = -dt^2 + t^{2k} \left(\frac{\beta}{x^2 + 1} dx^2 + \frac{x^2 + 1}{\beta} dy^2 + dz^2 \right)$$

inside of some ball, where k and $\beta > 0$ are real numbers. The curvature of this spacetime is $12k^2t^{-2} - 2t^{-2k}\beta^{-1} - 6kt^{-2}$, while the spatial slices have the constant curvature $-2\beta^{-1}$. We consider semilinear equation in this spacetime

$$\psi_{tt} - t^{-2k} \left(\frac{x^2 + 1}{\beta} \partial_x^2 \psi + \frac{2x}{\beta} \partial_x \psi + \frac{\beta}{x^2 + 1} \partial_y^2 \psi + \partial_z^2 \psi \right) \psi + 2t^{-1}\psi_t = |\psi|^p, \quad t > 0. \quad (1.14)$$

The modification outside of some ball is similar to the one mentioned in Example 2. The equation (1.14) is a covariant wave equation if $k = 2/3$. It is easy to verify that Theorem 1.3 can be applied to the problem for this equation.

Example 5. Consider the spacetime with the metric

$$ds^2 = -dt^2 + t^{2k} \left(\frac{\beta}{e^{-x^2} + 1} dx^2 + \frac{e^{-x^2} + 1}{\beta} dy^2 + dz^2 \right)$$

inside of some ball, where k and $\beta > 0$ are real numbers. The curvature of the spacetime is

$$2\beta^{-1}t^{-2(k+1)} \left(3k(2k - 1)\beta t^{2k} + t^2 e^{-x^2} (1 - 2x^2) \right),$$

while the spatial slices have the curvature $2\beta^{-1}t^{-2k}e^{-x^2}(1-2x^2)$. Theorem 1.3 can be applied to the semilinear equation of (1.7) in this spacetime, where

$$A(x, D)\psi = \frac{e^{-x^2} + 1}{\beta} \partial_x^2 \psi + \frac{-2xe^{-x^2}}{\beta} \partial_x \psi + \frac{\beta}{e^{-x^2} + 1} \partial_y^2 \psi + \partial_z^2 \psi.$$

The equation (1.7) in this spacetime is a covariant wave equation if $k = 2/3$. It will be interesting to replace requirement on the coefficients of $A(x, D)$ to be constant outside of a ball with a condition on their rate of convergence to the constants at infinity.

The last two examples belong to more general class of equations written in the background given by the following metric

$$ds^2 = -dt^2 + t^{2k} (G_1(x, y, z)dx^2 + G_2(x, y, z)dy^2 + G_3(x, y, z)dz^2)$$

such that $G_1(x, y, z)G_2(x, y, z)G_3(x, y, z) = \text{constant} \neq 0$.

This paper is organized as follows. In Section 2 we introduce the basic ideas of the proof of Theorem 1.1 and give main tools which will be also used in the next sections. In Section 3 we prove Theorem 1.3. The existence of the local in time solution is proved in Section 4. Section 5 is devoted to the uniqueness problem and the finite speed of propagation property.

2 Proof of Theorem 1.1

The number $p_0(n)$ is defined as a positive root of the equation (1.3), that is,

$$p_0(n) = \frac{n + 13 + \sqrt{n^2 + 34n + 193}}{2(n + 3)}.$$

It is easily seen that

$$p_0(n) < \frac{2n + 10}{n + 3} \quad \text{for all } n \geq 4,$$

and that

$$p_0(n) < 1 + \frac{6}{n} \quad \text{for all } n \leq 4.$$

If we denote

$$\mathcal{L} := \partial_t^2 - t^{-4/3}A(x, D_x) + 2t^{-1}\partial_t, \quad \mathcal{S} := \partial_t^2 - t^{-4/3}A(x, D_x), \quad (2.1)$$

then we can easily check for $t \neq 0$ the following operator identity

$$t^{-1} \circ \mathcal{S} \circ t = \mathcal{L}. \quad (2.2)$$

The last equation suggests a partial Liouville transform of an unknown function ψ with u

$$\psi = t^{-1}u.$$

Then the problem for u is:

$$\begin{cases} u_{tt} - t^{-4/3}A(x, D_x)u = t^{1-p}|u|^p, & t > 0, x \in \mathbb{R}^n, \\ \lim_{t \rightarrow 0^+} u(x, t) = \varphi_0(x), & x \in \mathbb{R}^n, \\ \lim_{t \rightarrow 0^+} \left(u_t(x, t) + 3t^{-1/3}A(x, D_x)\varphi_0(x) \right) = \varphi_1(x), & x \in \mathbb{R}^n. \end{cases} \quad (2.3)$$

Recall (1.4), (1.5), and that the coefficients $a(x), a_{kj}(x) \in C^\infty(\mathbb{R}^n)$ are constant outside of some ball $B_{R_A}(0)$. Denote

$$F(t) = \int_{\mathbb{R}^n} a(x)u(x, t) dx.$$

Then $F \in C^2(0, T)$ provided that the function u is defined for all $(x, t) \in \mathbb{R}^n \times (0, T)$, and

$$\lim_{t \rightarrow 0^+} F(t) = \int_{\mathbb{R}^n} a(x)\varphi_0(x) dx = C_0,$$

while

$$\begin{aligned} \lim_{t \rightarrow 0^+} F'(t) &= \lim_{t \rightarrow 0^+} \int_{\mathbb{R}^n} a(x) \left[u_t(x, t) + 3t^{-1/3}A(x, D_x)\varphi_0(x) - 3t^{-1/3}A(x, D_x)\varphi_0(x) \right] dx \\ &= \lim_{t \rightarrow 0^+} \int_{\mathbb{R}^n} a(x) \left[u_t(x, t) + 3t^{-1/3}A(x, D_x)\varphi_0(x) \right] dx = \int_{\mathbb{R}^n} a(x)\varphi_1(x) dx = C_1. \end{aligned}$$

Thus

$$F \in C^1[0, \infty) \cap C^2(0, \infty).$$

From the equation we have

$$F'' = t^{1-p} \int_{\mathbb{R}^n} a(x)|u(x, t)|^p dx \geq 0 \quad \text{for all } t > 0.$$

Furthermore,

$$\begin{aligned} F(t) &= F(\varepsilon) + (t - \varepsilon)F'(\varepsilon) + \int_\varepsilon^t \int_\varepsilon^{t_1} F''(t_2) dt_2 dt_1 \\ &\geq F(\varepsilon) + (t - \varepsilon)F'(\varepsilon) \geq 0 \quad \text{for all } t \geq \varepsilon. \end{aligned}$$

By letting $\varepsilon \rightarrow 0^+$ we obtain

$$F(t) \geq tC_1 + C_0 \geq 0 \quad \text{for all } t \geq 0$$

provided that $C_0 \geq 0$ and $C_1 \geq 0$. We can assume also that $\text{supp } \varphi_i \subseteq B_R(0) := \{x \in \mathbb{R}^n \mid |x| \leq R\}$, $i = 0, 1$ and $R \geq R_A$. On the other hand, using the compact support of $u(\cdot, t)$ and Hölder's inequality we obtain with $\phi(t) = 3t^{1/3}$

$$\left| \int_{\mathbb{R}^n} a(x)u(x, t) dx \right|^p \lesssim (R + \phi(t))^{n(p-1)} \left(\int_{|x| \leq R + \phi(t)} a(x)|u(x, t)|^p dx \right).$$

Here and henceforth, if A and B are two non-negative quantities, we use $A \lesssim B$ ($A \gtrsim B$) to denote the statement that $A \leq CB$ ($AC \geq B$) for some absolute constant $C > 0$. Hence

$$F''(t) \gtrsim (R + \phi(t))^{-(n+3)(p-1)} |F(t)|^p \quad \text{for all } t \geq 0. \quad (2.4)$$

If $1 < p < 1 + \frac{6}{n}$ and $C_1 > 0$, then we can apply Kato's lemma (see, e.g., [19, Lemma 2.1]) since

$$p - 1 > \frac{(n+3)(p-1)}{3} - 2 \iff p < \frac{6}{n} + 1$$

that proves that solution blows up for such p .

Next we consider the case of $1 < p \leq (2n + 10)/(n + 3)$ and $p < p_0(n)$. For $\varphi_0 \in C_0^{[\frac{n}{2}]+3}(\mathbb{R}^n)$, according to [21], the solution of the problem

$$\begin{cases} \mathcal{S}u = 0, & x \in \mathbb{R}^n, \quad t > 0, \\ \lim_{t \rightarrow 0} u(x, t) = \varphi_0(x), & \lim_{t \rightarrow 0} \left(u_t(x, t) + 3t^{-1/3} A(x, D_x) \varphi_0(x) \right) = 0, \quad x \in \mathbb{R}^n, \end{cases} \quad (2.5)$$

is given by the function

$$u(x, t) = v_{\varphi_0}(x, 3t^{1/3}) - 3t^{1/3} (\partial_r v_{\varphi_0})(x, 3t^{1/3}),$$

where $v_{\varphi}(x, 3t^{1/3})$ is the value of the solution $v(x, r)$ to the Cauchy problem

$$v_{rr} - A(x, D_x)v = 0, \quad v(x, 0) = \varphi(x), \quad v_t(x, 0) = 0,$$

taken at the point $(x, r) = (x, 3t^{1/3})$. Hence, if we assume that $A(x, D_x)\varphi = \varphi$, then we obtain

$$v_{\varphi}(x, t) = \cosh(t)\varphi(x)$$

and, consequently,

$$u(x, t) = \left(\cosh(3t^{1/3}) - 3t^{1/3} \sinh(3t^{1/3}) \right) \varphi(x).$$

The second independent solution with separated variables is

$$w(x, t) = \left(\sinh(3t^{1/3}) - 3t^{1/3} \cosh(3t^{1/3}) \right) \varphi(x).$$

Thus, the function $v(x, t) = u(x, t) - w(x, t)$, that is,

$$v(x, t) = \left(3\sqrt[3]{t} + 1 \right) \exp \left(-3\sqrt[3]{t} \right) \varphi(x) = (\phi(t) + 1) \exp(-\phi(t)) \varphi(x)$$

solves the problem (2.5) with $\varphi_0 = \varphi$. Moreover, v is such that

$$v(x, 0) = \varphi(x), \quad \lim_{t \rightarrow \infty} v(x, t) = 0.$$

The following lemma generalizes corresponding result from [18].

Lemma 2.1 *There is a smooth function $\varphi(x)$ such that*

$$A(x, D_x)\varphi(x) = \varphi(x) \quad \text{for all } x \in \mathbb{R}^n$$

and

$$\varphi(x) = \int_{\mathbb{S}^{n-1}} e^{x\omega} d\omega \quad \text{for all } x, \quad |x| \geq R_A + 1.$$

Moreover,

$$\varphi(x) \sim C_n |x|^{-(n-1)/2} e^{|x|} \quad \text{as } |x| \rightarrow \infty.$$

Proof. We have

$$\Delta \int_{\mathbb{S}^{n-1}} e^{x\omega} d\omega = \int_{\mathbb{S}^{n-1}} e^{x\omega} d\omega \quad \text{for all } x \in \mathbb{R}^n,$$

where Δ is the Laplace operator. It is well known [18] that

$$\varphi_L(x) := \int_{\mathbb{S}^{n-1}} e^{x\omega} d\omega \sim C_n |x|^{-(n-1)/2} e^{|x|} \quad \text{as } |x| \rightarrow \infty.$$

To find the function $\varphi(x)$ we solve the Dirichlet problem for the elliptic equation

$$A(x, D_x)\varphi(x) - \varphi(x) = 0 \quad \text{in } B_{R_A+1}(0), \quad \varphi(x) = \varphi_L(x) \quad \text{on } \partial B_{R_A+1}(0) = \mathbb{S}_{R_A+1}^{n-1}$$

(see, e.g. [10, Sec 9.6]). We set also $\varphi(x) = \varphi_L(x)$ if $|x| \geq R_A$. The lemma is proved. \square

Thus, the function $v(x, t)$ is the ‘‘low frequency’’ solution of the linear equation

$$v_{tt} - t^{-4/3} A(x, D_x)v = 0.$$

Next we define the function $F_1(t)$,

$$F_1(t) := \int_{\mathbb{R}^n} a(x)u(x, t)v(x, t) dx,$$

that is, the projection of the solution on the ‘‘low frequency’’ eigenspace of the problem for the operator $A(x, D_x)$. Here $F_1 \in C^2(0, T)$. We estimate the function F_1 from above as follows

$$F''(t) \gtrsim \left(\int_{|x| \leq R+\phi(t)} |v(x, t)|^{p/(p-1)} dx \right)^{1-p} t^{1-p} |F_1(t)|^p. \quad (2.6)$$

To find out the properties of $F_1(t)$ we need the following lemma.

Lemma 2.2 *The function*

$$\lambda(t) = (\phi(t) + 1) \exp(-\phi(t))$$

solves the equation

$$\lambda''(t) - t^{-4/3} \lambda(t) = 0$$

and has the following properties:

$$\begin{aligned} (i) \quad \lambda'(t) &= -\frac{9}{\phi(t)} \exp(-\phi(t)) \leq 0, \\ (ii) \quad \lim_{t \rightarrow 0} \lambda(t) &= 1, \quad \lim_{t \rightarrow \infty} \lambda(t) = 0, \quad \lim_{t \rightarrow \infty} \lambda'(t) = 0, \\ (iii) \quad \frac{\lambda'(t)}{\lambda(t)} &= -\frac{9}{\phi(t)(\phi(t) + 1)}. \end{aligned}$$

Proof. It can be verified by straightforward calculations. \square

Next we turn to the function $\varphi(x)$. The following lemma is an analog of Lemma 2.3 [18].

Lemma 2.3 *Assume that $p > 1$. Then*

$$\int_{|x| \leq \tau} |\varphi(x)|^{p/(p-1)} dx \leq c_{R\tau} \tau^{\frac{n-1}{2} \frac{p-2}{p-1}} e^{\tau \frac{p}{p-1}} \quad \text{for all } \tau \geq R_A + 1.$$

Proof. Indeed, for $\tau \geq R_A + 1$ we have

$$\begin{aligned}
\int_{|x| \leq \tau} |\varphi(x)|^{p/(p-1)} dx &= \int_{|x| \leq R_A+1} |\varphi(x)|^{p/(p-1)} dx + \int_{R_A+1 \leq |x| \leq \tau} |\varphi(x)|^{p/(p-1)} dx \\
&= \int_{|x| \leq R_A+1} |\varphi(x)|^{p/(p-1)} dx + \int_{R_A+1 \leq |x| \leq \tau} |\varphi_L(x)|^{p/(p-1)} dx \\
&= \int_{|x| \leq R_A+1} |\varphi(x)|^{p/(p-1)} dx + \int_{|x| \leq \tau} |\varphi_L(x)|^{p/(p-1)} dx .
\end{aligned}$$

The application of Lemma 2.3 [18] completes the proof. \square

Lemma 2.4 Assume that $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$, and that

$$\int_{\mathbb{R}^n} a(x)\varphi_1(x)\varphi(x) dx \geq 18 \int_{\mathbb{R}^n} a(x)\varphi_0(x)\varphi(x) dx > 0,$$

then

$$F_1(t) \gtrsim (9\sqrt[3]{t^2} - 1) \int_{\mathbb{R}^n} a(x)\varphi_1(x)\varphi(x) dx \quad \text{for all } t > 1.$$

Proof. We have

$$F_1(0) = \lim_{t \rightarrow 0^+} \int_{\mathbb{R}^n} a(x)u(x, t)v(x, t) dx = \int_{\mathbb{R}^n} a(x)\varphi_0(x)\varphi(x) dx \geq c_0 > 0.$$

For every $\epsilon > 0$ we have

$$\begin{aligned}
0 &= \int_{\epsilon}^t \int_{\mathbb{R}^n} a(x)(u_{tt}(x, \tau) - \tau^{-4/3}A(x, D_x)u - \tau^{1-p}|u|^p)v(x, \tau) dx d\tau \\
&= \int_{\epsilon}^t \int_{\mathbb{R}^n} a(x)u_{tt}(x, \tau)v(x, \tau) dx d\tau - \int_{\epsilon}^t \int_{\mathbb{R}^n} \tau^{-4/3}u(x, \tau)a(x)A(x, D_x)v(x, \tau) dx d\tau \\
&\quad - \int_{\epsilon}^t \int_{\mathbb{R}^n} \tau^{1-p}a(x)|u(x, \tau)|^p v(x, \tau) dx d\tau .
\end{aligned}$$

Further,

$$\begin{aligned}
&\int_{\epsilon}^t \int_{\mathbb{R}^n} a(x)u_{tt}(x, \tau)v(x, \tau) dx d\tau \\
&= \int_{\mathbb{R}^n} a(x)u_t(x, \tau)v(x, \tau) dx \Big|_{\epsilon}^t - \int_{\mathbb{R}^n} a(x)u(x, \tau)v_t(x, \tau) dx \Big|_{\epsilon}^t \\
&\quad + \int_{\epsilon}^t \int_{\mathbb{R}^n} u(x, \tau)\tau^{-4/3}a(x)A(x, D_x)v(x, \tau) dx d\tau .
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^n} a(x)u_t(x, \tau)v(x, \tau) dx \Big|_{\epsilon}^t - \int_{\mathbb{R}^n} a(x)u(x, \tau)v_t(x, \tau) dx \Big|_{\epsilon}^t \\
&= \int_{\epsilon}^t \int_{\mathbb{R}^n} \tau^{1-p}a(x)|u(x, \tau)|^p v(x, \tau) dx d\tau .
\end{aligned}$$

The last equation implies

$$\begin{aligned} & \left(\frac{d}{d\tau} \int_{\mathbb{R}^n} a(x)u(x, \tau)v(x, \tau) dx - 2 \int_{\mathbb{R}^n} a(x)u(x, \tau)v_t(x, \tau) dx \right) \Big|_{\varepsilon}^t \\ &= \int_{\varepsilon}^t \int_{\mathbb{R}^n} \tau^{1-p} a(x)|u(x, \tau)|^p v(x, \tau) dx d\tau. \end{aligned}$$

It follows

$$\begin{aligned} & \frac{d}{dt} F_1(t) - 2 \frac{\lambda_t(t)}{\lambda(t)} \int_{\mathbb{R}^n} a(x)u(x, t)\lambda(t)\varphi(x) dx \\ &= \frac{d}{dt} F_1(t) \Big|_{\varepsilon} - 2 \frac{\lambda_t(\varepsilon)}{\lambda(\varepsilon)} \int_{\mathbb{R}^n} a(x)u(x, \varepsilon)\lambda(\varepsilon)\varphi(x) dx \\ & \quad + \int_{\varepsilon}^t \int_{\mathbb{R}^n} \tau^{1-p} a(x)|u(x, \tau)|^p v(x, \tau) dx d\tau. \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{d}{dt} F_1(t) - 2 \frac{\lambda_t(t)}{\lambda(t)} F_1(t) &= \frac{d}{dt} F_1(t) \Big|_{\varepsilon} - 2 \frac{\lambda_t(\varepsilon)}{\lambda(\varepsilon)} F_1(\varepsilon) \\ & \quad + \int_{\varepsilon}^t \int_{\mathbb{R}^n} \tau^{1-p} a(x)|u(x, \tau)|^p v(x, \tau) dx d\tau. \end{aligned}$$

It follows

$$\begin{aligned} & \frac{d}{dt} \left(F_1(t) \exp \left(- \int_{\varepsilon}^t 2 \frac{\lambda_t(\tau)}{\lambda(\tau)} d\tau \right) \right) \\ &= \exp \left(- \int_{\varepsilon}^t 2 \frac{\lambda_t(\tau)}{\lambda(\tau)} d\tau \right) \\ & \quad \times \left\{ \frac{d}{dt} F_1(t) \Big|_{\varepsilon} - 2 \frac{\lambda_t(\varepsilon)}{\lambda(\varepsilon)} F_1(\varepsilon) + \int_{\varepsilon}^t \int_{\mathbb{R}^n} \tau^{1-p} a(x)|u(x, \tau)|^p v(x, \tau) dx d\tau \right\}, \end{aligned}$$

that is

$$\begin{aligned} & \frac{d}{dt} \left(F_1(t) \left(\frac{\lambda(t)}{\lambda(\varepsilon)} \right)^{-2} \right) \\ &= \left(\frac{\lambda(t)}{\lambda(\varepsilon)} \right)^{-2} \left\{ \frac{d}{dt} F_1(t) \Big|_{\varepsilon} - 2 \frac{\lambda_t(\varepsilon)}{\lambda(\varepsilon)} F_1(\varepsilon) + \int_{\varepsilon}^t \int_{\mathbb{R}^n} \tau^{1-p} a(x)|u(x, \tau)|^p v(x, \tau) dx d\tau \right\}. \end{aligned}$$

We integrate it and obtain

$$\begin{aligned} F_1(t) &= \left(\frac{\lambda(t)}{\lambda(\varepsilon)} \right)^2 \left[F_1(\varepsilon) + \int_{\varepsilon}^t \left(\frac{\lambda(s)}{\lambda(\varepsilon)} \right)^{-2} \right. \\ & \quad \left. \times \left\{ \frac{d}{dt} F_1(t) \Big|_{\varepsilon} - 2 \frac{\lambda_t(\varepsilon)}{\lambda(\varepsilon)} F_1(\varepsilon) + \int_{\varepsilon}^s \int_{\mathbb{R}^n} \tau^{1-p} a(x)|u(x, \tau)|^p v(x, \tau) dx d\tau \right\} ds \right]. \end{aligned} \tag{2.7}$$

On the other hand, according to (iii) of Lemma 2.2 we have $\frac{\lambda_t(t)}{\lambda(t)} = -\frac{3}{\sqrt[3]{t}(3\sqrt[3]{t}+1)}$. Consider the term

$$\begin{aligned} \frac{d}{dt} F_1(t) \Big|_{\varepsilon} - 2 \frac{\lambda_t(\varepsilon)}{\lambda(\varepsilon)} F_1(\varepsilon) &= \int_{\mathbb{R}^n} a(x)u_t(x, \varepsilon)\lambda(\varepsilon)\varphi(x) dx + \int_{\mathbb{R}^n} a(x)u(x, \varepsilon)\lambda_t(\varepsilon)\varphi(x) dx \\ & \quad + \frac{6}{\sqrt[3]{\varepsilon}(3\sqrt[3]{\varepsilon}+1)} \int_{\mathbb{R}^n} a(x)u(x, \varepsilon)\lambda(\varepsilon)\varphi(x) dx. \end{aligned}$$

We can rewrite it as follows

$$\begin{aligned}
& \frac{d}{dt}F_1(t)\Big|_\varepsilon + \frac{6}{\sqrt[3]{\varepsilon}(3\sqrt[3]{\varepsilon}+1)}F_1(\varepsilon) \\
= & \int_{\mathbb{R}^n} a(x) \left\{ u_t(x, \varepsilon) + 3\varepsilon^{-1/3}A(x, D_x)\varphi_0(x) \right\} v(x, \varepsilon) dx \\
& - \int_{\mathbb{R}^n} 3a(x)\varepsilon^{-1/3}A(x, D_x)\varphi_0(x)v(x, \varepsilon) dx \\
& - \int_{\mathbb{R}^n} \frac{3}{\sqrt[3]{\varepsilon}(3\sqrt[3]{\varepsilon}+1)}a(x)u(x, \varepsilon)v(x, \varepsilon) dx + \frac{6}{\sqrt[3]{\varepsilon}(3\sqrt[3]{\varepsilon}+1)} \int_{\mathbb{R}^n} a(x)u(x, \varepsilon)v(x, \varepsilon) dx \\
= & \int_{\mathbb{R}^n} a(x)\{u_t(x, \varepsilon) + 3\varepsilon^{-1/3}A(x, D_x)\varphi_0(x)\}v(x, \varepsilon) dx \\
& + \int_{\mathbb{R}^n} 3\varepsilon^{-1/3}a(x)\left\{-\varphi_0(x) + \frac{1}{(3\sqrt[3]{\varepsilon}+1)}u(x, \varepsilon)\right\}v(x, \varepsilon) dx.
\end{aligned}$$

Hence, taking into account the initial conditions for u , we derive

$$\lim_{\varepsilon \rightarrow 0^+} \left(\frac{d}{dt}F_1(t)\Big|_\varepsilon - 2\frac{\lambda_t(\varepsilon)}{\lambda(\varepsilon)}F_1(\varepsilon) \right) = \int_{\mathbb{R}^n} a(x)\varphi_1(x)\varphi(x) dx - 9 \int_{\mathbb{R}^n} a(x)\varphi_0(x)\varphi(x) dx.$$

Now

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0^+} \left(\frac{\lambda(t)}{\lambda(\varepsilon)} \right)^2 \left[F_1(\varepsilon) + \int_\varepsilon^t \left(\frac{\lambda(s)}{\lambda(\varepsilon)} \right)^{-2} \left\{ \frac{d}{dt}F_1(t)\Big|_\varepsilon - 2\frac{\lambda_t(\varepsilon)}{\lambda(\varepsilon)}F_1(\varepsilon) \right\} ds \right] \\
= & \lambda(t)^2 F_1(0) + \lim_{\varepsilon \rightarrow 0^+} \left(\frac{\lambda(t)}{\lambda(\varepsilon)} \right)^2 \int_\varepsilon^t \left(\frac{\lambda(s)}{\lambda(\varepsilon)} \right)^{-2} \left\{ \frac{d}{dt}F_1(t)\Big|_\varepsilon - 2\frac{\lambda_t(\varepsilon)}{\lambda(\varepsilon)}F_1(\varepsilon) \right\} ds \\
= & \lambda(t)^2 F_1(0) + \left\{ \int_{\mathbb{R}^n} a(x)\varphi_1(x)\varphi(x) dx - 9 \int_{\mathbb{R}^n} a(x)\varphi_0(x)\varphi(x) dx \right\} \lambda^2(t) \int_0^t \lambda^{-2}(s) ds \\
= & (3\sqrt[3]{t}+1)^2 \exp(-6\sqrt[3]{t}) \int_{\mathbb{R}^n} \varphi_0(x)\varphi(x) dx \\
& + \left\{ \int_{\mathbb{R}^n} a(x)\varphi_1(x)\varphi(x) dx - 9 \int_{\mathbb{R}^n} a(x)\varphi_0(x)\varphi(x) dx \right\} \\
& \times (3\sqrt[3]{t}+1)^2 \exp(-6\sqrt[3]{t}) \int_0^t (3\sqrt[3]{s}+1)^{-2} \exp(6\sqrt[3]{s}) ds.
\end{aligned}$$

On the other hand

$$\int_0^t (3\sqrt[3]{s}+1)^{-2} \exp(6\sqrt[3]{s}) ds = \frac{1}{18} \left(\exp(6\sqrt[3]{t}) \frac{3\sqrt[3]{t}-1}{3\sqrt[3]{t}+1} + 1 \right)$$

implies

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0^+} \left(\frac{\lambda(t)}{\lambda(\varepsilon)} \right)^2 \left[F_1(\varepsilon) + \int_\varepsilon^t \left(\frac{\lambda(s)}{\lambda(\varepsilon)} \right)^{-2} \left\{ \frac{d}{dt}F_1(t)\Big|_\varepsilon - 2\frac{\lambda_t(\varepsilon)}{\lambda(\varepsilon)}F_1(\varepsilon) \right\} ds \right] \\
= & (3\sqrt[3]{t}+1)^2 \exp(-6\sqrt[3]{t}) \int_{\mathbb{R}^n} a(x)\varphi_0(x)\varphi(x) dx \\
& + \left\{ \int_{\mathbb{R}^n} a(x)\varphi_1(x)\varphi(x) dx - 9 \int_{\mathbb{R}^n} \varphi_0(x)\varphi(x) dx \right\} \\
& \times (3\sqrt[3]{t}+1)^2 \frac{1}{18} \left(\frac{3\sqrt[3]{t}-1}{3\sqrt[3]{t}+1} + \exp(-6\sqrt[3]{t}) \right).
\end{aligned}$$

Due to the conditions of the lemma,

$$\int_{\mathbb{R}^n} a(x)\varphi_1(x)\varphi(x) dx - 9 \int_{\mathbb{R}^n} a(x)\varphi_0(x)\varphi(x) dx \geq \frac{1}{2} \int_{\mathbb{R}^n} a(x)\varphi_1(x)\varphi(x) dx > 0.$$

Then, from (2.7), by letting $\varepsilon \rightarrow 0$, we derive

$$\begin{aligned} F_1(t) &\geq \left(\frac{\lambda(t)}{\lambda(\varepsilon)} \right)^2 \left[F_1(\varepsilon) + \int_{\varepsilon}^t \left(\frac{\lambda(s)}{\lambda(\varepsilon)} \right)^{-2} \left\{ \frac{d}{ds} F_1(s) \Big|_{\varepsilon} - 2 \frac{\lambda_t(\varepsilon)}{\lambda(\varepsilon)} F_1(\varepsilon) \right\} ds \right] \\ &\geq (3\sqrt[3]{t} + 1)^2 \exp\left(-6\sqrt[3]{t}\right) \int_{\mathbb{R}^n} a(x)\varphi_0(x)\varphi(x) dx \\ &\quad + (3\sqrt[3]{t} + 1)^2 \frac{1}{18} \left(\frac{3\sqrt[3]{t} - 1}{3\sqrt[3]{t} + 1} + \exp\left(-6\sqrt[3]{t}\right) \right) \frac{1}{2} \int_{\mathbb{R}^n} a(x)\varphi_1(x)\varphi(x) dx \\ &\geq (3\sqrt[3]{t} + 1)^2 \exp\left(-6\sqrt[3]{t}\right) \int_{\mathbb{R}^n} a(x)\varphi_0(x)\varphi(x) dx \\ &\quad + (9\sqrt[3]{t^2} - 1) \frac{1}{36} \int_{\mathbb{R}^n} a(x)\varphi_1(x)\varphi(x) dx. \end{aligned}$$

Lemma is proved. □

Furthermore, the inequality (2.6) implies

$$F''(t) \geq \lambda^{-p}(t) \left(\int_{|x| \leq R + \phi(t)} |\varphi(x)|^{p/(p-1)} dx \right)^{1-p} t^{1-p} |F_1(t)|^p \quad \text{for all } t \geq R_A + 1.$$

According to the last lemma

$$\begin{aligned} &F''(t) \\ &\geq c_R \lambda^{-p}(t) (R + \phi(t))^{-\frac{n-1}{2}(p-2)} e^{-\phi(t)p} t^{1-p} |F_1(t)|^p \\ &\geq c_R (R + \phi(t))^{-p - \frac{n-1}{2}(p-2)} t^{1-p} \left| (9t^{\frac{2}{3}} - 1) \int_{\mathbb{R}^n} a(x)\varphi_1(x)\varphi(x) dx \right|^p \quad \text{for all } t \geq R_A + 1. \end{aligned}$$

Finally

$$F''(t) \geq C_R (R + \phi(t))^{-p - \frac{n-1}{2}(p-2)} t^{1-p + \frac{2}{3}p} \left| \int_{\mathbb{R}^n} a(x)\varphi_1(x)\varphi(x) dx \right|^p \quad \text{for all } t \geq R_A + 1. \quad (2.8)$$

For $t > 1$ and arbitrary $\varepsilon \in (0, 1)$, it follows

$$\begin{aligned} F(t) &= F(\varepsilon) + \int_{\varepsilon}^1 \left\{ F'(\varepsilon) + \int_{\varepsilon}^{t_1} F''(t_2) dt_2 \right\} dt_1 + \int_1^t \left\{ F'(\varepsilon) + \int_{\varepsilon}^{t_1} F''(t_2) dt_2 \right\} dt_1 \\ &\geq F(\varepsilon) + \int_{\varepsilon}^1 F'(\varepsilon) dt_1 + \int_1^t F'(\varepsilon) dt_1 + \int_1^t \left\{ \int_1^{t_1} F''(t_2) dt_2 \right\} dt_1 \\ &\geq F(\varepsilon) + (t - \varepsilon)F'(\varepsilon) + \int_1^t \left\{ \int_1^{t_1} F''(t_2) dt_2 \right\} dt_1. \end{aligned}$$

By letting $\varepsilon \rightarrow 0$ and using (2.8) we derive

$$\begin{aligned} F(t) &\geq tF'(0) + F(0) \\ &\quad + c_R \left| \int_{\mathbb{R}^n} a(x)\varphi_1(x)\varphi(x) dx \right|^p \int_1^t \int_1^{t_2} (R + \phi(t_1))^{-p - \frac{n-1}{2}(p-2)} t_1^{1 - \frac{1}{3}p} dt_1 dt_2. \end{aligned}$$

Set (see (2.4))

$$r = \frac{1}{6} [2n + 16 - (n + 3)p], \quad q = \frac{(n + 3)(p - 1)}{3}.$$

We need $r \geq 1$ that is, $p \leq (2n + 10)/(n + 3)$. The Kato's lemma (see, e.g., [19, Lemma 2.1]), concerning differential inequalities

$$\begin{aligned} F(t) &\geq c_0(1 + t)^r \quad \text{for large } t, \\ F''(t) &\geq (1 + t)^{-q}|F(t)|^p \quad \text{for large } t, \end{aligned}$$

conditions are $r \geq 1$, $p > 1$ and

$$(p - 1)r > q - 2 \iff (n + 3)p^2 - (n + 13)p - 2 < 0.$$

Due to the definition of $p_{cr}(n)$ we obtain $p < p_{cr}(n)$. The theorem is proved. \square

Corollary 2.5 *For the covariant semilinear wave equation with $n = 3$ and $F(\psi) = |\psi|^p$ assume that $1 < p < 3$. Then for every arbitrary small number $\varepsilon > 0$ and an arbitrary number s there exist functions $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^3)$, $\text{supp } \varphi_0, \varphi_1 \subseteq \{x \in \mathbb{R}^3 \mid |x| \leq R\}$ with norms satisfying inequality*

$$\|\varphi_0\|_{H_{(s)}(\mathbb{R}^3)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^3)} < \varepsilon$$

such that the solution of the problem (1.2) with support in $\{(x, t) \mid t > 0, x \in B_{R+\phi(t)s_A}(0)\}$ blows up in finite time.

Now we analyze the conditions of the theorem. From the graph it follows that for $n \leq 4$ there is a small data blowing up solution if $1 < p < 1 + \frac{6}{n}$. For the dimensions $n \geq 5$ such solution exists if $1 < p < \frac{n+13+\sqrt{n^2+34n+193}}{2(n+3)}$.

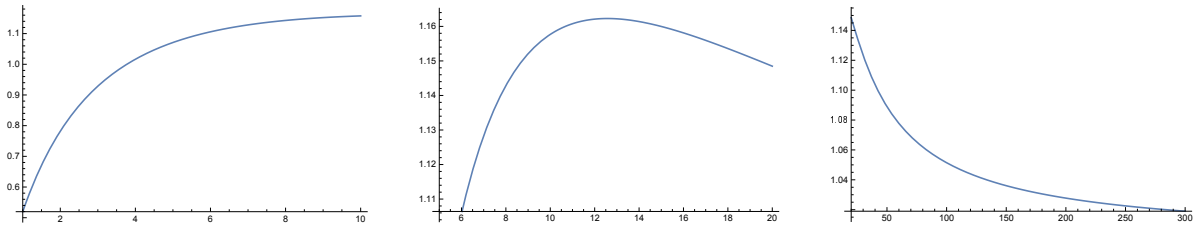


Figure 1: $\frac{n+13+\sqrt{n^2+34n+193}}{2(n+3)}/(1 + \frac{6}{n})$, $n \in [1, 10]$ $n \in [4, 20]$ $n \in [20, 300]$

3 Equation without singularity. Proof of Theorem 1.3

Theorem 1.3 shows that the blow-up, which is stated in Theorem 1.1, is caused by the semilinear term. Consider the Cauchy problem (1.7). Let $p_0(n, k)$ be a positive root of the equation (1.8). (For the graph $p = p_0(n, k)$ see Figure 2.) The equation of (1.7) is strictly hyperbolic for every bounded interval of time and it has smooth coefficients. Consequently, for every smooth initial functions φ_0 and φ_1 the problem (1.7) has a local solution. According to Theorem 1.3 for $n = 3$ and $p < 3$ a local in time solution, in general, cannot be prolonged to the global solution.

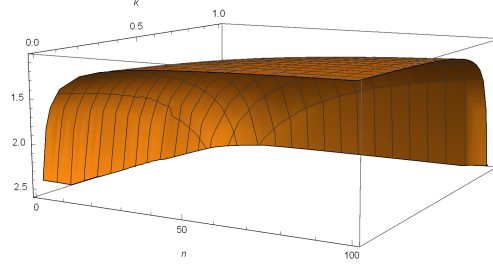


Figure 2: Function $p_0(n, k)$ for $0 \leq k \leq 1$ and $1 \leq n \leq 100$

Proof of Theorem 1.3. We use operators \mathcal{L} and \mathcal{S} which are introduced above in (2.1): $\mathcal{L} := \partial_t^2 - t^{-2k}A(x, D_x) + 2t^{-1}\partial_t$, $\mathcal{S} := \partial_t^2 - t^{-2k}A(x, D_x)$, and for $t \neq 0$ the operator identity (2.2). The last identity suggests a change of unknown function ψ with u such that $\psi = t^{-1}u$. The problem for u is as follows:

$$\begin{cases} u_{tt} - t^{-2k}A(x, D_x)u = t^{1-p}|u|^p, & t > 1, x \in \mathbb{R}^n, \\ u(x, 1) = u_0(x), \quad u_0(x) := \varphi_0(x), & x \in \mathbb{R}^n, \\ u_t(x, 1) = u_1(x), \quad u_1(x) := \varphi_0(x) + \varphi_1(x), & x \in \mathbb{R}^n. \end{cases} \quad (3.1)$$

Denote

$$F(t) = \int_{\mathbb{R}^n} a(x)u(x, t) dx.$$

Then $F \in C^2[1, T]$, provided that the function u is defined for all $(x, t) \in \mathbb{R}^n \times [1, T]$, and

$$F(1) = \int_{\mathbb{R}^n} a(x)u_0(x) dx = C_0 > 0, \quad F'(1) = \int_{\mathbb{R}^n} a(x)u_1(x) dx = C_1.$$

From the equation of (3.1) we have

$$F''(t) = t^{1-p} \int_{\mathbb{R}^n} a(x)|u(x, t)|^p dx \geq 0 \quad \text{for all } t > 1. \quad (3.2)$$

Furthermore,

$$\begin{aligned} F(t) &= F(1) + (t-1)F'(1) + \int_1^t \int_1^{t_1} F''(t_2) dt_2 dt_1 \\ &\geq F(1) + (t-1)F'(1) \geq 0 \quad \text{for all } t \geq 1, \end{aligned}$$

provided that $C_0 \geq 0$ and $C_1 \geq 0$. Hence

$$F(t) \geq (t-1) \int_{\mathbb{R}^n} u_1(x) dx + \int_{\mathbb{R}^n} u_0(x) dx \geq 0 \quad \text{for all } t \geq 1.$$

Assume that $\text{supp } u_i \subseteq B_R(0)$, $i = 0, 1$. On the other hand, using the compact support of $u(\cdot, t)$ and Hölder's inequality with $\phi(t) := \frac{1}{1-k}t^{1-k}$ we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^n} a(x)u(x, t) dx \right|^p &\leq \left(\int_{|x| \leq R+\phi(t)-\phi(1)} 1 dx \right)^{p-1} \left(\int_{|x| \leq R+\phi(t)-\phi(1)} a(x)^p |u(x, t)|^p dx \right) \\ &\lesssim (R + \phi(t))^{n(p-1)} \left(\int_{|x| \leq R+\phi(t)-\phi(1)} a(x)|u(x, t)|^p dx \right). \end{aligned}$$

Hence from (3.2) we derive

$$F''(t) \geq (1+t)^{1-p-n(p-1)(1-k)} |F(t)|^p \quad \text{for all } t \geq 1.$$

We set

$$r = 1, \quad q := (p-1) + n(p-1)(1-k) = (p-1)(1+n(1-k)).$$

Consider the first case of $1 < p < 1 + \frac{2}{n(1-k)}$. If $1 < p < 1 + \frac{2}{n(1-k)}$ and $C_1 > 0$, then we can apply Kato's lemma (see, e.g., [19, Lemma 2.1]) since

$$p-1 > (p-1)(1+n(1-k)) - 2 \iff p < 1 + \frac{2}{n(1-k)}.$$

Thus, the solution blows up.

Consider the second case. For this case we choose

$$v(x, t) := \tilde{\lambda}(t)\varphi(x), \quad \tilde{\lambda}(t) := \frac{1}{K_{\frac{1}{2-2k}}(\phi(1))} \sqrt{t} K_{\frac{1}{2-2k}}(\phi(t)),$$

where $K_a(z)$ is the modified Bessel function of the second kind. The function $\tilde{\lambda} = \tilde{\lambda}(t)$ solves the equation

$$\tilde{\lambda}_{tt} - t^{-2k}\tilde{\lambda} = 0.$$

It is easy to verify the following limit

$$\lim_{t \rightarrow \infty} \sqrt{t} K_{\frac{1}{2-2k}}(\phi(t)) = 0.$$

Hence

$$v(x, 1) = \varphi(x), \quad \lim_{t \rightarrow \infty} v(x, t) = 0.$$

We skip the proof of the next lemma.

Lemma 3.1 *There is a number $\Lambda_0 > 0$ such that*

$$\Lambda_1(k) := -\tilde{\lambda}_t(1) = \frac{K_{\frac{1-2k}{2-2k}}\left(\frac{1}{1-k}\right)}{K_{\frac{1}{2-2k}}\left(\frac{1}{1-k}\right)} > \Lambda_0 \quad \text{for all } k \in [0, 1).$$

Assume that $u_0, u_1 \in C_0^\infty$, $\text{supp } u_0, \text{supp } u_1 \subseteq \{x \in \mathbb{R}^n \mid |x| \leq R\}$. Now we turn to the function

$$F_1(t) := \int_{\mathbb{R}^n} a(x)u(x, t)v(x, t) dx$$

and obtain

$$|F_1(t)|^p \lesssim \left(\int_{|x| \leq R+\phi(t)-\phi(1)} |v(x, t)|^{p/(p-1)} dx \right)^{p-1} t^{p-1} F''(t) \quad \text{for all } t > 1.$$

The last estimate implies

$$F''(t) \geq \left(\int_{|x| \leq R+\phi(t)-\phi(1)} |v(x, t)|^{p/(p-1)} dx \right)^{1-p} t^{1-p} |F_1(t)|^p \quad \text{for all } t > 1. \quad (3.3)$$

Lemma 3.2 Assume that $u_0, u_1 \in C_0^\infty$, $\text{supp } u_0, \text{supp } u_1 \subseteq B_R(0) \subseteq \mathbb{R}^n$, and

$$\Lambda_1(k) \int_{\mathbb{R}^n} a(x)u_0(x)\varphi(x)dx + \int_{\mathbb{R}^n} a(x)u_1(x)\varphi(x)dx \geq c_0 \int_{\mathbb{R}^n} a(x)u_0(x)\varphi(x)dx > 0.$$

Then there exists a sufficiently large $T > 1$ such that for the solution $u = u(x, t)$ of the problem (3.1) with the support in $\{x \in \mathbb{R}^n \mid |x| \leq R + \phi(t) - \phi(1)\}$ one has

$$F_1(t) \geq \frac{1}{16}t^k \left\{ \Lambda_1(k) \int_{\mathbb{R}^n} a(x)u_0(x)\varphi(x)dx + \int_{\mathbb{R}^n} a(x)u_1(x)\varphi(x)dx \right\} \quad \text{for all } t > T.$$

Proof. We have

$$F_1(1) = \int_{\mathbb{R}^n} a(x)u(x, 1)v(x, 1) dx = \int_{\mathbb{R}^n} a(x)u_0(x)\varphi(x) dx \geq c_0 > 0$$

and

$$\begin{aligned} 0 &= \int_1^t \int_{\mathbb{R}^n} a(x)(u_{tt}(x, \tau) - \tau^{-2k}A(x, D_x)u - \tau^{1-p}|u|^p)v(x, \tau) dx d\tau \\ &= \int_1^t \int_{\mathbb{R}^n} a(x)u_{tt}(x, \tau)v(x, \tau) dx d\tau - \int_1^t \int_{\mathbb{R}^n} \tau^{-2k}a(x)uA(x, D_x)v(x, \tau) dx d\tau \\ &\quad - \int_1^t \int_{\mathbb{R}^n} \tau^{1-p}a(x)|u|^pv(x, \tau) dx d\tau. \end{aligned}$$

Further,

$$\begin{aligned} &\int_1^t \int_{\mathbb{R}^n} a(x)u_{tt}(x, \tau)v(x, \tau) dx d\tau \\ &= \int_{\mathbb{R}^n} a(x)u_t(x, \tau)v(x, \tau) dx \Big|_{\tau=1}^{\tau=t} - \int_{\mathbb{R}^n} a(x)u(x, \tau)v_t(x, \tau) dx \Big|_{\tau=1}^{\tau=t} \\ &\quad + \int_1^t \int_{\mathbb{R}^n} u(x, \tau)t^{-2k}a(x)A(x, D_x)v(x, \tau) dx d\tau. \end{aligned}$$

Hence,

$$\begin{aligned} &\int_{\mathbb{R}^n} a(x)u_t(x, \tau)v(x, \tau) dx \Big|_{\tau=1}^{\tau=t} - \int_{\mathbb{R}^n} a(x)u(x, \tau)v_t(x, \tau) dx \Big|_{\tau=1}^{\tau=t} \\ &= \int_1^t \int_{\mathbb{R}^n} \tau^{1-p}a(x)|u(x, \tau)|^pv(x, \tau) dx d\tau \end{aligned}$$

implies

$$\begin{aligned} &\left(\frac{d}{d\tau} \int_{\mathbb{R}^n} a(x)u(x, \tau)v(x, \tau) dx - 2 \int_{\mathbb{R}^n} a(x)u(x, \tau)v_t(x, \tau) dx \right) \Big|_{\tau=1}^{\tau=t} \\ &= \int_1^t \int_{\mathbb{R}^n} \tau^{1-p}a(x)|u(x, \tau)|^pv(x, \tau) dx d\tau \end{aligned}$$

and

$$\begin{aligned} &\frac{d}{dt}F_1(t) - 2\frac{\tilde{\lambda}_t(t)}{\tilde{\lambda}(t)} \int_{\mathbb{R}^n} a(x)u(x, t)\tilde{\lambda}(t)\varphi(x) dx \\ &= \frac{d}{dt}F_1(t) \Big|_1 - 2\frac{\tilde{\lambda}_t(1)}{\tilde{\lambda}(1)} \int_{\mathbb{R}^n} a(x)u(x, 1)\tilde{\lambda}(1)\varphi(x) dx \\ &\quad + \int_1^t \int_{\mathbb{R}^n} \tau^{1-p}a(x)|u(x, \tau)|^pv(x, \tau) dx d\tau. \end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{\tilde{\lambda}_t(t)}{\tilde{\lambda}(t)} &= -\frac{t^{-k}K_{\frac{1-2k}{2-2k}}(\phi(t))}{K_{\frac{1}{2-2k}}(\phi(t))} < 0 \quad \text{for all } t > 0, \\ \lim_{t \rightarrow \infty} \frac{\tilde{\lambda}_t(t)}{\tilde{\lambda}(t)} &= 0, \quad \frac{\tilde{\lambda}_t(1)}{\tilde{\lambda}(1)} = \tilde{\lambda}_t(1) = -\frac{K_{\frac{1-2k}{2-2k}}\left(\frac{1}{1-k}\right)}{K_{\frac{1}{2-2k}}\left(\frac{1}{1-k}\right)}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\frac{d}{dt}F_1(t) - 2\frac{\tilde{\lambda}_t(t)}{\tilde{\lambda}(t)}F_1(t) \\ &= \frac{d}{dt}F_1(t)\Big|_1 - 2\frac{\tilde{\lambda}_t(1)}{\tilde{\lambda}(1)}F_1(1) + \int_1^t \int_{\mathbb{R}^n} \tau^{1-p}a(x)|u(x, \tau)|^p v(x, \tau) dx d\tau,\end{aligned}$$

that is,

$$\begin{aligned}&\frac{d}{dt}\left(F_1(t)\left(\tilde{\lambda}(t)\right)^{-2}\right) \\ &= \left(\tilde{\lambda}(t)\right)^{-2}\left\{\frac{d}{dt}F_1(t)\Big|_1 + 2\Lambda_1(k)F_1(1) + \int_1^t \int_{\mathbb{R}^n} \tau^{1-p}a(x)|u(x, \tau)|^p v(x, \tau) dx d\tau\right\},\end{aligned}$$

where, due to Lemma 3.1, $\Lambda_1(k) = -\tilde{\lambda}_t(1) = \frac{K_{\frac{1-2k}{2-2k}}\left(\frac{1}{1-k}\right)}{K_{\frac{1}{2-2k}}\left(\frac{1}{1-k}\right)} > \Lambda_0 > 0$. We integrate the last relation

$$\begin{aligned}F_1(t)\left(\tilde{\lambda}(t)\right)^{-2} &= F_1(1) + \int_1^t \left(\tilde{\lambda}(s)\right)^{-2}\left\{\frac{d}{dt}F_1(t)\Big|_{t=1} \right. \\ &\quad \left. + 2\Lambda_1(k)F_1(1) + \int_1^s \int_{\mathbb{R}^n} \tau^{1-p}a(x)|u(x, \tau)|^p v(x, \tau) dx d\tau\right\} ds.\end{aligned}$$

Finally,

$$\begin{aligned}F_1(t) &= \left(\tilde{\lambda}(t)\right)^2 \left[F_1(1) + \int_1^t \left(\tilde{\lambda}(s)\right)^{-2} \right. \\ &\quad \left. \times \left\{ \frac{d}{dt}F_1(t)\Big|_{t=1} + 2\Lambda_1(k)F_1(1) + \int_1^s \int_{\mathbb{R}^n} \tau^{1-p}a(x)|u(x, \tau)|^p v(x, \tau) dx d\tau \right\} ds \right].\end{aligned}$$

Consider two first terms of the integrand

$$\begin{aligned}&\frac{d}{dt}F_1(t)\Big|_{t=1} + 2\Lambda_1F_1(1) \\ &= \int_{\mathbb{R}^n} a(x)u_0(x)v_t(x, 1)dx + \int_{\mathbb{R}^n} a(x)u_1(x)v(x, 1)dx + 2\Lambda_1 \int_{\mathbb{R}^n} a(x)u_0(x)v(x, 1)dx \\ &= \Lambda_1(k) \int_{\mathbb{R}^n} a(x)u_0(x)\varphi(x) dx + \int_{\mathbb{R}^n} a(x)u_1(x)\varphi(x) dx.\end{aligned}$$

Then

$$\begin{aligned}
& \left(\tilde{\lambda}(t) \right)^2 \left[F_1(1) + \int_1^t \left(\tilde{\lambda}(s) \right)^{-2} \left\{ \frac{d}{dt} F_1(t) \Big|_{t=1} + 2\Lambda_1(k) F_1(1) \right\} ds \right] \\
&= \left(\tilde{\lambda}(t) \right)^2 F_1(1) \\
&+ \left[\Lambda_1(k) \int_{\mathbb{R}^n} a(x) u_0(x) \varphi(x) dx + \int_{\mathbb{R}^n} a(x) u_1(x) \varphi(x) dx \right] \left(\tilde{\lambda}(t) \right)^2 \int_1^t \left(\tilde{\lambda}(s) \right)^{-2} ds.
\end{aligned} \tag{3.4}$$

The following lemma completes the proof of Lemma 3.2. \square

Lemma 3.3 *There is a number $T_1 > 0$ such that*

$$\tilde{\lambda}^2(t) \int_T^t \tilde{\lambda}^{-2}(s) ds \geq \frac{1}{32} t^k \quad \text{for all } t \geq T_1.$$

Proof. For all $T > 1$ we have

$$\tilde{\lambda}^2(t) \int_1^t \tilde{\lambda}^{-2}(s) ds = \tilde{\lambda}^2(t) \int_1^T \tilde{\lambda}^{-2}(s) ds + \tilde{\lambda}^2(t) \int_T^t \tilde{\lambda}^{-2}(s) ds \quad \text{for all } t \geq T. \tag{3.5}$$

For large t there is the following asymptotic

$$\sqrt{t} K_{\frac{1}{2-2k}}(\phi(t)) = \sqrt{\frac{\pi}{2}} \sqrt{1-k} e^{-\phi(t)} t^{k/2} (1 + o(1)).$$

Consider the second integral; for the sufficiently large T we have

$$\begin{aligned}
& \tilde{\lambda}^2(t) \int_T^t \tilde{\lambda}^{-2}(s) ds \\
&\geq \frac{1}{2} e^{-2\frac{t^{1-k}}{1-k}} t^{2k} \int_T^t e^{2\frac{s^{1-k}}{1-k}} s^{-2k} ds \\
&= \frac{1}{2} e^{-2\frac{t^{1-k}}{1-k}} t^{2k} \frac{1}{4} \left(2e^{\frac{2t^{1-k}}{1-k}} t^{-k} + ke^{\frac{2t^{1-k}}{1-k}} t^{-1} + 2^{\frac{1}{1-k}} k \left(\frac{1}{k-1} \right)^{\frac{k-2}{k-1}} \Gamma \left(\frac{1}{k-1}, \frac{2t^{1-k}}{k-1} \right) \right. \\
&\quad \left. - 2e^{-\frac{2T^{1-k}}{k-1}} T^{-k} - \frac{ke^{-\frac{2T^{1-k}}{k-1}}}{T} - 2^{\frac{1}{1-k}} k \left(\frac{1}{k-1} \right)^{\frac{k-2}{k-1}} \Gamma \left(\frac{1}{k-1}, \frac{2T^{1-k}}{k-1} \right) \right) \quad \text{for all } t \geq T,
\end{aligned}$$

where $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$ is the incomplete gamma function. (See, e.g., [1, Sec.6.9.2].) On the other hand, since $k = 1 - \varepsilon$, $\varepsilon > 0$, we obtain for the incomplete gamma function the following asymptotic formula (see [1, Sec.6.13.1])

$$\begin{aligned}
\left(\frac{1}{k-1} \right)^{\frac{k-2}{k-1}} \Gamma \left(\frac{1}{k-1}, \frac{2t^{1-k}}{k-1} \right) &= 2^{\frac{3-2k}{k-1}} e^{-\frac{2t^{1-k}}{k-1}} t^{k-2} \left(2 + O(t^{k-1}) \right) \\
&\leq ce^{-\frac{2t^{1-k}}{k-1}} t^{-1-\varepsilon} \quad \text{for all } t \geq T.
\end{aligned}$$

Consequently, for the sufficiently large $T_1 > T$ we obtain

$$\begin{aligned}
\tilde{\lambda}^2(t) \int_T^t \tilde{\lambda}^{-2}(s) ds &\geq \frac{1}{2} e^{-2\frac{t^{1-k}}{1-k}} t^{2k} \frac{1}{4} \left(e^{\frac{2t^{1-k}}{1-k}} t^{-k} - 2e^{\frac{2T^{1-k}}{1-k}} T^{-k} \right. \\
&\quad \left. - \frac{ke^{-\frac{2T^{1-k}}{k-1}}}{T} - 2^{\frac{1}{1-k}} k \left(\frac{1}{k-1} \right)^{\frac{k-2}{k-1}} \Gamma \left(\frac{1}{k-1}, \frac{2T^{1-k}}{k-1} \right) \right) \\
&\geq \frac{1}{16} t^k \quad \text{for all } t \geq T_1.
\end{aligned}$$

The estimate for the first term of (3.5) is evident. Lemma is proved. \square

On the other hand, according to (3.3), we have

$$F''(t) \gtrsim \tilde{\lambda}^{-p}(t) \left(\int_{|x| \leq R + \phi(t) - \phi(1)} |\varphi(x)|^{p/(p-1)} dx \right)^{1-p} t^{1-p} |F_1(t)|^p \quad \text{for large } t,$$

and, consequently, the asymptotic of $\tilde{\lambda}(t)$, Lemma 3.2, and Lemma 2.3 imply

$$\begin{aligned} F''(t) &\gtrsim \tilde{\lambda}^{-p}(t) \left(\int_{|x| \leq R + \phi(t) - \phi(1)} |\varphi(x)|^{p/(p-1)} dx \right)^{1-p} t^{1-p} |F_1(t)|^p \\ &\gtrsim c_R (R + \phi(t) - \phi(1))^{-\frac{n-1}{2}(p-2)} t^{1-p} t^{\frac{kp}{2}} \\ &\quad \times \left| \left\{ \Lambda_1(k) \int_{\mathbb{R}^n} a(x) u_0(x) \varphi(x) dx + \int_{\mathbb{R}^n} a(x) u_1(x) \varphi(x) dx \right\} \right|^p \quad \text{for } t \geq T. \end{aligned}$$

Here $T > 1$ is a sufficiently large number. It follows

$$\begin{aligned} F(t) &= F(1) + \int_1^T \left\{ F'(1) + \int_T^{t_1} F''(t_2) dt_2 \right\} dt_1 + F'(T)(t - T) + \int_T^t \int_T^{t_1} F''(t_2) dt_2 dt_1 \\ &\gtrsim F(1) + F'(1)(T - 1) + (t - T) \left\{ F'(1) + \int_1^T F''(t_1) dt_1 \right\} \\ &\quad + \left| \left\{ \Lambda_1(k) \int_{\mathbb{R}^n} a(x) u_0(x) \varphi(x) dx + \int_{\mathbb{R}^n} a(x) u_1(x) \varphi(x) dx \right\} \right|^p \\ &\quad \times \int_T^t \int_T^{t_1} c_R (R + \phi(t_2) - \phi(1))^{-\frac{n-1}{2}(p-2)} t_2^{1-p} t_2^{\frac{kp}{2}} dt_2 dt_1, \end{aligned}$$

where $F'(1) = \int_{\mathbb{R}^n} a(x) u_0(x) \varphi(x) dx + \int_{\mathbb{R}^n} a(x) u_1(x) \varphi(x) dx$. Thus,

$$\begin{aligned} F(t) &\gtrsim F(1) + F'(1)(t - 1) + \left| \left\{ \Lambda_1(k) \int_{\mathbb{R}^n} a(x) u_0(x) \varphi(x) dx + \int_{\mathbb{R}^n} a(x) u_1(x) \varphi(x) dx \right\} \right|^p \\ &\quad \times \int_T^t \int_T^{t_1} \phi(t_2)^{-\frac{n-1}{2}(p-2)} t_2^{1-p} t_2^{\frac{kp}{2}} dt_2 dt_1. \end{aligned}$$

Set

$$r = -(1 - k) \frac{n - 1}{2} (p - 2) + 1 - p + \frac{kp}{2} + 2, \quad q = (p - 1)(1 + n(1 - k)).$$

We need $r \geq 1$, that is,

$$p \leq 2 + \frac{2k}{n + 1 - kn}.$$

We check the condition $(p - 1)r > q - 2$ of the Kato's lemma (see, e.g., [19, Lemma 2.1]), that is,

$$p^2(n + 1 - kn) - p(2k + n + 3 - kn) - 2(1 - k) < 0.$$

Since $k < 1$, we conclude $1 < p < p_0(n, k)$. Theorem is proved. \square

For the semilinear generalized Tricomi equation $\partial_t^2 u - t^m \Delta u = |u|^p$ with increasing coefficient, that is with $m \in \mathbb{N}$, the critical exponent $p_{crit}(m, n)$ and conformal exponent $p_{conf}(m, n)$ are suggested in [12]. Then, there are interesting articles on the non-linear higher-order degenerate hyperbolic equations [13], the low regularity solution problem for the semilinear mixed type equation

[15], and the local existence and singularity structures of low regularity solution to the semilinear generalized Tricomi equation with discontinuous initial data [14].

The Cauchy problem for the damped linear wave equations with time-dependent propagation speed and dissipations, $u_{tt} - a(t)^2 \Delta u + b(t)u_t = 0$, where $a \in L^1(0, \infty)$, is considered in [6]. An interesting example of the quasilinear equation $u_{tt} - t^{-4} \exp(-2t^{-1}) \Delta u - (u_{tt})^2 + t^{-4} \exp(-2t^{-1}) (\nabla u)^2 = 0$ without global solvability for arbitrarily small initial data is given in [17]. See also [4, 19] for more examples of such quasilinear equation.

4 Local in time solution

In this section we prove a local in time existence of the waves propagating in the Einstein-de Sitter spacetime. The initial data are prescribed at the plane $t = 0$ where the coefficients are singular. We discuss only the massless fields. Denote by G a solution operator of the problem

$$\begin{cases} \psi_{tt} - t^{-4/3} A(x, D_x) \psi + 2t^{-1} \psi_t = f, & t > 0, x \in \mathbb{R}^n, \\ \lim_{t \rightarrow 0^+} t \psi(x, t) = \varphi_0(x), & x \in \mathbb{R}^n, \\ \lim_{t \rightarrow 0^+} (t \psi_t(x, t) + \psi(x, t) + 3t^{-1/3} A(x, D_x) \varphi_0(x)) = \varphi_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (4.1)$$

with $\varphi_0(x) = \varphi_1(x) = 0$, that is $\psi = G[f]$. Let ψ_0 is the solution of the problem (4.1) with $f = 0$. Then any solution ψ of the problem

$$\begin{cases} \psi_{tt} - t^{-4/3} A(x, D_x) \psi + 2t^{-1} \psi_t = F(\psi), & t > 0, x \in \mathbb{R}^n, \\ \lim_{t \rightarrow 0^+} t \psi(x, t) = \varphi_0(x), & x \in \mathbb{R}^n, \\ \lim_{t \rightarrow 0^+} (t \psi_t(x, t) + \psi(x, t) + 3t^{-1/3} A(x, D_x) \varphi_0(x)) = \varphi_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (4.2)$$

solves also the linear integral equation

$$\psi(x, t) = \psi_0(x, t) + G[F(\psi(\cdot, \tau))](x, t), \quad t > 0. \quad (4.3)$$

We define the solution of (4.2) as a solution of the integral equation (4.3). Let $\alpha_0(n) = (-(n+3) + \sqrt{n^2 + 30n + 81}) / (2(n+3))$ be a positive solution of the equation

$$(n+3)\alpha^2 + (n+3)\alpha - 6 = 0.$$

Theorem 4.1 *Consider the problem (4.2) for $F(\psi) = |\psi|^{1+\alpha}$ or $F(\psi) = |\psi|^\alpha \psi$, and with the elliptic operator $A(x, D_x)$ having the properties (1.4)-(1.6). Assume that $0 < \alpha < \alpha_0(n)$. For every given $\varphi_0(x), \varphi_1(x)$, there exists $T = T(\varphi_0, \varphi_1)$ such that the problem (4.2) has a solution $\psi \in C^2((0, T(\varphi_0, \varphi_1)]; L^q(\mathbb{R}^n))$, where $q = 2 + \alpha$.*

Proof. The following estimate is an analog of [7] (see (3.6),(3.7) and Prop. 3.3) and can be proved by means of Theorem 3.1 [3] and the representation formulas of [21]:

$$\begin{aligned} \|\psi(\cdot, t)\|_{L^q(\mathbb{R}^n)} &\leq C t^{\frac{1}{3}(-1-n(1/p-1/q))} \left(t^{-\frac{2}{3}} \|\varphi_0\|_{L^p(\mathbb{R}^n)} + \|A(x, D_x) \varphi_0\|_{L^p(\mathbb{R}^n)} \right) \\ &\quad + C t^{\frac{1}{3}(-n(1/p-1/q))} \|\varphi_1\|_{L^p(\mathbb{R}^n)} \\ &\quad + t^{\frac{1}{3}(-n(1/p-1/q))} \int_0^t \tau \|\psi|^{1+\alpha}(\cdot, \tau)\|_{L^p(\mathbb{R}^n)} d\tau \quad \text{for all } t \in (0, T]. \end{aligned}$$

In particular, for $q = \alpha + 2$ and $p = (\alpha + 2)/(\alpha + 1)$ we obtain

$$\begin{aligned} t^{1+\frac{n\alpha}{3(\alpha+2)}} \|\psi(\cdot, t)\|_{L^q(\mathbb{R}^n)} &\leq C \left(\|\varphi_0\|_{L^p(\mathbb{R}^n)} + \|A(x, D_x)\varphi_0\|_{L^p(\mathbb{R}^n)} + \|\varphi_1\|_{L^p(\mathbb{R}^n)} \right) \\ &\quad + Ct \int_0^t \tau \|\psi(\cdot, \tau)\|_{L^p(\mathbb{R}^n)}^{1+\alpha} d\tau \\ &\leq C \left(\|\varphi_0\|_{L^p(\mathbb{R}^n)} + \|A(x, D_x)\varphi_0\|_{L^p(\mathbb{R}^n)} + \|\varphi_1\|_{L^p(\mathbb{R}^n)} \right) \\ &\quad + Ct \int_0^t \tau \|\psi(\cdot, \tau)\|_{L^q(\mathbb{R}^n)}^{1+\alpha} d\tau \end{aligned}$$

for all $t \in (0, T]$. Then, it follows

$$\begin{aligned} t^{1+\frac{n\alpha}{3(\alpha+2)}} \|\psi(\cdot, t)\|_{L^q(\mathbb{R}^n)} &\leq C \left(\|\varphi_0\|_{L^p(\mathbb{R}^n)} + \|A(x, D_x)\varphi_0\|_{L^p(\mathbb{R}^n)} + \|\varphi_1\|_{L^p(\mathbb{R}^n)} \right) \\ &\quad + Ct \int_0^t \tau^{-\frac{n\alpha(\alpha+1)}{3(\alpha+2)}-\alpha} \left(\tau^{1+\frac{n\alpha}{3(\alpha+2)}} \|\psi(\cdot, \tau)\|_{L^q(\mathbb{R}^n)} \right)^{1+\alpha} d\tau \end{aligned}$$

for all $t \in (0, T]$. Since $t\psi$ is continuous at $t = 0$ and $\alpha < \alpha(n)$, we obtain

$$\begin{aligned} t^{1+\frac{n\alpha}{3(\alpha+2)}} \|\psi(\cdot, t)\|_{L^q(\mathbb{R}^n)} &\leq C \left(\|\varphi_0\|_{L^p(\mathbb{R}^n)} + \|A(x, D_x)\varphi_0\|_{L^p(\mathbb{R}^n)} + \|\varphi_1\|_{L^p(\mathbb{R}^n)} \right) \\ &\quad + Ct \max_{\tau \in [0, t]} \left(\tau^{1+\frac{n\alpha}{3(\alpha+2)}} \|\psi(\cdot, \tau)\|_{L^q(\mathbb{R}^n)} \right)^{1+\alpha} \int_0^t \tau^{-\frac{n\alpha(\alpha+1)}{3(\alpha+2)}-\alpha} d\tau \end{aligned}$$

for all $t \in (0, T]$. Hence, for $\alpha < \alpha_0(n)$ we have

$$\begin{aligned} t^{1+\frac{n\alpha}{3(\alpha+2)}} \|\psi(\cdot, t)\|_{L^q(\mathbb{R}^n)} &\leq C \left(\|\varphi_0\|_{L^p(\mathbb{R}^n)} + \|A(x, D_x)\varphi_0\|_{L^p(\mathbb{R}^n)} + \|\varphi_1\|_{L^p(\mathbb{R}^n)} \right) \\ &\quad + Ct^{2-\frac{n\alpha(\alpha+1)}{3(\alpha+2)}-\alpha} \max_{\tau \in [0, t]} \left(\tau^{1+\frac{n\alpha}{3(\alpha+2)}} \|\psi(\cdot, \tau)\|_{L^q(\mathbb{R}^n)} \right)^{1+\alpha} d\tau \end{aligned}$$

for all $t \in (0, T]$. If we consider the map S defined as follows

$$S[\psi](x, t) := \psi_0(x, t) + G[|\psi(\cdot, \tau)|^p](x, t), \quad \forall t \in [0, T],$$

then the last estimate implies that S is a contraction for small T . Indeed, for ψ_1 and ψ_2 we obtain

$$\begin{aligned} &\max_{t \in [0, T]} t^{1+\frac{n\alpha}{3(\alpha+2)}} \|\psi_1(\cdot, t) - \psi_2(\cdot, t)\|_{L^q(\mathbb{R}^n)} \\ &\leq c \max_{t \in [0, T]} \left(t^{1+\frac{n\alpha}{3(\alpha+2)}} \|\psi_1(\cdot, t) - \psi_2(\cdot, t)\|_{L^q(\mathbb{R}^n)} \right)^{\alpha+1} T^{2-\frac{n\alpha(\alpha+1)}{3(\alpha+2)}-\alpha}. \end{aligned}$$

The theorem is proved. \square

Thus, for $n = 3$ we have the following range of α of the nonlinear term $\alpha + 1 \in (1, (\sqrt{5} + 1)/2)$.

5 Uniqueness. Finite speed of propagation property

In [7] and [21] the representations for the solutions of the initial value problem for the equations with singular coefficients are given. Because of that particular type of singularity in the coefficients one cannot apply the known uniqueness theorems (see, e.g., [2]). The uniqueness must be established independently of the representation formulas. For the case of $A(x, D_x) = \Delta$ it was done in [8].

In this section we prove the uniqueness of the solution and then the finite speed of propagation property.

Suppose that

$$A(x, D_x) = \sum_{|\alpha| \leq 2} a_\alpha(x) \partial_x^\alpha$$

is negative elliptic operator with smooth coefficients $a_\alpha(x) \in C^\infty(\mathbb{R}^n)$ such that

$$A(x, D_x) = A(\infty, D_x) \quad \text{for all } x \in \mathbb{R}^n \quad |x| \geq R_A,$$

and

$$\sum_{|\alpha|=2} a_\alpha(x) \xi^\alpha > 0 \quad \text{if } \xi \in \mathbb{R}^n, \quad \xi \neq 0, \quad x \in \mathbb{R}^n.$$

Theorem 5.1 *Assume that $A(x, D_x)$ is elliptic negative self-adjoint operator. The solution ψ of the problem*

$$\begin{cases} \psi_{tt} - t^{-4/3} A(x, D_x) \psi + 2t^{-1} \psi_t = f, & t > 0, \quad x \in \mathbb{R}^n, \\ \lim_{t \rightarrow 0^+} t \psi(x, t) = \varphi_0(x), & x \in \mathbb{R}^n, \\ \lim_{t \rightarrow 0^+} \left(t \psi_t(x, t) + \psi(x, t) + 3t^{-1/3} A(x, D_x) \varphi_0(x) \right) = \varphi_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (5.1)$$

is unique in $C^2((0, T]; \mathcal{D}'(\mathbb{R}^n))$.

Proof. It suffices to prove the uniqueness in the problem

$$\begin{cases} u_{tt} - t^{-4/3} A(x, D_x) u = 0, & \text{in } t > 0, \quad x \in \mathbb{R}^n, \\ u(x, 0) = 0, \quad u_t(x, 0) = 0, & \text{in } \mathbb{R}^n, \end{cases}$$

where $u = t\psi \in C^1([0, T]; \mathcal{D}'(\mathbb{R}^n)) \cap C^2((0, T]; \mathcal{D}'(\mathbb{R}^n))$. We choose an arbitrary $T > 0$ and for the function $\varphi \in C_0^\infty(\mathbb{R}^n)$ consider the Cauchy problem

$$\begin{cases} v_{tt} - t^{-4/3} A(x, D_x) v = 0, & \text{in } t \in (0, T], \quad x \in \mathbb{R}^n, \\ v(x, T) = 0, \quad v_t(x, T) = \varphi(x), & \text{in } \mathbb{R}^n. \end{cases} \quad (5.2)$$

Since the operator $\mathcal{S} = \partial_t^2 - t^{-4/3} A(x, D_x)$ is strictly hyperbolic for $t > 0$, there is a unique solution $v \in C^\infty((0, T] \times \mathbb{R}^n)$. This solution obeys finite speed of propagation, consequently there is a ball $B \subseteq \mathbb{R}^n$ of the finite radius R , such that $\text{supp } v \subseteq [0, T] \times B$.

Then we define operator $\sqrt{-A(x, D_x)}$ (see, e.g., [16, Ch.XII]), which is a pseudodifferential operator. The solution $v(x, t)$ of the problem (5.2) can be written in terms of the Fourier integral operators as follows

$$\begin{aligned} v(x, t) &= \frac{i}{18} (\sqrt{-A(x, D_x)})^{-3} \\ &\times \left\{ \left(i\phi(t) \sqrt{-A(x, D_x)} - 1 \right) \left(i\phi(T) \sqrt{-A(x, D_x)} + 1 \right) e^{-i(\phi(T) - \phi(t)) \sqrt{-A(x, D_x)}} \right. \\ &\left. - \left(i\phi(t) \sqrt{-A(x, D_x)} + 1 \right) \left(i\phi(T) \sqrt{-A(x, D_x)} - 1 \right) e^{i(\phi(T) - \phi(t)) \sqrt{-A(x, D_x)}} \right\} \varphi(x) \end{aligned}$$

as well as

$$\begin{aligned}
& v(x, t) \\
&= \frac{1}{9}(\sqrt{-A(x, D_x)})^{-3} \left\{ \left(\phi(t)\phi(T)\sqrt{-A(x, D_x)} - 1 \right) \sin \left(\sqrt{-A(x, D_x)} (\phi(T) - \phi(t)) \right) \right. \\
&\quad \left. - \sqrt{-A(x, D_x)} (\phi(t) - \phi(T)) \cos \left(\sqrt{-A(x, D_x)} (\phi(T) - \phi(t)) \right) \right\} \varphi(x).
\end{aligned}$$

Thus, the solution is given by the Fourier integral operators of order -1 . In particular, for the derivative we obtain

$$\begin{aligned}
v_t(x, t) &= \frac{1}{2\phi(t)}(\sqrt{-A(x, D_x)})^{-1} \left\{ e^{-i(\phi(t)-\phi(T))\sqrt{-A(x, D_x)}} \left(\sqrt{-A(x, D_x)}\phi(T) + i \right) \right. \\
&\quad \left. + e^{i(\phi(t)-\phi(T))\sqrt{-A(x, D_x)}} \left(\sqrt{-A(x, D_x)}\phi(T) - i \right) \right\} \varphi(x) \\
&= \frac{1}{\phi(t)} \left\{ \cos \left((\phi(t) - \phi(T)) \sqrt{-A(x, D_x)} \right) \phi(T) \right. \\
&\quad \left. + (\sqrt{-A(x, D_x)})^{-1} \sin \left((\phi(t) - \phi(T)) \sqrt{-A(x, D_x)} \right) \right\} \varphi(x).
\end{aligned}$$

One can easily check the following limits

$$\begin{aligned}
\lim_{t \rightarrow 0^+} v(x, t) &= \frac{1}{18}(\sqrt{-A(x, D_x)})^{-3} \\
&\times \left\{ e^{-i\sqrt{-A(x, D_x)}\phi(T)} \sqrt{-A(x, D_x)}\phi(T) - i e^{-i\sqrt{-A(x, D_x)}\phi(T)} \right. \\
&\quad \left. + e^{i\sqrt{-A(x, D_x)}\phi(T)} \sqrt{-A(x, D_x)}\phi(T) + i e^{i\sqrt{-A(x, D_x)}\phi(T)} \right\} \varphi(x) \\
&= -\frac{1}{9}(A(x, D_x))^{-1} \left\{ \cos \left(\phi(T)\sqrt{-A(x, D_x)} \right) \phi(T) \right. \\
&\quad \left. - (\sqrt{-A(x, D_x)})^{-1} \sin \left(\phi(T)\sqrt{-A(x, D_x)} \right) \right\} \varphi(x)
\end{aligned}$$

and

$$\begin{aligned}
& \lim_{t \rightarrow 0^+} \phi(t)v_t(x, t) \\
&= \left\{ \phi(T) \cos \left(\phi(T)\sqrt{-A(x, D_x)} \right) + (\sqrt{-A(x, D_x)})^{-1} \sin \left(\phi(T)\sqrt{-A(x, D_x)} \right) \right\} \varphi(x).
\end{aligned}$$

In particular, it follows

$$v, t^{1/3}v_t \in C([0, T]; C^\infty(K)),$$

where $K \subseteq \mathbb{R}^n$ is a compact. We denote $\langle u, \varphi \rangle$ the pairing of the distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ and a test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Consider the functions $\langle u(\cdot, t), v(\cdot, t) \rangle$, $\langle u_t(\cdot, t), v_t(\cdot, t) \rangle$, and

$\langle u(\cdot, t), A(\cdot, D_x)v(\cdot, t) \rangle$. We can assume that $\text{supp } u \subseteq [0, T] \times B_u$, where B_u is a compact and it contains B . Then we can estimate these functions as follows

$$|\langle u(\cdot, t), v(\cdot, t) \rangle| + |\langle u(\cdot, t), A(\cdot, D_x)v(\cdot, t) \rangle| \leq ct \quad \text{for all } t \in [0, T].$$

Hence,

$$\int_0^T t^{-4/3} |\langle u(\cdot, t), A(\cdot, D_x)v(\cdot, t) \rangle| dt < \infty \quad \text{and} \quad \int_0^T |\langle u_{tt}(\cdot, t), v(\cdot, t) \rangle| dt < \infty,$$

as well as

$$\int_0^T |\langle u_t(\cdot, t), v_t(\cdot, t) \rangle| dt < \infty.$$

Hence, taking into account that u solves equation without source term, we obtain

$$\int_0^T \langle u_{tt}(\cdot, t), v(\cdot, t) \rangle dt - \int_0^T t^{-4/3} \langle u(\cdot, t), A(\cdot, D_x)v(\cdot, t) \rangle dt = 0.$$

Applying the integration by parts, taking into account that v solves equation without source term, we derive

$$\langle u(\cdot, T), \varphi(\cdot) \rangle = 0$$

for arbitrary $\varphi \in C_0^\infty(\mathbb{R}^n)$, which completes the proof of the theorem. \square

Theorem 5.1 allows us to prove the finite speed of propagation property in the Cauchy problem.

Theorem 5.2 *The solution $\psi \in C^2((0, T]; \mathcal{D}'(\mathbb{R}^n))$ of the problem (5.1) obeys finite speed of propagation, that is, for every given $T > 0$ and the open ball $B_R(x_0) = \{x \in \mathbb{R}^n; |x - x_0| < R\}$, if*

$$\varphi_0 = \varphi_1 = 0 \quad \text{on } B_{R+3T^{1/3}s_A}(x_0) \quad \text{and} \quad f = 0 \quad \text{on } \bigcup_{t \in [0, T]} B_{R+3(T^{1/3}-t^{1/3})s_A}(x_0),$$

then

$$\psi(T) = 0 \quad \text{on } B_R(x_0) = \{x \in \mathbb{R}^n; |x - x_0| < R\}.$$

Here

$$s_A = \max_{x \in \mathbb{R}^n, \xi \in \mathbb{R}^n, |\xi|=1} \sum_{|\alpha|=2} a_\alpha(x) \xi^\alpha.$$

Proof. It suffices to use the finite speed of propagation in the problem for the auxiliary function v in the proof of the previous theorem. \square

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