k-Hyponormality of powers of multivariable weighted shifts

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k-HYPONORMALITY OF POWERS

OF MULTIVARIABLE WEIGHTED SHIFTS

A Thesis

by

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k-HYPONORMALITY OF POWERS

OF MULTIVARIABLE WEIGHTED SHIFTS

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ABSTRACT

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In mathematics, weighted shifts, also known as shift operators or translation operators, are examples of linear operators. Single and multivariable weighted shifts have played an important role in the study of (joint) k-hyponormality to LPCS. They have also played a significant role in the study of cyclicity and reflexivity, in the study of C^*-algebras generated by multiplication operators on Bergman spaces, as fertile ground to test new hypotheses, and as canonical models for theories of dilation and positivity. We will first consider the hyponormality of powers of commuting multivariable weighted shifts. Specifically, if we let $W_{(\alpha,\beta)} := (T_1, T_2)$ denote a commuting 2-variable weighted shift, we will prove that: (i) $W_{(\alpha,\beta)}$ is hyponormal but $W_{(\alpha,\beta)}^{(2,1)} \equiv (T_1^2, T_2)$ is not hyponormal; and (ii) $W_{(\alpha,\beta)}$ is not hyponormal but $W_{(\alpha,\beta)}^{(2,1)}$ is hyponormal In this work, we have expanded the results just mentioned above using the Smul'jan's test for the positivity of 2×2 block operators and the new direct sum decomposition for powers of 2-variable weighted shifts. As a consequence, we show that the 2-hyponormality of 2-variable weighted shifts are not invariant under powers.

DEDICATION

The completion of my graduate studies would not have been possible without the support and love of my father, Adolfo R. Martinez, my mother, Irma Martinez, and my sister, Mariela Martinez. Their unconditional support has kept me motivated, inspired, and relentless in completing my graduate work. Thank you for your love and patience. All the sacrifices you have made for me will never be forgotten. I am blessed to have you in my life.

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CHAPTER I

INTRODUCTION

In mathematics, weighted shifts, also known as shift operators or translation operators, are examples of linear operators. The Lifting Problem for Commuting Subnormals (LPCS) asks for necessary and sufficient conditions for a pair of subnormal operators on Hilbert space to admit commuting normal extensions. Single and multivariable weighted shifts have played an important role in the study of (joint) k-hyponormality to LPCS (cf. [7], [8], [9], [11], [16], [17], [25], [32], [33], [34]). They have also played a significant role in the study of cyclicity and reflexivity, in the study of C^*-algebras generated by multiplication operators on Bergman spaces, as fertile ground to test new hypotheses, and as canonical models for theories of dilation and positivity (cf. [4], [23], [25], [28]). For a general operator T on a Hilbert space, it is well known that the hyponormality of T does not imply the hyponormality of T^2. In the multivariable case, we can consider these analogous results. In [8, Theorem 2.7], the authors first considered this problem for the case of 1-hyponormality. Specifically, they showed that there exists a commuting hyponormal 2-variable weighted shift $W_{(\alpha,\beta)}$ such that: (i) $W_{(\alpha,\beta)}$ is hyponormal but $W_{(\alpha,\beta)}^{(2,1)} \equiv (T_1^2, T_2)$ is not hyponormal; and (ii) $W_{(\alpha,\beta)}$ is not hyponormal but $W_{(\alpha,\beta)}^{(2,1)}$ is hyponormal. Thus it is natural to consider

Problem 1 *For some $h \geq 2$, $\ell \geq 1$ does exist $W_{(\alpha,\beta)} \in \mathfrak{H}_0$ for which the hyponormality of $W_{(\alpha,\beta)}^{(h,\ell)} := (T_1^h, T_2^\ell)$, but $W_{(\alpha,\beta)}^{(h+1,\ell)} := (T_1^{h+1}, T_2^\ell) \notin \mathfrak{H}_1$, and the hyponormality of $W_{(\alpha,\beta)}^{(h+1,\ell)}$, but*

$W^{(h,\ell)}_{(\alpha,\beta)} \notin \mathfrak{H}_1$?

Since k-hyponormality lies between hyponormality and subnormality, we can also consider

Problem 2 *Given $k \geq 1$, assume that for all $h \geq 2$ and $\ell \geq 1$, the k-hyponormality of $W^{(h,\ell)}_{(\alpha,\beta)}$. Does it follow that the k-hyponormality of $W_{(\alpha,\beta)}$?*

In this work we consider these problems just given above and give answers for them using the Smul'jan's test for the positivity of 2×2 block operators and the new direct sum decomposition for powers of 2-variable weighted shifts. We use $\mathfrak{H}_0$ to denote the set of commuting pairs of subnormal operators on a Hilbert space $\mathcal{H}$. We denote the class of subnormal pairs by $\mathfrak{H}_\infty$ and, for any integer $k \geq 1$, the class of k-hyponormal pairs in $\mathfrak{H}_0$ by $\mathfrak{H}_k$. Clearly, $\mathfrak{H}_\infty \subseteq \cdots \subseteq \mathfrak{H}_k \subseteq \cdots \subseteq \mathfrak{H}_2 \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_0$. The main results in [16] and [7] show that these inclusions are all proper.

CHAPTER II

PRELIMINARY AND NOTATIONS

In order to begin we must give some background to understand what are weighted shifts, multivariable weighted shifts, hyponormal, $k-$hyponormal and subnormal operators. The foundation of this study begins with defining Hilbert spaces and operators.

Hilbert Space and Operators

Let $\mathcal{H}$ be a complex *Hilbert space* with *inner product* $\langle \cdot, \cdot \rangle$. We say that $\{e_n\}$ is an *orthonormal basis* for $\mathcal{H}$ if (i) $\langle e_n, e_m \rangle = \delta_{nm}$, n,$m \in \mathbb{Z}_+$; and (ii) every vector $x \in \mathcal{H}$ can be written as $x = \sum_{n=0}^{\infty} \langle x, e_n \rangle e_n$, or alternatively, if $x = 0$ whenever $\langle x, e_n \rangle = 0$ (all n). The canonical example of Hilbert space is $\ell^2(\mathbb{Z}_+)$ which consists, by definition, of all infinite sequences $(\alpha_0, \alpha_1, \alpha_2, \cdots)$ of complex numbers such that $\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty$. The vector operations are coordinate-wise and the inner product is defined by

$$\langle (\alpha_0, \alpha_1, \alpha_2, \cdots), (\beta_0, \beta_1, \beta_2, \cdots) \rangle := \sum_{n=0}^{\infty} \alpha_n \beta_n^*.$$

where * denotes complex conjugation. The orthonormal basis of $\ell^2(\mathbb{Z}_+)$ consists of $\{e_n\}_{n=0}^{\infty}$, where $e_n := (0, \cdots, 0, \underset{n}{1}, 0, \cdots)$. If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, a linear transformation (we call it an *operator*) T from $\mathcal{H}$ into $\mathcal{K}$ is *bounded* if there exists a positive number m such that $\|Tf\| \leq m\,\|f\|$ for all f in $\mathcal{H}$; the norm of T, in symbols $\|T\|$, is the infimum of all such values of m. T is called *contraction* if $\|T\| \leq 1$, and T is *bounded below* if there exist a positive number m such that $\|Tf\| \geq m\,\|f\|$ for all f in $\mathcal{H}$. We denote by $N(T)$ and $R(T)$ the null space and range of T, respectively. It is trivial to verify that if T is bounded below, then T is indeed one-to-one. Given a bounded linear operator $T : \mathcal{H} \to \mathcal{K}$, the inner product $\langle Tf, g\rangle$ makes sense whenever f is in $\mathcal{H}$ and g in $\mathcal{K}$; the inner product is formed in $\mathcal{K}$. For fixed g the inner product defines a bounded linear function of f, and, consequently, by the Riesz Representation Theorem, it is identically equal to $\langle f, \widetilde{g}\rangle$ for some $\widetilde{g}$ in $\mathcal{H}$. The mapping from g to $\widetilde{g}$ is the *adjoint* of T; it is bounded linear operator T^* from $\mathcal{K}$ to $\mathcal{H}$, with $\|T^*\| = \|T\|$. By definition

$$\langle Tf, g\rangle = \langle f, T^*g\rangle$$

whenever $f \in \mathcal{H}$ and $g \in \mathcal{K}$. For further reference, we list here some notation used throughout:

$\mathbb{Z}$	integers
$\mathbb{Z}_+$	nonnegative integers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers

Weighted Shifts

A *weighted shift* is the product of a *shift* (one-sided or two-sided) and a compatible *diagonal operator*. More explicitly, suppose that $\{e_n\}$ is an orthonormal basis $(n = 0, 1, 2, \cdots,$

or else $n = 0, \pm 1, \pm 2, \cdots$), and suppose that $(\alpha_n)_{n=0}^{\infty}$ is a bounded sequence of complex numbers (called *weights*). The most important example of a weighted shift is the *unilateral shift*. It is the operator U_+ on $\ell^2(\mathbb{Z}_+)$ defined by

$$U_+((\alpha_0, \alpha_1, \alpha_2, \cdots)) = (0, \alpha_0, \alpha_1, \alpha_2, \cdots).$$

A close relative of the unilateral shift is the *bilateral shift*. To define it, let $\ell^2(\mathbb{Z})$ be the Hilbert space of all two-way (bilateral) square-summable sequences. The elements of $\ell^2(\mathbb{Z})$ are most conveniently written in the form

$$(\cdots, \alpha_{-2}, \alpha_{-1}, (\alpha_0), \alpha_1, \alpha_2, \cdots);$$

the term in parentheses indicates the one corresponding to the index 0. The bilateral shift is the operator U on $\ell^2(\mathbb{Z})$ defined by

$$U((\cdots, \alpha_{-2}, \alpha_{-1}, (\alpha_0), \alpha_1, \alpha_2, \cdots)) := (\cdots, \alpha_{-3}, \alpha_{-2}, (\alpha_{-1}), \alpha_0, \alpha_1, \cdots).$$

Linearity is obvious, and $\|U\| = \|U_+\| = 1$; the bilateral shift, like the unilateral one, is an *isometry,* i.e., $\|U(\alpha)\| = \|\alpha\|$ (all $\alpha \in \ell^2(\mathbb{Z})$). Since the range of the bilateral shift is the entire space $\ell^2(\mathbb{Z})$, it is even unitary. If e_n is the vector $(\cdots, \xi_{-1}, (\xi_0), \xi_1, \cdots)$ for which $\xi_n = 1$ and $\xi_i = 0$ whenever $i \neq n$ ($n = 0, \pm 1, \pm 2, \cdots$), then the e_n's form an orthonormal basis for $\ell^2(\mathbb{Z})$. The effect of U on this basis is described by

$$Ue_n = e_{n+1} \ (n = 0, \pm 1, \pm 2, \cdots).$$

A weighted shift W_α, $\alpha \equiv (\alpha_0, \alpha_1, \alpha_2, \cdots)$, is an operator of the form U_+P where U_+ is the (unweighted) unilateral shift and P is a diagonal operator with diagonal $\{\alpha_n\}$ ($Pe_n = \alpha_n e_n$). The *moments* of W_α are usually defined by $\gamma_0 := 1$, $\gamma_{n+1} := \alpha_n^2 \gamma_n$ $(n \geq 0)$ ([28]), but we reserve this term for the sequence $\gamma_n := (\alpha_0 \cdots \alpha_{n-1})^2$ $(n \geq 0)$. W_α is called *flat* if $\alpha_0 \leq \alpha_1 = \alpha_2 = \cdots$. We shall often attempt to identify a weighted shift with the defining sequence. To avoid confusion, we shall use the notations $shift(\alpha_0, \alpha_1, \alpha_2, \cdots)$ and $diag(\alpha_0, \alpha_1, \alpha_2, \cdots)$ to denote the weighted shift W_α and the diagonal operator associated with the sequence $\alpha_0, \alpha_1, \alpha_2, \cdots$, respectively.

Hyponormality and Subnormality

An operator T is *positive* (in symbols, $T \geq 0$) if $\langle Tf, f\rangle \geq 0$ for all $f \in \mathcal{H}$. T is *normal* if $TT^* = T^*T$, and *subnormal* if it has a normal extension. More precisely, an operator S on Hilbert space $\mathcal{H}$ is subnormal if there exists a normal operator T on Hilbert space $\mathcal{K}$ such that $\mathcal{H}$ is a subspace of $\mathcal{K}$, $\mathcal{H}$ is invariant under T (i.e., $T(\mathcal{H}) \subseteq \mathcal{H}$), and the restriction of T to $\mathcal{H}$ coincides with S. Every normal operator is trivially subnormal. On finite-dimensional spaces, every subnormal operator is normal. A typical example of a subnormal operator is the unilateral shift; the bilateral shift is a normal extension. An operator T such that $T^*T \geq TT^*$ is said to be *hyponormal*.

Proposition 3 *If W_α is hyponormal, then $\alpha_n \leq \alpha_{n+1}$, for all $n \geq 0$.*

On a finite-dimensional space, every hyponormal operator is normal, so hyponormal, subnormal and normal operators are identical notions on finite-dimensional spaces. In general, it is well known that normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal. The converse implications are false, and recently the classes of k-hyponormal and weakly k-hyponormal operators have been studied in attempt to " bridge the gap" between subnormality and hyponormality. For $S, T \in \mathcal{B}(\mathcal{H})$, let $[S,T] := ST - TS$, we say that an n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ of operators on $\mathcal{H}$ is hyponormal if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of n copies of $\mathcal{H}$. More precisely, if $\mathbf{T} \equiv (T_1, \cdots, T_n)$, we let

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \cdots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix} \geq 0$$

For $\mathbf{T} \equiv (T_1, \cdots, T_n)$, $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal. $\mathbf{T}$ is said to be *subnormal* if $\mathbf{T}$ is the restriction of a normal $n-$tuple to a common invariant subspace.

k-hyponormality

For $k \geq 1$ and $T \in \mathcal{B}(\mathcal{H})$, T is *k-hyponormal* if $(I, T, \ldots, T^k)$ is hyponormal. Additionally, T is *weakly k-hyponormal* if $p(T)$ is hyponormal for every polynomial p of degree k or less. Thus k-hyponormal $\Rightarrow$ weakly k-hyponormal, and "hyponormal", "1-hyponormal" and "weakly 1-hyponormal" are identical ([2]). On the other hand, results in ([12]), ([6]) and ([26]) show that weakly 2-hyponormal $\not\Rightarrow$ 2-hyponormal. The Bram-Halmos characterization of subnormality ([4, III.1.9]) can be paraphrased as follow: $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([12, Proposition 1.9]). In particular, each subnormal operator is *polynomially hyponormal* (i.e., weakly k-hyponormal for every $k \geq 1$). The converse implication, whether T polynomially hyponormal $\Rightarrow T$ subnormal, was a long-standing problem in operator theory which remained unsolved until recently. In ([26]), S. McCullough and V. Paulsen showed that to answer this question it suffices to resolve it for the class of unilateral weighted shifts. The following diagram show us an easy representation of the above relation:

$$
\begin{array}{ccccccccccc}
\text{subnormal} & \Leftrightarrow & \infty\text{-hypon.} & \Rightarrow \cdots & \Rightarrow & \text{3-hypon.} & \Rightarrow & \text{2-hypon.} & \searrow & \\
\uparrow\not\downarrow & & & & & & & & & \text{hypon.} \\
\begin{array}{c}\text{polyn.}\\ \text{hypon.}\end{array} & \Leftrightarrow & \begin{array}{c}\text{weekly}\\ \infty\text{-hypo.}\end{array} & \Rightarrow \cdots & \Rightarrow & \begin{array}{c}\text{weekly}\\ \text{3-hypo.}\end{array} & \Rightarrow & \begin{array}{c}\text{weekly}\\ \text{2-hypo.}\end{array} & \nearrow &
\end{array}
$$

We call weakly 2-hyponormal operators *quadratically hyponormal.*

Theorem 4 *([6]) Let $W_\alpha e_i = \alpha_i e_{i+1}$ $(i \geq 0)$ be a hyponormal weighted shift, and let $k \geq 1$. The following statements are equivalent:*

(i) W_α is k-hyponormal.

(ii) The matrix

$$(([W_\alpha^{*j}, W_\alpha^i]e_{n+j}, e_{n+i}))_{i,j=1}^k$$

is positive for all $n \geq -1$.

(iii) The matrix

$$(\gamma_n \gamma_{n+i+j} - \gamma_{n+i}\gamma_{n+j})_{i,j=1}^k$$

is positive for all $n \geq 0$, where as usual $\gamma_0 = 1$, $\gamma_k = \alpha_0^2 \cdot \cdots \cdot \alpha_{k-1}^2$ $(k \geq 1)$.

(iv) *The Hankel matrix*

$$(\gamma_{n+i+j-2})_{i,j=1}^{k+1}$$

is positive for all $n \geq 0$.

Proposition 5 *If* $\mathbf{T} \equiv (T_1, \cdots, T_n)$, *then normal* $\Rightarrow$ *subnormal* $\Rightarrow$ *(jointly) hyponormal* $\Rightarrow$ *weakly hyponormal.*

Proposition 6 *([12]) Let* $\mathbf{T} \equiv (T_1, T_2)$ *be a pair of operators on* $\mathcal{H}$*.*

Then $\mathbf{T}$ *is weakly hyponormal if and only if*

(i) T_1 *is hyponormal*

(ii) T_2 *is hyponormal*

(iii) $|\langle [T_2^*, T_1]x, x\rangle|^2 \leq \langle [T_1^*, T_1]x, x\rangle \langle [T_2^*, T_2]x, x\rangle$, *for all* $x \in \mathbb{C}$.

Proposition 7 *([12]) Let* $\mathbf{T} \equiv (T_1, T_2)$ *be a pair of operators on* $\mathcal{H}$*.*

Then $\mathbf{T}$ *is hyponormal if and only if*

(i) T_1 *is hyponormal*

(ii) T_2 *is hyponormal*

(iii) $|\langle [T_2^*, T_1]y, x\rangle|^2 \leq \langle [T_1^*, T_1]x, x\rangle \langle [T_2^*, T_2]y, y\rangle$, *for all* $x, y \in \mathbb{C}$.

Corollary 8 *([12]) Let* $\mathbf{T} \equiv (T_1, T_2)$ *be a pair of operators on* $\mathcal{H}$*.*

Then $\mathbf{T}$ *is hyponormal if and only if*

(i) T_1 *is hyponormal*

(ii) T_2 *is hyponormal*

(iii) there exists a contraction $D \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, *such that*

$$[T_2^*, T_1]^{\frac{1}{2}} = [T_1^*, T_1]^{\frac{1}{2}} D [T_2^*, T_2]^{\frac{1}{2}}.$$

Corollary 9 *([12]) Let* $\mathbf{T} \equiv (T_1, T_2)$ *be a pair of operators on* $\mathcal{H}$*.*

Then $\mathbf{T}$ *is hyponormal if and only if*

(i) T_1 *is hyponormal*

(ii) T_2 *is hyponormal*

(iii) there exists a $V : \mathcal{H}_2 \to \mathcal{H}_1 \ominus N([T_1^*, T_1]^{\frac{1}{2}})$*, such that*

$$[T_2^*, T_1] = [T_1^*, T_1]^{\frac{1}{2}} V \text{ and } [T_2^*, T_2] \geq V^* V.$$

Subnormal Weighted Shifts

Subnormal weighted shifts were characterized by J. Stampfli ([29]), who gave an explicit geometric construction of the minimal normal extension in terms of the weight sequence. An analytic description of subnormality for weighted shifts, which we now recall, was obtained by C. Berger (cf. [23] and [4]).

Theorem 10 *(Berger's Theorem)([4]) Let* $\{e_n : n \geq 0\}$ *be orthonormal basis for* $\mathcal{H}$ *and let* T *be a weighted shift relative to this basis with weight sequence* $\{\alpha_n : n \geq 0\}$*,where* $\sup_n \alpha_n = 1$*. The following statements are equivalent.*

(i) T *is subnormal;*

(ii) there is a Borel probability measure μ *on* $\left[0, \|T\|^2\right]$ *with* $\|T\|^2$ *in its support such that*

$$\gamma_n := (\alpha_0 \cdots \alpha_{n-1})^2 = \int_0^{\|T\|^2} t^n d\mu(t) \text{ (all } n \geq 0).$$

In this case, μ is the unique compactly supported solution to the Stieltjes moment problem with data $\{\gamma_n\}_{n=0}^{\infty}$, and it thus follows from the theory of moments that for every $k \geq 1$ and $n \geq 1$, the Hankel matrices

$$\{\gamma_{n+p+q}\}_{p,q=0}^{\infty}$$

are all positive. In the sequel, we refer to μ as the Berger measure of T (W_α).

Theorem 11 *([6]) Let T be a weighted shift relative to the orthonormal basis $\{e_n : n \geq 0\}$ for $\mathcal{H}$ with weight sequence $\{\alpha_n : n \geq 0\}$.*

The following statements are equivalent.

(i) T is subnormal;

(ii) $\{\gamma_{i+j}(T)\}_{i,j\geq 0}^{\infty} \geq 0$ and $\{\gamma_{i+j+1}(T)\}_{i,j\geq 0}^{\infty} \geq 0$.

Theorem 12 *([1]) A contraction T is subnormal if and only if*

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k} (T^*)^k T^k \geq 0 \ (\text{all } m \geq 1).$$

Definition 13 *Let $\mathbb{Z}_+^n := \underbrace{\mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+}_{n\text{-times}}$ and let $\ell^2(\mathbb{Z}_+^n)$ be the Hilbert space of square summable complex sequence indexed by $\mathbb{Z}_+^n$. An n-tuple $\mathbf{T} \equiv (T_1, \cdots, T_n)$ is called multivariable weighted shift if $\mathbf{T}$ is an n-tuple of weighted shifts, i.e., $T_i e_{\mathbf{k}} = \alpha_{\mathbf{k}}^{(i)} e_{\mathbf{k}+\varepsilon_i}$, where $\mathbf{k} \in \mathbb{Z}_+^n, \alpha_{\mathbf{k}}^{(i)} > 0$ (all i and $\mathbf{k}$), $\varepsilon_i = (0, \ldots, \underset{i\text{-th}}{1}, \ldots, 0)$ and $\{e_{\mathbf{k}} : \mathbf{k} \in \mathbb{Z}_+^n\}$ is orthonormal basis for $\ell^2(\mathbb{Z}_+^n)$.*

Special case($n = 2$).

Let $\alpha_{\mathbf{k}}^{(1)} := \alpha_{\mathbf{k}}$, $\alpha_{\mathbf{k}}^{(2)} := \beta_{\mathbf{k}}$, $\mathbb{Z}_+^2 = \mathbb{Z}_+ \times \mathbb{Z}_+$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences double-indexed by $\mathbb{Z}_+^2$, where $\mathbf{k} := (k_1, k_2) \in \mathbb{Z}_+^2$, $\alpha_{\mathbf{k}} > 0$,

$\beta_{\mathbf{k}} > 0$, *and* $\varepsilon_1 := (1,0)$, $\varepsilon_2 := (0,1)$. *Then* $T := (T_1, T_2)$ *is called 2-variable weighted shift (see Figure given below) if*

$$T_1 e_{\mathbf{k}} = \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}$$

$$T_2 e_{\mathbf{k}} = \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2}$$

$$T_1 T_2 = T_2 T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \ (all\ \mathbf{k}).$$

1

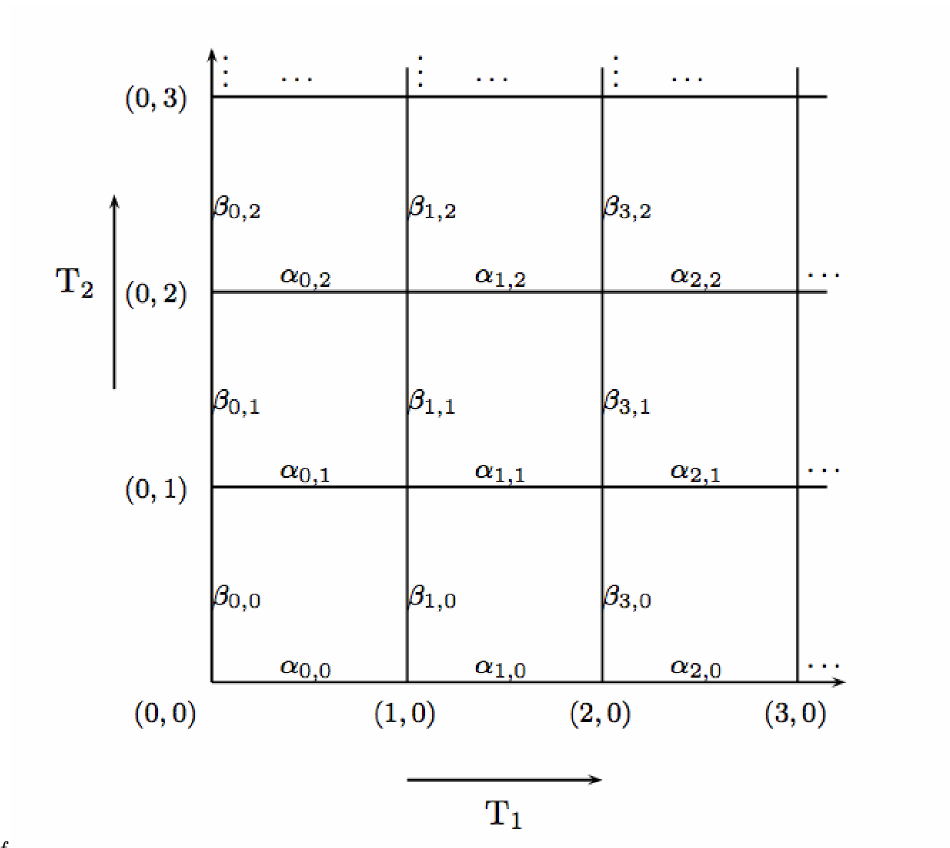

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Figure 1: Weight diagram of the general 2-variable weighted shift

Theorem 14 *(Berger's Theorem)([24])* $\mathbf{T} \equiv (T_1, \cdots, T_n)$ *has commuting normal extension if and only if there is a probability measure* μ *defined on the n-dimensional rectangle* $R = [0, a_1] \times \cdots \times [0, a_n]$ *where* $a_i = \|T_i\|^2$ *such that* $\gamma_{\mathbf{k}} = \int_R t^{\mathbf{k}} d\mu(t) := \int_R t_1^{k_1} \cdots t_n^{k_n} d\mu(t)$, *for all* $\mathbf{k} \geq 0$*. (Here* $\mathbf{k} \geq 0$ *means* $k_i \geq 0$ *(all* $i = 1, \ldots, n$*).)*

Definition 15 *([1]) A commuting pair of contractions* $\mathbf{T} \equiv (T_1, T_2)$ *is subnormal if and only if*

$$\sum_{0 \leq p_1 \leq k_1, 0 \leq p_2 \leq k_2}^{\mathbf{k}} (-1)^{p_1+p_2} \binom{k_1}{p_1} \binom{k_2}{p_2} (T_1^*)^{p_1} (T_2^*)^{p_2} T_1^{p_1} T_2^{p_2} \geq 0 \text{ (all } \mathbf{k} \geq 0).$$

Theorem 16 *([5]) For* $\mathbf{T} := (T_1, \cdots, T_n)$ *a multivariable weighted shift as above,*

$$T_i e_{\mathbf{k}} = \alpha_{\mathbf{k}}^{(i)} e_{\mathbf{k}+\varepsilon_i},$$

$$[\mathbf{T}^*, \mathbf{T}] \geq 0 \Leftrightarrow (([T_j^*, T_i] e_{\mathbf{k}+\varepsilon_j}, e_{\mathbf{k}+\varepsilon_i}))_{i,j=1}^n \geq 0 \text{ (all } \mathbf{k} \in \mathbf{Z}_+^n).$$

$$\Leftrightarrow (\alpha_{\mathbf{k}+\varepsilon_j}^{(i)}, \alpha_{\mathbf{k}+\varepsilon_i}^{(j)} - \alpha_{\mathbf{k}}^{(i)}, \alpha_{\mathbf{k}}^{(j)})_{i,j=1}^n \geq 0 \text{ (all } \mathbf{k} \in \mathbf{Z}_+^n).$$

CHAPTER III

MAIN RESULTS

In our research we were able to give a positive answer to Problem 1 and a negative answer to Problem 2 when $k = 3$. A 2-variable weighted shift $W_{(\alpha,\beta)}$ was constructed such that the 3-hyponormality of $W_{(\alpha,\beta)}$ but $W^{(2,1)}_{(\alpha,\beta)} \notin \mathfrak{H}_3$, and the 3-hyponormality of $W^{(2,1)}_{(\alpha,\beta)}$ but $W_{(\alpha,\beta)} \notin \mathfrak{H}_3$.

We start with a subnormal weighted shift $W_x \equiv shift(x_0, x_1, x_2 \cdots)$ given in [7]. For $0 < y < 1$, let $x \equiv \{x_n\}_{n=0}^{\infty}$ be defined by

$$x_n := \begin{cases} y\sqrt{\frac{3}{4}} & \text{if } n = 0 \\ \frac{\sqrt{(n+1)(n+3)}}{(n+2)} & \text{if } n \geq 1 \end{cases} \tag{1}$$

with Berger measure

$$d\xi_x(s) := (1-y^2)d\delta_0(s) + \frac{y^2}{2}ds + \frac{y^2}{2}d\delta_1(s) \;\; \text{([7, Proposition 4.2]).}$$

We now list several Lemmas below which are needed in the proof of our main results. To detect hyponormality for 2-variable weighted shifts we use the simple criterion involving a base point $\mathbf{k}$ in $\mathbb{Z}^2_+$ and its five neighboring points in $\mathbf{k} + \mathbb{Z}^2_+$ at path distance at most 2.

Lemma 17 *([5]) (Six-point Test) Let* $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ *be a 2-variable weighted shift, with* *weight sequences α and β. Then*

$$[\mathbf{T}^*, \mathbf{T}] \geq 0 \Longleftrightarrow (([T_j^*, T_i]e_{\mathbf{k}+\varepsilon_j}, e_{\mathbf{k}+\varepsilon_i}))_{i,j=1}^2 \geq 0 \ (\textit{for all } \mathbf{k} \in \mathbb{Z}_+^2)$$

$$\Longleftrightarrow H(k_1, k_2) := \begin{pmatrix} \alpha_{\mathbf{k}+\boldsymbol{\varepsilon}_1}^2 - \alpha_{\mathbf{k}}^2 & \alpha_{\mathbf{k}+\boldsymbol{\varepsilon}_2}\beta_{\mathbf{k}+\boldsymbol{\varepsilon}_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\boldsymbol{\varepsilon}_2}\beta_{\mathbf{k}+\boldsymbol{\varepsilon}_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta_{\mathbf{k}+\boldsymbol{\varepsilon}_2}^2 - \beta_{\mathbf{k}}^2 \end{pmatrix} \geq 0.$$

The following are criterions on k-hyponormality for 2-variable weighted shifts.

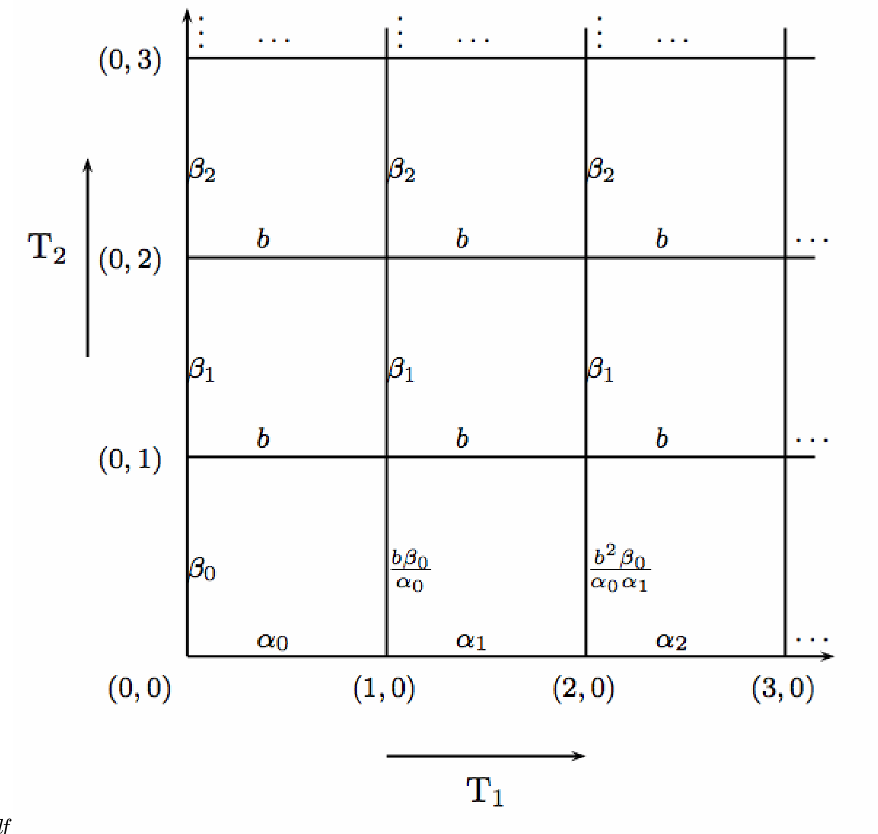
Lemma 18 *([7]) If $W_{(\alpha,\beta)}$ is 2-variable weighted shift with weight sequence $\alpha \equiv \{\alpha_{\mathbf{k}}\}$ and $\beta \equiv \{\beta_{\mathbf{k}}\}$, then the following are equivalent:*

(i) $W_{(\alpha,\beta)}$ is k-hyponormal;

(ii) $(\gamma_{\mathbf{k}}\gamma_{\mathbf{k}+(n,m)+(p,q)} - \gamma_{\mathbf{k}+(n,m)}\gamma_{\mathbf{k}+(p,q)})_{\substack{1 \leq n+m \leq k \\ 1 \leq p+q \leq k}} \geq 0$ for all $\mathbf{k} \in \mathbb{Z}_+^2$;

(iii) $M_{\mathbf{k}}(k) := (\gamma_{\mathbf{k}+(n,m)+(p,q)})_{\substack{0 \leq n+m \leq k \\ 0 \leq p+q \leq k}} \geq 0$ for all $\mathbf{k} \in \mathbb{Z}_+^2$.

2



2.pdf

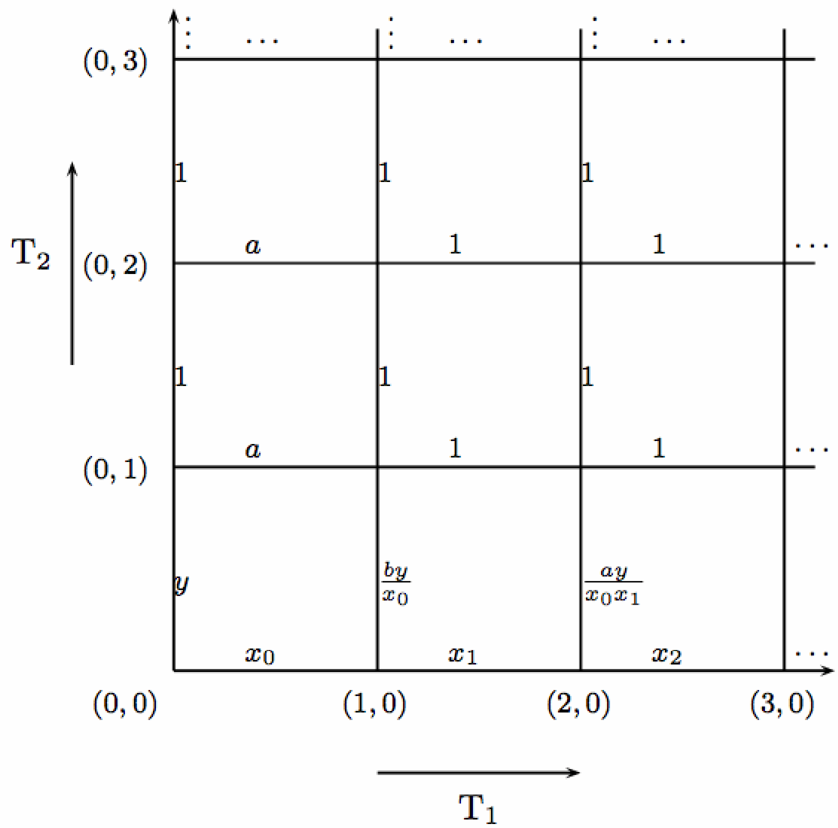
Figure 2: Weight diagram of the 2-variable weighted shifts in Lemma 19.

Lemma 19 *([33]) Let* $W_{(\alpha,\beta)} \in \mathfrak{H}_0$ *be a 2-variable weighted shift whose weight diagram is given in Figure 2 and let*

$$W_{(\alpha,\beta)}|_{\mathcal{M}_1} \cong (I \otimes shift(\beta_1, \beta_2, \cdots), U_+ \otimes bI).$$

Assume that $\|W_\alpha\| = b > 0$*, where* $W_\alpha \equiv shift(\alpha_0, \alpha_1, \alpha_2, \cdots)$*. Then the Berger measure* μ_α *of* W_α *has an atom at* $\{b^2\}$ *if and only if* $W_{(\alpha,\beta)} \in \mathfrak{H}_1$ *if and only if* $W_{(\alpha,\beta)} \in \mathfrak{H}_\infty$.

3



3.pdf

Figure 3: Weight diagram of the 2-variable weighted shifts in Theorem 20.

We now consider the 2-variable weighted shift given in Figure 1 (i), where $W_x \equiv shift(x_0, x_1, x_2, \cdots)$ is as in (1). Then we have

Theorem 20 *Let $W_{(\alpha,\beta)}$ be the 2-variable weighted shift whose weight diagram is given in Figure 3. Then for $h, \ell \geq 1$, we have that $W_{(\alpha,\beta)}^{(h,\ell)}$ is hyponormal if and only if $Ay^2 \leq B$, where*

$$A \ : \ = (2h+1)(h+2)^2 + 4(h+1)^3 - 4a^2(h+1)(h+2)(2h+1)$$

$$\textit{and } B \ : \ = 4(h+1)^2 \left(h+1-a^4-2a^4h\right).$$

Proof. For this, we let $\mathcal{H}_{(m,n)} := \bigvee_{i,j=0}^{\infty}\{e_{(hi+m,\ell j+n)} : h,\ell \geq 1\}$, for $0 \leq m \leq h-1$ and $0 \leq n \leq \ell - 1$. Then $\ell^2(\mathbb{Z}_+^2) \equiv \bigoplus_{m=0}^{h-1}\bigoplus_{n=0}^{\ell-1}\mathcal{H}_{(m,n)}$ and $\mathcal{H}_{(m,n)}$ reduces T_1^h and T_2^ℓ. Thus, if a 2-variable weighted shift $\mathbf{T}$ is given in Figure 1 (i), we can write

$$W_{(\alpha,\beta)}^{(h,\ell)} \cong (W_{\alpha(h:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_0}) \bigoplus \bigoplus_{i=1}^{h-1}(W_{\alpha(h:i)} \oplus (I \otimes U_+), T_2|_{\mathcal{H}_i}),$$

where

$$\begin{aligned} W_{\alpha(h:0)} &= shift(\sqrt{\gamma_h}, \sqrt{\frac{\gamma_{2h}}{\gamma_h}}, \sqrt{\frac{\gamma_{3h}}{\gamma_{2h}}}, \cdots),\ \mathcal{H}_0 := \bigoplus_{n=0}^{\ell-1}\mathcal{H}_{(0,n)}, \\ W_{\alpha(h:i)} &= shift(\sqrt{\frac{\gamma_{(i+1)h}}{\gamma_{ih}}}, \sqrt{\frac{\gamma_{(i+2)h}}{\gamma_{(i+1)h}}}, \cdots) \text{ and } \mathcal{H}_i := \bigoplus_{n=0}^{\ell-1}\mathcal{H}_{(i,n)}. \end{aligned}$$

By Lemma 19 we note that $(W_{\alpha(h:i)} \oplus (I \otimes U_+), T_2|_{\mathcal{H}_i})$ $(1 \leq i \leq h-1)$ is subnormal, because $W_{\alpha(h:i)}$ has an atom at $\{1\}$. Thus the hyponormality of (T_1^h, T_2^ℓ) is equivalent to the hyponormality of $(W_{\alpha(h:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_0})$. Observe that

$$(W_{\alpha(h:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_0}) \cong \bigoplus_{n=0}^{\ell-1}(W_{\alpha(h:0)} \oplus (I \otimes S_a), T_2^\ell|_{\mathcal{H}_{(0,n)}})$$

and

$$\bigoplus_{n=0}^{\ell-1}(W_{\alpha(h:0)} \oplus (I \otimes S_a), T_2^\ell|_{\mathcal{H}_{(0,n)}})$$

$$\cong (W_{\alpha(h:0)} \oplus (I \otimes S_a), T_2^\ell|_{\mathcal{H}_{(0,0)}}) \bigoplus \bigoplus_{n=0}^{\ell-1}(I \otimes S_a, U_+ \otimes I).$$

Because the second summand, $\bigoplus_{n=0}^{\ell-1}(I \otimes S_a, U_+ \otimes I)$, is subnormal with the Berger measure $((1-a^2)\delta_0 + a^2\delta_1) \times \delta_1$, the hyponormality of (T_1^h, T_2^ℓ) is equivalent to the hyponormality of the first summand, $(W_{\alpha(h:0)} \oplus (I \otimes S_a), T_2^{\ell}|_{\mathcal{H}_{(0,0)}})$. Observe that

$$(W_{\alpha(h:0)} \oplus (I \otimes S_a), T_2^{\ell}|_{\mathcal{H}_{(0,0)}}) \cong (W_{\alpha(h:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_{(0,0)}}).$$

Therefore, we have that the hyponormality of (T_1^h, T_2^{ℓ}) is equivalent to the hyponormality of $(W_{\alpha(h:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_{(0,0)}})$. Now, to check the hyponormality of $(W_{\alpha(h:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_{(0,0)}})$, by Lemma 17 and Lemma 19, it suffices to apply the Six-point Test at $\mathbf{k} = (0,0)$. We thus have

$$H_{(W_{\alpha(h:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_{(0,0)}})}(0,0)$$

$$\equiv \begin{pmatrix} \frac{\gamma_{2h}(W_x)}{\gamma_h(W_x)} - \gamma_h(W_x) & \frac{a^2 y}{\sqrt{\gamma_h(W_x)}} - y \cdot \sqrt{\gamma_h(W_x)} \\ \frac{a^2 y}{\sqrt{\gamma_h(W_x)}} - y \cdot \sqrt{\gamma_h(W_x)} & 1 - y^2 \end{pmatrix} \geq 0$$

By a direct calculation shows that

$$H_{(W_{\alpha(h:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_{(0,0)}})}(0,0) \geq 0 \Longleftrightarrow y^2 A \leq B.$$

Therefore, for all $h, \ell \geq 1$ we have that $W_{(\alpha,\beta)}^{(h,\ell)}$ is hyponormal if and only if $Ay^2 \leq B$, as desired. ■

From Theorem 20 we now construct an example which gives a positive answer to Problem 1.

Example 21 *Let $W_{(\alpha,\beta)}$ be the 2-variable weighted shift whose weight diagram is given in*

Figure 1 (i), where $0 < a < \sqrt{\frac{2}{3}} \simeq 0.816$. *For* $h \geq 1$ *we let*

$$g(h,a) := \frac{B}{A}.$$

Then we have

(i) For all $h \geq 3$ *and* $\ell \geq 1$*, if* $\sqrt{g(h+1,\frac{4}{5})} < y \leq \sqrt{g(h,\frac{4}{5})}$*, then* $W_{(\alpha,\beta)}^{(h,\ell)}$ *is hyponormal, but* $W_{(\alpha,\beta)}^{(h+1,\ell)} \notin \mathfrak{H}_1$*;*

(ii) For all $h \geq 2$ *and* $\ell \geq 1$*, if* $\sqrt{g(h,\frac{1}{10})} \leq y < \sqrt{g(h+1,\frac{1}{10})}$*, then* $W_{(\alpha,\beta)}^{(h,\ell)} \notin \mathfrak{H}_1$*, but* $W_{(\alpha,\beta)}^{(h+1,\ell)}$ *is hyponormal.*

Proof. By simple calculation, we first note that $A, B > 0$ on $\left(0, \sqrt{\frac{2}{3}}\right)$, so the function $g(h,a)$ is well defined.

For (i), let $a = \frac{4}{5}$. Then we have $g(h,\frac{4}{5}) = \frac{4(h+1)^2(113h+369)}{25(22h^3+77h^2+152h+72)}$. For all $3 \leq h < \infty$, we observe that $\frac{d}{dh}g(h,\frac{4}{5}) = \frac{-4(h+1)(7h-18)(627h^2+427h+288)}{25(22h^3+77h^2+152h+72)^2} < 0$. Thus, $g(h,\frac{4}{5})$ is decreasing on $[3,\infty)$. Therefore, by Theorem 20, if $\sqrt{g(h+1,\frac{4}{5})} < y \leq \sqrt{g(h,\frac{4}{5})}$, then we have $W_{(\alpha,\beta)}^{(h+1,\ell)}$ is hyponormal, but $W_{(\alpha,\beta)}^{(h+1,\ell)} \notin \mathfrak{H}_1$, as desired.

For (ii), let $a = \frac{1}{10}$. Then we have $g(h,\frac{1}{10}) = \frac{(h+1)^2(9998h+9999)}{100(148h^3+518h^2+593h+198)}$. For all $2 \leq h < \infty$ we note that $\frac{d}{dh}g(h,\frac{1}{10}) = \frac{(h+1)(49h+99)(15096h^2+15196h+99)}{100(148h^3+518h^2+593h+198)^2} > 0$. Thus, $g(h,\frac{1}{10})$ is increasing on $[2,\infty)$. Therefore, by Theorem 20 again, if $\sqrt{g(h,\frac{1}{10})} \leq y < \sqrt{g(h+1,\frac{1}{10})}$, then we have $W_{(\alpha,\beta)}^{(h,\ell)} \notin \mathfrak{H}_1$, but $W_{(\alpha,\beta)}^{(h+1,\ell)}$ is hyponormal, as desired. ■

Remark 22 *(i) From Example 21, we note that for all* $h \geq 2$ *and* $\ell \geq 1$ *there exists a commuting* 2*-variable weighted shift* $W_{(\alpha,\beta)} \in \mathfrak{H}_0$ *for which* $W_{(\alpha,\beta)}^{(h,\ell)}$ *is hyponormal, but* $W_{(\alpha,\beta)}^{(h+1,\ell)} \notin \mathfrak{H}_1$*, and* $W_{(\alpha,\beta)}^{(h+1,\ell)}$ *is hyponormal, but* $W_{(\alpha,\beta)}^{(h,\ell)} \notin \mathfrak{H}_1$*.*

(ii) Looking at Theorem 20 and Example 21, it seems natural to conjecture that these similar results should work for the k-hyponormality ($k \geq 4$). However, it is highly nontrivial to establish these similar results in the k-hyponormality ($k \geq 4$)., when k is very huge. Because, a, y, h, ℓ are arbitrary unknown variables, so it becomes unwieldy to check the positivity of $M_{(0,0)}(k)\left(W_{(\alpha,\beta)}^{(h,\ell)}\right)$ in Lemma 18 (iii).

Form Problem 1 and Remark 22, it is natural to consider

Problem 23 *For some $h \geq 2$, $\ell \geq 1$, $k \geq 4$ does there exist a 2-variable weighted shift $W_{(\alpha,\beta)}$ for which $W_{(\alpha,\beta)}^{(h,\ell)}$ is k-hyponormal, but $W_{(\alpha,\beta)}^{(h+1,\ell)} := (T_1^{h+1}, T_2^{\ell}) \notin \mathfrak{H}_k$, and $W_{(\alpha,\beta)}^{(h+1,\ell)}$ is k-hyponormal, but $W_{(\alpha,\beta)}^{(h,\ell)} \notin \mathfrak{H}_k$?*

We now consider a negative answer to Problem 2 when $k = 3$. We begin with

Lemma 24 *(cf. [30]) Let $M \equiv \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ be a 2×2 operator matrix, where A and C are square matrices and B is a rectangular matrix. Then*

$$M \geq 0 \iff \text{there exists } W \text{ such that } \begin{cases} A \geq 0 \\ B = AW \\ C \geq W^* A W. \end{cases}$$

We thus have

Theorem 25 *Let $W_{(\alpha,\beta)}$ be the 2-variable weighted shift whose weight diagram is given in Figure 1 (i) (where $0 < a \leq \frac{3}{4}$ and W_x is as in (1)). Then we have*

(i) $W_{(\alpha,\beta)}$ is 3-hyponormal if and only if $(1007 - 2400a^2 - 960a^4)\, y^2 \leq (512 - 960a^2)$;

(ii) $W_{(\alpha,\beta)}^{(2,1)}$ is 3-hyponormal if and only if $(32444 - 79716a^2 - 33075a^4)\, y^2 \leq (17150 - 33075a^2)$.

Proof. (i) By Lemma 19, we first note that $W_{(\alpha,\beta)}$ on $\mathcal{M}$ (resp. $\mathcal{N}$) is subnormal, because $shift(x_0, x_1, \cdots)$ (resp. $shift(y, 1, 1\cdots)$) has an atom at $\{1\}$. Thus, to verify the 3-hyponormality of $W_{(\alpha,\beta)}$, it suffices to apply Lemma 18 (iii) to $W_{(\alpha,\beta)}$ at $\mathbf{k} = (0,0)$. Note that the moments associated with $W_{(\alpha,\beta)}$ of order $\mathbf{k}$ are

$$\gamma_{\mathbf{k}}(W_{(\alpha,\beta)}) = \begin{cases} 1, & \text{if } k_1 = 0 \text{ and } k_2 = 0 \\ \gamma_{k_1 h}(W_{(\alpha,\beta)}), & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ y^2, & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \frac{y^2}{2}, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases} \tag{2}$$

Since the third and sixth, and fifth, eighth and ninth rows of $M_{(0,0)}(3)\left(W_{(\alpha,\beta)}\right)$ are identical, respectively, if we multiply $\frac{1}{y^2}$ and then apply row and column operations to $M_{(0,0)}(3)\left(W_{(\alpha,\beta)}\right)$, then we have $M_{(0,0)}(3)\left(W_{(\alpha,\beta)}\right) \geq 0 \Longleftrightarrow \widetilde{M_{(0,0)}}\left(W_{(\alpha,\beta)}\right) \geq 0$, where

$$\widetilde{M_{(0,0)}}\left(W_{(\alpha,\beta)}\right)$$

$$:= \begin{pmatrix} \begin{pmatrix} \frac{1}{y^2} & \frac{3}{4} & \frac{2}{3} & \frac{5}{8} \\ \frac{3}{4} & \frac{2}{3} & \frac{5}{8} & \frac{3}{5} \\ \frac{2}{3} & \frac{5}{8} & \frac{3}{5} & \frac{7}{12} \\ \frac{5}{8} & \frac{3}{5} & \frac{7}{12} & \frac{4}{7} \end{pmatrix} & \begin{pmatrix} 1 & a^2 \\ a^2 & a^2 \\ a^2 & a^2 \\ a^2 & a^2 \end{pmatrix} \\ \begin{pmatrix} 1 & a^2 & a^2 & a^2 \\ a^2 & a^2 & a^2 & a^2 \end{pmatrix} & \begin{pmatrix} 1 & a^2 \\ a^2 & a^2 \end{pmatrix} \end{pmatrix} \tag{3}$$

$$\equiv \begin{pmatrix} A\left(W_{(\alpha,\beta)}\right) & B\left(W_{(\alpha,\beta)}\right) \\ B^*\left(W_{(\alpha,\beta)}\right) & C\left(W_{(\alpha,\beta)}\right) \end{pmatrix} \geq 0,$$

We now apply Lemma 24 to $\widetilde{M_{(0,0)}}\left(W_{(\alpha,\beta)}\right)$. Since $A\left(W_{(\alpha,\beta)}\right)$ is invertible, we have

$$\widetilde{M_{(0,0)}}\left(W_{(\alpha,\beta)}\right) \geq 0$$
$$\Longleftrightarrow Q\left(W_{(\alpha,\beta)}\right) := C\left(W_{(\alpha,\beta)}\right) - W^*\left(W_{(\alpha,\beta)}\right) A\left(W_{(\alpha,\beta)}\right) W\left(W_{(\alpha,\beta)}\right) \geq 0,$$

where

$$W\left(W_{(\alpha,\beta)}\right) := \begin{pmatrix} \frac{8y^2\left(65a^2-64\right)}{495y^2-512} & \frac{8a^2y^2}{495y^2-512} \\ \frac{-120\left(15a^2y^2-30y^2+16a^2\right)}{495y^2-512} & \frac{120a^2\left(15y^2-16\right)}{495y^2-512} \\ \frac{-120\left(5a^2y^2+56y^2-64a^2\right)}{495y^2-512} & \frac{-120a^2\left(61y^2-64\right)}{495y^2-512} \\ \frac{280\left(10a^2y^2+13y^2-24a^2\right)}{495y^2-512} & \frac{280a^2\left(23y^2-24\right)}{495y^2-512} \end{pmatrix}$$

and

$$B\left(W_{(\alpha,\beta)}\right) = A\left(W_{(\alpha,\beta)}\right) W\left(W_{(\alpha,\beta)}\right).$$

A direct calculation shows that

$$\begin{aligned}
&Q\left(W_{(\alpha,\beta)}\right)\\
:\ &= \begin{pmatrix} \frac{y^2\left(512-960a^4-1007y^2+1040a^2y^2+400a^4y^2\right)}{512-495y^2} & \frac{a^2y^2\left(512-960a^2-487y^2+920a^2y^2\right)}{512-495y^2} \\ \frac{a^2y^2\left(512-960a^2-487y^2+920a^2y^2\right)}{512-495y^2} & \frac{a^2y^2\left(512-960a^2-495y^2+928a^2y^2\right)}{512-495y^2} \end{pmatrix}\\
=\ :& \begin{pmatrix} q_{11}\left(Q\left(W_{(\alpha,\beta)}\right)\right) & q_{12}\left(Q\left(W_{(\alpha,\beta)}\right)\right) \\ q_{21}\left(Q\left(W_{(\alpha,\beta)}\right)\right) & q_{22}\left(Q\left(W_{(\alpha,\beta)}\right)\right) \end{pmatrix}.
\end{aligned}$$

From the argument above, we note that

$$W_{(\alpha,\beta)} \text{ is 3-hyponormal} \Longleftrightarrow \widetilde{M_{(0,0)}}\left(W_{(\alpha,\beta)}\right) \Longleftrightarrow Q\left(W_{(\alpha,\beta)}\right) \geq 0.$$

Observe that $Q\left(W_{(\alpha,\beta)}\right) \geq 0$ if and only if $\left\{q_{11}\left(Q\left(W_{(\alpha,\beta)}\right)\right) \geq 0 \text{ and } \det N\left(W_{(\alpha,\beta)}\right) \geq 0\right\}$ if and only if $(1007 - 2400a^2 - 960a^4)\, y^2 \leq (512 - 960a^2)$. Thus, we have the desired result.

(ii) Recall that for $n = 0, 1$, $\mathcal{H}_n \equiv \bigvee_{i=0}^{\infty}\{e_{(2i+n,j)} : j = 0, 1, 2, \cdots\}$ and $\ell^2(\mathbb{Z}_+^2) \equiv \mathcal{H}_0 \bigoplus \mathcal{H}_1$.

Note that

$$W_{(\alpha,\beta)}^{(2,1)} \equiv (T_1^2, T_2) \cong W_{(\alpha,\beta)}^{(2,1)}|_{\mathcal{H}_0} \bigoplus W_{(\alpha,\beta)}^{(2,1)}|_{\mathcal{H}_1}.$$

By Lemma 19, we have that $W_{(\alpha,\beta)}^{(2,1)}|_{\mathcal{H}_1} \cong (W_{\alpha(2:1)} \oplus (I \otimes U_+), T_2|_{\mathcal{H}_1})$ is subnormal, because $W_{\alpha(2:1)} = shift(x_1x_2, x_3x_4, \cdots)$ has an atom at $\{1\}$. Hence, we have $W_{(\alpha,\beta)}^{(2,1)}$ is 3-hyponormal if and only if $W^{(2,1)}_{(\alpha,\beta)}|_{\mathcal{H}_0} := (T_1^2, T_2)|_{\mathcal{H}_0}$ is 3-hyponormal. Let $\mathcal{M}_1(0)$ (resp. $\mathcal{N}_1(0)$) be the subspace of $\mathcal{H}_0$ spanned by canonical orthonormal basis vectors associated to indices $\mathbf{k} = (k_1, k_2)$ with $k_1 \geq 0$ and $k_2 \geq 1$ (resp. $k_1 \geq 1$ and $k_2 \geq 0$). By Lemma 19, we note that $W^{(2,1)}_{(\alpha,\beta)}|_{\mathcal{H}_0}$ on $\mathcal{M}_1(0)$ (resp. $\mathcal{N}_1(0)$) is subnormal, because $shift(x_2x_3, x_4x_5, \cdots)$ (resp. $shift(y, 1, 1\cdots)$) has an atom at $\{1\}$. Thus, to verify the 3-hyponormality of $W^{(2,1)}_{(\alpha,\beta)}$, it suffices to apply Lemma 18 (iii) to $W^{(2,1)}_{(\alpha,\beta)}|_{\mathcal{H}_0}$ at $\mathbf{k} = (0,0)$. Note that the moments associated with $(T_1^2, T_2)|_{\mathcal{H}_0}$ of order $\mathbf{k}$ are

$$\gamma_{\mathbf{k}}\left(W^{(2,1)}_{(\alpha,\beta)}|_{\mathcal{H}_0}\right) = \begin{cases} 1, & \text{if } k_1 = 0 \text{ and } k_2 = 0 \\ \prod_{i=1}^{k_1} x_{2(i-1)}^2 x_{2i-1}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ y^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \frac{289}{400} y^2, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1 \end{cases}.$$

Since the third and sixth, and fifth, eighth and ninth rows of $M_{(0,0)}(3)|_{\mathcal{H}_0}$ are identical, if we multiply $\frac{1}{y^2}$ and then apply row and column operations to $M_{(0,0)}(3)|_{\mathcal{H}_0}$, then we have

$$M_{(0,0)}(3)|_{\mathcal{H}_0} \geq 0 \Longleftrightarrow \widetilde{M_{(0,0)}}\left(W^{(2,1)}_{(\alpha,\beta)}|_{\mathcal{H}_0}\right) \geq 0,$$

where

$$\widetilde{M_{(0,0)}}\left(W_{(\alpha,\beta)}^{(2,1)}|\mathcal{H}_0\right)$$

$$:=\left(\begin{array}{cc}\left(\begin{array}{cccc}\frac{1}{y^2} & \frac{2}{3} & \frac{3}{5} & \frac{4}{7} \\ \frac{2}{3} & \frac{3}{5} & \frac{4}{7} & \frac{5}{9} \\ \frac{3}{5} & \frac{4}{7} & \frac{5}{9} & \frac{6}{11} \\ \frac{4}{7} & \frac{5}{9} & \frac{6}{11} & \frac{7}{13}\end{array}\right) & \left(\begin{array}{cc}1 & a^2 \\ a^2 & a^2 \\ a^2 & a^2 \\ a^2 & a^2\end{array}\right) \\ \left(\begin{array}{cccc}1 & a^2 & a^2 & a^2 \\ a^2 & a^2 & a^2 & a^2\end{array}\right) & \left(\begin{array}{cc}1 & a^2 \\ a^2 & a^2\end{array}\right)\end{array}\right)$$

$$\equiv\left(\begin{array}{cc}A\left(W_{(\alpha,\beta)}^{(2,1)}|\mathcal{H}_0\right) & B\left(W_{(\alpha,\beta)}^{(2,1)}|\mathcal{H}_0\right) \\ B^*\left(W_{(\alpha,\beta)}^{(2,1)}|\mathcal{H}_0\right) & C\left(W_{(\alpha,\beta)}^{(2,1)}|\mathcal{H}_0\right)\end{array}\right)\geq 0,$$

We now apply Lemma 24 to $\widetilde{M_{(0,0)}}\left(W_{(\alpha,\beta)}^{(2,1)}|\mathcal{H}_0\right)$ again. A same calculation shown in (ii) above shows that

$$Q\left(W_{(\alpha,\beta)}^{(2,1)}|\mathcal{H}_0\right)$$

$$=:\left(\begin{array}{cc}q_{11}\left(W_{(\alpha,\beta)}^{(2,1)}|\mathcal{H}_0\right) & q_{12}\left(W_{(\alpha,\beta)}^{(2,1)}|\mathcal{H}_0\right) \\ q_{21}\left(W_{(\alpha,\beta)}^{(2,1)}|\mathcal{H}_0\right) & q_{22}\left(W_{(\alpha,\beta)}^{(2,1)}|\mathcal{H}_0\right)\end{array}\right).$$

where

$$\begin{aligned}q_{11}\left(W_{(\alpha,\beta)}^{(2,1)}|\mathcal{H}_0\right) &:= \frac{y^2\left(17150-33075a^4-32444y^2+34860a^2y^2+11781a^4y^2\right)}{2(8575-7647y^2)}, \\ q_{12}\left(W_{(\alpha,\beta)}^{(2,1)}|\mathcal{H}_0\right) = q_{21}\left(W_{(\alpha,\beta)}^{(2,1)}|\mathcal{H}_0\right) &:= \frac{a^2y^2\left(17150-33075a^2-15014y^2+29211a^2y^2\right)}{2(8575-7647y^2)} \\ q_{22}\left(W_{(\alpha,\beta)}^{(2,1)}|\mathcal{H}_0\right) &:= \frac{a^2y^2\left(17150-33075a^2-15294y^2+29491a^2y^2\right)}{2(8575-7647y^2)}.\end{aligned}$$

Thus, by the argument given above, we have that $W_{(\alpha,\beta)}^{(2,1)}$ is 3-hyponormal if and only if $\left(32444 - 79716a^2 - 33075a^4\right) y^2 \leq \left(17150 - 33075a^2\right)$ ■

Corollary 26 *By Theorem 25, we note that*

(i) if $a = \frac{1}{2}$*, then* $W_{(\alpha,\beta)}$ *is* 3*-hyponormal but* $W_{(\alpha,\beta)}^{(2,1)} \notin \mathfrak{H}_3$ *when* $0 < y \leq \sqrt{\frac{272}{467}} \approx 0.763$.

(ii) If $a = \frac{3}{4}$*, then* $W_{(\alpha,\beta)} \notin \mathfrak{H}_3$ *but* $W_{(\alpha,\beta)}^{(2,1)}$ *is* 3*-hyponormal when* $0.868 \approx \sqrt{\frac{74480}{98875}} < y < 1$.

CHAPTER IV

FURTHER STUDY

In Theorem 25 we have shown that given $k = 3$ and $a = \frac{1}{2}$, then $W_{(\alpha,\beta)}$ is 3-hyponormal but $W^{(2,1)}_{(\alpha,\beta)} \notin \mathfrak{H}_3$ when $0 < y \leq \sqrt{\frac{272}{467}} \approx 0.763$. Also, if $a = \frac{3}{4}$, then $W_{(\alpha,\beta)} \notin \mathfrak{H}_3$ but $W^{(2,1)}_{(\alpha,\beta)}$ is 3-hyponormal when $0.868 \approx \sqrt{\frac{74480}{98875}} < y < 1$. Now an obvious question is that given $k \geq 4$ and the k-hyponormality of $W_{(\alpha,\beta)}$, does it follow that $W^{(h,\ell)}_{(\alpha,\beta)} \notin \mathfrak{H}_k$ for all $h, \ell \geq 1$? Looking at Theorem 25, it seems natural to conjecture that a similar result should work for the k-hyponormality ($k \geq 4$). That is, perhaps we have the necessary and sufficient conditions for the k-hyponormality $W_{(\alpha,\beta)}$ $\left(\text{resp. } k\text{-hyponormality of } W^{(2,1)}_{(\alpha,\beta)}\right)$ for $k \geq 4$, especially based on the determinent of $A\left(W_{(\alpha,\beta)}\right)$ $\left(\text{resp. } A\left(W^{(2,1)}_{(\alpha,\beta)}|_{\mathcal{H}_0}\right)\right)$ in the Proof of Theorem 25. However, it is highly nontrivial to establish the necessary and sufficient condition for the k-hyponormality of them. For, $A\left(W_{(\alpha,\beta)}\right)$ is just a Hankel matrix without a common well known pattern, so it becomes unwieldy to check its determinent or invertibility. in the future study, we will continue to extend our result when $k \geq 4$ using the techniques of a Mathematical Induction, nonlinear optimization problems and a Moore-Penrose inverse of a matrix with computer programs such as "Mathematica" and "Matlab".

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BIOGRAPHICAL SKETCH

Mayra O. Martinez graduated with honors from Roma High School in 2004. She then attended the University of Texas-Pan American for her undergraduate degree in which she double majored in Biology and Mathematics with a minor in Chemistry. Throughout her undergraduate studies, Mayra participated in the Weiner's Society of Mathematicians. Also, she was a violinist for the UTPA Mariachi Femenil. She completed her undergraduate coursework in May 2009. In August 2010, she started her graduate coursework for the pursuit of a Master of Science in Mathematics. She was employed by the UTPA department of Mathematics as a Graduate Teaching Assistant which included duties such as tutoring and teaching an undergraduate course. She was a recipient of the Summer Research Initiative grant in which she participated in research work along with her mentor Dr. Jasang Yoon for a year. Her obligations included giving several lectures over her research to other universities. She graduated with honors in May 2011.