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Blocks in the category of finite-dimensional representations of principal W-algebra for $Q(2)$

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Abstract. We describe the blocks in the category of finite-dimensional representations of the principal finite W-algebra for the Lie superalgebra $Q(2)$.

Introduction

In the classical case when $\mathfrak{g}$ is a complex semi-simple Lie algebra and e is a nilpotent element in $\mathfrak{g}$, a *finite W-algebra* for $\mathfrak{g}$ is a quantization of the Poisson structure on the Slodowy slice (a transversal slice to the orbit of e in the adjoint representation). The general definition of a finite W-algebra was given by A. Premet in [16]. For a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a reductive even part $\mathfrak{g}_{\bar{0}}$, the finite W-algebra is associated with an even nilpotent element $e \in \mathfrak{g}_{\bar{0}}$. It is denoted by W.

In the case when $\mathfrak{g} = \mathfrak{gl}(m|n)$ and e is the even principal nilpotent, J. Brown, J. Brundan and S. Goodwin classified irreducible representations of W and explored the connection with the category $\mathcal{O}$ for $\mathfrak{g}$ using coinvariants functor [1, 2].

In [13] we considered W associated with an even nilpotent element $\varphi \in \mathfrak{g}_{\bar{0}}^* \subset \mathfrak{g}^*$ in the coadjoint representation (this means that for the algebraic reductive group $G_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$, the closure of the $G_{\bar{0}}$-orbit of φ in $\mathfrak{g}_{\bar{0}}^*$ contains zero). We proved that if φ is the *principal* nilpotent, i.e. the dimension of the even part of the annihilator of φ in $\mathfrak{g}$ is minimal, and $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2|2n)$ or $Q(n)$, then every simple W-module is finite-dimensional. In [15] we classified simple W-modules for $\mathfrak{g} = Q(n)$ associated with the principal nilpotent coadjoint orbits (Theorem 4.6). The technique we used is completely different from one used in [1] due to the lack of triangular decomposition of W in our case. Instead, we described the restriction of simple $U(\mathfrak{h})$-modules to W and proved that any simple W-module occurs as a constituent of this restriction.

We consider the category $W-\mathrm{mod}$ of finite-dimensional W-modules. The natural problem is to describe the *blocks* in this category. In this work we make a step in this direction and describe the blocks in the case when $\mathfrak{g} = Q(2)$ (Theorem 8). Our results should have applications to the study of primitive ideals of $U(\mathfrak{g})$ in the sense of I. Losev (see [8, 9, 10]). In the super case the theory of the primitive ideals is even more complicated (see [3]). We also intend to apply these results to classify simple modules for super-Yangians of type Q.

All results are joint work with V. Serganova.

1. Finite W-algebra for $Q(n)$

In this paper we consider the Lie superalgebra $\mathfrak{g} = Q(n)$ defined as follows (see [6]). Equip $\mathbb{C}^{n|n}$ with the odd operator ζ such that $\zeta^2 = -\mathrm{Id}$. Then $Q(n)$ is the centralizer of ζ in the Lie superalgebra $\mathfrak{gl}(n|n)$. It is easy to see that $Q(n)$ consists of matrices of the form $\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right)$, where A, B are $n\times n$ matrices. We fix the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ to be the set of matrices with diagonal A and B. By n^+ (respectively, $\mathfrak{n}^-$) we denote the nilpotent subalgebras consisting of matrices with strictly upper triangular (respectively, low triangular) A and B. The Lie superalgebra $\mathfrak{g}$ has the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and we set $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$.

Denote by W the finite W-algebra associated with a principal even nilpotent element φ in the coadjoint representation of $Q(n)$. Let us recall the definition (see [16]). Let $\{e_{i,j}, f_{i,j} \,|\, i,j = 1,\dots,n\}$ denote the basis consisting of elementary even and odd matrices:

$$e_{i,j} = \left(\begin{array}{c|c} E_{ij} & 0 \\ \hline 0 & E_{ij} \end{array}\right), \quad f_{i,j} = \left(\begin{array}{c|c} 0 & E_{ij} \\ \hline E_{ij} & 0 \end{array}\right).$$

Choose $\varphi \in \mathfrak{g}^*$ such that

$$\varphi(f_{i,j}) = 0, \quad \varphi(e_{i,j}) = \delta_{i,j+1}.$$

Let I_φ be the left ideal in $U(\mathfrak{g})$ generated by $x - \varphi(x)$ for all $x \in \mathfrak{n}^-$. Let $\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/I_\varphi$ be the natural projection. Then

$$W = \{\pi(y) \in U(\mathfrak{g})/I_\varphi \mid \mathrm{ad}(x)y \in I_\varphi \text{ for all } x \in \mathfrak{n}^-\}.$$

Using identification of $U(\mathfrak{g})/I_\varphi$ with the Whittaker module $U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}_\varphi \simeq U(\mathfrak{b}) \otimes \mathbb{C}$ we can consider W as a subalgebra of $U(\mathfrak{b})$. The natural projection $\vartheta : U(\mathfrak{b}) \to U(\mathfrak{h})$ with the kernel $\mathfrak{n}^+U(\mathfrak{b})$ is called the *Harish-Chandra homomorphism.* It is proven in [13] that the restriction of ϑ to W is injective.

Set

$$\xi_i := (-1)^{i+1} f_{i,i}, \; x_i := \xi_i^2 = e_{i,i},$$

then

$$U(\mathfrak{h}) \simeq \mathbb{C}[\xi_1, \dots, \xi_n]/(\xi_i\xi_j + \xi_j\xi_i)_{i<j\le n}.$$

We will identify W with $\vartheta(W)$ and use the generators in W introduced in [14] (Corollary 5.15):

$$u_k(0) := [\sum_{1\le i_1<i_2<\ldots<i_k\le n} (x_{i_1} + (-1)^{k+1}\xi_{i_1}) \cdots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})]_{even},$$

$$u_k(1) := [\sum_{1\le i_1<i_2<\ldots<i_k\le n} (x_{i_1} + (-1)^{k+1}\xi_{i_1}) \cdots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})]_{odd}.$$

For convenience we assume $u_k(0) = u_k(1) = 0$ for $k > n$.

2. Simple modules over associative superalgebras

We work in the category of vector superspaces over $\mathbb{C}$. We denote the *parity* of a homogeneous vector v of a superspace by $\bar{v} \in \mathbb{Z}_2$. All tensor products are over $\mathbb{C}$ unless specified otherwise.

Let $\mathcal{A}$ be a superalgebra. By an $\mathcal{A}$-module M we mean a $\mathbb{Z}_2$-graded left $\mathcal{A}$-module. A submodule of M is a $\mathbb{Z}_2$-graded submodule. By Π we denote the functor of parity switch $\Pi(M) = M \otimes \mathbb{C}^{0|1}$. For a module M over an associate superalgebra $\mathcal{A}$, ΠM has the same underlying vector space but with the opposite $\mathbb{Z}$-grading. The new action of $a \in \mathcal{A}$ on $m \in \Pi M$ is given in terms of the old action by $a \cdot m := (-1)^{\bar{a}} am$.

Recall that if M is a simple finite-dimensional $\mathcal{A}$-module over some associative superalgebra $\mathcal{A}$, then by Schur's Lemma $\mathrm{End}_{\mathcal{A}}(M)$ is either one-dimensional, or two-dimensional and has basis $\{\mathrm{Id}_M, \epsilon_M\}$, where ϵ_M is a (unique up to a sign) odd involution on M: $\epsilon_M^2 = \mathrm{Id}_M$. Note that ϵ_M provides an $\mathcal{A}$ isomorphism $M \longrightarrow \Pi(M)$. We say that M is an *irreducible of* M*-type* in the former case and an *irreducible of* Q*-type* in the latter (see [7, 4]).

Let $\mathcal{A}$ and $\mathcal{B}$ be two superalgebras. The tensor product $\mathcal{A} \otimes \mathcal{B}$ is again a superalgebra, where multiplication is given by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\bar{b}_1 \bar{a}_2} a_1 a_2 \otimes b_1 b_2$$

for $a_i \in \mathcal{A}, b_i \in \mathcal{B}$. Let M and N be two modules over associative superalgebras $\mathcal{A}$ and $\mathcal{B}$. Then $M \otimes N$ is naturally a module over $\mathcal{A} \otimes \mathcal{B}$ where

$$(a \otimes b)(m \otimes n) = (-1)^{\bar{b}\bar{m}} am \otimes bn,$$

where $a \in \mathcal{A}, b \in \mathcal{B}$ and $m \in M, n \in N$. If M and N are two simple finite-dimensional modules over associative superalgebras $\mathcal{A}$ and $\mathcal{B}$, then the module $M \otimes N$ might be not simple. In fact, if M and N are both of M-type, then $M \otimes N$ is simple of M-type. If one of these modules is of M-type, and the other is of Q-type, then $M \otimes N$ is simple of Q-type. However, if M and N are both of Q-type, then $M \otimes N$ is not simple. Let ϵ_M and ϵ_N be odd involutions of M and N, respectively. Then the map $\epsilon_M \otimes \epsilon_N$ defined by

$$(\epsilon_M \otimes \epsilon_N)(m \otimes n) = (-1)^{\bar{m}} \epsilon_M(m) \otimes \epsilon_N(n)$$

is an even $\mathcal{A} \otimes \mathcal{B}$-automorphism of $M \otimes N$, and its square is $-\mathrm{Id}_{M \otimes N}$. In this case $M \otimes N$ decomposes into a direct sum of two $\mathcal{A} \otimes \mathcal{B}$-submodules, which are formed by the $\pm\mathbf{i}$-eigenspaces of $\epsilon_M \otimes \epsilon_N$. We can choose either submodule and denote it by $M \boxtimes N$. Then

$$M \otimes N \simeq M \boxtimes N \oplus \Pi(M \boxtimes N).$$

Both submodules are simple and of M-type.

3. Irreducible representations of W

3.1. Representations of $U(\mathfrak{h})$

Let $\mathfrak{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n$. We call $\mathfrak{s}$ *regular* if $s_i \neq 0$ for all $i \leq n$ and *typical* if $s_i + s_j \neq 0$ for all $i \neq j \leq n$.

It follows from the representation theory of Clifford algebras that all irreducible representations of $U(\mathfrak{h})$ up to change of parity can be parameterized by $\mathfrak{s} \in \mathbb{C}^n$. Indeed, let M be an irreducible representation of $U(\mathfrak{h})$. By Schur's lemma every x_i acts on M as a scalar operator $s_i\mathrm{Id}$. Let $I_{\mathfrak{s}}$ denote the ideal in $U(\mathfrak{h})$ generated by $x_i - s_i$, then the quotient algebra $U(\mathfrak{h})/I_{\mathfrak{s}}$ is isomorphic to the Clifford superalgebra $C_{\mathfrak{s}}$ (we consider Clifford algebras as superalgebras with the natural $\mathbb{Z}_2$-grading) associated with the quadratic form:

$$B_{\mathfrak{s}}(\xi_i, \xi_j) = \delta_{ij} s_i.$$

Then M is a simple $C_{\mathfrak{s}}$-module.

The radical $R_{\mathfrak{s}}$ of $C_{\mathfrak{s}}$ is generated by the kernel of the form $B_{\mathfrak{s}}$. Let $m(\mathfrak{s})$ be the number of non-zero coordinates of $\mathfrak{s}$, then $C_{\mathfrak{s}}/R_{\mathfrak{s}}$ is isomorphic to the matrix superalgebra $M(2^{\frac{m}{2}-1}|2^{\frac{m}{2}-1})$ for even m and to the superalgebra $M(2^{\frac{m-1}{2}}) \otimes \mathbb{C}[\epsilon]/(\epsilon^2 - 1)$ for odd m.

Therefore $C_{\mathfrak{s}}$ has one (up to isomorphism) simple $\mathbb{Z}_2$-graded module $V(\mathfrak{s})$ of type Q for odd $m(\mathfrak{s})$, and two simple modules $V(\mathfrak{s})$ and $\Pi V(\mathfrak{s})$ of type M for even $m(\mathfrak{s})$ (see [11]). In the case when $\mathfrak{s}$ is regular, the form $B_{\mathfrak{s}}$ is non-degenerate and the dimension of $V(\mathfrak{s})$ equals 2^k, where $k = \lceil n/2 \rceil$. In general, $\dim V(\mathfrak{s}) = 2^{\lceil m(\mathfrak{s})/2 \rceil}$.

Let $i+j=n$. We have the natural embedding of the Lie superalgebras $Q(i)\oplus Q(j)\hookrightarrow Q(n)$. If $\mathfrak{h}_r$ denotes the Cartan subalgebra of $Q(r)$, the above embedding induces the isomorphism

$$U(\mathfrak{h})\simeq U(\mathfrak{h}_i)\otimes U(\mathfrak{h}_j).$$

It induces an isomorphism of $U(\mathfrak{h})$-modules

$$V(\mathfrak{s})\simeq V(s_1,\dots,s_i)\boxtimes V(s_{i+1},\dots,s_n).$$

3.2. Restriction from $U(\mathfrak{h})$ to W

We denote by the same symbol $V(\mathbf{s})$ the restriction to W of the $U(\mathfrak{h})$-module $V(\mathfrak{s})$. We proved the following two statements in [15].

Proposition 1 ([15], Proposition 4.1). Let S be a simple W-module. Then S is a simple constituent of $V(\mathfrak{s})$ for some $\mathfrak{s}\in\mathbb{C}^n$.

Theorem 2 ([15], Theorem 4.2). If $\mathfrak{s}$ is typical, then $V(\mathfrak{s})$ is a simple W-module.

3.3. Simple W-modules for $n=2$

Let $n=2$. Let $s_1\neq -s_2$ and assume that $s_1,s_2\neq 0$. Then by Theorem 2, $V(\mathfrak{s})$ is simple as W-module. First, we describe the action of $U(\mathfrak{h})$ in $V(s_1,s_2)\simeq V(s_1)\boxtimes V(s_2)$. Note that

$$U(\mathfrak{h})\simeq U(\mathfrak{h}_1)\otimes U(\mathfrak{h}_1),$$

where $\mathfrak{h}_1$ is the Cartan subalgebra of $Q(1)$. Clearly, $U(\mathfrak{h}_1)\cong\mathbb{C}[\xi]$. Let $x=\xi^2$. For some suitable bases in $V(s_1)$ and $V(s_2)$, namely, $V(s_1)=<v_1\mid v_2>$, $V(s_2)=<w_1\mid w_2>$, where $\bar v_1=\bar w_1=0$ and $\bar v_2=\bar w_2=1$, the action of $U(\mathfrak{h}_1)$ in $V(s_i)$ is given by

$$\xi\mapsto\begin{pmatrix}0 & \sqrt{s_i}\\ \sqrt{s_i} & 0\end{pmatrix},\quad x\mapsto\begin{pmatrix}s_i & 0\\ 0 & s_i\end{pmatrix}\qquad\text{for } i=1,2.$$

We identify the elements ξ_i,x_i of $U(\mathfrak{h})$ is follows:

$$\xi_1\leftrightarrow\xi\otimes 1,\quad \xi_2\leftrightarrow 1\otimes\xi,\quad x_1\leftrightarrow x\otimes 1,\quad x_2\leftrightarrow 1\otimes x.$$

Then $V(s_1)\otimes V(s_2)=V(s_1,s_2)\oplus\Pi V(s_1,s_2)$, where

$$V(s_1,s_2)=<v_1\otimes w_1+\mathbf{i}v_2\otimes w_2\mid v_2\otimes w_1+\mathbf{i}v_1\otimes w_2>,\tag{1}$$

$$\Pi V(s_1,s_2)=<-v_1\otimes w_1+\mathbf{i}v_2\otimes w_2\mid v_2\otimes w_1-\mathbf{i}v_1\otimes w_2>.\tag{2}$$

Hence the action of $U(\mathfrak{h})$ in $V(s_1,s_2)$ is given by the following formulas in basis (1):

$$\xi_1\mapsto\begin{pmatrix}0 & \sqrt{s_1}\\ \sqrt{s_1} & 0\end{pmatrix},\quad \xi_2\mapsto\begin{pmatrix}0 & \sqrt{s_2}\mathbf{i}\\ -\sqrt{s_2}\mathbf{i} & 0\end{pmatrix}.$$

Note that W is generated by ϕ_0,ϕ_1,z_0 and z_1, where

$$\phi_0:=u_1(1)=\xi_1+\xi_2,\ \phi_1:=-u_2(1)=x_2\xi_1-x_1\xi_2,\ z_0=u_1(0)=x_1+x_2,\ z_1=u_2(0)=x_1x_2-\xi_1\xi_2.$$

Then we obtain the following formulas for the action of the generators of W:

$$\phi_0\mapsto\begin{pmatrix}0 & \sqrt{s_1}+\sqrt{s_1}\mathbf{i}\\ \sqrt{s_1}-\sqrt{s_2}\mathbf{i} & 0\end{pmatrix},\quad \phi_1\mapsto\sqrt{s_1s_2}\begin{pmatrix}0 & \sqrt{s_2}-\sqrt{s_1}\mathbf{i}\\ \sqrt{s_2}+\sqrt{s_1}\mathbf{i} & 0\end{pmatrix},\tag{3}$$

$$z_0 \mapsto (s_1+s_2)\mathrm{Id}, \quad z_1 \mapsto \begin{pmatrix} s_1s_2+\sqrt{s_1s_2}\mathbf{i} & 0 \\ 0 & s_1s_2-\sqrt{s_1s_2}\mathbf{i} \end{pmatrix}. \tag{4}$$

Note that formulas (3) and (4) hold when $s_1 \neq -s_2$.
Assume that $s_1 = -s_2$. If $s_1, s_2 = 0$ then $V(\mathfrak{s})$ is isomorphic to $\mathbb{C} \oplus \Pi\mathbb{C}$, where $\mathbb{C}$ is the trivial module. If $s_1 \neq 0$, we choose $\sqrt{s_1}, \sqrt{s_2}$ so that $\sqrt{s_2} = \sqrt{s_1}\mathbf{i}$. Note that the choice of sign controls the choice of the parity of $V(\mathfrak{s})$. The following exact sequence easily follows from (3) and (4):

$$0 \to \Pi\Gamma_{-s_1^2+s_1} \to V(\mathfrak{s}) \to \Gamma_{-s_1^2-s_1} \to 0, \tag{5}$$

where Γ_t is the simple module of dimension $(1|0)$ on which ϕ_0, ϕ_1 and z_0 act by zero and z_1 acts by the scalar t. The sequence splits only in the case $s_1 = 0$, when $\Gamma_0 \simeq \mathbb{C}$ is trivial. Thus, using Proposition 1, Theorem 2 and (5) we obtain
Lemma 3. If $n = 2$, then every simple W-module is isomorphic to one of the following
(1) $V(s_1, s_2)$ or $\Pi V(s_1, s_2)$ for $s_1 \neq -s_2$, $s_1, s_2 \neq 0$;
(2) $V(s, 0)$ if $s \neq 0$;
(3) Γ_t or $\Pi\Gamma_t$.

3.4. Invariance under permutations
Theorem 4 ([15], Theorem 4.4). Let $\mathfrak{s}' = \sigma(\mathfrak{s})$ for some permutation of coordinates.
(1) If $\mathfrak{s}$ is typical, then $V(\mathfrak{s})$ is isomorphic to $V(\mathfrak{s}')$ as a W-module.
(2) If $\mathfrak{s}$ is arbitrary, then $[V(\mathfrak{s})] = [V(\mathfrak{s}')]$ or $[\Pi V(\mathfrak{s}')]$, where $[X]$ denotes the class of X in the Grothendieck group.
Proof. We will prove the statement for $n = 2$. Assume first that $s_2 \neq -s_1$. In this case $V(s_1, s_2)$ is a $(1|1)$-dimensional simple W-module.

Let

$$D = \begin{pmatrix} \sqrt{s_2}+\sqrt{s_1}\mathbf{i} & 0 \\ 0 & \sqrt{s_1}+\sqrt{s_2}\mathbf{i} \end{pmatrix}.$$

Then by direct computation we have

$$D\phi_0D^{-1} = \begin{pmatrix} 0 & \sqrt{s_2}+\sqrt{s_1}\mathbf{i} \\ \sqrt{s_2}-\sqrt{s_1}\mathbf{i} & 0 \end{pmatrix} \text{ and } D\phi_1D^{-1} = \sqrt{s_1s_2}\begin{pmatrix} 0 & \sqrt{s_1}-\sqrt{s_2}\mathbf{i} \\ \sqrt{s_1}+\sqrt{s_2}\mathbf{i} & 0 \end{pmatrix}.$$

Therefore D defines an isomorphism between $V(s_1, s_2)$ and $V(s_2, s_1)$.

Now consider the case $s_1 = -s_2$. Then the structure of $V(s_1, -s_1)$ is given by the exact sequence (5). Let $V(\mathfrak{s}') = V(-s_1, s_1)$, then analogously we have the exact sequence

$$0 \to \Pi\Gamma_{-s_1^2-s_1} \to V(\mathfrak{s}') \to \Gamma_{-s_1^2+s_1} \to 0. \tag{6}$$

The statement (2) now follows directly from comparison of (5) and (6). The proof for an arbitrary n see in [15].

□

4. Central characters

The center of $U(\mathfrak{g})$ for $\mathfrak{g} = Q(n)$ is described in [18]. The center of $U(\mathfrak{h})$ coincides with $\mathbb{C}[x_1, \ldots, x_n]$ and the image of the center of $U(\mathfrak{g})$ under the Harish-Chandra homomorphism is generated by the polynomials $p_k = x_1^{2k+1} + \ldots + x_n^{2k+1}$ for all $k \in \mathbb{N}$. These polynomials are called Q-symmetric polynomials.

In [13] we proved that the center Z of W coincides with the image of the center of $U(\mathfrak{g})$ and hence can be also identified with the ring of Q-symmetric polynomials.

Every $\mathfrak{s}$ defines the central character $\chi_{\mathfrak{s}} : Z \to \mathbb{C}$. Furthermore, it follows from the description of simple W-modules in [15] (Theorem 4.6) that every simple W-module admits central character $\chi_{\mathfrak{s}}$ for some $\mathfrak{s}$. For every $\mathfrak{s} = (s_1, \dots, s_n)$ we define the *core* $c(\mathfrak{s}) = (s_{i_1}, \dots, s_{i_m})$ as a subsequence obtained from $\mathfrak{s}$ by removing all $s_j = 0$ and all pairs (s_i, s_j) such that $s_i + s_j = 0$. Up to a permutation this result does not depend on the order of removing. Thus, the core is well defined up to permutation. We call m the length of the core. The notion of core is very useful for describing the blocks in the category of finite-dimensional $Q(n)$-modules, see [12, 17].

Example 5. Let $\mathfrak{s} = (1, 0, 3, -1, -1)$, then $c(\mathfrak{s}) = (3, -1)$.

The following is a reformulation of the central character description in [18].

Lemma 6. Let $\mathfrak{s}, \mathfrak{s}' \in \mathbb{C}^n$. Then $\chi_{\mathfrak{s}} = \chi_{\mathfrak{s}'}$ if and only if $\mathfrak{s}$ and $\mathfrak{s}'$ have the same core (up to permutation).

It follows from Lemma 6 that the core depends only on the central character $\chi_{\mathfrak{s}}$, we denote it $c(\chi)$.

5. The category of finite-dimensional W-modules and blocks

Let $W-\mathrm{mod}$ be the *category of finite-dimensional W-modules.* A W-module M has *generalized central character* χ, if for any $z \in Z$ and $m \in M$, there exists $n \in \mathbb{Z}_{\geq 0}$ such that $(z - \chi(z))^n \cdot m = 0$. Let $W^\chi - \mathrm{mod}$ be the full *subcategory of modules admitting generalized central character* χ. The category $W - \mathrm{mod}$ is the direct sum of the subcategories $W^\chi - \mathrm{mod}$, as χ ranges over the central characters.

The *blocks* in the category $W - \mathrm{mod}$ are equivalence class of linked objects. Each block lies in a single $W^\chi - \mathrm{mod}$, however, different blocks can belong to the same $W^\chi - \mathrm{mod}$, see [5].

5.1. Blocks in the category of finite-dimensional W-modules for $Q(2)$

Lemma 7. Let $n = 2$. A simple W-module S belongs to $W^\chi - \mathrm{mod}$ if and only if one of the following three cases takes place:

(1) $S \simeq V(s_1, s_2)$ for $s_1 \neq s_2, s_1, s_2 \neq 0$ and $c(\chi) = (s_1, s_2)$,

(2) $S \simeq V(s, 0)$ for $s \neq 0$ and $c(\chi) = (s)$,

(3) $S \simeq \Gamma_t$ or $\Pi\Gamma_t$ and $\chi = 0$.

Proof. We have to compute the central character of the simple W-module. For a Q-symmetric polynomial $p_k = x_1^{2k+1} + x_2^{2k+1}$ we have that

$$p_k(S) = \begin{cases} s_1^{2k+1} + s_2^{2k+1} & \text{if } (1) \\ s^{2k+1} & \text{if } (2) \\ 0 & \text{if } (3) \end{cases}$$

Since p_k generate the center of W the statement follows.

□

Theorem 8. (1) Each simple W-module $V(s_1, s_2)$ for $s_1 \neq s_2, s_1, s_2 \neq 0$ forms a block in W^χ-mod, where $c(\chi) = (s_1, s_2)$.

(2) Each simple W-module $V(s, 0)$ for $s \neq 0$ forms a block in W^χ-mod, where $c(\chi) = (s)$.

(3) The blocks in the subcategory W^χ-mod, where $\chi = 0$, are described as follows. Let $a \in \mathbb{C}$. Define

$$a_n = a - n^2 + n\sqrt{1 - 4a} \text{ for } n = 0, \pm 1, \pm 2, \dots \tag{7}$$

Then Γ_a lies in the block formed by Γ_{a_n} if n is even and $\Pi\Gamma_{a_n}$, if n is odd. $\Pi\Gamma_a$ lies in the block formed by $\Pi\Gamma_{a_n}$ if n is even and Γ_{a_n}, if n is odd.

Proof. Statements (1) and (2) follow from Lemma 6 and Lemma 7. To prove (3), first we will show that Γ_a is linked to $\Pi\Gamma_b$ if and only if

$$b = a - 1 \pm \sqrt{1-4a}. \tag{8}$$

Recall that W is generated by ϕ_0, ϕ_1, z_0 and z_1, where

$$\phi_0 = \xi_1 + \xi_2,\ \phi_1 = x_2\xi_1 - x_1\xi_2,\ z_0 = x_1 + x_2,\ z_1 = x_1x_2 - \xi_1\xi_2.$$

We have

$$[z_1, \phi_1] = 2z_1\phi_0 + 2\phi_1, \tag{9}$$

$$[z_1, \phi_0] = -2\phi_1. \tag{10}$$

Suppose that Γ_a is linked to $\Pi\Gamma_b$. The generators ϕ_0 and z_1 act in the vector superspace $\Gamma_a \oplus \Pi\Gamma_b$ as follows:

$$\phi_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad z_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Then from (10)

$$\phi_1 = \begin{pmatrix} 0 & \frac{b-a}{2} \\ 0 & 0 \end{pmatrix} \text{ and hence } [z_1, \phi_1] = \begin{pmatrix} 0 & \frac{(a-b)(b-a)}{2} \\ 0 & 0 \end{pmatrix}.$$

On the other hand, from (9)

$$[z_1, \phi_1] = 2\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 0 & \frac{b-a}{2} \\ 0 & 0 \end{pmatrix}.$$

Hence

$$\frac{(a-b)(b-a)}{2} = 2a + b - a.$$

Then

$$b^2 + (2-2a)b + (a^2 + 2a) = 0. \tag{11}$$

This implies (8).

Conversely, if $a \neq 0$, set $s_1 = \frac{1-\sqrt{1-4a}}{2}$ and consider $V(-s_1, s_1)$ $(s_1 \neq 0)$. Recall that we have the non-split exact sequence (6):

$$0 \to \Pi\Gamma_{-s_1^2-s_1} \to V(-s_1, s_1) \to \Gamma_{-s_1^2+s_1} \to 0,$$

which becomes

$$0 \to \Pi\Gamma_b \to V(-s_1, s_1) \to \Gamma_a \to 0, \tag{12}$$

with $b = a - 1 + \sqrt{1-4a}$.
If we set $s_1 = \frac{1+\sqrt{1-4a}}{2}$, we obtain (12) with $b = a - 1 - \sqrt{1-4a}$.
If $a = 0$, set $s_1 = 1$ in (12). Then

$$0 \to \Pi\Gamma_{-2} \to V(-1, 1) \to \Gamma_0 \to 0.$$

Let V be a $(1|1)$-dimensional module on which z_0, z_1 and ϕ_1 act by zero and ϕ_0 acts by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$0 \to \Gamma_0 \to V \to \Pi\Gamma_0 \to 0.$$

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Finally, the sum of the roots of equation (11) is $2a - 2$. This gives the relation

$$a_{n-1} + a_{n+1} = 2a_n - 2 \quad (a_n = a). \tag{13}$$

Then (8) and (13) imply (7).

□

Example 9.
(1) $a = 0$, then $a_n = n(1 - n)$ and Γ_0 lies in the block

$$\ldots, \Gamma_{-30}, \Pi\Gamma_{-20}, \Gamma_{-12}, \Pi\Gamma_{-6}, \Gamma_{-2}, , \Pi\Gamma_0, \Gamma_0, \Pi\Gamma_{-2}, \Gamma_{-6}, \Pi\Gamma_{-12}, \Gamma_{-20}, \Pi\Gamma_{-30}, \ldots$$

(2) $a = \frac{1}{4}$, then $a_n = \frac{1}{4} - n^2$ and $\Gamma_{\frac{1}{4}}$ lies in the block

$$\Gamma_{\frac{1}{4}}, \Pi\Gamma_{-\frac{3}{4}}, \Gamma_{-\frac{15}{4}}, \ldots$$

(3) $a = 1$, then $a_n = 1 - n^2 + n\sqrt{-3}$ and Γ_1 lies in the block

$$\ldots, \Pi\Gamma_{-3\sqrt{-3}-8}, \Gamma_{-2\sqrt{-3}-3}, \Pi\Gamma_{-\sqrt{-3}}, \Gamma_1, \Pi\Gamma_{\sqrt{-3}}, \Gamma_{2\sqrt{-3}-3}, \Pi\Gamma_{3\sqrt{-3}-8}, \ldots$$

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