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On the finite W-algebra for the Lie superalgebra Q(N) in the non-regular case

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ON THE FINITE W-ALGEBRA FOR THE LIE SUPERALGEBRA $Q(N)$ IN THE NON-REGULAR CASE

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Abstract. In this paper we study the finite W-algebra for the queer Lie superalgebra $Q(n)$ associated with the non-regular even nilpotent coadjoint orbits in the case when the corresponding nilpotent element has Jordan blocks each of size l. We prove that this finite W-algebra is isomorphic to a quotient of the super-Yangian of $Q(\frac{n}{l})$

1. Introduction

A finite W-algebra is a certain associative algebra attached to a pair $(\mathfrak{g}, e)$ where $\mathfrak{g}$ is a complex semisimple Lie algebra and $e \in \mathfrak{g}$ is a nilpotent element. Geometrically a finite W algebra is a quantization of the Poisson structure on the so-called Slodowy slice (a transversal slice to the orbit of e in the adjoint representation).

In the case when $e = 0$ the finite W-algebra coincides with the universal enveloping algebra $U(\mathfrak{g})$ and in the case when e is a *regular* nilpotent element, the corresponding W-algebra coincides with the center of $U(\mathfrak{g})$. The latter case was studied by B. Kostant [6] who was motivated by applications to generalized Toda lattices. The general definition of a finite W-algebra was given by A. Premet in [15] (see also [7]). In the case of Lie superalgebras, finite W-algebras have been extensively studied by mathematicians and physicists in [1, 2, 10-14, 18-20].

E. Ragoucy and P. Sorba first observed that in the case when $\mathfrak{g}$ is the general linear Lie algebra and e consists of n Jordan blocks each of size l, the finite W-algebra for $\mathfrak{g}$ is isomorphic to the truncated Yangian of level l associated to $\mathfrak{gl}(n)$, which is a certain quotient of the Yangian Y_n for $\mathfrak{gl}(n)$ [16]. J. Brundan and A. Kleshchev generalized this result to an arbitrary nilpotent e, and obtained a realization of the finite W-algebra for the general linear Lie algebra as a quotient of a so-called shifted Yangian [3] (see also [4]).

For the general linear Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m|n)$, a connection between finite W-algebras for $\mathfrak{g}$ and super-Yangians was firstly observed by C. Briot and E. Ragoucy [1]. In a more recent article, J. Brown, J. Brundan and S. Goodwin described principal finite W-algebras for $\mathfrak{gl}(m|n)$ associated to regular (principal) nilpotent e as truncations of shifted super-Yangians of $\mathfrak{gl}(1|1)$ [2]. After that, Y. Peng described the finite W-algebra for $\mathfrak{g} = \mathfrak{gl}(m|n)$ associated to an e in the case when the Jordan type of e satisfies the following condition: $e = e_m \oplus e_n$, where e_m is principal nilpotent in $\mathfrak{gl}(m|0)$ and the sizes of the Jordan blocks of e_n are all greater or equal to m [10].

For a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a reductive even part $\mathfrak{g}_{\bar{0}}$ we denote by W_χ the finite W-algebra associated to an even nilpotent element $\chi \in \mathfrak{g}_{\bar{0}}^* \subset \mathfrak{g}^*$ in the coadjoint representation. If $\mathfrak{g}$ admits an even non-degenerate invariant form, then $\mathfrak{g} \cong \mathfrak{g}^*$ and $\chi(x) = (e|x)$ for some nilpotent $e \in \mathfrak{g}_{\bar{0}}$. If $\mathfrak{g}$ is the queer Lie superalgebra $Q(n)$, then it admits an odd non-degenerate invariant form. In this case $\mathfrak{g} \cong \Pi(\mathfrak{g}^*)$ and $\chi(x) = (E|x)$ for some nilpotent $E \in \mathfrak{g}_{\bar{1}}$.

In [12] we studied in detail $Q(n)$ in the regular case. In particular, we proved that W_χ for $Q(n)$ associated to a regular nilpotent χ is isomorphic to a quotient of the super-Yangian of $Q(1)$ (Theorem 6.2). An interesting problem is to extend this result to $Q(n)$ associated to an *arbitrary* even nilpotent χ.

In this work we consider the case when the corresponding nilpotent element has Jordan blocks each of size l. We construct a set of generators of W_χ (Theorem 5.1) and prove that W_χ is isomorphic to a quotient of the super-Yangian of $Q(\frac{n}{l})$ (Theorem 4.2 and Corollary 5.9). This proves the conjecture, which we formulated in [14].

A. Premet has proved that if $\mathfrak{g}$ is a semi-simple Lie algebra, then the associated graded algebra $Gr_K W_\chi$ with respect to the Kazhdan filtration is isomorphic to $S(\mathfrak{g}^\chi)$ (the symmetric algebra of the annihilator $\mathfrak{g}^\chi$ of χ in $\mathfrak{g}$) (see [15]).

In [12] we formulated the following conjecture (Conjecture 2.8): Assume that $\mathfrak{g}$ is a Lie superalgebra with a reductive even part $\mathfrak{g}_{\bar{0}}$ endowed with a $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$, which is good for χ. Then if $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even, we have that $Gr_K W_\chi \simeq S(\mathfrak{g}^\chi)$ and if $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is odd, then $Gr_K W_\chi \simeq S(\mathfrak{g}^\chi) \otimes \mathbb{C}[\xi]$, where $\mathbb{C}[\xi]$ is the exterior algebra generated by one element ξ.

For $\mathfrak{g} = \mathfrak{gl}(m|n)$ and a regular nilpotent χ this conjecture is proven in [2]. Recently Y. Zeng and B. Shu have proved this conjecture for a basic Lie superalgebra $\mathfrak{g}$ over $\mathbb{C}$ of any type except $D(2,1;\alpha)$, where $\alpha \notin \bar{\mathbb{Q}}$ ([20], Theorem 0.1). In [13] it is proven for $D(2,1;\alpha)$ and a regular nilpotent χ. We proved this conjecture for $\mathfrak{g} = Q(n)$ in the regular case in [12] (Corollary 4.9). It follows from our results, that the conjecture is also true in the case that we consider in this paper (Corollary 5.5).

Notice that in Theorem 4.2 we realized the finite W-algebra for $Q(n)$ inside $U(Q(\frac{n}{l}))^{\otimes l}$ as

$$W_\chi \cong U^{\otimes l} \circ \Delta_l^{op}(Y(Q(\frac{n}{l}))),$$

where Δ^{op} is the opposite comultiplication in $Y(Q(\frac{n}{l}))$ and U is its homomorphism into $U(Q(\frac{n}{l}))$ defined in [9]. Then we obtained a realization of the same subspace as

$$W_\chi \cong ev^{\otimes l} \circ \Delta_l(Y(Q(\frac{n}{l}))),$$

where Δ is the comultiplication and ev is the evaluation homomorphism (Corollary 5.12). Note that the latter realization is in the spirit of [3] and [2], where the authors used (shifted) truncated (super)-Yangians.

2. Finite W-algebras for Lie superalgebras

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra with reductive even part $\mathfrak{g}_{\bar{0}}$. Let $\chi \in \mathfrak{g}_{\bar{0}}^* \subset \mathfrak{g}^*$ be an even nilpotent element in the coadjoint representation, i.e. the closure of the $G_{\bar{0}}$-orbit of χ in $\mathfrak{g}_{\bar{0}}^*$ (where $G_{\bar{0}}$ is the algebraic reductive group of $\mathfrak{g}_{\bar{0}}$) contains zero.

Definition 2.1. *The annihilator of* χ *in* $\mathfrak{g}$ is

$$\mathfrak{g}^\chi = \{x \in \mathfrak{g} \mid \chi([x, \mathfrak{g}]) = 0\}.$$

Definition 2.2. *A good* $\mathbb{Z}$*-grading for* χ is a $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ satisfying the following two conditions:

(1) $\chi(\mathfrak{g}_j) = 0$ if $j \neq -2$;
(2) $\mathfrak{g}^\chi$ belongs to $\bigoplus_{j \geq 0} \mathfrak{g}_j$.

Note that $\chi([\cdot, \cdot])$ defines a non-degenerate skew-symmetric even bilinear form on $\mathfrak{g}_{-1}$. Let $\mathfrak{l}$ be a maximal isotropic subspace with respect to this form. We consider a nilpotent subalgebra $\mathfrak{m} = (\oplus_{j \leq -2} \mathfrak{g}_j) \bigoplus \mathfrak{l}$ of $\mathfrak{g}$. The restriction of χ to $\mathfrak{m}$, $\chi : \mathfrak{m} \longrightarrow \mathbb{C}$, defines a one-dimensional representation $\mathbb{C}_\chi =< v >$ of $\mathfrak{m}$. Let I_χ be the left ideal of $U(\mathfrak{g})$ generated by $a - \chi(a)$ for all $a \in \mathfrak{m}$.

Definition 2.3. The induced $\mathfrak{g}$-module

$$Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi \cong U(\mathfrak{g})/I_\chi$$

is called *the generalized Whittaker module.*

Definition 2.4. *The finite* W*-algebra* associated to the nilpotent element χ is

$$W_\chi := \mathrm{End}_{U(\mathfrak{g})}(Q_\chi)^{op}.$$

As in the Lie algebra case, the superalgebras W_χ are all isomorphic for different choices of good $\mathbb{Z}$-gradings and maximal isotropic subspaces $\mathfrak{l}$ [19].
If $\mathfrak{g}$ admits an even non-degenerate $\mathfrak{g}$-invariant supersymmetric bilinear form, then $\mathfrak{g} \simeq \mathfrak{g}^*$ and $\chi(x) = (e|x)$ for some nilpotent $e \in \mathfrak{g}_{\bar{0}}$ (i.e. $\mathrm{ad}e$ is a nilpotent endomorphism of $\mathfrak{g}$). By the Jacobson–Morozov theorem e can be included in $\mathfrak{sl}(2) =< e, h, f >$. As in the Lie algebra case, the linear operator $\mathrm{ad}h$ defines a Dynkin $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$, where

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \mid \mathrm{ad}h(x) = jx\}.$$

As follows from the representation theory of $\mathfrak{sl}(2)$, the Dynkin $\mathbb{Z}$-grading is good for χ. Let $\mathfrak{g}^e := \mathrm{Ker}(\mathrm{ad}e)$. Clearly, $\mathfrak{g}^e = \mathfrak{g}^\chi$. Note that as in the Lie algebra case, $\dim \mathfrak{g}^e = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1$.

Note that by Frobenius reciprocity

$$\mathrm{End}_{U(\mathfrak{g})}(Q_\chi) = \mathrm{Hom}_{U(\mathfrak{m})}(\mathbb{C}_\chi, Q_\chi).$$

That defines an identification of W_χ with the subspace

$$Q_\chi^{\mathfrak{m}} = \{u \in Q_\chi \mid au = \chi(a)u \text{ for all } a \in \mathfrak{m}\}.$$

In what follows we denote by $\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/I_\chi$ the natural projection. By above

$$W_\chi = \{\pi(y) \in U(\mathfrak{g})/I_\chi \mid (a - \chi(a))y \in I_\chi \text{ for all } a \in \mathfrak{m}\},$$

or, equivalently,

$$W_\chi = \{\pi(y) \in U(\mathfrak{g})/I_\chi \mid \operatorname{ad}(a)y \in I_\chi \text{ for all } a \in \mathfrak{m}\}. \tag{2.1}$$

The algebra structure on W_χ is given by

$$\pi(y_1)\pi(y_2) = \pi(y_1y_2)$$

for $y_i \in U(\mathfrak{g})$ such that $\operatorname{ad}(a)y_i \in I_\chi$ for all $a \in \mathfrak{m}$ and $i = 1, 2$.

Definition 2.5. A nilpotent $\chi \in \mathfrak{g}_{\bar 0}^*$ is called *regular* nilpotent if $G_{\bar 0}$-orbit of χ has maximal dimension, i.e. the dimension of $\mathfrak{g}_{\bar 0}^\chi$ is minimal. (Equivalently, a nilpotent $e \in \mathfrak{g}_{\bar 0}$ is *regular* nilpotent, if $\mathfrak{g}_{\bar 0}^e$ attains the minimal dimension, which is equal to $\operatorname{rank}\mathfrak{g}_{\bar 0}$.)

Theorem 2.6. *(B. Kostant, [6]) For a reductive Lie algebra $\mathfrak{g}$ and a regular nilpotent element $e \in \mathfrak{g}$, the finite W-algebra W_χ is isomorphic to the center of $U(\mathfrak{g})$.*

This theorem does not hold for Lie superalgebras, since W_χ must have a non-trivial odd part, and the center of $U(\mathfrak{g})$ is even.

Definition 2.7. Define a $\mathbb{Z}$-grading on $T(\mathfrak{g})$ by setting the degree of $g \in \mathfrak{g}_j$ to be $j + 2$. This induces a filtration on $U(\mathfrak{g})$ and therefore on $U(\mathfrak{g})/I_\chi$ which is called the *Kazhdan filtration*. We will denote by Gr_K the corresponding graded algebras. Since by (2.1) $W_\chi \subset U(\mathfrak{g})/I_\chi$, we have an induced filtration on W_χ.

Theorem 2.8. *(A. Premet, [15]) Let $\mathfrak{g}$ be a semi-simple Lie algebra. Then the associated graded algebra $Gr_K W_\chi$ is isomorphic to $S(\mathfrak{g}^\chi)$.*

To generalize this result to the super case, we assume that $\mathfrak{l}'$ is some subspace in $\mathfrak{g}_{-1}$ satisfying the following two properties:

(1) $\mathfrak{g}_{-1} = \mathfrak{l} \oplus \mathfrak{l}'$,

(2) $\mathfrak{l}'$ contains a maximal isotropic subspace with respect to the form defined by $\chi([\cdot,\cdot])$ on $\mathfrak{g}_{-1}$.

If $\dim(\mathfrak{g}_{-1})_{\bar 1}$ is even, then $\mathfrak{l}'$ is a maximal isotropic subspace. If $\dim(\mathfrak{g}_{-1})_{\bar 1}$ is odd, then $\mathfrak{l}^\perp \cap \mathfrak{l}'$ is one-dimensional and we fix $\theta \in \mathfrak{l}^\perp \cap \mathfrak{l}'$ such that $\chi([\theta, \theta]) = 2$. It is clear that $\pi(\theta) \in W_\chi$ and $\pi(\theta)^2 = 1$.

Let $\mathfrak{p} = \bigoplus_{j \geq 0} \mathfrak{g}_j$. By the PBW theorem, $U(\mathfrak{g})/I_\chi \simeq S(\mathfrak{p} \oplus \mathfrak{l}')$ as a vector space. Therefore $Gr_K(U(\mathfrak{g})/I_\chi)$ is isomorphic to $S(\mathfrak{p} \oplus \mathfrak{l}')$ as a vector space. The good $\mathbb{Z}$-grading of $\mathfrak{g}$ induces the grading on $S(\mathfrak{p} \oplus \mathfrak{l}')$. For any $X \in U(\mathfrak{g})/I_\chi$ let $Gr_K(X)$ denote the corresponding element in $Gr_K(U(\mathfrak{g})/I_\chi)$, and $P(X)$ denote the highest weight component of $Gr_K(X)$ in this $\mathbb{Z}$-grading.

Theorem 2.9. *([12], Proposition 2.7) Let $y_1, \dots, y_p$ be a basis in $\mathfrak{g}^\chi$ homogeneous in the good $\mathbb{Z}$-grading. Assume that there exist $Y_1, \dots, Y_p \in W_\chi$ such that $P(Y_i) = y_i$ for all $i = 1, \dots, p$. Then*
(a) if $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even, then $Y_1, \dots, Y_p$ generate W_χ, and if $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is odd, then $Y_1, \dots, Y_p$ and $\pi(\theta)$ generate W_χ;
(b) if $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even, then $Gr_K W_\chi \simeq S(\mathfrak{g}^\chi)$, and if $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is odd, then $Gr_K W_\chi \simeq S(\mathfrak{g}^\chi) \otimes \mathbb{C}[\xi]$, where $\mathbb{C}[\xi]$ is the exterior algebra generated by one element ξ.

3. The queer superalgebra $Q(n)$

Recall that the *queer* Lie superalgebra is defined as follows

$$Q(n) := \{\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) \mid A, B \text{ are } n \times n \text{ matrices}\}.$$

Let $\mathrm{otr}\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) = \mathrm{tr} B$.

Remark 3.1. $Q(n)$ has one-dimensional center $< z >$, where $z = 1_{2n}$. Let

$$SQ(n) = \{X \in Q(n) \mid \mathrm{otr} X = 0\}.$$

The Lie superalgebra $\tilde{Q}(n) := SQ(n)/ < z >$ is simple for $n \geq 3$, see [5].

Note that $\mathfrak{g} = Q(n)$ admits an *odd* non-degenerate $\mathfrak{g}$-invariant supersymmetric bilinear form

$$(x|y) := \mathrm{otr}(xy) \text{ for } x, y \in \mathfrak{g}.$$

Therefore, we identify the coadjoint module $\mathfrak{g}^*$ with $\Pi(\mathfrak{g})$, where Π is the change of parity functor.
Let $e_{i,j}$ and $f_{i,j}$ be standard bases in $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ respectively:

$$e_{i,j} = \left(\begin{array}{c|c} E_{ij} & 0 \\ \hline 0 & E_{ij} \end{array}\right), \quad f_{i,j} = \left(\begin{array}{c|c} 0 & E_{ij} \\ \hline E_{ij} & 0 \end{array}\right),$$

where E_{ij} are elementary $n \times n$ matrices.

Let $\mathfrak{sl}(2) =< e, h, f >$, where

$$e = \sum_{p=1}^{\frac{n}{l}} \sum_{i=1}^{l-1} e_{l(p-1)+i,l(p-1)+i+1},$$

$$f = \sum_{p=1}^{\frac{n}{l}} \sum_{i=1}^{l-1} i(l-i) e_{l(p-1)+i+1,l(p-1)+i},$$

$$h = \sum_{p=1}^{\frac{n}{l}} \sum_{i=1}^{l} (l-2i+1) e_{l(p-1)+i,l(p-1)+i}.$$

Note that e is an even nilpotent element in $Q(n)$, which is represented by a nilpotent $n \times n$-matrix whose Jordan blocks are all of the same size l. Note also that ad$h$ defines an even Dynkin $\mathbb{Z}$-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{k=1-l}^{l-1} \mathfrak{g}_{2k},$$

where

$$\mathfrak{g}_{2k} =< e_{l(p-1)+i,l(q-1)+i+k} \mid f_{l(p-1)+i,l(q-1)+i+k} >,$$
$$\mathfrak{g}_{-2k} =< e_{l(p-1)+i+k,l(q-1)+i} \mid f_{l(p-1)+i+k,l(q-1)+i} >,$$
$$\text{where } k = 0, 1, \ldots, l-1, i = 1, \ldots, l-k \text{ and } 1 \leq p, q \leq \frac{n}{l}.$$

Let $E = \sum_{p=1}^{\frac{n}{l}} \sum_{i=1}^{l-1} f_{l(p-1)+i,l(p-1)+i+1}$. Since we have an isomorphism $\mathfrak{g}^* \simeq \Pi(\mathfrak{g})$, an even nilpotent $\chi \in \mathfrak{g}^*$ can be defined by $\chi(x) := (x|E)$ for $x \in \mathfrak{g}$. Note that the Dynkin $\mathbb{Z}$-grading is good for χ. We have that

$$\mathfrak{g}^\chi = \mathfrak{g}^E =< \sum_{i=1}^{l-k} e_{l(p-1)+i,l(q-1)+i+k} \mid \sum_{i=1}^{l-k} (-1)^{i+k-1} f_{l(p-1)+i,l(q-1)+i+k} >,$$
$$\text{where } k = 0, 1, \ldots, l-1, \text{ and } 1 \leq p, q \leq \frac{n}{l}.$$

Thus $\dim(\mathfrak{g}^E) = (\frac{n^2}{l}|\frac{n^2}{l})$. Note that as in the Lie algebra case, $\dim(\mathfrak{g}^E) = \dim \mathfrak{g}_0$, since the $\mathbb{Z}$-grading is even, and $\mathfrak{g}^E \subseteq \bigoplus_{k=0}^{l-1} \mathfrak{g}_{2k}$. Note also that χ is regular nilpotent if and only if e has a single Jordan block, i.e. $l = n$.

Let

$$\mathfrak{m} = \bigoplus_{j=1}^{l-1} \mathfrak{g}_{-2j}.$$

Note that $\mathfrak{m}$ is generated by $e_{l(p-1)+i+1,l(q-1)+i}$ and $f_{l(p-1)+i+1,l(q-1)+i}$, where $i = 1,\dots,l-1$, and $1 \le p,q \le \frac{n}{l}$. We have that

$$\begin{aligned} &\chi(e_{l(p-1)+i+1,l(q-1)+i}) = \delta_{p,q}, \text{ for } i = 1,\dots,l-1, \\ &\chi(e_{l(p-1)+i+k,l(q-1)+i}) = 0 \text{ for } k \ge 2, i = 1,\dots,l-k, \\ &\chi(f_{l(p-1)+i+k,l(q-1)+i}) = 0 \text{ for } k \ge 1, i = 1,\dots,l-k. \end{aligned} \tag{3.1}$$

The left ideal I_χ and W_χ are defined now as usual. Moreover,

$$\mathfrak{p} := \bigoplus_{j=0}^{l-1} \mathfrak{g}_{2j}$$

is a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{n}$, where

$$\mathfrak{n} := \bigoplus_{j=1}^{l-1} \mathfrak{g}_{2j}.$$

Note that since the $\mathbb{Z}$-grading of $\mathfrak{g}$ is even, then the algebra W_χ can be regarded as a *subalgebra* of $U(\mathfrak{p})$. In fact, from the PBW theorem,

$$U(\mathfrak{g}) = U(\mathfrak{p}) \oplus I_\chi.$$

The projection $pr : U(\mathfrak{g}) \longrightarrow U(\mathfrak{p})$ along this direct sum decomposition induces an isomorphism: $U(\mathfrak{g})/I_\chi \xrightarrow{\sim} U(\mathfrak{p})$. Let

$$U(\mathfrak{p})^+ := \oplus_{i>0} U(\mathfrak{p})_{2i}.$$

It is a two sided ideal in $U(\mathfrak{p})$ and $U(\mathfrak{p})/U(\mathfrak{p})^+ \cong U(\mathfrak{g}_0)$. Let $\vartheta : U(\mathfrak{p}) \longrightarrow U(\mathfrak{g}_0)$ be the natural projection. Its restriction to W_χ is the *Harish-Chandra homomorphism*

$$\vartheta : W_\chi \longrightarrow U(\mathfrak{g}_0),$$

which is injective by [12] (Theorem 3.1).

4. Super-Yangian of $Q(n)$

Super-Yangian $Y(Q(n))$ was introduced by M. Nazarov in [8]. Recall that $Y(Q(n))$ is the associative unital superalgebra over $\mathbb{C}$ with the countable set of generators

$$T_{i,j}^{(m)} \text{ where } m = 1,2,\dots \text{ and } i,j = \pm1,\pm2,\dots,\pm n.$$

The $\mathbb{Z}_2$-grading of the algebra $Y(Q(n))$ is defined as follows:

$$p(T_{i,j}^{(m)}) = p(i) + p(j), \text{ where } p(i) = 0 \text{ if } i > 0, \text{ and } p(i) = 1 \text{ if } i < 0.$$

To write down defining relations for these generators we employ the formal series in $Y(Q(n))[[u^{-1}]]$:

$$T_{i,j}(u) = \delta_{i,j} \cdot 1 + T_{i,j}^{(1)} u^{-1} + T_{i,j}^{(2)} u^{-2} + \dots. \tag{4.1}$$

Then for all possible indices i,j,k,l we have the relations

$$\begin{aligned}(u^2-v^2)[T_{i,j}(u),T_{k,l}(v)]\cdot(-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)}\\ =(u+v)(T_{k,j}(u)T_{i,l}(v)-T_{k,j}(v)T_{i,l}(u))\\ -(u-v)(T_{-k,j}(u)T_{-i,l}(v)-T_{k,-j}(v)T_{i,-l}(u))\cdot(-1)^{p(k)+p(l)},\end{aligned}\tag{4.2}$$

where v is a formal parameter independent of u, so that (4.2) is an equality in the algebra of formal Laurent series in u^{-1}, v^{-1} with coefficients in $Y(Q(n))$.
For all indices i, j we also have the relations

$$T_{i,j}(-u)=T_{-i,-j}(u).\tag{4.3}$$

Note that the relations (4.2) and (4.3) are equivalent to the following defining relations:

$$\begin{aligned}&([T_{i,j}^{(m+1)},T_{k,l}^{(r-1)}]-[T_{i,j}^{(m-1)},T_{k,l}^{(r+1)}])\cdot(-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)}=\\ &T_{k,j}^{(m)}T_{i,l}^{(r-1)}+T_{k,j}^{(m-1)}T_{i,l}^{(r)}-T_{k,j}^{(r-1)}T_{i,l}^{(m)}-T_{k,j}^{(r)}T_{i,l}^{(m-1)}\\ &+(-1)^{p(k)+p(l)}(-T_{-k,j}^{(m)}T_{-i,l}^{(r-1)}+T_{-k,j}^{(m-1)}T_{-i,l}^{(r)}+T_{k,-j}^{(r-1)}T_{i,-l}^{(m)}-T_{k,-j}^{(r)}T_{i,-l}^{(m-1)}),\end{aligned}\tag{4.4}$$

$$T_{-i,-j}^{(m)}=(-1)^m T_{i,j}^{(m)},\tag{4.5}$$

where $m, r = 1, \dots$ and $T_{i,j}^{(0)} = \delta_{i,j}$.

Recall that $Y(Q(n))$ is a Hopf superalgebra with comultiplication given by the formula

$$\Delta(T_{i,j}^{(r)})=\sum_{s=0}^{r}\sum_{k}(-1)^{(p(i)+p(k))(p(j)+p(k))}T_{i,k}^{(s)}\otimes T_{k,j}^{(r-s)}.\tag{4.6}$$

The opposite comultiplication is given by

$$\Delta^{op}(T_{i,j}^{(r)})=\sum_{s=0}^{r}\sum_{k}T_{k,j}^{(r-s)}\otimes T_{i,k}^{(s)}.\tag{4.7}$$

Combine the series (4.1) into the single element

$$T(u)=\sum_{i,j}E_{i,j}\otimes T_{i,j}(u)$$

of the algebra $\mathrm{End}(\mathbb{C}^{n|n})\otimes Y(Q(n))[[u^{-1}]]$. The element $T(u)$ is invertible and we put

$$T(u)^{-1}=\sum_{i,j}E_{i,j}\otimes\tilde{T}_{i,j}(u).\tag{4.8}$$

The assignment $T_{i,j}(u) \mapsto \tilde{T}_{i,j}(u)$ defines the *antipodal map*

$$S : Y(Q(n)) \longrightarrow Y(Q(n)), \tag{4.9}$$

which is an anti-automorphism of the $\mathbb{Z}_2$-graded algebra $Y(Q(n))$. Recall that an *anti-homomorphism* $\varphi : \mathcal{A} \to \mathcal{B}$ of associative Lie superalgebras is a linear map, which preserves the $\mathbb{Z}_2$-grading and satisfies for any homogeneous $X, Y \in \mathcal{A}$

$$\varphi(XY) = (-1)^{p(X)p(Y)}\varphi(Y)\varphi(X).$$

Note that $T(u) = I + A$, where

$$I = \sum_{i,j} E_{i,j} \otimes \delta_{i,j}, \quad A = \sum_{i,j}\sum_{r>0} E_{i,j} \otimes T_{i,j}^{(r)} u^{-r}.$$

Then

$$T(u)^{-1} = I - A + A^2 - A^3 + \dots$$

Note that

$$(E_{i,j} \otimes T_{i,j}^{(r_1)} u^{-r_1})(E_{j,k} \otimes T_{j,k}^{(r_2)} u^{-r_2}) = (-1)^{(p(i)+p(j))(p(j)+p(k))}(E_{i,k} \otimes T_{i,j}^{(r_1)} T_{j,k}^{(r_2)} u^{-r_1-r_2}).$$

Hence, if $r \geq 1$, then

(4.10)

$$S(T_{i,j}^{(r)}) = -T_{i,j}^{(r)} + \sum_{m=2}^{\infty}(-1)^m\Big(\sum_{i_1,\dots i_{m-1} \in \{\pm 1, \dots \pm n\}}\Big(\sum_{\substack{r_1+\dots+r_m=r\\ r_1,\dots,r_m>0}}(-1)^{\nu(i,i_1,\dots,i_{m-1},j)} T_{i,i_1}^{(r_1)} T_{i_1,i_2}^{(r_2)} \dots T_{i_{m-1},j}^{(r_m)}\Big)\Big),$$

where

$$\nu(i_0, i_1, \dots, i_{m-1}, i_m) = \sum_{0 \leq k < s \leq m-1} (p(i_k) + p(i_{k+1})(p(i_s) + p(i_{s+1})).$$

If $r = 0$, then $S(T_{i,j}^{(0)}) = \delta_{i,j}$.

The evaluation homomorphism $ev : Y(Q(n)) \to U(Q(n))$ is defined as follows

$$T_{i,j}^{(1)} \mapsto -e_{j,i}, \quad T_{-i,j}^{(1)} \mapsto -f_{j,i} \text{ for } i,j > 0, \quad T_{i,j}^{(0)} \mapsto \delta_{i,j}, \quad T_{i,j}^{(r)} \mapsto 0 \text{ for } r > 1. \tag{4.11}$$

Recall that for any Lie superalgebra $\mathfrak{g}$, the principal anti-automorphism α of the enveloping superalgebra $U(\mathfrak{g})$ is defined by the assignment $\alpha : X \mapsto -X$ for all $X \in \mathfrak{g}$. Then $\bar{ev} := \alpha \circ ev$ in an anti-homomorphism $\bar{ev} : Y(Q(n)) \to U(Q(n))$, and it is given as follows

$$T_{i,j}^{(1)} \mapsto e_{j,i}, \quad T_{-i,j}^{(1)} \mapsto f_{j,i} \text{ for } i,j > 0, \quad T_{i,j}^{(0)} \mapsto \delta_{i,j}, \quad T_{i,j}^{(r)} \mapsto 0 \text{ for } r > 1. \tag{4.12}$$

In [17] A. Sergeev recursively defined the elements $e_{i,j}^{(m)}$ and $f_{i,j}^{(m)}$ of $U(Q(n))$:

$$e_{i,j}^{(m)} = \sum_{k=1}^{n} e_{i,k} e_{k,j}^{(m-1)} + (-1)^{m+1} \sum_{k=1}^{n} f_{i,k} f_{k,j}^{(m-1)}, \tag{4.13}$$
$$f_{i,j}^{(m)} = \sum_{k=1}^{n} e_{i,k} f_{k,j}^{(m-1)} + (-1)^{m+1} \sum_{k=1}^{n} f_{i,k} e_{k,j}^{(m-1)},$$

where $e_{i,j}^{(0)} = \delta_{i,j}$ and $f_{i,j}^{(0)} = 0$. Then

$$[e_{i,j}, e_{k,l}^{(m)}] = \delta_{j,k} e_{i,l}^{(m)} - \delta_{i,l} e_{k,j}^{(m)}, \quad [e_{i,j}, f_{k,l}^{(m)}] = \delta_{j,k} f_{i,l}^{(m)} - \delta_{i,l} f_{kj}^{(m)}, \tag{4.14}$$
$$[f_{i,j}, e_{k,l}^{(m)}] = (-1)^{m+1} \delta_{j,k} f_{i,l}^{(m)} - \delta_{i,l} f_{k,j}^{(m)},$$
$$[f_{i,j}, f_{k,l}^{(m)}] = (-1)^{m+1} \delta_{j,k} e_{i,l}^{(m)} + \delta_{i,l} e_{k,j}^{(m)}.$$

Lemma 4.1. *There exists a homomorphism $U : Y(Q(n)) \to U(Q(n))$ defined as follows*

$$T_{i,j}^{(r)} \mapsto (-1)^r e_{j,i}^{(r)}, \text{ if } i > 0, j > 0, r > 0,$$
$$T_{i,j}^{(r)} \mapsto (-1)^r f_{j,-i}^{(r)}, \text{ if } i < 0, j > 0, r > 0, T_{i,j}^{(0)} \mapsto \delta_{i,j}. \tag{4.15}$$

Proof. It follows from [9] (Proposition 1.6) that one can define an anti-homomorphism $\omega : Y(Q(n)) \to U(Q(n))$ by

$$T_{i,j}^{(r)} \mapsto e_{j,i}^{(r)}, \text{ if } i > 0, j > 0, r > 0,$$
$$T_{i,j}^{(r)} \mapsto f_{j,-i}^{(r)}, \text{ if } i < 0, j > 0, r > 0, T_{i,j}^{(0)} \mapsto \delta_{i,j}. \tag{4.16}$$

On the other hand, there exists an anti-automorphism β of $Y(Q(n))$ defined as follows:

$$\beta : T_{i,j}^{(r)} \longrightarrow (-1)^r T_{i,j}^{(r)}. \tag{4.17}$$

One can easily verify that β preserves the relation (4.2). In fact, according to (4.3) and (4.5) $\beta(T_{i,j}(u)) = T_{i,j}(-u)$. One can apply β to equation (4.2) and identify: $u \leftrightarrow -v, v \leftrightarrow -u, i \leftrightarrow k, j \leftrightarrow l$. Hence $U := \omega \circ \beta$ is a homomorphism. □

Finally, observe that the maps

$$\Delta_l : Y(Q(n)) \longrightarrow Y(Q(n))^{\otimes l},$$
$$\Delta_l^{op} : Y(Q(n)) \longrightarrow Y(Q(n))^{\otimes l},$$

where

$$\Delta_l := \Delta_{l-1,l} \circ \cdots \circ \Delta_{2,3} \circ \Delta,$$
$$\Delta_l^{op} := \Delta_{l-1,l}^{op} \circ \cdots \circ \Delta_{2,3}^{op} \circ \Delta^{op}$$

are homomorphisms of associative algebras.

We will prove the following statement.

Theorem 4.2. *Let e be an even nilpotent element in $Q(n)$ whose Jordan blocks are each of size l. Then the finite W-algebra for $Q(n)$ is isomorphic to the image of $Y(Q(\frac{n}{l}))$ under the homomorphism*

$$U^{\otimes l}\circ\Delta_l^{op}: Y(Q(\frac{n}{l}))\longrightarrow (U(Q(\frac{n}{l})))^{\otimes l}.$$

Remark 4.3. In [12] we proved this theorem in the case when e is a regular even nilpotent element, i.e. $l=n$.

5. Generators of W_χ for the queer Lie superalgebra $Q(n)$

In this section we construct some generators of W_χ. In particular, we will prove that W_χ is finitely generated. We use the elements $e_{i,j}^{(m)}$ and $f_{i,j}^{(m)}$ defined in (4.13).

Theorem 5.1. *$\pi(e^{(l+k-1)}_{lp,l(q-1)+1})$ and $\pi(f^{(l+k-1)}_{lp,l(q-1)+1})$ for $p,q=1,\dots,\frac{n}{l}$ and $k=1,\dots,l$ generate W_χ.*

Lemma 5.2. *Let $1\le r\le l-1$. Then*

$$\pi(e^{(r)}_{l(p-1)+m,l(q-1)+1})=\begin{cases}\delta_{p,q} & if \quad m=r+1,\\ 0 & if \quad r+2\le m\le l,\end{cases} \tag{5.1}$$

$$\pi(f^{(r)}_{l(p-1)+m,l(q-1)+1})=0,\ \ if\, r+1\le m\le l.$$

Proof. We will prove the statement by induction on r. For $r=1$ we have that

$$\pi(e^{(1)}_{l(p-1)+m,l(q-1)+1})=\pi(e_{l(p-1)+m,l(q-1)+1}),\quad \pi(f^{(1)}_{l(p-1)+m,l(q-1)+1})=\pi(f_{l(p-1)+m,l(q-1)+1}).$$

Then (5.1) follows from (3.1). Assume that (5.1) holds for r. From (4.13) we have that

$$e^{(r+1)}_{l(p-1)+m,l(q-1)+1}=\sum_{s=1}^{\frac{n}{l}}\Big[\sum_{k=1}^{l}e_{l(p-1)+m,l(s-1)+k}e^{(r)}_{l(s-1)+k,l(q-1)+1}+$$
$$(-1)^r\sum_{k=1}^{l}f_{l(p-1)+m,l(s-1)+k}f^{(r)}_{l(s-1)+k,l(q-1)+1}\Big],$$
$$f^{(r+1)}_{l(p-1)+m,l(q-1)+1}=\sum_{s=1}^{\frac{n}{l}}\Big[\sum_{k=1}^{l}e_{l(p-1)+m,l(s-1)+k}f^{(r)}_{l(s-1)+k,l(q-1)+1}+$$
$$(-1)^r\sum_{k=1}^{l}f_{l(p-1)+m,l(s-1)+k}e^{(r)}_{l(s-1)+k,l(q-1)+1}\Big].$$

Note that from (4.14)

$$[e_{l(p-1)+m,l(s-1)+k}, e^{(r)}_{l(s-1)+k,l(q-1)+1}] = e^{(r)}_{l(p-1)+m,l(q-1)+1},$$
$$[e_{l(p-1)+m,l(s-1)+k}, f^{(r)}_{l(s-1)+k,l(q-1)+1}] = f^{(r)}_{l(p-1)+m,l(q-1)+1},$$

$$[f_{l(p-1)+m,l(s-1)+k}, e^{(r)}_{l(s-1)+k,l(q-1)+1}] = (-1)^{r+1} f^{(r)}_{l(p-1)+m,l(q-1)+1},$$
$$[f_{l(p-1)+m,l(s-1)+k}, f^{(r)}_{l(s-1)+k,l(q-1)+1}] = (-1)^{r+1} e^{(r)}_{l(p-1)+m,l(q-1)+1}.$$

Hence

$$e^{(r+1)}_{l(p-1)+m,l(q-1)+1} = \sum_{s=1}^{\frac{n}{l}} \Big[\sum_{k=1}^{m-1} (e^{(r)}_{l(s-1)+k,l(q-1)+1} e_{l(p-1)+m,l(s-1)+k} + e^{(r)}_{l(p-1)+m,l(q-1)+1})+$$
$$\sum_{k=m}^{l} e_{l(p-1)+m,l(s-1)+k} e^{(r)}_{l(s-1)+k,l(q-1)+1} + (-1)^r \Big(\sum_{k=1}^{m-1} (-f^{(r)}_{l(s-1)+k,l(q-1)+1} f_{l(p-1)+m,l(s-1)+k}+$$
$$(-1)^{r+1} e^{(r)}_{l(p-1)+m,l(q-1)+1}) + \sum_{k=m}^{l} f_{l(p-1)+m,l(s-1)+k} f^{(r)}_{l(s-1)+k,l(q-1)+1} \Big) \Big],$$

$$f^{(r+1)}_{l(p-1)+m,l(q-1)+1} = \sum_{s=1}^{\frac{n}{l}} \Big[\sum_{k=1}^{m-1} (f^{(r)}_{l(s-1)+k,l(q-1)+1} e_{l(p-1)+m,l(s-1)+k} + f^{(r)}_{l(p-1)+m,l(q-1)+1})+$$
$$\sum_{k=m}^{l} e_{l(p-1)+m,l(s-1)+k} f^{(r)}_{l(s-1)+k,l(q-1)+1} + (-1)^r \Big(\sum_{k=1}^{m-1} (e^{(r)}_{l(s-1)+k,l(q-1)+1} f_{l(p-1)+m,l(s-1)+k}+$$
$$(-1)^{r+1} f^{(r)}_{l(p-1)+m,l(q-1)+1}) + \sum_{k=m}^{l} f_{l(p-1)+m,l(s-1)+k} e^{(r)}_{l(s-1)+k,l(q-1)+1} \Big) \Big].$$

Then

$$\pi(e^{(r+1)}_{l(p-1)+m,l(q-1)+1}) = \sum_{s=1}^{\frac{n}{l}} \Big[\sum_{k=1}^{m-1} \pi(e^{(r)}_{l(s-1)+k,l(q-1)+1}) \pi(e_{l(p-1)+m,l(s-1)+k})+$$
$$\sum_{k=m}^{l} \Big(\pi(e_{l(p-1)+m,l(s-1)+k}) \pi(e^{(r)}_{l(s-1)+k,l(q-1)+1}) + (-1)^r \pi(f_{l(p-1)+m,l(s-1)+k}) \pi(f^{(r)}_{l(s-1)+k,l(q-1)+1}) \Big)+$$
$$(-1)^{r+1} \Big(\sum_{k=1}^{m-1} \pi(f^{(r)}_{l(s-1)+k,l(q-1)+1}) \pi(f_{l(p-1)+m,l(s-1)+k}) \Big) \Big],$$

$$\pi(f^{(r+1)}_{l(p-1)+m,l(q-1)+1}) = \sum_{s=1}^{\frac{n}{l}} \Big[\sum_{k=1}^{m-1} \pi(f^{(r)}_{l(s-1)+k,l(q-1)+1})\pi(e_{l(p-1)+m,l(s-1)+k}) +$$
$$\sum_{k=m}^{l} \Big(\pi(e_{l(p-1)+m,l(s-1)+k})\pi(f^{(r)}_{l(s-1)+k,l(q-1)+1}) + (-1)^r \pi(f_{l(p-1)+m,l(s-1)+k})\pi(e^{(r)}_{l(s-1)+k,l(q-1)+1}) \Big) +$$
$$(-1)^r \Big(\sum_{k=1}^{m-1} \pi(e^{(r)}_{l(s-1)+k,l(q-1)+1})\pi(f_{l(p-1)+m,l(s-1)+k}) \Big) \Big].$$

Then by (3.1)

$$\pi(e^{(r+1)}_{l(p-1)+m,l(q-1)+1}) = \pi(e^{(r)}_{l(p-1)+m-1,l(q-1)+1}) + \sum_{s=1}^{\frac{n}{l}} \Big[\sum_{k=m}^{l} \Big(\pi(e_{l(p-1)+m,l(s-1)+k})\pi(e^{(r)}_{l(s-1)+k,l(q-1)+1}) +$$
$$(-1)^r \pi(f_{l(p-1)+m,l(s-1)+k})\pi(f^{(r)}_{l(s-1)+k,l(q-1)+1}) \Big) \Big],$$
$$\pi(f^{(r+1)}_{l(p-1)+m,l(q-1)+1}) = \pi(f^{(r)}_{l(p-1)+m-1,l(q-1)+1}) + \sum_{s=1}^{\frac{n}{l}} \Big[\sum_{k=m}^{l} \Big(\pi(e_{l(p-1)+m,l(s-1)+k})\pi(f^{(r)}_{l(s-1)+k,l(q-1)+1}) +$$
$$(-1)^r \pi(f_{l(p-1)+m,l(s-1)+k})\pi(e^{(r)}_{l(s-1)+k,l(q-1)+1}) \Big) \Big].$$

Let $m \geq r+2$. Then by induction hypothesis,

$$\pi(e^{(r)}_{l(s-1)+k,l(q-1)+1}) = \pi(f^{(r)}_{l(s-1)+k,l(q-1)+1}) = 0 \text{ for } k = m, \dots, l.$$

If $m = r+2$, then $\pi(e^{(r+1)}_{l(p-1)+m,l(q-1)+1}) = \pi(e^{(r)}_{l(p-1)+r+1,l(q-1)+1}) = \delta_{p,q}$. If $m \geq r+3$, then $\pi(e^{(r+1)}_{l(p-1)+m,l(q-1)+1}) = \pi(e^{(r)}_{l(p-1)+m-1,l(q-1)+1}) = 0$. Also, if $m \geq r+2$, then $\pi(f^{(r+1)}_{l(p-1)+m,l(q-1)+1}) = \pi(f^{(r)}_{l(p-1)+m-1,l(q-1)+1}) = 0$. Hence (5.1) holds for $r+1$. $\square$

Lemma 5.3. *Let $1 \leq m \leq l$, and $1 \leq p, q, \leq \frac{n}{l}$. Then*

$$\pi(e^{(m)}_{l(p-1)+m,l(q-1)+1}) = \sum_{k=1}^{m} \pi(e_{l(p-1)+k,l(q-1)+k}), \tag{5.2}$$
$$\pi(f^{(m)}_{l(p-1)+m,l(q-1)+1}) = \sum_{k=1}^{m} (-1)^{k-1} \pi(f_{l(p-1)+k,l(q-1)+k}).$$

Proof. We proceed by induction on m. If $m=1$, then (5.2) obviously holds. Assume that (5.2) holds for m. By (4.13) and (4.14) we have that

$$e^{(m+1)}_{l(p-1)+m+1,l(q-1)+1} =$$
$$\sum_{s=1}^{\frac{n}{l}}\Big[\sum_{k=1}^{l} e_{l(p-1)+m+1,l(s-1)+k}e^{(m)}_{l(s-1)+k,l(q-1)+1} + (-1)^m\sum_{k=1}^{l} f_{l(p-1)+m+1,l(s-1)+k}f^{(m)}_{l(s-1)+k,l(q-1)+1}\Big] =$$
$$\sum_{s=1}^{\frac{n}{l}}\Big[\sum_{k=1}^{m}\Big(e^{(m)}_{l(s-1)+k,l(q-1)+1}e_{l(p-1)+m+1,l(s-1)+k} + e^{(m)}_{l(p-1)+m+1,l(q-1)+1}\Big)+$$
$$\sum_{k=m+1}^{l} e_{l(p-1)+m+1,l(s-1)+k}e^{(m)}_{l(s-1)+k,l(q-1)+1} + (-1)^m\Big(\sum_{k=1}^{m}(-f^{(m)}_{l(s-1)+k,l(q-1)+1}f_{l(p-1)+m+1,l(s-1)+k}+$$
$$(-1)^{m+1}e^{(m)}_{l(p-1)+m+1,l(q-1)+1}) + \sum_{k=m+1}^{l} f_{l(p-1)+m+1,l(s-1)+k}f^{(m)}_{l(s-1)+k,l(q-1)+1}\Big)\Big].$$

We also have

$$f^{(m+1)}_{l(p-1)+m+1,l(q-1)+1} =$$
$$\sum_{s=1}^{\frac{n}{l}}\Big[\sum_{k=1}^{l} e_{l(p-1)+m+1,l(s-1)+k}f^{(m)}_{l(s-1)+k,l(q-1)+1} + (-1)^m\sum_{k=1}^{l} f_{l(p-1)+m+1,l(s-1)+k}e^{(m)}_{l(s-1)+k,l(q-1)+1}\Big] =$$
$$\sum_{s=1}^{\frac{n}{l}}\Big[\sum_{k=1}^{m}\Big(f^{(m)}_{l(s-1)+k,l(q-1)+1}e_{l(p-1)+m+1,l(s-1)+k} + f^{(m)}_{l(p-1)+m+1,l(q-1)+1}\Big)+$$
$$\sum_{k=m+1}^{l} e_{l(p-1)+m+1,l(s-1)+k}f^{(m)}_{l(s-1)+k,l(q-1)+1} + (-1)^m\Big(\sum_{k=1}^{m}\Big(e^{(m)}_{l(s-1)+k,l(q-1)+1}f_{l(p-1)+m+1,l(s-1)+k}+$$
$$(-1)^{m+1}f^{(m)}_{l(p-1)+m+1,l(q-1)+1}\Big) + \sum_{k=m+1}^{l} f_{l(p-1)+m+1,l(s-1)+k}e^{(m)}_{l(s-1)+k,l(q-1)+1}\Big)\Big].$$

Then using (3.1) and (5.1) we obtain that

$$\pi(e^{(m+1)}_{l(p-1)+m+1,l(q-1)+1}) = \pi(e^{(m)}_{l(p-1)+m,l(q-1)+1}) + \pi(e_{l(p-1)+m+1,l(q-1)+m+1}),$$
$$\pi(f^{(m+1)}_{l(p-1)+m+1,l(q-1)+1}) = \pi(f^{(m)}_{l(p-1)+m,l(q-1)+1}) + (-1)^m\pi(f_{l(p-1)+m+1,l(q-1)+m+1}).$$

By induction hypothesis we have

$$\pi(e^{(m+1)}_{l(p-1)+m+1,l(q-1)+1}) = \sum_{k=1}^{m} \pi(e_{l(p-1)+k,l(q-1)+k}) + \pi(e_{l(p-1)+m+1,l(q-1)+m+1}) = \sum_{k=1}^{m+1} \pi(e_{l(p-1)+k,l(q-1)+k}),$$

$$\pi(f^{(m+1)}_{l(p-1)+m+1,l(q-1)+1}) = \sum_{k=1}^{m} (-1)^{k-1}\pi(f_{l(p-1)+k,l(q-1)+k}) + (-1)^{m}\pi(f_{l(p-1)+m+1,l(q-1)+m+1}) = \sum_{k=1}^{m+1} (-1)^{k-1}\pi(f_{l(p-1)+k,l(q-1)+k}).$$

□

Consider the Kazhdan filtration on $U(\mathfrak{p})$. By definition, the graded algebra $Gr_K U(\mathfrak{p})$ is isomorphic to $S(\mathfrak{p})$. Moreover, $Gr_K U(\mathfrak{p}) \simeq S(\mathfrak{p})$ is a commutative graded ring, where the grading is induced from the Dynkin $\mathbb{Z}$-grading of $\mathfrak{g}$. For any $X \in U(\mathfrak{p})$ let $Gr_K(X)$ denote the corresponding element in $Gr_K U(\mathfrak{p})$ and $P(X)$ denote the highest weight component of $Gr_K(X)$ in the Dynkin $\mathbb{Z}$-grading. For $X \in U(\mathfrak{p})$, we denote by $\deg P(X)$ the Kazhdan degree of $Gr_K(X)$ and by $\mathrm{wt} P(X)$ the weight of the highest weight component of $Gr_K(X)$.

Lemma 5.4.

$$P\Big(\pi(e^{(l+k)}_{lp,l(q-1)+1})\Big) = \sum_{i=1}^{l-k} e_{l(p-1)+i,l(q-1)+i+k}, \tag{5.3}$$

$$P\Big(\pi(f^{(l+k)}_{lp,l(q-1)+1})\Big) = \sum_{i=1}^{l-k} (-1)^{i+k-1} f_{l(p-1)+i,l(q-1)+i+k}, \tag{5.4}$$

where $k = 0, 1, \ldots, l-1$.

Proof. We will prove a more general statement. We claim that for $0 \le k \le l-1$ and $1 \le m \le l$

$$P\Big(\pi(e^{(m+k)}_{l(p-1)+m,l(q-1)+1})\Big) = \sum_{i=1}^{r} e_{l(p-1)+i,l(q-1)+i+k}, \tag{5.5}$$

$$P\Big(\pi(f^{(m+k)}_{l(p-1)+m,l(q-1)+1})\Big) = \sum_{i=1}^{r} (-1)^{i+k-1} f_{l(p-1)+i,l(q-1)+i+k}, \tag{5.6}$$

where $r = \min\{m, l-k\}$.

In particular,

$$\deg P\Big(\pi(e^{(m+k)}_{l(p-1)+m,l(q-1)+1})\Big)=\deg P\Big(\pi(f^{(m+k)}_{l(p-1)+m,l(q-1)+1})\Big)=2k+2,$$
$$\mathrm{wt} P\Big(\pi(e^{(m+k)}_{l(p-1)+m,l(q-1)+1})\Big)=\mathrm{wt} P\Big(\pi(f^{(m+k)}_{l(p-1)+m,l(q-1)+1})\Big)=2k.$$

We proceed to the proof of (5.5) by induction on k and m. Note that if $k=0$, then (5.5) holds for any $1\leq m\leq l$ by (5.2). Assume that if $k\leq b-1$, then (5.5) holds for any $1\leq m\leq l$. Let $k=b$. Show that (5.5) holds for $m=1$. Note that by (4.13)

$$e^{(1+b)}_{l(p-1)+1,l(q-1)+1}=\sum_{s=1}^{\frac{n}{l}}\Big[\Big(\sum_{a=1}^{l}e_{l(p-1)+1,l(s-1)+a}e^{(b)}_{l(s-1)+a,l(q-1)+1}\Big)+$$
$$(-1)^b\Big(\sum_{a=1}^{l}f_{l(p-1)+1,l(s-1)+a}f^{(b)}_{l(s-1)+a,l(q-1)+1}\Big)\Big].$$

Let $X=\pi(e_{l(p-1)+1,l(s-1)+a}e^{(b)}_{l(s-1)+a,l(q-1)+1})$, $Y=\pi(f_{l(p-1)+1,l(s-1)+a}f^{(b)}_{l(s-1)+a,l(q-1)+1})$ where $a=1,\ldots,b$. Note that

$$\deg P\Big(\pi(e_{l(p-1)+1,l(s-1)+a})\Big)=\deg P\Big(\pi(f_{l(p-1)+1,l(s-1)+a})\Big)=2a,$$
$$\mathrm{wt} P\Big(\pi(e_{l(p-1)+1,l(s-1)+a})\Big)=\mathrm{wt} P\Big(\pi(f_{l(p-1)+1,l(s-1)+a})\Big)=2a-2.$$

By induction hypothesis,

$$\deg P\Big(\pi(e^{(b)}_{l(s-1)+a,l(q-1)+1})\Big)=\deg P\Big(\pi(f^{(b)}_{l(s-1)+a,l(q-1)+1})\Big)=2(b-a)+2,$$
$$\mathrm{wt} P\Big(\pi(e^{(b)}_{l(s-1)+a,l(q-1)+1})\Big)=\mathrm{wt} P\Big(\pi(f^{(b)}_{l(s-1)+a,l(q-1)+1})\Big)=2(b-a).$$

Then

$$\deg P(X)=2b+2,\quad \mathrm{wt} P(X)=2b-2,$$
$$\deg P(Y)=2b+2,\quad \mathrm{wt} P(Y)=2b-2.$$

Let $X=\sum_{s=1}^{\frac{n}{l}}\pi(e_{l(p-1)+1,l(s-1)+b+1}e^{(b)}_{l(s-1)+b+1,l(q-1)+1})$. Then by (5.1) $X=\pi(e_{l(p-1)+1,l(q-1)+b+1})$. Hence

$$\deg P(X)=2b+2,\quad \mathrm{wt} P(X)=2b.$$

Finally, by (5.1)

$$\pi(e^{(b)}_{l(p-1)+b+i,l(q-1)+1})=0 \text{ for } i=2,\ldots,l-b,\quad \pi(f^{(b)}_{l(p-1)+b+i,l(q-1)+1})=0 \text{ for } i=1,\ldots,l-b.$$

Hence

$$P(\pi(e^{(1+b)}_{l(p-1)+1,l(q-1)+1}))=e_{l(p-1)+1,l(q-1)+b+1}.$$

Let $k=b$ and assume that (5.5) holds for $m \leq c$. Show that it holds for $m=c+1$. Note that (4.13)

$$e^{(c+1+b)}_{l(p-1)+c+1,l(q-1)+1} = \sum_{s=1}^{\frac{n}{l}} \Big[\Big(\sum_{a=1}^{l} e_{l(p-1)+c+1,l(s-1)+a} e^{(c+b)}_{l(s-1)+a,l(q-1)+1}\Big)+ (-1)^{c+b}\Big(\sum_{a=1}^{l} f_{l(p-1)+c+1,l(s-1)+a} f^{(c+b)}_{l(s-1)+a,l(q-1)+1}\Big)\Big].$$

Thus

$$\pi(e^{(c+1+b)}_{l(p-1)+c+1,l(q-1)+1}) = \sum_{s=1}^{\frac{n}{l}} \Big[\sum_{a=1}^{c-1} \pi(e_{l(p-1)+c+1,l(s-1)+a} e^{(c+b)}_{l(s-1)+a,l(q-1)+1})+$$

$$\pi(e_{l(p-1)+c+1,l(s-1)+c} e^{(c+b)}_{l(s-1)+c,l(q-1)+1}) + \sum_{i=1}^{b} \pi(e_{l(p-1)+c+1,l(s-1)+c+i} e^{(c+b)}_{l(s-1)+c+i,l(q-1)+1})+$$

$$\pi(e_{l(p-1)+c+1,l(s-1)+c+b+1} e^{(c+b)}_{l(s-1)+c+b+1,l(q-1)+1}) + \sum_{i=2}^{l-c-b} \pi(e_{l(p-1)+c+1,l(s-1)+c+b+i} e^{(c+b)}_{l(s-1)+c+b+i,l(q-1)+1})+$$

$$(-1)^{c+b}\Big(\sum_{a=1}^{c} \pi(f_{l(p-1)+c+1,l(s-1)+a} f^{(c+b)}_{l(s-1)+a,l(q-1)+1}) + \sum_{i=1}^{b} \pi(f_{l(p-1)+c+1,l(s-1)+c+i} f^{(c+b)}_{l(s-1)+c+i,l(q-1)+1})+$$

$$\sum_{i=1}^{l-c-b} \pi(f_{l(p-1)+c+1,l(s-1)+c+b+i} f^{(c+b)}_{l(s-1)+c+b+i,l(q-1)+1})\Big)\Big].$$

Let $X=\pi(e_{l(p-1)+c+1,l(s-1)+i} e^{(c+b)}_{l(s-1)+i,l(q-1)+1})$, where $i=1,\dots,c-1$, and
$Y=\pi(f_{l(p-1)+c+1,l(s-1)+i} f^{(c+b)}_{l(s-1)+i,l(q-1)+1})$, where $i=1,\dots,c$. By (4.14) and (3.1)
$X=\pi(e^{(c+b)}_{l(s-1)+i,l(q-1)+1} e_{l(p-1)+c+1,l(s-1)+i} + e^{(c+b)}_{l(p-1)+c+1,l(q-1)+1}) = \pi(e^{(c+b)}_{l(p-1)+c+1,l(q-1)+1})$,
$Y=\pi(-f^{(c+b)}_{l(s-1)+i,l(q-1)+1} f_{l(p-1)+c+1,l(s-1)+i} + (-1)^{c+b+1} e^{(c+b)}_{l(p-1)+c+1,l(q-1)+1}) =$
$\pi((-1)^{c+b+1} e^{(c+b)}_{l(p-1)+c+1,l(q-1)+1})$.

By induction hypothesis

$$\text{(5.7)} \qquad \begin{aligned} &\deg P(X) = \deg P(Y) = 2b, \\ &\mathrm{wt} P(X) = \mathrm{wt} P(Y) = 2b-2. \end{aligned}$$

Let $X=\pi(e_{l(p-1)+c+1,l(s-1)+c} e^{(c+b)}_{l(s-1)+c,l(q-1)+1})$. Then by (4.14) and (3.1)

$$X=\pi(e^{(c+b)}_{l(s-1)+c,l(q-1)+1} e_{l(p-1)+c+1,l(s-1)+c} + e^{(c+b)}_{l(p-1)+c+1,l(q-1)+1}) = \pi(e^{(c+b)}_{l(p-1)+c,l(q-1)+1} + e^{(c+b)}_{l(p-1)+c+1,l(q-1)+1}).$$

By induction hypothesis

$$\deg P(\pi(e^{(c+b)}_{l(p-1)+c,l(q-1)+1})) = 2b+2, \quad \mathrm{wt}P(\pi(e^{(c+b)}_{l(p-1)+c,l(q-1)+1})) = 2b, \tag{5.8}$$

$$\deg P(\pi(e^{(c+b)}_{l(p-1)+c+1,l(q-1)+1})) = 2b, \quad \mathrm{wt}P(\pi(e^{(c+b)}_{l(p-1)+c+1,l(q-1)+1})) = 2b-2. \tag{5.9}$$

Let $X = \pi(e_{l(p-1)+c+1,l(s-1)+c+i}e^{(c+b)}_{l(s-1)+c+i,l(q-1)+1})$, $Y = \pi(f_{l(p-1)+c+1,l(s-1)+c+i}f^{(c+b)}_{l(s-1)+c+i,l(q-1)+1})$ for $i = 1,\dots,b$. Then by induction hypothesis

$$\deg P(X) = \deg P(Y) = 2b+2, \quad \mathrm{wt}P(X) = \mathrm{wt}P(Y) = 2b-2. \tag{5.10}$$

Let $X = \pi(e_{l(p-1)+c+1,l(s-1)+c+b+1}e^{(c+b)}_{l(s-1)+c+b+1,l(q-1)+1})$. Hence by (5.1) $X = \pi(e_{l(p-1)+c+1,l(q-1)+c+b+1})$. Then

$$\deg P(X) = 2b+2, \quad \mathrm{wt}P(X) = 2b. \tag{5.11}$$

Finally, by (5.1) $\pi(e_{l(p-1)+c+1,l(s-1)+c+b+i}e^{(c+b)}_{l(s-1)+c+b+i,l(q-1)+1}) = 0$ for $i = 2,\dots,l-c-b$ and $\pi(f_{l(p-1)+c+1,l(s-1)+c+b+i}f^{(c+b)}_{l(s-1)+c+b+i,l(q-1)+1}) = 0$ for $i = 1,\dots,l-c-b$. From (5.7)-(5.11) one can see that the highest degree component in $\pi(e^{(c+1+b)}_{l(p-1)+c+1,l(q-1)+1})$ has degree $2b+2$, and its highest weight component has weight $2b$. In fact, if $c \geq l-b$, then by (5.8) this component is $P(\pi(e^{(c+b)}_{l(p-1)+c,l(q-1)+1}))$. By induction hypothesis $P(\pi(e^{(c+b)}_{l(p-1)+c,l(q-1)+1})) = \sum_{i=1}^{l-b} e_{l(p-1)+i,l(q-1)+i+b}$. If $c < l-b$, then $P(\pi(e^{(c+b)}_{l(p-1)+c,l(q-1)+1})) = \sum_{i=1}^{c} e_{l(p-1)+i,l(q-1)+i+b}$. Note that in this case $\pi(e^{(c+1+b)}_{l(p-1)+c+1,l(q-1)+1})$ has an additional element $\pi(e_{l(p-1)+c+1,l(q-1)+c+b+1})$ of degree $2b+2$ and weight $2b$ according to (5.11). Clearly, $P(\pi(e_{l(p-1)+c+1,l(q-1)+c+b+1})) = e_{l(p-1)+c+1,l(q-1)+c+b+1}$ and $P(\pi(e^{(c+b)}_{l(p-1)+c,l(q-1)+1}))+P(\pi(e_{l(p-1)+c+1,l(q-1)+c+b+1})) \neq 0$. Hence

$$P(\pi(e^{(c+1+b)}_{l(p-1)+c+1,l(q-1)+1})) = P(\pi(e^{(c+b)}_{l(p-1)+c,l(q-1)+1})) + P(\pi(e_{l(p-1)+c+1,l(q-1)+c+b+1})) = \sum_{i=1}^{c+1} e_{l(p-1)+i,l(q-1)+i+b}.$$

Then in either case,

$$P(\pi(e^{(c+1+b)}_{l(p-1)+c+1,l(q-1)+1})) = \sum_{i=1}^{r} e_{l(p-1)+i,l(q-1)+i+b}, \text{ where } r = \min\{c+1, l-b\}.$$

Thus if $0 \le k \le l-1$ and $1 \le m \le l$, then (5.5) holds. Similarly, one can prove that (5.6) holds. In particular, if $m = l$ and $k = 0, \dots, l-1$, we obtain (5.3) and (5.4). □

The statement of Theorem 5.1 follows from Lemma 5.4 and Theorem 2.9 (a). Lemma 5.4 and Theorem 2.9 (b) imply the following

Corollary 5.5.

$$Gr_K W_\chi \simeq S(\mathfrak{g}^\chi).$$

This implies that Conjecture 2.8 in [12] is true in this case.
Let

$$x^i_{p,q} = e_{l(p-1)+i,l(q-1)+i}, \quad \xi^i_{p,q} = (-1)^{i+1} f_{l(p-1)+i,l(q-1)+i}$$

for $p, q = 1, \dots, \frac{n}{l}$ and $i = 1, \dots, l$. Then

$$\mathfrak{g}_0 =< x^i_{p,q} \mid \xi^i_{p,q} >,$$
$$\text{where } i = 1, \dots, l, \text{ and } 1 \le p, q \le \frac{n}{l}.$$

Theorem 5.6.

$$\vartheta(\pi(e^{(l+k-1)}_{lp,l(q-1)+1})) = [\sum_{1 \le p_1,p_2,\dots,p_{k-1} \le \frac{n}{l}} \sum_{l \ge i_1 \ge i_2 \ge \dots \ge i_k \ge 1} (x^{i_1}_{p,p_1} + (-1)^{k+1}\xi^{i_1}_{p,p_1})(x^{i_2}_{p_1,p_2} + (-1)^k \xi^{i_2}_{p_1,p_2}) \dots (x^{i_k}_{p_{k-1},q} + \xi^{i_k}_{p_{k-1},q})]_{even},$$

$$\vartheta(\pi(f^{(l+k-1)}_{lp,l(q-1)+1})) = [\sum_{1 \le p_1,p_2,\dots,p_{k-1} \le \frac{n}{l}} \sum_{l \ge i_1 \ge i_2 \ge \dots \ge i_k \ge 1} (x^{i_1}_{p,p_1} + (-1)^{k+1}\xi^{i_1}_{p,p_1})(x^{i_2}_{p_1,p_2} + (-1)^k \xi^{i_2}_{p_1,p_2}) \dots (x^{i_k}_{p_{k-1},q} + \xi^{i_k}_{p_{k-1},q})]_{odd}$$

Proof. We will prove by induction on s and r that if $s \ge 0$ and $1 \le r \le l$, then

$$\vartheta(\pi(e^{(r+s)}_{l(p-1)+r,l(q-1)+1})) + \vartheta(\pi(f^{(r+s)}_{l(p-1)+r,l(q-1)+1})) = \sum_{1 \le p_1,p_2,\dots,p_s \le \frac{n}{l}} [\sum_{r \ge i_1 \ge i_2 \ge \dots \ge i_{s+1} \ge 1} (x^{i_1}_{p,p_1} + (-1)^s \xi^{i_1}_{p,p_1}) \dots (x^{i_{s+1}}_{p_s,q} + \xi^{i_{s+1}}_{p_s,q})]. \tag{5.12}$$

Note that if $s = 0$, then (5.12) holds for any $1 \le r \le l$, since

$$\vartheta(\pi(e^{(r)}_{l(p-1)+r,l(q-1)+1})) + \vartheta(\pi(f^{(r)}_{l(p-1)+r,l(q-1)+1})) =$$
$$\vartheta(\sum_{i=1}^r \pi(e_{l(p-1)+i,l(q-1)+i})) + \vartheta(\sum_{i=1}^r (-1)^{i-1}\pi(f_{l(p-1)+i,l(q-1)+i})) = \sum_{i=1}^r (x^i_{p,q} + \xi^i_{p,q}).$$

Assume that if $s \leq k-1$, then (5.12) holds for any $1 \leq r \leq l$. Let $s = k$, show that (5.12) holds for $r = 1$. We have

$$\vartheta(\pi(e^{(1+k)}_{l(p-1)+1,l(q-1)+1})) + \vartheta(\pi(f^{(1+k)}_{l(p-1)+1,l(q-1)+1})) =$$

$$\sum_{p_1=1}^{\frac{n}{l}} \vartheta(\pi(e_{l(p-1)+1,l(p_1-1)+1})\pi(e^{(k)}_{l(p_1-1)+1,l(q-1)+1}) + (-1)^k\pi(f_{l(p-1)+1,l(p_1-1)+1})\pi(f^{(k)}_{l(p_1-1)+1,l(q-1)+1}))+$$

$$\sum_{p_1=1}^{\frac{n}{l}} \vartheta(\pi(e_{l(p-1)+1,l(p_1-1)+1})\pi(f^{(k)}_{l(p_1-1)+1,l(q-1)+1}) + (-1)^k\pi(f_{l(p-1)+1,l(p_1-1)+1})\pi(e^{(k)}_{l(p_1-1)+1,l(q-1)+1})) =$$

$$\sum_{p_1=1}^{\frac{n}{l}} (e_{l(p-1)+1,l(p_1-1)+1} + (-1)^k f_{l(p-1)+1,l(p_1-1)+1})(\vartheta(\pi(e^{(k)}_{l(p_1-1)+1,l(q-1)+1}) + \vartheta(\pi(f^{(k)}_{l(p_1-1)+1,l(q-1)+1})) =$$

$$\sum_{p_1=1}^{\frac{n}{l}} (x^1_{p,p_1} + (-1)^k\xi^1_{p,p_1}) \sum_{1\leq p_2,\dots,p_k\leq\frac{n}{l}} (x^1_{p_1,p_2} + (-1)^{k-1}\xi^1_{p_1,p_2})\dots(x^1_{p_k,q} + \xi^1_{p_k,q}) =$$

$$\sum_{l\leq p_1,\dots,p_k\leq\frac{n}{l}} (x^1_{p,p_1} + (-1)^k\xi^1_{p,p_1})\dots(x^1_{p_k,q} + \xi^1_{p_k,q}).$$

Let $s = k$ and assume that (5.12) holds for $r \leq m$. Show that it holds for $r = m+1$. By induction hypothesis we have

$$\vartheta(\pi(e^{(m+1+k)}_{l(p-1)+m+1,l(q-1)+1})) + \vartheta(\pi(f^{(m+1+k)}_{l(p-1)+m+1,l(q-1)+1})) =$$

$$\vartheta(\pi(e_{l(p-1)+m+1,l(p-1)+m}))\vartheta(\pi(e^{(m+k)}_{l(p-1)+m,l(q-1)+1}))+$$

$$\sum_{p_1=1}^{\frac{n}{l}} \Big(\vartheta(\pi(e_{l(p-1)+m+1,l(p_1-1)+m+1}))\vartheta(\pi(e^{(m+k)}_{l(p_1-1)+m+1,l(q-1)+1}))\Big)+$$

$$(-1)^{m+k}\sum_{p_1=1}^{\frac{n}{l}} \Big(\vartheta(\pi(f_{l(p-1)+m+1,l(p_1-1)+m+1}))\vartheta(\pi(f^{(m+k)}_{l(p_1-1)+m+1,l(q-1)+1}))\Big)+$$

$$\vartheta(\pi(e_{l(p-1)+m+1,l(p-1)+m}))\vartheta(\pi(f^{(m+k)}_{l(p-1)+m,l(q-1)+1}))+$$

$$\sum_{p_1=1}^{\frac{n}{l}} \Big(\vartheta(\pi(e_{l(p-1)+m+1,l(p_1-1)+m+1}))\vartheta(\pi(f^{(m+k)}_{l(p_1-1)+m+1,l(q-1)+1}))\Big)+$$

$$(-1)^{m+k}\sum_{p_1=1}^{\frac{n}{l}} \Big(\vartheta(\pi(f_{l(p-1)+m+1,l(p_1-1)+m+1}))\vartheta(\pi(e^{(m+k)}_{l(p_1-1)+m+1,l(q-1)+1}))\Big) =$$

$$\vartheta(\pi(e^{(m+k)}_{l(p-1)+m,l(q-1)+1})) + \vartheta(\pi(f^{(m+k)}_{l(p-1)+m,l(q-1)+1}))+$$
$$\sum_{p_1=1}^{\frac{n}{l}} \Big([\vartheta(\pi(e_{l(p-1)+m+1,l(p_1-1)+m+1})) + (-1)^{m+k}\vartheta(\pi(f_{l(p-1)+m+1,l(p_1-1)+m+1}))]\times$$
$$[\vartheta(\pi(e^{(m+k)}_{l(p_1-1)+m+1,l(q-1)+1})) + \vartheta(\pi(f^{(m+k)}_{l(p_1-1)+m+1,l(q-1)+1}))]\Big) =$$
$$\sum_{l\le p_1,\dots,p_k\le \frac{n}{l}} \Big(\sum_{m\ge i_1\ge i_2\ge \dots\ge i_{k+1}\ge 1} (x^{i_1}_{p,p_1} + (-1)^k\xi^{i_1}_{p,p_1})\dots(x^{i_{k+1}}_{p_k,q} + \xi^{i_{k+1}}_{p_k,q})\Big)+$$
$$\sum_{p_1=1}^{\frac{n}{l}} (x^{m+1}_{p,p_1} + (-1)^k\xi^{m+1}_{p,p_1}) \sum_{l\le p_2,\dots,p_k\le \frac{n}{l}} \Big(\sum_{m+1\ge i_1\ge i_2\ge\dots\ge i_k\ge 1} (x^{i_1}_{p_1,p_2} + (-1)^{k-1}\xi^{i_1}_{p_1,p_2})\dots(x^{i_k}_{p_k,q} + \xi^{i-k}_{p_k,q})\Big) =$$
$$\sum_{l\le p_1,\dots,p_k\le \frac{n}{l}} \Big(\sum_{m+1\ge i_1\ge i_2\ge\dots\ge i_{k+1}\ge 1} (x^{i_1}_{p,p_1} + (-1)^k\xi^{i_1}_{p,p_1})\dots(x^{i_{k+1}}_{p_k,q} + \xi^{i_{k+1}}_{p_k,q})\Big).$$

Thus (5.12) holds. In particular, if $r = l$ and $s = k-1$, we obtain the statement of Theorem 5.6. □

At the moment we are interested in $Y(Q(\frac{n}{l}))$.

Theorem 5.7. *Let* $1 \le p, q \le \frac{n}{l}$.

$$U^{\otimes l}\circ\Delta^{op}_l(T^{(r)}_{q,p}) = (-1)^r[\sum_{1\le p_1,p_2,\dots,p_{r-1}\le\frac{n}{l}}$$
$$\sum_{l\ge i_1\ge i_2\ge\dots\ge i_r\ge 1} (x^{i_1}_{p,p_1} + (-1)^{r+1}\xi^{i_1}_{p,p_1})(x^{i_2}_{p_1,p_2} + (-1)^r\xi^{i_2}_{p_1,p_2})\dots(x^{i_r}_{p_{r-1},q} + \xi^{i_r}_{p_{r-1},q})]_{even},$$
$$U^{\otimes l}\circ\Delta^{op}_l(T^{(r)}_{-q,p}) = (-1)^r[\sum_{1\le p_1,p_2,\dots,p_{r-1}\le\frac{n}{l}}$$
$$\sum_{l\ge i_1\ge i_2\ge\dots\ge i_r\ge 1} (x^{i_1}_{p,p_1} + (-1)^{r+1}\xi^{i_1}_{p,p_1})(x^{i_2}_{p_1,p_2} + (-1)^r\xi^{i_2}_{p_1,p_2})\dots(x^{i_r}_{p_{r-1},q} + \xi^{i_r}_{p_{r-1},q})]_{odd}$$

Proof. Let $1 \le p, q \le \frac{n}{l}$. Set

$$T^{(r)+}_{q,p} = T^{(r)}_{q,p} + T^{(r)}_{-q,p},$$
$$T^{(r)-}_{q,p} = T^{(r)}_{q,p} - T^{(r)}_{-q,p}.$$

One can easily verify the following recursive relations

$$\Delta^{op}_l(T^{(r)+}_{q,p}) = \sum_{s=0}^{r}\Big(\sum_{k=1}^{\frac{n}{l}}\Big(T^{(r-s)}_{k,p} + (-1)^sT^{(r-s)}_{-k,p}\Big)\otimes\Delta^{op}_{l-1}\Big(T^{(s)+}_{q,k}\Big)\Big). \quad (5.13)$$

As a direct consequence of (4.13) we obtain

$$U(T^{(r)+}_{q,p}) = (-1)^r \sum_{1\le p_1,\dots,p_{r-1}\le \frac{n}{l}} \left(e_{p,p_1} + (-1)^{r-1} f_{p,p_1}\right)\left(e_{p_1,p_2} + (-1)^{r-2} f_{p_1,p_2}\right)\dots\left(e_{p_{r-1},q} + f_{p_{r-1},q}\right), \tag{5.14}$$

$$U(T^{(r)-}_{q,p}) = (-1)^r \sum_{1\le p_1,\dots,p_{r-1}\le \frac{n}{l}} \left(e_{p,p_1} + (-1)^{r} f_{p,p_1}\right)\left(e_{p_1,p_2} + (-1)^{r-1} f_{p_1,p_2}\right)\dots\left(e_{p_{r-1},q} - f_{p_{r-1},q}\right). \tag{5.15}$$

Lemma 5.8. *Identify $U(\mathfrak{g}_0) \subset U(Q(n))$ with $U(Q(\frac{n}{l}))^{\otimes l}$ by setting*

$$x^i_{p,q} \mapsto 1^{\otimes l-i} \otimes e_{p,q} \otimes 1^{\otimes i-1},$$
$$\xi^i_{p,q} \mapsto 1^{\otimes l-i} \otimes f_{p,q} \otimes 1^{\otimes i-1}.$$

Then

$$U^{\otimes l} \circ \Delta^{op}_l (T^{(r)+}_{q,p}) = (-1)^r [\sum_{1\le p_1,p_2,\dots,p_{r-1}\le \frac{n}{l}} \sum_{l\ge i_1\ge i_2\ge\dots\ge i_r\ge 1} (x^{i_1}_{p,p_1} + (-1)^{r+1}\xi^{i_1}_{p,p_1})(x^{i_2}_{p_1,p_2} + (-1)^{r}\xi^{i_2}_{p_1,p_2})\dots(x^{i_r}_{p_{r-1},q} + \xi^{i_r}_{p_{r-1},q})].$$

Proof. Follows from (5.13), (5.14) and (5.15). □

This completes the proof of Theorem 5.7. □

Corollary 5.9. *There exists a surjective homomorphism:*

$$\varphi : Y(Q(\frac{n}{l})) \longrightarrow W_\chi$$

defined as follows:

$$\varphi(T^{(r)}_{q,p}) = (-1)^r \pi(e^{(l+r-1)}_{lp,l(q-1)+1}), \quad \varphi(T^{(r)}_{-q,p}) = (-1)^r \pi(f^{(l+r-1)}_{lp,l(q-1)+1}), \text{ for } r = 1, 2, \dots.$$

Proof. Recall that the Harish-Chandra homomorphism $\vartheta : W_\chi \to U(\mathfrak{g}_0)$ is injective ([12]). We have

$$(-1)^r\vartheta(\pi(e^{(l+r-1)}_{lp,l(q-1)+1})) = U^{\otimes l}\circ\Delta^{op}_l(T^{(r)}_{q,p}), \quad (-1)^r\vartheta(\pi(f^{(l+r-1)}_{lp,l(q-1)+1})) = U^{\otimes l}\circ\Delta^{op}_l(T^{(r)}_{-q,p}).$$

Hence $\varphi = \vartheta^{-1}\circ U^{\otimes l}\circ \Delta^{op}_l$ is a surjective homomorphism $\varphi : Y(Q(\frac{n}{l})) \longrightarrow W_\chi$. □

This proves Theorem 4.2.

Theorem 5.10.

$$U^{\otimes k}\circ\Delta^{op}_k(Y(Q(\frac{n}{l}))) = ev^{\otimes k}\circ\Delta_k(Y(Q(\frac{n}{l}))). \tag{5.16}$$

Proof. First, we will prove the following

Lemma 5.11.

$$\bar{ev} \circ S(T_{q,p}^{(r)+}) = U(T_{q,p}^{(r)+}). \tag{5.17}$$

Proof. According to (4.10) and (4.12) we have that

$$\bar{ev} \circ S(T_{p,q}^{(r)}) = (-1)^r \bar{ev}((\sum_{i_1,\dots i_{r-1} \in \{\pm 1,\dots \pm n\}} (-1)^{\nu(p,i_1,\dots,i_{r-1},q)} T_{p,i_1}^{(1)} T_{i_1,i_2}^{(1)} \dots T_{i_{r-1},q}^{(1)})). \tag{5.18}$$

We proceed by induction on r. The statement is obviously true if $r = 1$. Assume that (5.17) holds for r. Then according to (5.18)

$$\bar{ev} \circ S(T_{q,p}^{(r+1)}) = (-1)^{r+1} \bar{ev}(\sum_{i_1,\dots,i_r \in \{\pm 1,\dots,\pm \frac{n}{l}\}} (-1)^{\nu(q,i_1,\dots,i_r,p)} T_{q,i_1}^{(1)} T_{i_1,i_2}^{(1)} \dots T_{i_{r-1},i_r}^{(1)} T_{i_r,p}^{(1)}) =$$

$$(-1)^{r+1} \sum_{i_r=1}^{\frac{n}{l}} \bar{ev}(\sum_{i_1,\dots,i_{r-1} \in \{\pm 1,\dots,\pm \frac{n}{l}\}} (-1)^{\nu(q,i_1,\dots,i_r)} T_{q,i_1}^{(1)} T_{i_1,i_2}^{(1)} \dots T_{i_{r-1},i_r}^{(1)} T_{i_r,p}^{(1)}) +$$

$$(-1)^{r+1} \sum_{i_r=1}^{\frac{n}{l}} \bar{ev}(\sum_{i_1,\dots,i_{r-1} \in \{\pm 1,\dots,\pm \frac{n}{l}\}} (-1)^{\nu(q,i_1,\dots,-i_r)+1} T_{q,i_1}^{(1)} T_{i_1,i_2}^{(1)} \dots T_{i_{r-1},-i_r}^{(1)} T_{-i_r,p}^{(1)}) =$$

$$(-1)^{r+1} \sum_{i_r=1}^{\frac{n}{l}} \bar{ev}(T_{i_r,p}^{(1)}) \bar{ev}(\sum_{i_1,\dots,i_{r-1} \in \{\pm 1,\dots,\pm \frac{n}{l}\}} (-1)^{\nu(q,i_1,\dots,i_r)} T_{q,i_1}^{(1)} T_{i_1,i_2}^{(1)} \dots T_{i_{r-1},i_r}^{(1)}) +$$

$$(-1)^{r+1} \sum_{i_r=1}^{\frac{n}{l}} \bar{ev}(T_{-i_r,p}^{(1)}) \bar{ev}(\sum_{i_1,\dots,i_{r-1} \in \{\pm 1,\dots,\pm \frac{n}{l}\}} (-1)^{\nu(-q,-i_1,\dots,i_r)} T_{-q,-i_1}^{(1)} T_{-i_1,-i_2}^{(1)} \dots T_{-i_{r-1},i_r}^{(1)})(-1)^r =$$

$$(-1)^{r+1} \sum_{i_r=1}^{\frac{n}{l}} e_{p,i_r} (\bar{ev} \circ S)(T_{q,i_r}^{(r)})(-1)^r + f_{p,i_r} (\bar{ev} \circ S)(T_{-q,i_r}^{(r)}).$$

Similarly,

$$\bar{ev} \circ S(T_{-q,p}^{(r+1)}) = (-1)^{r+1} \bar{ev}(\sum_{i_1,\dots,i_r \in \{\pm 1,\dots,\pm \frac{n}{l}\}} (-1)^{\nu(-q,i_1,\dots,i_r,p)} T_{-q,i_1}^{(1)} T_{i_1,i_2}^{(1)} \dots T_{i_{r-1},i_r}^{(1)} T_{i_r,p}^{(1)}) =$$

$$(-1)^{r+1} \sum_{i_r=1}^{\frac{n}{l}} \bar{ev}(\sum_{i_1,\dots,i_{r-1} \in \{\pm 1,\dots,\pm \frac{n}{l}\}} (-1)^{\nu(-q,i_1,\dots,i_r)} T_{-q,i_1}^{(1)} T_{i_1,i_2}^{(1)} \dots T_{i_{r-1},i_r}^{(1)} T_{i_r,p}^{(1)}) +$$

$$(-1)^{r+1} \sum_{i_r=1}^{\frac{n}{l}} \bar{ev}(\sum_{i_1,\dots,i_{r-1} \in \{\pm 1,\dots,\pm \frac{n}{l}\}} (-1)^{\nu(-q,i_1,\dots,-i_r)} T_{-q,i_1}^{(1)} T_{i_1,i_2}^{(1)} \dots T_{i_{r-1},-i_r}^{(1)} T_{-i_r,p}^{(1)}) =$$

$$(-1)^{r+1}\sum_{i_r=1}^{\frac{n}{l}}\bar{ev}(T^{(1)}_{i_r,p})\bar{ev}(\sum_{i_1,\dots,i_{r-1}\in\{\pm1,\dots,\pm\frac{n}{l}\}}(-1)^{\nu(-q,i_1,\dots,i_r)}T^{(1)}_{-q,i_1}T^{(1)}_{i_1,i_2}\dots T^{(1)}_{i_{r-1},i_r})+$$

$$(-1)^{r+1}\sum_{i_r=1}^{\frac{n}{l}}\bar{ev}(T^{(1)}_{-i_r,p})\bar{ev}(\sum_{i_1,\dots,i_{r-1}\in\{\pm1,\dots,\pm\frac{n}{l}\}}(-1)^{\nu(q,-i_1,\dots,i_r)}T^{(1)}_{q,-i_1}T^{(1)}_{-i_1,-i_2}\dots T^{(1)}_{-i_{r-1},i_r})(-1)^r=$$

$$(-1)^{r+1}\sum_{i_r=1}^{\frac{n}{l}}e_{p,i_r}(\bar{ev}\circ S)(T^{(r)}_{-q,i_r})(-1)^r+f_{p,i_r}(\bar{ev}\circ S)(T^{(r)}_{q,i_r}).$$

Hence

$$\bar{ev}\circ S(T^{(r+1)+}_{q,p})=(-1)^{r+1}\sum_{i_r=1}^{\frac{n}{l}}e_{p,i_r}(\bar{ev}\circ S)(T^{(r)+}_{q,i_r})(-1)^r+f_{p,i_r}(\bar{ev}\circ S)(T^{(r)+}_{q,i_r})=$$

$$(-1)^{r+1}\sum_{i_r=1}^{\frac{n}{l}}(e_{p,i_r}+(-1)^r f_{p,i_r})((-1)^r U(T^{(r)+}_{q,i_r})=$$

$$(-1)^{r+1}\sum_{i_r=1}^{\frac{n}{l}}(e_{p,i_r}+(-1)^r f_{p,i_r})\sum_{1\le p_1,\dots,p_{r-1}\le\frac{n}{l}}(e_{i_r,p_1}+(-1)^{r-1}f_{i_r,p_1})\dots(e_{p_{r-1},q}+f_{p_{r-1},q})=U(T^{(r+1)+}_{q,p}).$$

□

It follows from Lemma 5.11 that

$$\bar{ev}^{\otimes k}\circ S^{\otimes k}\otimes\Delta_k^{op}(T^{(r)+}_{q,p})=U^{\otimes k}\circ\Delta_k^{op}(T^{(r)+}_{q,p}).$$

Finally, observe that the following diagram, where $Y:=Y(Q(\frac{n}{l}))$ is commutative:

$$\begin{array}{ccccccc}
Y & \xrightarrow{\Delta} & Y\otimes Y & \xrightarrow{id\circ\Delta} & Y\otimes Y\otimes Y & \xrightarrow{id\circ id\circ\Delta} & \dots \\
{\scriptstyle S}\uparrow & & {\scriptstyle S\otimes S}\uparrow & & {\scriptstyle S\otimes S\otimes S}\uparrow & & {\scriptstyle S^{\otimes 4}}\uparrow \\
Y & \xrightarrow{\Delta^{op}} & Y\otimes Y & \xrightarrow{\Delta^{op}\circ id} & Y\otimes Y\otimes Y & \xrightarrow{\Delta^{op}\circ id\circ id} & \dots
\end{array}$$

Hence

$$\bar{ev}^{\otimes k}\circ\Delta_k\circ S(T^{(r)+}_{q,p})=U^{\otimes k}\circ\Delta_k^{op}(T^{(r)+}_{q,p}).\tag{5.19}$$

This completes the proof of Theorem 5.10. □

Corollary 5.12.

$$W_\chi\cong\bar{ev}^{\otimes l}\circ\Delta_l(Y(Q(\frac{n}{l})))$$

Proof. Follows from Theorem 4.2 and (5.16) where $k=l$. □

Theorem 5.13. *Let* $1 \le p, q \le \frac{n}{l}$.

$$ev^{\otimes l} \circ \Delta_l(T^{(r)}_{q,p}) = (-1)^r [\sum_{1\le p_1,p_2,\ldots,p_{r-1}\le \frac{n}{l}}$$

$$\sum_{1\le i_1<i_2<\ldots<i_r\le l} (x^{i_1}_{p,p_1} + (-1)^{r+1}\xi^{i_1}_{p,p_1})(x^{i_2}_{p_1,p_2} + (-1)^r \xi^{i_2}_{p_1,p_2})\ldots(x^{i_r}_{p_{r-1},q} + \xi^{i_r}_{p_{r-1},q})]_{even},$$

$$ev^{\otimes l} \circ \Delta_l(T^{(r)}_{-q,p}) = (-1)^r [\sum_{1\le p_1,p_2,\ldots,p_{r-1}\le \frac{n}{l}}$$

$$\sum_{1\le i_1<i_2<\ldots<i_r\le l} (x^{i_1}_{p,p_1} + (-1)^{r+1}\xi^{i_1}_{p,p_1})(x^{i_2}_{p_1,p_2} + (-1)^r \xi^{i_2}_{p_1,p_2})\ldots(x^{i_r}_{p_{r-1},q} + \xi^{i_r}_{p_{r-1},q})]_{odd}$$

Proof. According to (4.6), if $1 \le m \le l$, then

$$\Delta_m(T^{(r)}_{q,p}) = \sum_{s=0}^{r}\sum_{k=1}^{\frac{n}{l}} (-1)^{(p(q)+p(k))(p(p)+p(k))} \Delta_{m-1}(T^{(s)}_{q,k}) \otimes T^{(r-s)}_{k,p} =$$

$$\sum_{s=0}^{r}\sum_{k=1}^{\frac{n}{l}} \Big(1^{\otimes m-1} \otimes T^{(r-s)}_{k,p}\Big) \cdot \Big(\Delta_{m-1}(T^{(s)}_{q,k}) \otimes 1\Big).$$

Then

$$\Delta_m(T^{(r)+}_{q,p}) = \sum_{s=0}^{r}\sum_{k=1}^{\frac{n}{l}} \Big(1^{\otimes m-1} \otimes (T^{(r-s)}_{k,p} + (-1)^s T^{(r-s)}_{-k,p})\Big) \cdot \Big(\Delta_{m-1}(T^{(s)}_{q,k} + T^{(s)}_{-q,k}) \otimes 1\Big).$$

Hence, using induction on m we have that

$$ev^{\otimes m} \circ \Delta_m(T^{(r)+}_{q,p}) =$$

$$\sum_{s=r-1,r}\sum_{k=1}^{\frac{n}{l}} \Big(1^{\otimes m-1} \otimes ev(T^{(r-s)}_{k,p} + (-1)^s T^{(r-s)}_{-k,p})\Big) \cdot \Big(ev^{\otimes m-1}(\Delta_{m-1}(T^{(s)+}_{q,k})) \otimes 1\Big) =$$

$$(-1)^r [\sum_{k=1}^{\frac{n}{l}} (x^1_{p,k} + (-1)^{r-1}\xi^1_{p,k})\Big(\sum_{1\le p_2,\ldots,p_{r-1}\le \frac{n}{l}} (\sum_{2\le i_2<\ldots<i_r\le m} (x^{i_2}_{k,p_2} + (-1)^r \xi^{i_2}_{k,p_2})\ldots(x^{i_r}_{p_{r-1},q} + \xi^{i_r}_{p_{r-1},q}))\Big) +$$

$$\sum_{1\le p_1,p_2,\ldots,p_{r-1}\le \frac{n}{l}} \Big(\sum_{2\le i_1<\ldots<i_r\le m} (x^{i_1}_{p,p_1} + (-1)^{r+1}\xi^{i_1}_{p,p_1})(x^{i_2}_{p_1,p_2} + (-1)^r \xi^{i_2}_{p_1,p_2})\ldots(x^{i_r}_{p_{r-1},q} + \xi^{i_r}_{p_{r-1},q})\Big)] =$$

$$(-1)^r \sum_{1\le p_1,p_2,\ldots,p_{r-1}\le \frac{n}{l}} \Big(\sum_{1\le i_1<i_2<\ldots<i_r\le m} (x^{i_1}_{p,p_1} + (-1)^{r+1}\xi^{i_1}_{p,p_1})\ldots(x^{i_r}_{p_{r-1},q} + \xi^{i_r}_{p_{r-1},q})\Big).$$

If $m = l$ we obtain the proof of Theorem 5.13. ∎

Corollary 5.14. *Let $1 \le p, q \le \frac{n}{l}$. Then for $r \le l$*

$$\bar{ev}^{\otimes l} \circ \Delta_l(T^{(r)}_{q,p}) = [\sum_{1\le p_1,p_2,\dots,p_{r-1}\le \frac{n}{l}} \sum_{1\le i_1<i_2<\dots<i_r\le l} (x^{i_1}_{p,p_1} + (-1)^{r+1}\xi^{i_1}_{p,p_1})(x^{i_2}_{p_1,p_2} + (-1)^r\xi^{i_2}_{p_1,p_2})\dots(x^{i_r}_{p_{r-1},q} + \xi^{i_r}_{p_{r-1},q})]_{even},$$

$$\bar{ev}^{\otimes l} \circ \Delta_l(T^{(r)}_{-q,p}) = [\sum_{1\le p_1,p_2,\dots,p_{r-1}\le \frac{n}{l}} \sum_{1\le i_1<i_2<\dots<i_r\le l} (x^{i_1}_{p,p_1} + (-1)^{r+1}\xi^{i_1}_{p,p_1})(x^{i_2}_{p_1,p_2} + (-1)^r\xi^{i_2}_{p_1,p_2})\dots(x^{i_r}_{p_{r-1},q} + \xi^{i_r}_{p_{r-1},q})]_{odd}$$

and $\bar{ev}^{\otimes l} \circ \Delta_l(T^{(r)}_{\pm q,p}) = 0$ for $r > l$.

Proof. Recall that $\bar{ev} = \alpha \circ ev$, where α is the principal anti-automorphism of $U(\mathfrak{g})$: $\alpha(X) = -X$ for all $X \in \mathfrak{g}$. Then

$$\bar{ev}^{\otimes l} \circ \Delta_l(T^{(r)}_{q,p}) = \alpha^{\otimes l} \circ ev^{\otimes l} \circ \Delta_l(T^{(r)}_{q,p}) = (-1)^r ev^{\otimes l} \circ \Delta_l(T^{(r)}_{q,p}),$$
$$\bar{ev}^{\otimes l} \circ \Delta_l(T^{(r)}_{-q,p}) = \alpha^{\otimes l} \circ ev^{\otimes l} \circ \Delta_l(T^{(r)}_{-q,p}) = (-1)^r ev^{\otimes l} \circ \Delta_l(T^{(r)}_{-q,p}).$$

□

Corollary 5.15.

$$W_\chi \cong ev^{\otimes l} \circ \Delta_l(Y(Q(\frac{n}{l})))$$

Proof. Follows from Corollary 5.12 and Corollary 5.14. □

Definition 5.16. Let $1 \le p, q \le \frac{n}{l}$ and $r > 0$. Let

$$z^{(r)}_{q,p} = [\sum_{1\le p_1,p_2,\dots,p_{r-1}\le \frac{n}{l}} \sum_{l\ge i_1\ge i_2\ge\dots\ge i_r\ge 1} (x^{i_1}_{p,p_1} + (-1)^{r+1}\xi^{i_1}_{p,p_1})(x^{i_2}_{p_1,p_2} + (-1)^r\xi^{i_2}_{p_1,p_2})\dots(x^{i_r}_{p_{r-1},q} + \xi^{i_r}_{p_{r-1},q})]_{even},$$

$$z^{(r)}_{-q,p} = [\sum_{1\le p_1,p_2,\dots,p_{r-1}\le \frac{n}{l}} \sum_{l\ge i_1\ge i_2\ge\dots\ge i_r\ge 1} (x^{i_1}_{p,p_1} + (-1)^{r+1}\xi^{i_1}_{p,p_1})(x^{i_2}_{p_1,p_2} + (-1)^r\xi^{i_2}_{p_1,p_2})\dots(x^{i_r}_{p_{r-1},q} + \xi^{i_r}_{p_{r-1},q})]_{odd}$$

$$\tilde{z}^{(r)}_{q,p} = [\sum_{1\le p_1,p_2,\dots,p_{r-1}\le \frac{n}{l}} \sum_{1\le i_1<i_2<\dots<i_r\le l} (x^{i_1}_{p,p_1} + (-1)^{r+1}\xi^{i_1}_{p,p_1})(x^{i_2}_{p_1,p_2} + (-1)^{r}\xi^{i_2}_{p_1,p_2})\dots(x^{i_r}_{p_{r-1},q} + \xi^{i_r}_{p_{r-1},q})]_{even},$$

$$\tilde{z}^{(r)}_{-q,p} = [\sum_{1\le p_1,p_2,\dots,p_{r-1}\le \frac{n}{l}} \sum_{1\le i_1<i_2<\dots<i_r\le l} (x^{i_1}_{p,p_1} + (-1)^{r+1}\xi^{i_1}_{p,p_1})(x^{i_2}_{p_1,p_2} + (-1)^{r}\xi^{i_2}_{p_1,p_2})\dots(x^{i_r}_{p_{r-1},q} + \xi^{i_r}_{p_{r-1},q})]_{odd}$$

Theorem 5.17. *Let $1 \le p, q \le \frac{n}{l}$ and $r > 0$. Then*

$$\sum_{t+s=r}[\sum_{j>0}\Big((-1)^s z^{(s)}_{j,p} + (-1)^r z^{(s)}_{-j,p}\Big)\Big(\tilde{z}^{(t)}_{q,j} + \tilde{z}^{(t)}_{-q,j}\Big)] = 0. \tag{5.20}$$

Proof. Note that by (4.8)

$$\sum_{t+s=r}[\sum_{j\in\{\pm1,\pm2,\dots,\pm\frac{n}{l}\}} (-1)^{(p(q)+p(j))(p(j)+p(p))} T^{(t)}_{q,j}\tilde{T}^{(s)}_{j,p}] = \delta_{q,p}. \tag{5.21}$$

Applying the anti-homomorphism $\bar{ev}^{\otimes l}\circ\Delta_l$ to (5.21), we obtain that

$$\sum_{t+s=r}[\sum_{j\in\{\pm1,\pm2,\dots,\pm\frac{n}{l}\}} \bar{ev}^{\otimes l}\circ\Delta_l(\tilde{T}^{(s)}_{j,p})\bar{ev}^{\otimes l}\circ\Delta_l(T^{(t)}_{q,j})] = \delta_{q,p}.$$

Recall that $\tilde{T}^{(s)}_{j,p} = S(T^{(s)}_{j,p})$. Then by (5.19), if $r > 0$, then

$$\sum_{t+s=r}[\sum_{j\in\{\pm1,\pm2,\dots,\pm\frac{n}{l}\}} U^{\otimes l}\circ\Delta^{op}_l(T^{(s)}_{j,p})\bar{ev}^{\otimes l}\circ\Delta_l(T^{(t)}_{q,j})] = 0. \tag{5.22}$$

Similarly, we have that

$$\sum_{t+s=r}[\sum_{j\in\{\pm1,\pm2,\dots,\pm\frac{n}{l}\}} U^{\otimes l}\circ\Delta^{op}_l(T^{(s)}_{j,p})\bar{ev}^{\otimes l}\circ\Delta_l(T^{(t)}_{-q,j})] = 0. \tag{5.23}$$

Adding the equations (5.22) and (5.23) we obtain

$$\sum_{t+s=r}[\sum_{j\in\{1,2,\dots,\frac{n}{l}\}} U^{\otimes l}\circ\Delta^{op}_l\Big(T^{(s)}_{j,p} + (-1)^{r-s}T^{(s)}_{-j,p}\Big)\bar{ev}^{\otimes l}\circ\Delta_l\Big(T^{(t)}_{q,j} + T^{(t)}_{-q,j}\Big)] = 0. \tag{5.24}$$

Equation (5.24) is equivalent to (5.20). □

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