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VORONOI CONJECTURE FOR FIVE-DIMENSIONAL PARALLELOHEDRA

Alexey Garber
The University of Texas Rio Grande Valley

Alexander Magazinov

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ALEXEY GARBER AND ALEXANDER MAGAZINOV

Abstract. We prove the Voronoi conjecture for five-dimensional parallelohedra. Namely, we show that if a convex five-dimensional polytope P tiles $\mathbb{R}^5$ with translations, then P is an affine image of the Dirichlet-Voronoi cell for a five-dimensional lattice.

Contents

1. Introduction

Tilings serve as source of numerous patterns for art objects as well as inspiration for mathematical notions such as symmetry groups associated with the whole tiling or with underlying point set. Particularly, Hilbert's eighteenth problem [28] asks about finiteness of number of classes of space groups (discrete groups of isometries with compact fundamental region) and about existence of polytope that tiles the space with congruent copies without being a fundamental region of any space group; both parts of Hilbert's eighteenth problem were solved by Bieberbach [2, 3] and Reinhardt [43] respectively. However, even for small dimensions, it is complicated to give a complete classification of convex polytopes that can tile Euclidean space with congruent copies with or without restricting to fundamental regions of space groups.

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Without an attempt to make a complete survey, we mention a few results regarding such polygons and polytopes. For $\mathbb{R}^2$, a proof of completeness of the list of known convex pentagons that tile the plane was announced only recently by Rao [42]. For three-dimensional space, even the maximal number of faces of a polytope that tiles $\mathbb{R}^3$ with congruent copies is unknown; the best known example has 38 faces and is due to Engel [12], see also [26]. Engel's polytope not only tiles $\mathbb{R}^3$ but gives a regular tiling meaning that the symmetry group of the tiling acts transitively on the tile set. Also, there are polytopes in $\mathbb{R}^3$, for example the Schmitt-Conway-Danzer polytope [45, Sect. 7.2], that tile it only in aperiodic way assuming the each two tiles are congruent using a rigid motion symmetry while reflections are not allowed.

In this paper we restrict our attention to tilings of Euclidean space with convex polytopes where every two tiles are translation of each other, the tilings with *parallelohedra*. The systematic study of parallelohedra and their properties goes back do Minkowski [39], Fedorov [16], Voronoi [51], and Delone [5].

One of the most intriguing and still open conjectures in parallelohedra theory is the Voronoi conjecture [51] that connects d-dimensional paralleloherda with d-dimensional lattices and their Dirichlet-Voronoi cells. This conjecture originates from Voronoi's study of geometric theory of positive definite quadratic forms [51].

A *lattice* in $\mathbb{R}^d$ is (a translation) of the set of all integer linear combinations of some basis of $\mathbb{R}^d$. For a fixed lattice Λ, its *Dirichlet-Voronoi polytope*, or just *Voronoi polytope* is the set of of points that are closer to a fixed point $x \in \Lambda$ than to any other point of Λ.

Conjecture (G. Voronoi)**.** *For every d-dimensional parallelhedron P there exists a d-dimensional lattice Λ and an affine transformation $\mathcal{A}$ such that $\mathcal{A}(P)$ is the Dirichlet-Voronoi polytope of Λ.*

Thus, the Voronoi conjecture claims that Dirichlet-Voronoi polytopes for lattices are essentially the only polytopes that give the best bound for lattice covering/packing density. Similar questions for lattice packings and coverings are studied for spheres as well as for other convex bodies; we refer to work of Schürmann and Vallentin [44] on computational approaches to lattice sphere packings and coverings, review of Gruber [25] on lattice packings and coverings with convex bodies, and breakthrough works of Viazovska [50] and Cohn, Kumar, Miller, Radchenko and Viazovska [4] on densest sphere packings in dimensions 8 and 24 for more details and additional references.

Also, parallelohedra appear in the study of spectral sets in d-dimensional space. As it was recently shown by Lev and Matolcsi [33], if a convex body $\Omega \subset \mathbb{R}^d$ is a spectral set, i.e. if there is an orthogonal basis of exponential functions in $L^2(\Omega)$, then Ω is a parallelohedron as it was conjectured by Fuglede [17]. We also refer to works of Kolountzakis [31]; Kolountzakis and Papadimitrakis [32]; Iosevich, Katz, and Tao [30]; and Greenfeld and Lev [20] for more details on spectral sets and their properties that resemble properties of parallelohedra that we introduce in further sections.

It is worth noting that in the works on packings and coverings mostly Dirichlet-Voronoi parallelohedra appear while the Fuglede conjecture and corresponding results concern general parallelohedra. The Voronoi conjecture essentially claims that every parallelohedron is a Dirichlet-Voronoi parallelohedron.

The Voronoi conjecture is proved in small dimensions $d \leq 4$. Two-dimensional case is usually treated as folklore as it is easy to see that only parallelograms and centrally symmetric hexagons are two-dimensional parallelohedra, and three-dimensional case is usually attributed to Fedorov [16] who obtained a complete list of 5 combinatorial types of three-dimensional parallelohedra, see Figure 1. Delone [5] proved the Voronoi conjecture in $\mathbb{R}^4$ while also providing a list of 51 four-dimensional parallelohedra which was completed by Stogrin [46] who found the last 52nd four-dimensional parallelohedron.

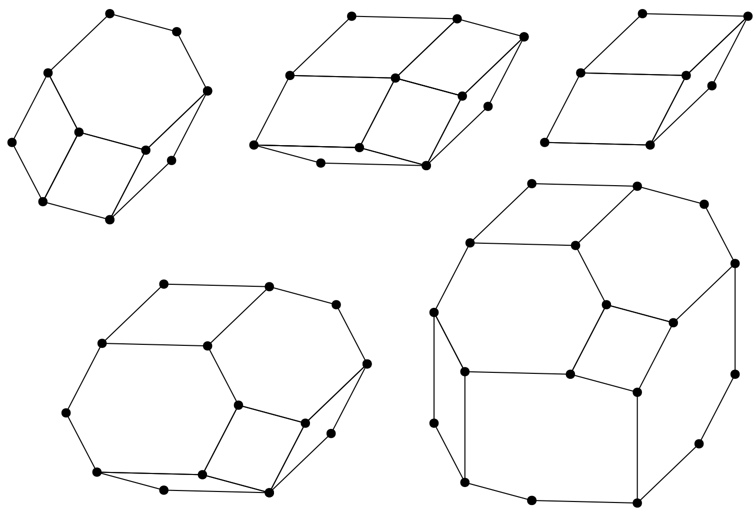

FIGURE 1. Five three-dimensional parallelohedra: hexagonal prism, rhombic dodecahedron, parallelepiped, elongated dodecahedron, and truncated octahedron.

Another series of results involves restrictions on local structure of face-to-face tiling by parallelohedra. Various types of combinatorial restrictions on the local structure around face of P imply that P satisfies Voronoi conjecture as shown by Voronoi [51], Zhitomirski [52], and Ordine [40]; we give more details on these results in Section 2. Face-to-face property of the tiling is crucial for these results, but as it was shown independently by Venkov [48] and McMullen [37] (see also [38]) if a convex polytope admits any tiling with parallel copies (for example, a brickwall tiling), then it admits a face-to-face tiling. We also would like to mention recent results of the authors with Gavrilyuk [18] and of Grishukhin [23] that prove the Voronoi conjecture for parallelohedra with global combinatorial properties.

Also, Erdahl [15] proved the Voronoi conjecture for parallelohedra that are zonotopes. This can be reformulated in terms of regularity for oriented matroids, see [7] for example. The paper [34] of the second author proves a generalization of Erdahl's result for extensions of Voronoi parallelohedra.

The main result of this paper is the proof of the Voronoi conjecture for five-dimensional parallelohedra, Theorem 4.1. This theorem also implies that the list of 110 244 five-dimensional Voronoi parallelohedra obtained in [10] is the complete list of combinatorial types of parallelohedra in $\mathbb{R}^5$ which is summarized in Corollary 4.2.

It should be mentioned that some sources refer to the paper of Engel [13] for a proof of Voronoi conjecture in $\mathbb{R}^5$. The main result stated in [13] for $\mathbb{R}^5$ claims that "every parallelohedron in $\mathbb{R}^5$ is combinatorially equivalent to a Voronoi parallelohedron". We have a strong doubt that this statement, and consequently the Voronoi conjecture in $\mathbb{R}^5$, has a rigorous justification in [13] as the methods used by Engel involve only zone contraction and zone extension procedures for Dirichlet-Voronoi parallelohedra of lattices represented using the cone of positive definite quadratic forms and studying faces of subcones that represent the same Delone tiling. These operations are equivalent to adding a segment as Minkowski sum or "subtracting" such segments if possible. However, in our opinion, the paper [13] does not contain a proof that every parallelhedron in $\mathbb{R}^5$ can be obtained from some Dirichlet-Voronoi parallelohedron using the operations of zone contraction and zone extension or can be found on the boundary of a secondary cone for primitive parallelohedron. This means that some five-dimensional parallelohedra could be missed by computations using an implementation of Engel's algorithm. The final judgment on the status of Engel's paper [13] and the results presented there is outside the scope of our work.

We also refer to [27, Section 3.2] as another source of known results on the Voronoi conjecture.

The paper is organized as follows. In Section 2 we introduce definitions and main concepts and present key known properties of parallelohedra that are used in our proof. In Section 3 we prove several lemmas that are crucial for our approach to five-dimensional parallelohedra. We pay special attention to combinatorics of local structure of parallelohedra tilings as this is the main tool that we use.

In Section 4 we provide an outline for the proof of the Voronoi conjecture in $\mathbb{R}^5$ and in Sections 5 through 11 we provide all the details for the proof.

The last Section 12 is devoted to discussion on paralleohedra and the Voronoi conjecture in higher dimensions.

2. Definitions and key properties

In this section we give an overview of known properties of parallelohedra and dual cells that we need further. In most cases we state the properties for d-dimensional parallelohedra without restricting to five-dimensional case.

Definition 2.1. A convex polytope P in $\mathbb{R}^d$ is called a *parallelohedron* if P tiles $\mathbb{R}^d$ with translated copies.

In the classical setting, the tiling with translated copies of P must be a face-to-face tiling. However as it was shown later the face-to-face restriction is redundant. Particularly, a convex d-dimensional polytope P is a parallelohedron if and only if P satisfies the following *Minkowski-Venkov conditions*.

(1) P is centrally symmetric;
(2) Each facet of P is centrally symmetric;
(3) Projection of P along any of its face of codimension 2 is a parallelogram or centrally symmetric hexagon.

Minkowski [39] proved that every convex polytope that tiles $\mathbb{R}^d$ with translated copies in face-to-face manner satisfies first two conditions. Venkov [48] proved that all three conditions are necessary and sufficient for a convex polytope P to tile $\mathbb{R}^d$ with translated copies in face-to-face or non-face-to-face way; McMullen [37] (see also [38]) obtained the results of Venkov independently. We also refer to work of Groemer [24] for necessity of first two of Minkowski-Venkov conditions in some cases of packings, not necessarily face-to-face. The first two Minkowski-Venkov conditions are also necessary for coverings with constant multiplicity as shown by Gravin, Robins, and Shiryaev [19].

For a fixed parallelohedron P there is a unique face-to-face tiling of $\mathbb{R}^d$ with translated copies of P assuming one copy is centered at the origin and from now on we will consider only the case of this particular tiling. In that case the centers of the polytopes of the tiling form a d-dimensional lattice.

Definition 2.2. We use the notations $\mathcal{T}_P$ and Λ_P for the tiling and the lattice respectively assuming P is centered at the origin. The lattice Λ_P is called the *lattice associated with* P, or the *lattice of the tiling* $\mathcal{T}_P$.

The tiling $\mathcal{T}_P$ is preserved under translations by vectors from Λ_P and by central symmetries in the points of $\frac{1}{2}\Lambda_P$ that preserve Λ_P.

2.1. **Dual cells.**

In the course of our proof we mainly study local combinatorics of the tiling $\mathcal{T}_P$. The main tool we use is the dual cell technique; the dual cell of a face F of $\mathcal{T}_P$ encodes which copies of P are incident to F.

Definition 2.3. Let F be a non-empty face of $\mathcal{T}_P$. The *dual cell* $\mathcal{D}(F)$ of F is the set of all centers of copies of P in $\mathcal{T}_P$ that are incident to F, so

$$\mathcal{D}(F) := \{x \in \Lambda_P | F \subseteq (P + x)\}.$$

If F is a face of codimension k, then we say that $\mathcal{D}(F)$ is a *dual cell of dimension* k, or *dual* k*-cell*.

If F is a facet, then $\mathcal{D}(F)$ contains exactly two points and a segment connecting these two points is called a *facet vector*. Facet vectors correspond to pairs of copies of P that share facets in $\mathcal{T}_P$.

The set of all dual cells of $\mathcal{T}_P$ inherits a face lattice structure dual to the face lattice structure of the tiling $\mathcal{T}_P$. Namely, if a face F is a subface of a face F', then the cell $\mathcal{D}(F')$ is a subcell of the cell $\mathcal{D}(F)$. Hence the set of all dual cells form a cell complex that we denote $\mathcal{C}_P$.

In a specific case when P is the Dirichlet-Voronoi cell for Λ_P, the dual cell of a face F is (the vertex set of) a face of the Delone tesselation for Λ_P. Particularly, the dual cells of vertices of $\mathcal{T}_P$ are the Delone polytopes for Λ_P and these dual cells tile $\mathbb{R}^d$. Consequently, if the Voronoi conjecture is true for P, then dual cells are affine images of vertex sets of faces of Delone polytopes with inherited face lattice, so the dual cell should carry the structure of convex polytopes. In certain cases this structure can be established without prior assumption that P satisfies the Voronoi conjecture.

Definition 2.4. Let $\mathcal{D}(F)$ be a dual k-cell. If the face lattice of $\mathcal{D}(F)$ within $\mathcal{C}_P$ coincides with the face lattice of the convex polytope $T := \operatorname{conv} \mathcal{D}(F)$, then we say that $\mathcal{D}(F)$ is *combinatorially equivalent* to T, or just that $\mathcal{D}(F)$ is *combinatorially* T.

We note that this definition requires that T is a k-dimensional polytope however this is not proved in general for every P and every k.

The theorem of Voronoi [51] can be formulated in terms of dual d-cells.

Theorem 2.5 (G. Voronoi)**.** *If all dual d-cells of $\mathcal{T}_P$ for d-dimensional P are combinatorially d-simplices, then the Voronoi conjecture is true for P.*

The Minkowski-Venkov conditions imply that there are only two types of dual 2-cells. For a fixed face F of codimension 2 of P, if a projection of P along F is a centrally symmetric hexagon, then the dual cell $\mathcal{D}(F)$ is combinatorially triangle, and if this projection is a parallelogram, then $\mathcal{D}(F)$ is combinatorially parallelogram, see Figure 2.

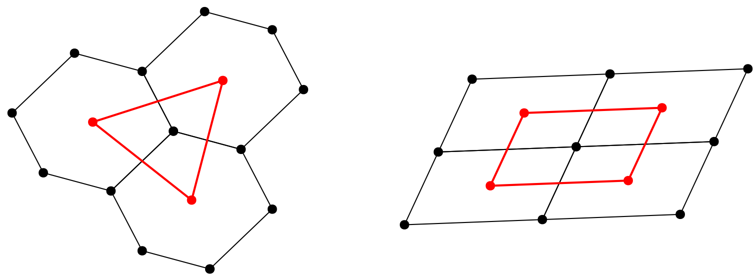

FIGURE 2. Two types of dual 2-cells. Original parallelohedra are black polygons and dual cells are red.

The theorem of Zhitomirski [52] can be stated in terms of dual cells as well.

Theorem 2.6 (O. Zhitmorski)**.** *If all dual 2-cells of $\mathcal{T}_P$ for d-dimensional P are combinatorially triangles, then the Voronoi conjecture is true for P.*

The complete list of dual 3-cells is also known. It was established by Delone [5] (see also [35]) as an intermediate step for his proof of the Voronoi conjecture in $\mathbb{R}^4$.

Theorem 2.7. *If F is a codimension 3 face of d-dimensional parallelohedron P, then $\mathcal{D}(F)$ is combinatorially equivalent to one of next five 3-dimensional polytopes.*

- *Tetrahedron;*
- *Octahedron;*
- *Pyramid over parallelogram;*
- *Triangular prism;*
- *Cube.*

The first three types of dual 3-cells above exhibit a "connectivity" property in the following sense. Each pair of edges within one dual 3-cell of that type can be connected by a path of triangular dual 2-cells. This property was exploited by Ordine [40] (see also [41]) in the following theorem.

Theorem 2.8 (A. Ordine)**.** *If all dual 3-cells of $\mathcal{T}_P$ for d-dimensional P are combinatorially tetrahedra, octahedra, or pyramids over parallelograms, then the Voronoi conjecture is true for P.*

The complete list of dual k-cells for $k > 3$ is not known however we expect that this list coincides with the list of lattice Delone polytopes of dimension k which are known for dimension $k \leq 6$, see [8] for details.

Definition 2.9. A non-empty face F of $\mathcal{T}_P$ is called a *contact face* if F is an intersection of two copies of P within $\mathcal{T}_P$. So for some $x, y \in \Lambda_P$

$$F = (P + x) \cap (P + y).$$

In that case the face F and its dual cell $\mathcal{D}(F)$ are centrally symmetric with respect to $\frac{x+y}{2} \in \frac{1}{2}\Lambda_P$ as this central symmetry preserves $\mathcal{T}_P$. The central symmetry of dual cells is a signature property of contact faces. If $\mathcal{D}(F)$ is centrally symmetric, then F is a contact face and centers of F and $\mathcal{D}(F)$ coincide.

Particularly, all facets of $\mathcal{T}_P$ are contact faces with dual cells being combinatorially segments. Among dual 2- and dual 3-cells, those combinatorially equivalent to parallelograms and to octahdera and parallelopipeds respectively are dual cells of contact faces while others are not.

The points of the shrinked lattice $\frac{1}{2}\Lambda_P$, half-lattice points, are in bijection with contact faces of $\mathcal{T}_P$. A point $x \in \frac{1}{2}\Lambda_P$ is in the relative interior of unique face F of $\mathcal{T}_P$. The central symmetry in x preserves $\mathcal{T}_P$ and hence it preserves the dual cell of F and F itself.

In addition to Euclidean space $\mathbb{R}^d$ and the lattice Λ_P we use two additional (finite) linear spaces. Namely, the *space of parity classes* $\Lambda_p := \Lambda_P/(2\Lambda_P)$ and the *space of half-lattice points* $\Lambda_{1/2} := \left(\frac{1}{2}\Lambda_P\right)/\Lambda_P$. For a point $x \in \Lambda_P$ we call the coset $x + \Lambda_P/(2\Lambda_P)$ the *parity class* of x.

As linear spaces both Λ_p and $\Lambda_{1/2}$ are isomorphic to $\mathbb{F}_2^d$, a d-dimensional linear space over two-element field $\mathbb{F}_2$, but they serve quite different roles. The space of parity classes gives us all possible options for various points in exhaustive approaches throughout Sections 5 through 11 and the space of half-lattice points is used to extract combinatorics of dual cell complex $\mathcal{C}_P$ and contact faces in particular.

We use notations $[x_1, x_2, \ldots, x_d]$ for elements of Λ_p and $\langle x_1, x_2, \ldots, x_d\rangle$ for elements of $\Lambda_{1/2}$ in coordinate representation.

The following lemma is a classical result in parallelohedra theory, see [6] for example.

Lemma 2.10. *If F is a face of $\mathcal{T}_P$, then $\mathcal{D}(F)$ contains at most one representative from each parity class.*

Remark. The following proof of this lemma is using a combinatorial approach that is later used to study other properties of dual cells in Section 3 and throughout our proof of the main result.

Suppose x and y belong to the same parity class and $x, y \in \mathcal{D}(F)$. The polytopes $P + x$ and $P + y$ have non-empty intersection, so they both must contain the midpoint $\frac{x+y}{2}$ of xy because the central symmetry with respect to $\frac{x+y}{2}$ preserves this intersection. However $\frac{x+y}{2} \in \Lambda_P$ and is an internal point of another copy of P which is impossible.

2.2. **Canonical scaling.**

One of the most used approaches to prove the Voronoi conjecture for a class of parallelohedra involves a proof of existence of canonical scaling for polytopes for that class. We use that approach in Sections 5 and 11.

Definition 2.11. Let $\mathcal{T}_P^{d-1}$ be the set of all facets of $\mathcal{T}_P$. For every facet F let n_F be one of its two unit normals. A function $s : \mathcal{T}_P^{d-1} \longrightarrow \mathbb{R}_+$ is called a *canonical scaling* if it satisfies the following conditions.

- If three facets F, G, and H are incident to a non-contact face of codimension 2 then for a certain choice of signs
$$\pm s(F)n_F \pm s(G)n_G \pm s(H)n_H = 0.$$
- If four facets F, G, H, and I are incident to a contact face of codimension 2 then for a certain choice of signs
$$\pm s(F)n_F \pm s(G)n_G \pm s(H)n_H \pm s(I)n_I = 0.$$

Effectively, the first condition dictates that the values of canonical scaling on three facets with common non-contact face of codimension 2 are proportional to absolute values of coefficients of their normals in the corresponding linear dependence, which is unique for three linearly dependent vectors in two-dimensional subspace orthogonal to $F \cap G \cap H$. The second condition only says that values of canonical scaling on two opposite facets at a contact face of codimension 2 are equal however it could be strengthened to equality of canonical scaling for every pair of parallel facets.

As was shown by Voronoi [51], a parallelohedron P satisfies the Voronoi conjecture if and only if $\mathcal{T}_P$ exhibits a canonical scaling. This equivalence was used by Voronoi, Zhitomirski, and Ordine to prove their theorems on Voronoi conjecture for respective classes of parallelohedra.

The first condition for canonical scaling can be transformed into the following notion.

Definition 2.12. Suppose F is a face of codimension 2 with triangular dual cell $\mathcal{D}(F) = ABC$. Each edge of ABC is a dual cell of a facet of $\mathcal{T}_P$; we denote normals of these facets as n_{AB}, n_{BC} and n_{CA}. There is a unique (up to non-zero factor) linear dependence between these normals, say

$$\alpha_{AB} n_{AB} + \alpha_{BC} n_{BC} + \alpha_{CA} n_{CA} = 0$$

with non-zero coefficients.

For a pair of facet vectors AB and AC that are incident to one triangular dual cell we define the *gain function* $\gamma(AB, AC)$ as

$$\gamma(AB, AC) := \frac{|\alpha_{AC}|}{|\alpha_{AB}|}.$$

This notion is naturally extended to a sequence of facet vectors $f_1, \ldots, f_k$ where each two consecutive facet vectors belong to one triangular dual cell as

$$\gamma(f_1, \ldots, f_k) := \gamma(f_1, f_2) \cdot \ldots \cdot \gamma(f_{k-1}, f_k).$$

As it was shown by Garber, Gavrilyuk and Magazinov in [18], a canonical scaling for P exists if and only if the gain function γ is 1 on every appropriate cycle within $\mathcal{T}_P$. We will use this property in Section 11.

2.3. **Free directions.**

Definition 2.13. Let P be a parallelohedron and let v be a non-zero vector. We say that P is *free* in the direction of v if there exists a segment I parallel to v such that the Minkowski sum $P + I$ is a parallelohedron of the same dimension as P. We say that the direction of v is a *free direction* for P as well as every non-zero segment parallel to v is a *free direction* for P.

Remark. If $P + I$ is a parallelohedron then the sum $P + I'$ is a paralleohedron for any segment I' parallel to I. Indeed, the combinatorics of $P + I$ and $P + I'$ is the same and the Minkowski–Venkov conditions for these polytopes can be satisfied only simultaneously.

Free directions of parallelohedra and their relation to the Voronoi conjecture are relatively well studied, we refer to papers of Magazinov [34], Horváth [29], Grishukhin [21] and references therein.

The following criterion can be used to determine whether the direction of segment I is a free direction for P. It was initially stated by Grishukhin [21] but a complete proof was given only in [11] by Dutour Sikirić, Grishukhin and Magazinov.

Lemma 2.14. *A non-zero vector v spans a free direction for P if and only if every triangle $xyz = \mathcal{D}(G)$, where G is a non-contact $(d-2)$-face of $\mathcal{T}_P$, satisfies the following condition. If $F(xy)$ is a $(d-1)$-face of $\mathcal{T}_P$ such that $\mathcal{D}(F(xy)) = xy$, and a similar definition applies for $F(xz)$ and $F(yz)$, then at least one of the faces $F(xy)$, $F(xz)$ and $F(yz)$ is parallel to v.*

The next lemma summarizes some useful combinatorial properties of a parallelohedron $P + I$ that have been established to date. We use these properties in Section 5.

Lemma 2.15. *Let P be a d-dimensional parallelohedron with a free direction I. If $P + I$ satisfies the Voronoi conjecture, then P satisfies the Voronoi conjecture.*

Proof. See [22, Theorem 4] or [47]. □

Remark. The proof by Grishukhin [22] relies on the technique of canonical scaling, while Végh [47] provided an explicit construction of the affine transformation from P to a Dirichlet-Voronoi polytope given a transformation for for $P + I$.

3. New lemmas

In this section we prove several new lemmas that we use in the proof of our main result.

First of all we formulate several properties of dual cells that are crucial for our approach to five-dimensional parallelohedra.

Lemma 3.1. *Let F and G be two faces of P and let H be the minimal face of P that contains both F and G. Then*

$$\mathcal{D}(H) = \mathcal{D}(F) \cap \mathcal{D}(G).$$

Proof. Let Q be the copy of P centered at a point of Λ_P. The polytope Q contains F and G if and only if $P \cap Q$ contains F and G. The intersection $P \cap Q$ is a face of P and it contains F and G if and only if it contains H. Hence Q contains F and G if and only if Q contains H. This implies the equality for dual cells. □

Definition 3.2. Let D be a dual cell. We define the *set of midpoints* for the dual cell D as the set of all classes of midpoints within D, so

$$M_D := \left\{ \frac{X+Y}{2} + \Lambda_P \,\middle|\, X, Y \in D \right\} \subseteq \Lambda_{1/2}.$$

Here $\frac{X+Y}{2}$ is the midpoint of the segment XY. Note, that we do not require X and Y to be different, so the class $\langle 0, 0, \ldots, 0 \rangle$ is always in M_D.

Next two lemmas transforms translation invariance of $\mathcal{T}_P$ into invariance of dual cells. Particularly, they use that if one representative of $\Lambda_{1/2}$ class is the center of a dual k-cell, then all points from that class are centers of translations of this k-cell.

Lemma 3.3. *Let D be a dual cell and let F be a contact face of P with the center c_F. Let x be the midpoint of a segment connecting two points of D. If x and c_F represent the same class in $\Lambda_{1/2}$ then D contains the translated copy $\mathcal{D}(F) + \overrightarrow{c_F x}$ of the dual cell of F.*

Proof. Let y and z be points in D such that $x = \frac{y+z}{2}$. Two polytopes $P + y$ and $P + z$ have a non-empty intersection, so their intersection is a contact face G of $\mathcal{T}_P$ with center x such that $\mathcal{D}(G)$ is a subcell of D because $(P + y) \cap (P + z)$ contains the face corresponding to D.

The translation by vector $\overrightarrow{c_F x}$ moves c_F to x and therefore moves the contact face F centered at c_F into the contact face G centered at x. Thus the translation of the dual cell $\mathcal{D}(F)$ is $\mathcal{D}(G)$ which is contained in D. □

Lemma 3.4. *Let D be a dual cell and let A and B be two points in one dual cell of P. Let x be the midpoint of a segment connecting two points of D. If x and the midpoint c of AB represent the same class in $\Lambda_{1/2}$ then D contains the translated copy $AB+\overrightarrow{cx}$ of the segment AB.*

Proof. We use Lemma 3.3 for the cell D and the face F which is the intersection of copies of P centered at A and B. □

Mostly we will use this lemma when F is a facet or, which is the same, when $\mathcal{D}(F)$ is the segment AB as in two lemmas below.

Lemma 3.5. *Let KL be a facet vector. If M and N are two points within one dual cell such that the midpoints of KL and MN belong to the same class in $\Lambda_{1/2}$, then $\overrightarrow{KL}=\pm\overrightarrow{MN}$.*

Proof. Two copies of P centered at M and N have a non-empty intersection F. We use the previous Lemma 3.3 for the dual cell KL and points M and N that both belong to $\mathcal{D}(F)$. The translation of $\mathcal{D}(F)$ must fit within KL which implies that $\mathcal{D}(F)$ contains exactly two points M and N and segments MN and KL are translations of each other. Hence $\overrightarrow{KL}=\pm\overrightarrow{MN}$. □

Lemma 3.6. *Let D be a dual cell. Suppose K, L and M are three points of D such that segments KL, LM, and MK are facet vectors. Then D does not contain a point from the parity class of $K+L+M$.*

Proof. Suppose D contains a point $N=K+L+M \pmod{2\Lambda_P}$. The midpoints of KL and MN differ by a vector of Λ_P because $\frac{K+L}{2}=\frac{M+N}{2} \pmod{\Lambda_P}$, hence Lemma 3.5 for the facet vector KL and pair of points M and N within D implies that $\overrightarrow{KL}=\pm\overrightarrow{MN}$.

Similarly $\overrightarrow{KM}=\pm\overrightarrow{LN}$ and $\overrightarrow{LM}=\pm\overrightarrow{KN}$ but all these three equalities cannot be satisfied simultaneously. □

Also we will use the following corollary of the criterion from Lemma 2.14 stated in terms of the set of midpoints of dual cell of an edge.

Lemma 3.7. *Let I be an edge of P. If there is a $(d-1)$-dimensional subspace π of $\Lambda_{1/2}$ such that each class of π is in $M_{\mathcal{D}(I)}$ or corresponds to non-facet contact face of P, then I is a free direction of P.*

Proof. Let KLM be any triangular dual cell of $\mathcal{T}_P$. Points K, L, and M belong to different parity classes so the midpoints $\frac{K+L}{2}$, $\frac{L+M}{2}$, and $\frac{M+K}{2}$ represent different classes in $\Lambda_{1/2}$. The sum

$$\frac{K+L}{2}+\frac{L+M}{2}+\frac{M+K}{2}=0\in\Lambda_{1/2},$$

hence three midpoints together with the origin fill a two-dimensional subspace of $\Lambda_{1/2}$. This two-dimensional subspace has a non-trivial intersection with π, so we can assume that $\frac{K+L}{2}\in\pi$.

The midpoint $\frac{K+L}{2}$ represents a facet, thus it coincides with the class of some midpoint of the dual cell of I. Lemma 3.3 implies that $\mathcal{D}(I)$ contains a translated copy of the edge KL which means that a translation of the facet corresponding to KL contains I. Therefore the facet corresponding to KL is parallel to I.

Now Lemma 2.14 implies that I is a free direction for P. □

4. Main theorem and the core of the proof

In this section we provide an outline for the proof of our main Theorem 4.1. In the following sections we fill in all the details for each specific step of the the proof.

Theorem 4.1. *The Voronoi conjecture is true in* $\mathbb{R}^5$.

Proof. This proof relies on several supplementary results that are proved in subsequent sections. However, whenever a proof of some implication is deferred, we give a reference to a particular section.

By Lemma 5.3, the main result of Section 5, a five-dimensional parallelohedron P satisfies the Voronoi conjecture if it has a free direction. Consequently, it will be sufficient to prove that every five-dimensional parallelohedron P satisfies at least one of the following properties:

(1) P has a free direction;
(2) P admits a canonical scaling.

Let P be a five-dimensional paralleloheron. By a result of Ordine [40] (see also [41]), if all dual 3-cells of P are either tetrahedra, octahedra or pyramids, then P admits a canonical scaling and therefore the Voronoi conjecture is true for P.

According to Corollary 6.2, the main result of Section 6, if P has a dual 3-cell combinatorially equivalent to a cube, then P has a free direction. In this case the Voronoi conjecture is true for P, too.

To this end, the situation that is still to be considered is as follows: at least one dual 3-cell for P is a triangular prism, while every other dual 3-cell is a tetrahedron, a pyramid, an octahedron, or a prism.

Let F be a 2-dimensional face of $\mathcal{T}_P$ whose dual cell $\mathcal{D}(F)$ is a triangular prism. By Lemmas 7.1 and 7.2, two main results of Section 7, P has a free direction unless F is a triangle, which we denote by xyz, and unless each of the dual 4-cells $\mathcal{D}(xy)$, $\mathcal{D}(xz)$ and $\mathcal{D}(yz)$ is either a pyramid over $\mathcal{D}(F)$ or a prism over a tetrahedron. Let $pr(F)$ denote the number of prismatic 4-cells among the dual cells $\mathcal{D}(xy)$, $\mathcal{D}(xz)$ and $\mathcal{D}(yz)$. We proceed by the case analysis.

Case 1 or **Prism-Prism-Prism case**. There exists F with $pr(F) = 3$. According to Lemma 8.3, the main result of Section 8, this is only possible if P is a direct sum of parallelohedra of smaller dimensions. Hence, in particular, P has a free direction and therefore satisfies the Voronoi conjecture.

Case 2 or **Prism-Prism-Pyramid case**. There exists F with $pr(F) = 2$. By Lemma 9.2, the main result of Section 9, P has a free direction and therefore satisfies the Voronoi conjecture.

Case 3 or **Prism-Pyramid-Pyramid case**. There exists F with $pr(F) = 1$. By Lemma 10.2, the main result of Section 10, at least one of the three sides of F gives a free direction for P. Therefore P satisfies the Voronoi conjecture.

Case 4 or **Pyramid-Pyramid-Pyramid case**. For every triangular face $F \subset P$ whose dual cell $\mathcal{D}(F)$ is a triangular prism it holds that $pr(F) = 0$. Then, by Lemma 11.4, the main result of Section 11, P necessarily admits a canonical scaling or has a free direction. In both cases P satisfies the Voronoi conjecture.

The proof is now finished, since all possible cases are considered. □

One particular corollary of Theorem 4.1 is that the list of Dirichlet-Voronoi parallelohedra from [10] is the complete list of combinatorial types of five-dimensional parallelohedra.

Corollary 4.2. *There are exactly 110 244 combinatorial types of parallelohedra in* $\mathbb{R}^5$.

5. Parallelohedra with free direction

In this section we prove that a parallelohedron in $\mathbb{R}^5$ with a free direction satisfies the Voronoi conjecture. Before proving that specific result for five-dimensional case, we prove a general statement for Voronoi parallelohedra with free direction.

Suppose P is d-dimensional parallelohedron with free direction I. In that case the projection of P along I is a $(d-1)$-dimensional parallelohedron due to result of Venkov [49].

Additionally we need the following notion of a (strong) equivalence for parallelhedra from [9].

Definition 5.1. Let P and P' be two d-dimensional parallelohedra. We say that P and P' are *equivalent*, if there is a combinatorial equivalence $\mathfrak{F}$ between $\mathcal{T}_P$ and $\mathcal{T}_{P'}$ that naturally induces a linear isomorphism of Λ_P to $\Lambda_{P'}$ restricting $\mathfrak{F}$ to copies of P and P' and then to their centers.

It is obvious that if two parallelohedra are equivalent in the sense of Definition 5.1, then they are combinatorially equivalent. For $d \leq 4$, the converse is true as well. Moreover, we are unaware of an example of two combinatorially equivalent parallelohedra that are not equivalent in the sense of Definition 5.1.

Theorem 5.2. *If a d-dimensional parallelohedron P has a free direction I and the projection of P along I satisfies the Voronoi conjecture, then $P+I$ is equivalent (in the sense of Definition 5.1) to the Voronoi parallelohedron for some d-dimensional lattice.*

Proof. Let $\mathcal{F}(I)$ be the set of all facet vectors of $P+I$ with corresponding facets parallel to I. According to result of Horváth [29], the set $\mathcal{F}(I)$ generates a $(d-1)$-dimensional sublattice Λ_I of Λ_{P+I}. The sublattice Λ_I coincides with the intersection $(\operatorname{lin}\Lambda_I)\cap\Lambda_{P+I}$ due to [34, Lemma 3.3] hence Λ_{P+I} splits into layers

$$\Lambda_{P+I} = \bigsqcup_{n\in\mathbb{Z}} \Lambda_I^n$$

where $\Lambda_I^n = nx + \Lambda_I$ for some fixed $x \in \Lambda_{P+I}$. Also, if two copies of $P+I$ have non-empty intersection, then their centers belong to the same or consecutive layers due to [34, Lemma 3.2].

Let Q be the projection of $P+I$ on $\operatorname{lin}\Lambda_I$ along I. We apply an affine transformation $\mathcal{A}$ with invariant subspace $\operatorname{lin}\Lambda_I$ that makes I orthogonal to $\operatorname{lin}\Lambda_I$ and transforms Q into the Dirichlet-Voronoi cell of $\mathcal{A}(\Lambda_I)$. Such a transformation exists because Q satisfies the Voronoi conjecture. This transformation does not change the combinatorial type of $P+I$ and$\mathcal{T}_{P+I}$, or the equivalence class according to Definition 5.1.

First, we notice that for any $k>0$ the polytope $P+kI$ is a parallelohedron and is equivalent to $P+I$, so we may assume that I is long enough so the affine space $\operatorname{lin}\mathcal{A}(\Lambda_I^0)$ is tiled by copies of $\mathcal{A}(P+I)$ centered at $\mathcal{A}(\Lambda_I^0)$ and long enough that in the Voronoi tiling of $\mathcal{A}(\Lambda_{P+I})$ (centers of) polytopes with non-empty intersection belong to the same or to adjacent layers $\mathcal{A}(\Lambda_I^n)$. We claim that in this case, parallelolhedron $\mathcal{A}(P+I)$ is equivalent to the Dirichlet-Voronoi cell DV_{P+I} of the lattice $\mathcal{A}(\Lambda_{P+I})$. We also can assume that $P+I$ is centered at the origin.

An m-dimensional face F of $\mathcal{A}(P+I)$ is an intersection of two sets of copies of $\mathcal{A}(P+I)$ centered in two consecutive layers as there is an obvious correspondence between faces of two tilings formed by polytopes in single layer; without loss of generality we can assume that these layers are $\mathcal{A}(\Lambda_I^0)$ and $\mathcal{A}(\Lambda_I^1)$. Let $x_1, \dots, x_k \in \mathcal{A}(\Lambda_I^0)$ be the centers in the 0th layer and $y_1, \dots, y_l \in \mathcal{A}(\Lambda_I^1)$ be the centers in the 1st layer; here $k, l \geq 1$.

Copies of $\mathcal{A}(Q)$ centered at $\mathcal{A}(\Lambda_I^0)$ give the Voronoi tiling of $\mathcal{A}(\Lambda_I^0)$. Thus, copies of $\mathcal{A}(Q)$ centered at $x_1, \dots, x_k$ intersect at a face F_I^0 of this Voronoi tiling. Moreover, all copies incident to F^0 have centers among $x_1, \dots, x_k$ as these points of $\mathcal{A}(\Lambda_I^0)$ are closest to every point of F_I^0, Similarly, F_I^1 is a face of the Voronoi tiling of $\mathcal{A}(\Lambda_I^1)$ given by intersection of copies of $\mathcal{A}(Q)$ centered at $y_1, \dots, y_l$.

In the Voronoi tiling of $\mathcal{A}(\Lambda_{P+I})$, the copies of DV_{P+I} centered at $x_1, \dots, x_k$ intersect at a face F^0 that is projected onto F_I^0 along I; also F_I^0 is a subset of F^0. Similarly, the copies centered at $y_1, \dots, y_l$ intersect at a face F^1 that is projected onto F_I^1 along I. The faces F^0 and F^1 must intersect between two layers as $F^0 + I \cdot \mathbb{R}$ and $F^1 + I \cdot \mathbb{R}$ both contain F and no other polytope of the Voronoi tiling of $\mathcal{A}(\Lambda_{P+I})$ can reach the intersection of $F^0 + I \cdot \mathbb{R}$ and $F^1 + I \cdot \mathbb{R}$ between 0th and 1st layers. The intersection gives an m-dimensional face of the copy of DV_{P+I} centered at the origin because the dimension of this face is exactly the dimension of $(F^0 + I \cdot \mathbb{R}) \cap (F^1 + I \cdot \mathbb{R})$. If the intersection has larger dimension, then the intersection of copies of $\mathcal{A}(P+I)$ centered at $x_1, \dots, x_k$ and $y_1, \dots, y_l$ would have larger dimension as well.

It is clear that this correspondence between faces of $\mathcal{A}(P+I)$ and faces of the copy of DV_{P+I} centered at the origin is a bijection. Moreover, if we propagate this correspondence for all faces of the tiling $\mathcal{T}_{\mathcal{A}(P+I)}$ we get that $\mathcal{A}(P+I)$ is equivalent to the Voronoi cell of $\mathcal{A}(\Lambda_{P+I})$ in the sense of Definition 5.1 as the induced bijection of the lattices is the identity isomrphism. □

Combining the previous theorem with Lemma 2.15, results of Delone [5] on 4-dimensional parallelohedra and results of Dutour Sikirić and the authors on five-dimensional combinatorially Voronoi parallelohedra [9] we get the main result of this section.

Lemma 5.3. *If a five-dimensional parallelohedron P has a free direction then P satisfies the Voronoi conjecture.*

Proof. Let I be a segment of a free direction for P so $P+I$ is a parallelohedron. The projection of P along I is a four-dimensional parallelohedron that satisfies the Voronoi conjecture according to [5]. Thus, $P+I$ is equivalent to a Voronoi parallelohedron for some lattice due to Theorem 5.2. The results of [9] imply that every five-dimensional parallelohedron equivalent to a Voronoi polytope in the sense of Definition 5.1 satisfies the Voronoi conjecture because for every lattice its Dirichlet-Voronoi cell satisfies a combinatorial condition from [18], so $P+I$ satisfies the Voronoi conjecture. Thus P satisfies the Voronoi conjecture due to Theorem 2.15. □

6. Parallelohedra with cubical dual 3-cells

In this section we prove that if a five-dimensional parallelohedron P has a dual 3-cell equivalent to a three-dimensional cube, then P has a free direction. In this and further sections we assume that $\Lambda_P = \mathbb{Z}^5$ as this can be achieved using an affine transformation.

Lemma 6.1. *If F is a two-dimensional face of P with dual cell $\mathcal{D}(F)$ equivalent to a cube, then every edge of F is a free direction for P.*

Proof. Let e be an edge of F. Then $\mathcal{D}(e)$ contains $\mathcal{D}(F)$ and these two dual cells do not coincide. Let A be any point in $\mathcal{D}(e) \setminus \mathcal{D}(F)$.

The points of $\mathcal{D}(F)$ represent 8 different parity classes within a three-dimensional affine subspace of $\mathbb{R}^5$. Therefore $\mathcal{D}(F)$ is a three-dimensional affine subspace of $\mathbb{Z}_p^5$ and $M_{\mathcal{D}(F)}$ is a three-dimensional linear subspace π of $\mathbb{Z}_{1/2}^5$.

The parity class of A differs from the parity classes of $\mathcal{D}(F)$ because these points are in one dual cell $\mathcal{D}(e)$. Therefore the set of 8 midpoints

$$\pi' := \left\{ \frac{A+X}{2} \,\middle|\, X \in \mathcal{D}(F) \right\} \subset \mathbb{Z}_{1/2}^5$$

is a translation of π that differs from π.

The union $\pi \cup \pi'$ is a four-dimensional linear subspace of $\mathbb{Z}_{1/2}^5$ and $\pi \cup \pi' \subseteq M_{\mathcal{D}(e)}$. Now Lemma 3.7 for the edge e and subspace $\pi \cup \pi'$ implies that e is a free direction for P. □

Corollary 6.2. *If a five-dimensional parallelohedron P has a dual 3-cell equivalent to a cube, then P satisfies the Voronoi conjecture.*

7. Parallelohedra with prismatic dual 3-cells and their properties

In this section we prove that if a five-dimensional parallelohedron P has the dual 3-cell of a face F equivalent to the triangular prism, then P has a free direction or F is a triangle. Moreover we show that the dual cells of edges of F are equivalent (not only as cell complexes but as geometrical vertex sets with inherited face structure) to prisms over tetrahedron or to pyramids over triangular prisms unless P has a free direction.

Suppose $\mathcal{D}(F) = XYZX'Y'Z'$ where XYZ and $X'Y'Z'$ are the bases of the prism, and $\overrightarrow{XX'} = \overrightarrow{YY'} = \overrightarrow{ZZ'}$. We note that the three-dimensional affine subspace of $\mathbb{Z}_p^5$ spanned by $\mathcal{D}(F)$ contains parity classes of $X, Y, Z, X', Y', Z', X+Y+Z$, and $X'+Y'+Z'$ and hence dual cells of edges and vertices of F contain only the prism $XYZX'Y'Z'$ in this affine span due to Lemmas 2.10 and 3.6.

Lemma 7.1. *The parallelohedron P has a free direction or F is a triangle.*

Proof. Suppose F is not a triangle so F is an n-gon for $n \geq 4$. For every edge $e_i, 1 \leq i \leq n$ of F, the dual cell $\mathcal{D}(e_i)$ contains an additional vertex A_i in a parity class outside of the three-dimensional affine subspace π_F of $\mathbb{Z}_p^5$ spanned by $XYZX'Y'Z'$. The space $\mathbb{Z}_p^5$ is split into four three-dimensional affine planes parallel to π_F including π_F itself. Since $n \geq 4$ and $A_i \notin \pi_F$, at least two points, say A_i and A_j corresponding to edges e_i and e_j belong to the same translation of π_F.

Without loss of generality we can assume that the points belong to the following parity classes in $\mathbb{Z}_p^5$

$$\begin{array}{ll} X \in [0,0,0,0,0], & X' \in [0,0,0,0,1], \\ Y \in [1,0,0,0,0], & Y' \in [1,0,0,0,1], \\ Z \in [0,1,0,0,0], & Z' \in [0,1,0,0,1], \end{array}$$

and $A_i \in [0,0,1,0,0]$. The plane π_F is given by the equation $x_3 = x_4 = 0$ and the translation of π_F that contains parity classes of A_i and A_j is given by $\pi_F' := \{x_3 = 1, x_4 = 0\}$, so $A_j \in [*,*,1,0,*]$ where each $*$ can be 0 or 1 independently.

There are three cases for the intersection of $\mathcal{D}(e_i)$ with the affine space π_F' in $\mathbb{Z}_p^5$.

Case 7.1.1: The intersection $\mathcal{D}(e_i) \cap \pi'_F$ contains two points that differ in a coordinate other than x_5. One of these points is $A_i \in [0,0,1,0,0]$; we denote the second one as A'_i and A'_i belongs to the parity class of the form $[1,0,1,0,*]$, $[0,1,1,0,*]$, or $[1,1,1,0,*]$.

The set of midpoints $M_{\mathcal{D}(F)}$ contains all 8 classes of the form $\langle *,*,0,0,*\rangle$ in $\mathbb{Z}^5_{1/2}$ (here each star is 0 or $\frac{1}{2}$ independently of others). The midpoints of segments connecting A_i with vertices of the prism $XYZX'Y'Z'$ represent 6 of 8 classes of the form $\langle *,*,\frac{1}{2},0,*\rangle$ except $\langle \frac{1}{2},\frac{1}{2},\frac{1}{2},0,0\rangle$ and $\langle \frac{1}{2},\frac{1}{2},\frac{1}{2},0,\frac{1}{2}\rangle$. These last two classes are present in the midpoints connecting A'_i with the vertices of $XYZX'Y'Z'$ for every possible choice of A'_i.

Thus, all points of the four-dimensional linear space $x_4 = 0$ in $\mathbb{Z}^5_{1/2}$ are in $M_{\mathcal{D}(e_i)}$ and e_i is a free direction for P according to Lemma 3.7.

Case 7.1.2: Each intersection $\mathcal{D}(e_i) \cap \pi'_F$ and $\mathcal{D}(e_j) \cap \pi'_F$ contains exactly two points that differ in the coordinate x_5. Since one point is $A_i \in [0,0,1,0,0]$, then the second one is $A'_i \in [0,0,1,0,1]$. We can also assume that $\mathcal{D}(e_j) \cap \pi'_F$ contains exactly two points from parity classes A_j and A'_j and $A'_j \in A_j + [0,0,0,0,1]$ as we can use **Case 7.1.1** for the edge e_j in case this intersection contains more than two points, or two points with another difference in $\mathbb{Z}^5_p$.

We first look on the dual cell $\mathcal{D}(e_i)$. The midpoints of A_iX and A'_iX' represent the same class $\langle 0,0,\frac{1}{2},0,0\rangle \in \mathbb{Z}^5_{1/2}$. If these midpoints coincide then $\overrightarrow{A_iA'_i} = \pm\overrightarrow{XX'}$. Otherwise, we use Lemma 3.4 and move the midpoint of A'_iX' onto the midpoint of A_iX and one of the points A'_i or X' must move onto A_i as this is the only point in the plane $\{x_3 = 1\} \subset \mathbb{Z}^5_p$ of $\mathcal{D}(e_i)$ other than A'_i. If A'_i is moved into A_i then $\overrightarrow{A_iA'_i} = \overrightarrow{XX'}$. If X' is moved into A_i then the midpoints of $A_iA'_i$ and XX' coincide in $\mathbb{R}^5$ and similar argumets applied to midpoints of A_iY and A'_iY' lead to a contradiction as the midpoints of XX' and YY' are different. Thus $\overrightarrow{A_iA'_i} = \pm\overrightarrow{XX'}$. Without loss of generality we can assume that $\overrightarrow{A_iA'_i} = \overrightarrow{XX'}$ as we can swap points A_i and A'_i and change coordinates (x_3 specifically) if needed.

Recall that $A_j \in [*,*,1,0,*]$. Since $\mathcal{D}(e_j)$ contains A'_j as well we can assume that fifth coordinate is 0 for A_j. Below we consider all 4 cases for remaining pair of coordinates of A_j. By similar arguments we used above, $\overrightarrow{A_jA'_j} = \pm\overrightarrow{XX'}$

Subcase 7.1.2.00: $A_j \in [0,0,1,0,0]$ and $A'_j \in [0,0,1,0,1]$. We note that $A_j \neq A_i$ as in that case the copy of P centered at A_i contains two edges of F and it must contain F as well, but this is false.

The midpoints of A_iX and A_jX represent the class $\langle 0,0,\frac{1}{2},0,0\rangle \in \mathbb{Z}^5_{1/2}$ so we can use Lemma 3.4 for the cell $\mathcal{D}(e_i)$ and points A_j and X. The translations of A_j and X are in $\mathcal{D}(e_i)$ and they stay within the plane $x_4 = 0$ of $\mathbb{Z}^5_p$, so one of translations coincides with A_i or A'_i to get $x_3 = \frac{1}{2}$ for the midpoint.

If any of the points is translated into A'_i then the second one is translated into the point symmetric to X' with respect to X (because $A_iXX'A'_i$ is a parallelogram), but this point does not belong to $\mathcal{D}(e_i)$. If A_j is translated into A_i then $A_j = A_i$ which is impossible.

The only possible case is when X is translated into A_i and A_j is translated into X. Thus X is the midpoint of A_iA_j. The same arguments for midpoints of A_iY and A_jY lead to conclusion that Y is the midpoint of A_iA_j which is impossible.

Subcase 7.1.2.10: $A_j \in [1,0,1,0,0]$ and $A'_j \in [1,0,1,0,1]$. The midpoints of A_iY and A_jX represent the same class $\langle \frac{1}{2},0,\frac{1}{2},0,0\rangle \in \mathbb{Z}^5_{1/2}$. We use Lemma 3.4 for the cell $\mathcal{D}(e_i)$ and points A_j and X. One of the points A_j or X is translated into A_i or A'_i. If any of the translations coincides with A'_i then the second point is translated in the point symmetric to Y' with respect to Y ($A_iYY'A'_i$ is a parallelogram), and this point is not in $\mathcal{D}(e_i)$.

If A_j or X is translated into A_i then $\overrightarrow{A_jX} = \pm\overrightarrow{A_iY}$. Similar arguments for the midpoints of A_iX and A_jY that represent the class $\langle 0,0,\frac{1}{2},0,0\rangle \in \mathbb{Z}^5_{1/2}$ give that $\overrightarrow{A_jY} = \pm\overrightarrow{A_iX}$. Both equalities are possible only if the midpoints of A_iA_j and XY coincide.

Similar arguments for the midpoints of A'_iY' and A'_jX' and the midpoints of A'_iX' and A'_jY' show that the midpoints of $A'_iA'_j$ and $X'Y'$ coincide. Therefore $\overrightarrow{A_jA'_j} = \overrightarrow{A_iA'_i} = \overrightarrow{XX'}$ and the midpoints of $A_iA'_j$, A'_iA_j and XY' coincide.

The parallelogram $XYY'X'$ is a dual cell of the tiling $\mathcal{T}_P$ (it is a face of prismatic dual 3-cell $XYZX'Y'Z'$). Let G be the 3-dimensional face of $\mathcal{T}_P$ with $\mathcal{D}(G) = XYY'X'$. The face G is centrally symmetric with respect to the midpoint of XY'. Also, the face G contains F and hence G contains edges e_i and e_j.

Let e'_j be the edge of G symmetric to e_j. The dual cell $\mathcal{D}(e'_j)$ is centrally symmetric to $\mathcal{D}(e_j)$ with respect to the midpoint of XY' and hence $\mathcal{D}(e'_j)$ contains X, Y, X', Y', A_i, and A'_i. Let H be the minimal face of G that contains e_i and e'_j. The dual cell of H is the intersection of the dual cells $\mathcal{D}(e_i)$ and $\mathcal{D}(e'_j)$ according to Lemma 3.1 so $\{X, Y, X', Y', A_i, A'_i\} \subseteq \mathcal{D}(H)$ and $\mathcal{D}(H)$ contains $\mathcal{D}(G)$ as a proper subset and H is a two-dimensional face of G.

The edges e_j and e'_j are parallel. If edges e_i and e_j are not parallel, then the line containing e_i intersects both lines containing e_j and e'_j, so the two-dimensional planes of faces F and H coincide. This is impossible as e_j and e'_j are opposite edges of G and hence cannot belong to one supporting plane of G. Thus, e_i and e_j are parallel. The union of sets of midpoints $M_{\mathcal{D}(e_i)}$ and $M_{\mathcal{D}(e_j)}$ contains all classes within $\mathbb{Z}^5_{1/2}$ satisfying $x_4 = 0$. The arguments similar to the proof of Lemma 3.7 show that every triangular dual 2-face has a facet parallel to e_i (and e_j). Lemma 2.14 implies that edge e_i is a free direction of P.

Subcase 7.1.2.01: $A_j \in [0,1,1,0,0]$ and $A'_j \in [0,1,1,0,1]$. This subcase becomes identical to **Subcase 7.1.2.10** if we swap Y and Y' with Z and Z'.

Subcase 7.1.2.11: $A_j \in [1,1,1,0,0]$ and $A'_j \in [1,1,1,0,1]$. This subcase becomes identical to **Subcase 7.1.2.10** if we swap X and X' with Z and Z'.

Case 7.1.3: One of the intersection $\mathcal{D}(e_i) \cap \pi'_F$ and $\mathcal{D}(e_j) \cap \pi'_F$ contains exactly one point; without loss of generality we can assume that this intersection is $\mathcal{D}(e_i) \cap \pi'_F$ and $A_i \in [0,0,1,0,0]$. Recall that $A_j \in [*,*,1,0,*]$; below we consider all 8 cases for unknown coordinates in the parity class of A_j.

In most cases below we translate a segment within $\mathcal{D}(e_j)$ with one endpoint A_j and the other endpoint in $XYZX'Y'Z'$ into the cell $\mathcal{D}(e_i)$ using Lemma 3.4. Since this segment is not parallel to π_F but parallel to $x_4 = 0$, the translation must have A_i as one endpoints.

Subcase 7.1.3.000: $A_j \in [0,0,1,0,0]$. The midpoints of A_jX and A_iX represent the class $\langle 0,0,\frac{1}{2},0,0\rangle \in \mathbb{Z}^5_{1/2}$; using Lemma 3.4 for the cell $\mathcal{D}(e_i)$ and points A_j and X we get that translated copy of A_jX is within $\mathcal{D}(e_i)$. One of translated points coincides with A_i. This cannot be A_j as in that case $A_i = A_j$ and the copy of P centered at A_i would contain two edges of F but not F itself. Thus translation of X is A_i and translation of A_j is X. So X is the midpoint of A_iA_j. Similar arguments for the midpoints of A_iY and A_jY give that Y is the midpoint of A_iA_j which is a contradiction.

Subcase 7.1.3.100: $A_j \in [1,0,1,0,0]$. The midpoints of A_jX and A_iY represent the class $\langle\frac{1}{2},0,\frac{1}{2},0,0\rangle \in \mathbb{Z}^5_{1/2}$. Using Lemma 3.4 we get that the segment A_jX is translated into the segment A_iY so $\overrightarrow{A_jX} = \pm\overrightarrow{A_iY}$. Similar arguments for midpoints of A_jY and A_iX give that $\overrightarrow{A_jY} = \pm\overrightarrow{A_iX}$. Both equalities are possible only if the midpoints of XY and A_iA_j coincide.

Similar arguments for the midpoints of A_jX' and A_iY' and for the midpoints of A_jY' and A_iX' give that the midpoints of $X'Y'$ and A_iA_j coincide which is impossible.

Subcase 7.1.3.010: $A_j \in [0,1,1,0,0]$. This subcase becomes identical to **Subcase 7.1.3.100** if we swap Y to Z and Y' to Z'.

Subcase 7.1.3.110: $A_j \in [1,1,1,0,0]$. This subcase becomes identical to **Subcase 7.1.3.100** if we swap X to Z and X' to Z'.

Subcase 7.1.3.001: $A_j \in [0,0,1,0,1]$. The midpoints of A_jX and A_iX' represent the same class $\langle 0,0,\frac{1}{2},0,\frac{1}{2}\rangle \in \mathbb{Z}^5_{1/2}$, therefore $\overrightarrow{A_jX} = \pm\overrightarrow{A_iX'}$. Also the midpoints of A_jX' and A_iX represent the same class $\langle 0,0,\frac{1}{2},0,0\rangle \in \mathbb{Z}^5_{1/2}$, therefore $\overrightarrow{A_jX'} = \pm\overrightarrow{A_iX}$. This is possible only if the midpoints of XX' and A_iA_j coincide.

Similar arguments for the midpoints of A_jY and A_iY' and for the midpoints of A_jY' and A_iY give that the midpoints of YY' and A_iA_j coincide which is impossible.

Subcase 7.1.3.101: $A_j \in [1,0,1,0,1]$. The midpoints of A_jX and A_iY' represent the same class $\langle \frac{1}{2},0,\frac{1}{2},0,\frac{1}{2}\rangle \in \mathbb{Z}^5_{1/2}$ and therefore $\overrightarrow{A_jX} = \pm\overrightarrow{A_iY'}$. Similarly, the midpoints of A_jY' and A_iX represent the same class $\langle 0,0,\frac{1}{2},0,0\rangle \in \mathbb{Z}^5_{1/2}$ and therefore $\overrightarrow{A_jX} = \pm\overrightarrow{A_iY'}$. This is possible only if the midpoints of A_iA_j and XY' coincide.

After that we use the same idea as in **Subcase 7.1.2.10**. Let G be the three-dimensional face of $\mathcal{T}_P$ with dual cell $XYY'X'$. Let e'_j be the edge of G symmetric to e_j. The dual cell $\mathcal{D}(e'_j)$ contains points X, Y, Y', X' and A_i. Let H be the minimal face of G that contains e_i and e'_j. By similar arguments, H is a two-dimensional face of G different from F. Again, similar arguments show that edges e_i and e_j are parallel.

As in **Subcase 7.1.2.10**, the union of the sets of midpoints $M_{\mathcal{D}(e_i)} \cup M_{\mathcal{D}(e_j)}$ contains all classes of $\mathbb{Z}^5_{1/2}$ satisfying $x_4 = 0$. This implies that e_i is a free direction of P.

Subcase 7.1.3.011: $A_j \in [0,1,1,0,1]$. This subcase becomes identical to **Subcase 7.1.3.101** if we swap Y to Z and Y' to Z'.

Subcase 7.1.3.111: $A_j \in [1,1,1,0,1]$. This subcase becomes identical to **Subcase 7.1.3.101** if we swap X to Z and X' to Z'.

As we see, if F is not a triangle, then in all possible cases P has one edge of F as a free direction. This concludes our proof. □

For the remaining part of the paper, F is a triangle xyz as P has a free direction and satisfies the Voronoi conjecture otherwise. The following two corollaries follow the ideas of the proof of Lemma 7.1 and limit the options for each dual cell and what could be "additional" vertices within each of these three dual cells. After that, in the remaining sections we consider all possible cases for dual cells of edges xy, yz, and zx of F.

Lemma 7.2. *Let e be an edge of $F = xyz$ with dual cell $\mathcal{D}(F) = XYZX'Y'Z'$. The parallelohedron P has a free direction or the dual 4-cell $\mathcal{D}(e)$ is equivalent to a prism over tetrahedron that has XYZ as its face or to a pyramid over $XYZX'Y'Z'$.*

Proof. As in the proof of Lemma 7.1, let π_F be the three-dimensional affine subspace of $\mathbb{Z}^5_p$ spanned by $XYZX'Y'Z'$. Each of dual cells $\mathcal{D}(xy)$, $\mathcal{D}(xz)$, and $\mathcal{D}(yz)$ have a point outside of π_F. If such points for two dual cells fall in one translated copy of π_F, then we use the same arguments as in the proof of Lemma 7.1 to show that P has a free direction. Otherwise, the dual cells have additional points in different translations of π_F.

Without loss of generality suppose $e = xy$. Recall that $\mathcal{D}(e)$ has exactly six points in π_F. If $\mathcal{D}(e)$ has more than two points outside of π_F or it has two points in π_F with parity classes differ by a vector other than $\pm\overrightarrow{XX'}$, then we can use the arguments from **Case 7.1.1** of Lemma 7.1 to show that e is a free direction for P. Thus, we have only two options $\mathcal{D}(e) = AXYZA'X'Y'Z'$ (A and A' are outside of π_F and $\overrightarrow{AA'} = \overrightarrow{XX'}$ in $\mathbb{Z}_p^5$) or $\mathcal{D}(e) = AXYZX'Y'Z'$ (A is outside π_F).

Case 7.2.1: $\mathcal{D}(e) = AXYZA'X'Y'Z'$. First, we use the arguments from **Case 7.1.2** of Lemma 7.1 to show that $\overrightarrow{AA'} = \overrightarrow{XX'}$ in $\mathbb{Z}^5$ after possible swap of A and A'. So geometrically, $\mathcal{D}(e)$ is a prism over tetrahedron $AXYZ$ and we need to recover the dual cells within $\mathcal{D}(e)$ to complete the proof for that case.

The parallelogram $XYY'X$ is a dual 2-cell of a three-dimensional face of $\mathcal{T}_P$. It belongs to two dual 3-cells of two-dimensional faces of $\mathcal{T}_P$ within $AXYZA'X'Y'Z'$; one of these cells is $XYZX'Y'Z'$. Denote the second cell as D. The cell D contains either A or A' as D cannot be a subset of $XYZX'Y'Z'$. The midpoints of AX' and $A'X$ coincide, so in both cases the second point belongs to D due to Lemma 3.4. By similar reasons, if D contains Z or Z' then it contains the other point and $D = \mathcal{D}(e)$ which is impossible, hence $D = AXYA'X'Y'$ is a cell equivalent to a prism over triangle. By similar reasons, the prisms $AXZA'X'Z'$ and $AYZA'Y'Z'$ are also subcells of $\mathcal{D}(e)$.

The triangular cell XYZ belongs to two dual 3-cells in $\mathcal{D}(e)$ as well; one of these 3-cells is $XYZX'Y'Z'$. If the second cell D contains A', then it contains A and X' as well (midpoints of AX' and $A'X$ coincide), but the intersection of $D \cap XYZX'Y'Z' = XYZ$. Hence D contains A only and $D = AXYZ$, the dual cell equivalent to a tetrahedron. By similar reasons there is a tetrahedral dual 3-cell $A'X'Y'Z'$ within $\mathcal{D}(e)$.

Summarizing, we found the following dual 3-cells within $\mathcal{D}(e)$: $AXYZ$, $A'X'Y'Z'$, $XYZX'Y'Z'$, $AXYA'X'Y'$, $AXZA'X'Z'$, and $AYZA'Y'Z'$. In this list every dual 2-cell belongs to exactly 2 dual 3-cells, hence it is a complete list of dual 3-cells within $\mathcal{D}(e)$ and $\mathcal{D}(e)$ is equivalent to a prism over tetrahedron $AXYZ$.

Case 7.2.2: $\mathcal{D}(e) = AXYZX'Y'Z'$. Similarly to the previous case we conclude that $AXYY'X'$ is the subcell (equivalent to a pyramid over parallelogram) of $\mathcal{D}(e)$ adjacent to $XYZX'Y'Z'$ by the parallelogram cell $XYY'X'$. By the similar reasons the pyramidal cells $AXZZ'X'$ and $AYZZ'Y'$ are subcells of $\mathcal{D}(e)$.

Also by similar reasons, the cells $AXYZ$ and $AX'Y'Z'$ are the only options for the second subcells of $\mathcal{D}(e)$ containing XYZ and $X'Y'Z'$ respectively. The complete list of dual 3-cells within $\mathcal{D}(e)$ now looks as $AXYZ$, $AX'Y'Z'$, $XYZX'Y'Z'$, $AXYY'X'$, $AXZZ'X'$, and $AYZZ'Y'$ and $\mathcal{D}(e)$ is equivalent to a pyramid over the prism $XYZX'Y'Z'$. □

Before formulating the next lemma we fix coordinate notations for parity classes of some points we have so far. We use these notations in the next lemma and in the next three sections. Recall that $F = xyz$ is a two-dimensional face of P with dual cell $\mathcal{D}(xyz) = XYZX'Y'Z'$. Let $A, B, C \notin XYZX'Y'Z'$ be three points such that $A \in \mathcal{D}(xy)$, $B \in \mathcal{D}(xz)$, and $C \in \mathcal{D}(yz)$.

Without loss of generality we can assume that the points belong to the following parity classes in $\mathbb{Z}_p^5$

$$\begin{array}{ll} X \in [0,0,0,0,0], & X' \in [0,0,0,0,1], \\ Y \in [1,0,0,0,0], & Y' \in [1,0,0,0,1], \\ Z \in [0,1,0,0,0], & Z' \in [0,1,0,0,1]. \end{array}$$

In that case the affine span of $\mathcal{D}(F)$ in $\mathbb{Z}_p^5$ is given by $x_3 = x_4 = 0$. So the points A, B, and C belong to affine planes $x_3 = 1, x_4 = 0$; $x_3 = 0, x_4 = 1$; and $x_3 = x_4 = 1$ in $\mathbb{Z}_p^5$ (may be not respectively).

Without loss of generality we can assume that $A \in [0,0,1,0,0]$ and the dual cell $\mathcal{D}(xy)$ may contain $A' \in [0,0,1,0,1]$ such that $\overrightarrow{AA'} = \overrightarrow{XX'}$. This follows from possible structure of dual cell $\mathcal{D}(xy)$ described in Lemma 7.2 after possible swap of A and A' and change of coordinates. Similarly, we can assume that $B \in [0,0,0,1,0]$ and the dual cell $\mathcal{D}(xz)$ may contain $B' \in [0,0,0,1,1]$ such that $\overrightarrow{BB'} = \overrightarrow{XX'}$. Finally, $C \in [*,*,1,1,*]$ and the dual cell $\mathcal{D}(yz)$ may contain $C' \in [*,*,1,1,*]$ such that $\overrightarrow{CC'} = \pm\overrightarrow{XX'}$.

The next lemma eliminates 6 options for C leaving only 2. Particularly, in the previous notations, $C \in [1,1,1,1,0]$ or $C \in [1,1,1,1,1]$.

Lemma 7.3. *Suppose P does not have a free direction. Let $F = xyz$ be a face of P with prismatic dual cell $XYZX'Y'Z'$.*

Let $A \in \mathcal{D}(xy)$, $B \in \mathcal{D}(xz)$, and $C \in \mathcal{D}(yz)$ be three points in the corresponding dual cells that are not in $XYZX'Y'Z'$. Then $A + B + C$ represents the parity class of $X + Y + Z$ or $X' + Y' + Z'$.

Proof. Recall that $C \in [*,*,1,1,*]$. Below we show that 6 of 8 cases are impossible.

Case 7.3.000: $C \in [0,0,1,1,0]$. The points A and C belong to the dual cell $\mathcal{D}(y)$ and BX is an edge (facet vector) of the dual cell $\mathcal{D}(xz)$ regardless of the type of dual cell of xz. The midpoints of AC and BX represent the class $\langle 0,0,0,\frac{1}{2},0\rangle \in \mathbb{Z}_{1/2}^5$ and using Lemma 3.4 for the cell BX and points A and C we get $\overrightarrow{AC} = \pm\overrightarrow{BX}$. Similarly, AX is an edge of $\mathcal{D}(xy)$ and $B, C \in \mathcal{D}(z)$ and midpoints of AX and BC represent $\langle 0,0,\frac{1}{2},0,0\rangle \in \mathbb{Z}_{1/2}^5$, and therefore $\overrightarrow{BC} = \pm\overrightarrow{AX}$. This is only possible if the midpoints of AB and CX coincide.

The cell $\mathcal{D}(yz)$ contains points C and X so Lemma 3.4 for this cell and points $A, B \in \mathcal{D}(x)$ implies that $\mathcal{D}(yz)$ contains A and B which is false.

Case 7.3.100: $C \in [1,0,1,1,0]$. This case becomes identical to **Case 7.3.000** if we swap X with Y.

Case 7.3.010: $C \in [0,1,1,1,0]$. This case becomes identical to **Case 7.3.000** if we swap X with Z.

Case 7.3.001: $C \in [0,0,1,1,1]$. Again, the points A and C belong to the dual cell $\mathcal{D}(y)$ and the midpoints of AC and BX' represent the same class $\langle 0,0,0,\frac{1}{2},\frac{1}{2}\rangle \in \mathbb{Z}_{1/2}^5$. We use Lemma 3.4 for the dual cell $\mathcal{D}(xz)$ (which contains B and X') and points A and C. After the translation the points A and C will be within $\mathcal{D}(xz)$. This is possible only if $\overrightarrow{AC} = \pm\overrightarrow{BX'}$ or $\mathcal{D}(xz)$ contains B' and $\overrightarrow{AC} = \pm\overrightarrow{B'X}$.

If $A' \notin \mathcal{D}(xy)$ and $B' \notin \mathcal{D}(xz)$, then $\overrightarrow{AC} = \pm\overrightarrow{BX'}$ and by similar reasons $\overrightarrow{BC} = \pm\overrightarrow{AX'}$ because these midpoints represent the class $\langle 0,0,\frac{1}{2},0,\frac{1}{2}\rangle \in \mathbb{Z}_{1/2}^5$, so the midpoints of AB and CX' coincide. The dual cell $\mathcal{D}(yz)$ contains C and X', so it must contain A and B as well which is false.

If $A' \in \mathcal{D}(xy)$ and $B' \notin \mathcal{D}(xz)$, then $\overrightarrow{AC} = \pm\overrightarrow{BX'}$. By similar reasons $\overrightarrow{A'C} = \pm\overrightarrow{BX}$ because the midpoints of $A'C$ and BX represent the class $\langle 0,0,0,\frac{1}{2},0\rangle \in \mathbb{Z}_{1/2}^5$. This is possible only if the midpoints of BC and AX' coincide. However the midpoint of AX' in that case belongs to the contact face with dual cell $AXX'A'$, so it cannot belong to either copy of P centered at B or C. Similar reasons work if $A' \notin \mathcal{D}(xy)$ and $B' \in \mathcal{D}(xz)$.

The last case is if $A' \in \mathcal{D}(xy)$ and $B' \in \mathcal{D}(xz)$. Then the midpoints of $A'C$ and BX represent the class $\langle 0,0,0,\frac{1}{2},0\rangle \in \mathbb{Z}^5_{1/2}$, so $\overrightarrow{A'C} = \pm\overrightarrow{BX}$ because BX is an edge of the dual cell $\mathcal{D}(xz)$. Similarly $\overrightarrow{B'C} = \pm\overrightarrow{AX}$. This is possible only if the midpoints of CX and $A'B$ coincide. The dual cell $\mathcal{D}(yz)$ contains C and X, so it must contain A' and B as well which is false.

Case 7.3.101: $C \in [1,0,1,1,1]$. This case becomes identical to **Case 7.3.001** if we swap X to Y and X' to Y'.

Case 7.3.011: $C \in [0,1,1,1,1]$. This case becomes identical to **Case 7.3.001** if we swap X to Z and X' to Z'.

The remaining two cases for C are $C \in [1,1,1,1,0]$ and $C \in [1,1,1,1,1]$. For the first option $A+B+C$ and $X+Y+Z$ represent the parity class $[1,1,0,0,0]$ and for the second option $A+B+C$ and $X'+Y'+Z'$ represent the parity class $[1,1,0,0,1]$. □

In the next four sections we consider all possible cases for dual cells of edges of F as outlined in the proof of Theorem 4.1.

8. Prism-Prism-Prism case

In this case we assume that dual cells of all edges xy, xz, and yz of F are prisms over tetrahedra. The results of Section 7 imply that we can consider only the case $\mathcal{D}(xyz) = XYZX'Y'Z'$, $\mathcal{D}(xy) = AXYZA'X'Y'Z'$, $\mathcal{D}(xz) = BXYZB'X'Y'Z'$, and $\mathcal{D}(yz) = CXYZC'X'Y'Z'$ where

$$\overrightarrow{XX'} = \overrightarrow{YY'} = \overrightarrow{ZZ'} = \overrightarrow{AA'} = \overrightarrow{BB'} = \pm\overrightarrow{CC'}$$

and the points represent the following parity classes

$$\begin{array}{ll} X \in [0,0,0,0,0], & X' \in [0,0,0,0,1], \\ Y \in [1,0,0,0,0], & Y' \in [1,0,0,0,1], \\ Z \in [0,1,0,0,0], & Z' \in [0,1,0,0,1], \\ A \in [0,0,1,0,0], & A' \in [0,0,1,0,1], \\ B \in [0,0,0,1,0], & B' \in [0,0,0,1,1], \end{array}$$

and C is in $[1,1,1,1,1]$ or in $[1,1,1,1,0]$. Then C' is in $[1,1,1,1,0]$ or in $[1,1,1,1,1]$ respectively.

We use the following "red Venkov graph" criterion for P to be decomposable in a direct sum of two parallelohedra of smaller dimensions. This criterion was proved by Ordine in [40]; we also refer to [34] and [36] for details.

Definition 8.1. Let P be a d-dimensional parallelohedron, $d \geq 2$. Let G_P be a graph with vertices corresponding to the pairs of opposite facets of P. Two vertices of G_P are connected with an edge if and only if two facets from two corresponding pairs of facets share a primitive face of codimension 2.

The graph G_P is called the *red Venkov graph* of P.

Theorem 8.2 (A. Ordine, [40]). *A parallelohedron P is a direct sum of two parallelohedra of smaller dimension if and only if the graph G_P is disconnected.*

Lemma 8.3. *If dual cells $\mathcal{D}(xy)$, $\mathcal{D}(xz)$, and $\mathcal{D}(yz)$ are equivalent to prisms over tetrahedra, then P satisfies the Voronoi conjecture.*

Proof. We claim that the vertex of G_P corresponding to the facet vector XX' is an isolated vertex.

Suppose XX' corresponds to a non-isolated vertex of G_P. Then XX' belongs to a triangular dual 2-cell TXX' of $\mathcal{T}_P$ for some $T \in \mathbb{Z}^5$. The midpoints of facet vectors TX and TX' represent non-zero classes of $\mathbb{Z}^5_{1/2}$ and differ by $\langle 0,0,0,0,\frac{1}{2}\rangle$. Below we show that for every choice of $a,b,c,d \in \{0,\frac{1}{2}\}$ except $a=b=c=d=0$ one of two classes of the form $\langle a,b,c,d,*\rangle$ does not represent a facet vector which gives a contradiction.

The first family of midpoints that are not midponts of facet vectors comes from parallelogram dual 2-cells within $\mathcal{D}(xy) = AXYZA'X'Y'Z'$ and $\mathcal{D}(xz) = BXYZB'X'Y'Z'$.

$\langle \frac{1}{2},0,0,0,0\rangle$ is the center of parallelogram cell $XYY'X'$,
$\langle 0,\frac{1}{2},0,0,0\rangle$ is the center of parallelogram cell $XZZ'X'$,
$\langle \frac{1}{2},\frac{1}{2},0,0,0\rangle$ is the center of parallelogram cell $YZZ'Y'$,
$\langle 0,0,\frac{1}{2},0,0\rangle$ is the center of parallelogram cell $AXX'A$,
$\langle \frac{1}{2},0,\frac{1}{2},0,0\rangle$ is the center of parallelogram cell $AYY'A'$,
$\langle 0,\frac{1}{2},\frac{1}{2},0,0\rangle$ is the center of parallelogram cell $AZZ'A'$,
$\langle 0,0,0,\frac{1}{2},0\rangle$ is the center of parallelogram cell $BXX'B$,
$\langle \frac{1}{2},0,0,\frac{1}{2},0\rangle$ is the center of parallelogram cell $BYY'B'$,
$\langle 0,\frac{1}{2},0,\frac{1}{2},0\rangle$ is the center of parallelogram cell $BZZ'B'$.

Similarly, four points C, X, X', and C' form a parallelogram dual 2-cell within $\mathcal{D}(yz) = CXYZC'X'Y'Z'$ with the center being the midpoint of CX or the midpoint of CX'. In both cases the center does not correspond to a facet vector and has the form $\langle \frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},*\rangle$. This and two similar parallelograms give three more non-facet midpoints.

$\langle \frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},*\rangle$ is the center of parallelogram cell with vertices C, X, X', and C',
$\langle 0,\frac{1}{2},\frac{1}{2},\frac{1}{2},*\rangle$ is the center of parallelogram cell with vertices C, Y, Y', and C',
$\langle \frac{1}{2},0,\frac{1}{2},\frac{1}{2},*\rangle$ is the center of parallelogram cell with vertices C, Z, Z', and C'.

The last block of three midpoints comes from points within one dual cell that geometrically form a parallelogram but not necessarily form a parallelogram dual 2-cell. The points A, B, B', and A' all belong to the dual cell $\mathcal{D}(x)$ and form a parallelogram with center in the midpoint of AB'. If this midpoint corresponds to a facet vector, then both AB' and $A'B$ must be facet vectors due to Lemma 3.3 which is impossible.

$\langle 0,0,\frac{1}{2},\frac{1}{2},0\rangle$ is the center of parallelogram with vertices $A,B,B',A' \in \mathcal{D}(x)$,
$\langle \frac{1}{2},\frac{1}{2},0,\frac{1}{2},*\rangle$ is the center of parallelogram with vertices $A,C,C',A' \in \mathcal{D}(y)$,
$\langle \frac{1}{2},\frac{1}{2},\frac{1}{2},0,*\rangle$ is the center of parallelogram with vertices $B,C,C',B' \in \mathcal{D}(z)$.

Thus, we have treated all 15 cases for a,b,c,d.

This implies that G_P is disconnected and hence by Theorem 8.2 $P = P_1 \oplus P_2$ for some parallelohedra P_1 and P_2 of dimension at most 4. The Voronoi conjecture is true for P_1 and for P_2 and therefore the Voronoi conjecture holds for P. $\square$

9. Prism-Prism-Pyramid case

In this case we assume that dual cells of edges xy and xz of F are prisms over tetrahedra and the dual cell of edge yz is a pyramid over triangular prism. The results of Section 7 imply that we can consider only the case $\mathcal{D}(xyz) = XYZX'Y'Z'$, $\mathcal{D}(xy) = AXYZA'X'Y'Z'$, $\mathcal{D}(xz) = BXYZB'X'Y'Z'$, and $\mathcal{D}(yz) = CXYZX'Y'Z'$ where

$$\overrightarrow{XX'} = \overrightarrow{YY'} = \overrightarrow{ZZ'} = \overrightarrow{AA'} = \overrightarrow{BB'}$$

and the points represent the following parity classes

$$\begin{array}{ll} X \in [0,0,0,0,0], & X' \in [0,0,0,0,1], \\ Y \in [1,0,0,0,0], & Y' \in [1,0,0,0,1], \\ Z \in [0,1,0,0,0], & Z' \in [0,1,0,0,1], \\ A \in [0,0,1,0,0], & A' \in [0,0,1,0,1], \\ B \in [0,0,0,1,0], & B' \in [0,0,0,1,1], \end{array}$$

and C is in $[1,1,0,1,1]$ or in $[1,1,1,1,1]$. The goal of this section is to show that this configuration is impossible unless P has a free direction.

Lemma 9.1. *The dual cell $\mathcal{D}(x)$ contains exactly 10 points so $\mathcal{D}(x) = ABXYZA'B'X'Y'Z'$.*

Proof. Suppose $\mathcal{D}(x)$ contains an additional point R. The point R cannot belong to a parity class of points A, B, X, Y, Z, A', B', X', Y', or Z'. Also we use Lemma 3.6 for 14 triangular dual 2-cells within dual cell $\mathcal{D}(x)$ and each triangle forbids a parity class for R.

Triangle	Forbidden parity class
XYZ	$X+Y+Z \in [1,1,0,0,0]$
AXY	$A+X+Y \in [1,0,1,0,0]$
AXZ	$A+X+Z \in [0,1,1,0,0]$
AYZ	$A+Y+Z \in [1,1,1,0,0]$
BXY	$B+X+Y \in [1,0,0,1,0]$
BXZ	$B+X+Z \in [0,1,0,1,0]$
BYZ	$B+Y+Z \in [1,1,0,1,0]$
$X'Y'Z'$	$X'+Y'+Z' \in [1,1,0,0,1]$
$A'X'Y'$	$A'+X'+Y' \in [1,0,1,0,1]$
$A'X'Z'$	$A'+X'+Z' \in [0,1,1,0,1]$
$A'Y'Z'$	$A'+Y'+Z' \in [1,1,1,0,1]$
$B'X'Y'$	$B'+X'+Y' \in [1,0,0,1,1]$
$B'X'Z'$	$B'+X'+Z' \in [0,1,0,1,1]$
$B'Y'Z'$	$B'+Y'+Z' \in [1,1,0,1,1]$

We have eliminated 24 options for the parity class of R (all except 8 points in the 3-dimensional plane $x_3 = x_4 = 1$ in $\mathbb{Z}_p^5$). The remaining 8 options are studied below.

Case 9.1.00110: $R \in [0,0,1,1,0]$. The midpoints of AX' and $B'R$ represent the parity class $\langle 0,0,\frac{1}{2},0,\frac{1}{2}\rangle \in \mathbb{Z}^5_{1/2}$. Therefore, the midpoint of $B'R$ corresponds to a contact dual 2-cell which is a translation of $AXX'A'$. Thus, the dual cell $\mathcal{D}(x)$ contains a point $R' \in [0,0,1,1,1]$. Moreover, since B is the only point of its parity class in $\mathcal{D}(x)$, the translation of $AXX'A'$ is the parallelogram $RBB'R'$ centered at the midpoint of $B'R$ and $\overrightarrow{RR'} = \overrightarrow{BB'}$.

The points R, Y, Y', R' form a parallelogram centered at the midpoint of $RY' \in \langle \frac{1}{2},0,\frac{1}{2},\frac{1}{2},0\rangle$. This parallelogram is not necessarily a dual cell, but its center does not belong to a facet vector as in that case both diagonals of this parallelograms are facet vectors and facet vectors cannot intersect. However, two segments CZ and CZ' are facet vectors of the cell $\mathcal{D}(yz)$ and their midpoints are in classes $\langle \frac{1}{2},0,\frac{1}{2},\frac{1}{2},0\rangle$ and $\langle \frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2}\rangle$ for both options for the parity class of C which gives a contradiction.

Case 9.1.10110: $R \in [1,0,1,1,0]$. This case becomes identical to **Case 9.1.00110** if we swap X and Y and X' and Y' for finding R' and use the same framework afterwards.

Case 9.1.01110: $R \in [0,1,1,1,0]$. This case becomes identical to **Case 9.1.00110** if we swap X and Z and X' and Z' for finding R' and use a similar framework afterwards..

Case 9.1.00111: $R \in [0,0,1,1,1]$. The midpoints of AX' and BR represent the parity class $\langle 0,0,\frac{1}{2},0,\frac{1}{2}\rangle \in \mathbb{Z}^5_{1/2}$. Thus, by the reasons similar to **Case 9.1.00110**, the cell $\mathcal{D}(x)$ contains a point $R' \in [0,0,1,1,0]$ which is impossible by **Case 9.1.00110**.

Case 9.1.10111: $R \in [1,0,1,1,1]$. This case becomes identical to **Case 9.1.00111** if we swap X' to Y' and use impossibility of **Case 9.1.10110**.

Case 9.1.01111: $R \in [0,1,1,1,1]$. This case becomes identical to **Case 9.1.00111** if we swap X' to Z' and use impossibility of **Case 9.1.01110**.

Case 9.1.1111*: $R \in [1,1,1,1,0]$ or $R \in [1,1,1,1,1]$. If C and R belong to the same parity class, then midpoints of CX (within $\mathcal{D}(yz)$) and RX (within $\mathcal{D}(x)$) belong to the same class in $\mathbb{Z}^5_{1/2}$ and CX is an edge of the cell $\mathcal{D}(yz)$. Thus by Lemma 3.4 $\overrightarrow{RX} = \pm\overrightarrow{CX}$. The points R and C are different as otherwise the copy of P centered at C contains all vertices of F but does not contain F itself, which is impossible. The only other option is when X is the midpoint of CR. Similar arguments for midpoints of CX' and RX' show that X' is the midpoint of CR which is a contradiction.

If C and R are in different parity classes then their classes differ by $[0,0,0,0,1]$. Therefore the midpoints of CX and RX' represent the same class in $\mathbb{Z}^5_{1/2}$ and CX is an edge of $\mathcal{D}(yz)$. Thus, $\overrightarrow{RX'} = \pm\overrightarrow{CX}$. Similar arguments for midpoints of CX' and RX give that $\overrightarrow{RX} = \pm\overrightarrow{CX'}$. This is only possible when midpoints of XX' and CR coincide. Same arguments for the midpoints of CY and RY' and for the midpoints of CY' and RY show that the midpoints of CR and YY' coincide which is a contradiction. ☐

Lemma 9.2. *If a triangular face xyz of P with prismatic dual cell has exactly two edges xy and xz with dual cells equivalent to prisms over tetrahedra, then P has a free direction.*

Proof. Suppose P does not have a free direction. $\mathcal{D}(x) = ABXYZA'B'X'Y'Z'$ according to Lemma 9.1. It might be useful to use Figure 4 to track dual cells of faces and edges we use in the arguments.

Parallelogram $XYY'X'$ is a dual cell of a face of the tiling $\mathcal{T}_P$. Let G be this face, so$\mathcal{D}(G) = XYY'X'$ and $\dim G = 3$. In particular, triangle $F = xyz$ is a face of G. Let H_{xy} be the face of G adjacent to F by xy. The dual cell of H_{xy} contains the points X, Y, Y' and X' and is a subcell of $\mathcal{D}(xy) = AXYZA'X'Y'Z'$. The only such 3-cell other than $\mathcal{D}(F)$ is $AXYA'X'Y'$ so $\mathcal{D}(H_{xy}) = AXYA'X'Y'$ which is equivalent to triangular prism. Thus H_{xy} is a triangle or P satisfies the Voronoi conjecture due to Lemma 7.1.

Let $H_{xy} = xyt$ for some point t. We look on the dual cell of the edge xt. This dual cell contains $AXYA'X'Y'$, the dual cell of xyt, and is contained in $ABXYZA'B'X'Y'Z'$, the dual cell of x obtained in Lemma 9.1. The dual cell $\mathcal{D}(xt)$ does not contain Z (or Z') as in that case the copy of P centered at Z (or Z') would contain all three vertices of xyt but not the triangle xyt itself. Hence, $\mathcal{D}(xt)$ contains B or B'. Since the intersection $\mathcal{D}(xz) = BXYZB'X'Y'Z'$ and $\mathcal{D}(xt)$ is a subcell of both, $\mathcal{D}(xt)$ contains both B and B' and $\mathcal{D}(xt) = ABXYA'B'X'Y'$.

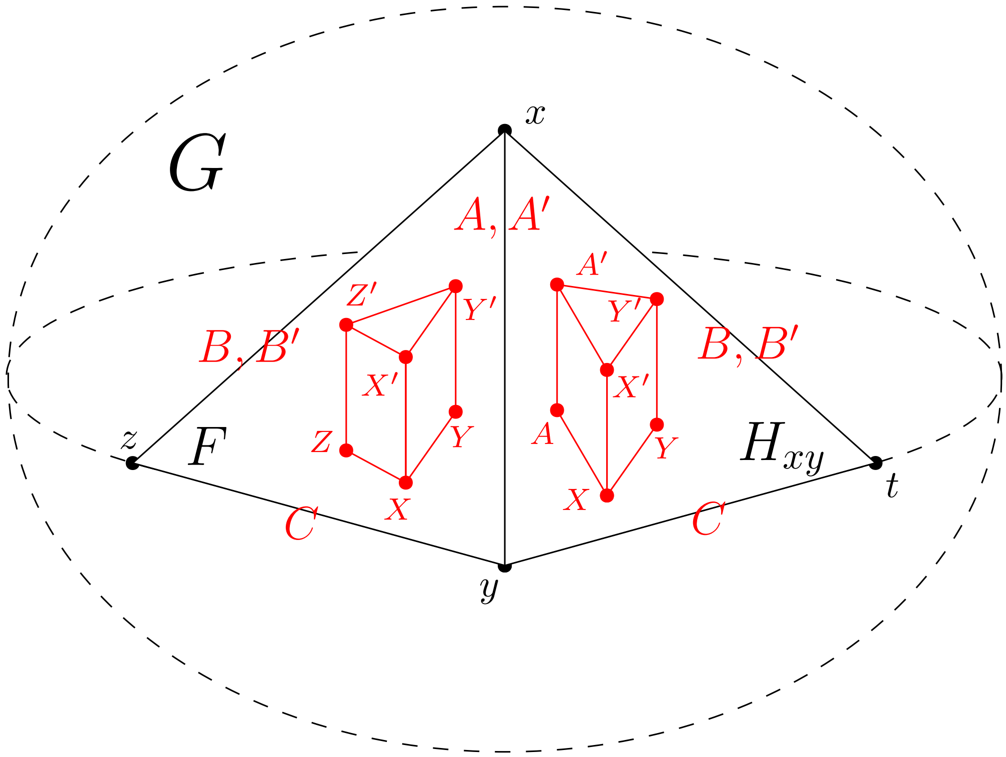

FIGURE 3. An illustration for the proof of Lemma 9.2. The face G with $\mathcal{D}(G) = XYY'X$ and its triangular faces xyz and xyt with prismatic dual cells. We put dual cells of two-dimensional faces inside corresponding triangles and show only additional points corresponding to edges. Dual cells are shown in red.

The dual cells $\mathcal{D}(xz) = BXYZB'X'Y'Z'$ and $D(xt) = ABXYA'B'X'Y'$ intersect by dual 3-cell $BXYB'X'Y'$ and therefore the edges xz and xt belong to two-dimensional face H_{xz} of G with the dual cell $\mathcal{D}(H_{xz}) = BXYB'X'Y'$. Since this dual cell is equivalent to a prism, the face H_{xz} is a triangle and $H_{xz} = xzt$ unless P has a free direction.

Next we identify the dual cell of the edge yt. The dual cell $\mathcal{D}(yt)$ contains $AXYA'X'Y'$, the dual cell of xyt. According to Lemma 7.2, the dual cell of yt contains one additional vertex R or two additional vertices R and R' that differ by $\overrightarrow{XX'}$. Lemma 7.3 for triangle xyt with prismatic dual cell $AXYA'X'Y'$ implies that $Z+B+R = A+X+Y$ or $Z+B+R = A'+X'+Y'$ in $\mathbb{Z}_p^5$. This means that $R \in [1,1,1,1,*] \in \mathbb{Z}^5$.

The points R, C, X, and X' are all in the dual cell $\mathcal{D}(x)$, and CXX' is a triangular dual 2-cell within the cell $\mathcal{D}(yz) = CXYZX'Y'Z'$ equivalent to a pyramid over triangular prism. Hence by Lemma 3.6, no point of the parity class $C + X + X' = C + [0,0,0,0,1]$ belong to $\mathcal{D}(x)$. Thus R represent the same parity class as C and the dual cell of yt does not contain another point R'. Also $R = C$ as $\mathcal{D}(z)$ can contain only one point from the parity class of C. So $\mathcal{D}(yt) = CAXYA'X'Y'$.

Similarly to edges xz and xt, the dual cells $\mathcal{D}(yz) = CXYZX'Y'Z'$ and $\mathcal{D}(yt) = CAXYA'X'Y'$ intersect by dual 3-cell $CXYX'Y'$ and therefore yz and yt belong to a two-dimensional face H_{yz} of G with $\mathcal{D}(H_{yz}) = CXYX'Y'$. Two 2-dimensional faces H_{yz} and H_{xz} have two vertices z and t in common, hence zt is an edge of both and $H_{yz} = yzt$. This means that the face G is a tetrahedron $xyzt$ as we identified four triangular faces $F = xyz$, $H_{xy} = xyt$, $H_{xz} = xzt$, and $H_{yz} = yzt$ of G.

However the dual cell $\mathcal{D}(G)$ is $XYY'X'$, so G is a contact face and must be centrally symmetric. Hence G cannot be a tetrahedron. □

10. Prism-Pyramid-Pyramid case

In this case we assume that dual cell of the edge xy of F is a prism over tetrahedron and the dual cells of edges xz and yz are pyramids over triangular prism $XYZX'Y'Z'$. The results of Section 7 imply that we can consider only the case $\mathcal{D}(xyz) = XYZX'Y'Z'$, $\mathcal{D}(xy) = AXYZA'X'Y'Z'$, $\mathcal{D}(xz) = BXYZX'Y'Z'$, and $\mathcal{D}(yz) = CXYZX'Y'Z'$ where

$$\overrightarrow{XX'} = \overrightarrow{YY'} = \overrightarrow{ZZ'} = \overrightarrow{AA'}$$

and the points represent the following parity classes

$$\begin{array}{ll} X \in [0,0,0,0,0], & X' \in [0,0,0,0,1], \\ Y \in [1,0,0,0,0], & Y' \in [1,0,0,0,1], \\ Z \in [0,1,0,0,0], & Z' \in [0,1,0,0,1], \\ A \in [0,0,1,0,0], & A' \in [0,0,1,0,1], \\ B \in [0,0,0,1,0], & \end{array}$$

and C is in $[1,1,1,1,1]$ or in $[1,1,1,1,0]$. The goal of this section is to show that this configuration is impossible unless P has a free direction; we use a framework similar to one used in Section 9.

Lemma 10.1. *The dual cell $\mathcal{D}(x)$ contains exactly 9 points so $\mathcal{D}(x) = BAXYZA'X'Y'Z'$ or P has a free direction.*

Proof. Suppose $\mathcal{D}(x)$ contains an additional point R. The point R cannot belong to a parity class of points B, A, X, Y, Z, A', X', Y', or Z'. Also we use Lemma 3.6 for 12 triangular dual 2-cells within dual cell $\mathcal{D}(x)$ and each triangle forbids a parity class for R.

Triangle	Forbidden parity class
XYZ	$X+Y+Z \in [1,1,0,0,0]$
AXY	$A+X+Y \in [1,0,1,0,0]$
AXZ	$A+X+Z \in [0,1,1,0,0]$
AYZ	$A+Y+Z \in [1,1,1,0,0]$
$X'Y'Z'$	$X'+Y'+Z' \in [1,1,0,0,1]$
$A'X'Y'$	$A'+X'+Y' \in [1,0,1,0,1]$
$A'X'Z'$	$A'+X'+Z' \in [0,1,1,0,1]$
$A'Y'Z'$	$A'+Y'+Z' \in [1,1,1,0,1]$
BXY	$B+X+Y \in [1,0,0,1,0]$
BXZ	$B+X+Z \in [0,1,0,1,0]$
BYZ	$B+Y+Z \in [1,1,0,1,0]$
BXX'	$B+X+X' \in [0,0,0,1,1]$.

We have eliminated 21 possible case for the parity class of R and the rest 11 cases are eliminated on the case-by-case basis.

Case 10.1.10011: $R \in [1,0,0,1,1]$. The midpoints of segments XY' and RB represent the same class $\langle \frac{1}{2},0,0,0,\frac{1}{2}\rangle \in \mathbb{Z}^5_{1/2}$. Using Lemma 3.3 for cell $\mathcal{D}(x)$ and the contact face with dual cell $XYY'X'$ we get that $\mathcal{D}(x)$ contains a point $R' \in [1,0,0,1,0]$. This case was eliminated above using Lemma 3.6 for points B, X, and Y.

Case 10.1.01011: $R \in [0,1,0,1,1]$. This case becomes identical to **Case 10.1.10011** if we swap Y to Z and Y' to Z'.

Case 10.1.11011: $R \in [1,1,0,1,1]$. This case becomes identical to **Case 10.1.10011** if we swap X to Z and X' to Z'.

Case 10.1.00110: $R \in [0,0,1,1,0]$. If AB is a facet vector of $\mathcal{T}_P$ then we get a contradiction with Lemma 3.6 for three points A, B, X connected with facet vectors and $R = A+B+X$ in $\mathbb{Z}_p^5$ as all four points belong to $\mathcal{D}(x)$. Similarly, the segment $A'B$ is not a facet vector as we get a contradiction with Lemma 3.6 for points A', B, X' and $R = A'+B+X'$ otherwise.

The set of midpoints $M_{\mathcal{D}(yz)}$ of the dual cell $\mathcal{D}(yz) = CXYZX'Y'Z'$ contains 14 classes of points satisfying $x_3 = x_4$ in $\mathbb{Z}_{1/2}^5$. The remaining two points in this 4-dimensional space are $\langle 0,0,\frac{1}{2},\frac{1}{2},0\rangle$ and $\langle 0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2}\rangle$ represented by midpoints of AB and $A'B$ respectively. These two points do not correspond to facet vectors and Lemma 3.7 implies that yz is a free direction for P.

Case 10.1.10110: $R \in [1,0,1,1,0]$. This case becomes identical to **Case 10.1.00110** if we swap X to Y and X' to Y'.

Case 10.1.01110: $R \in [0,1,1,1,0]$. This case becomes identical to **Case 10.1.00110** if we swap X to Z and X' to Z'.

Case 10.1.00111: $R \in [0,0,1,1,1]$. The midpoints of RB and AX' represent the same class $\langle 0,0,\frac{1}{2},0,\frac{1}{2}\rangle \in \mathbb{Z}_{1/2}^5$. Using Lemma 3.3 for dual cell $\mathcal{D}(x)$ and the face with dual cell $AXX'A'$ we get that $\mathcal{D}(x)$ contains a point $R' \in [0,0,1,1,0]$. From **Case 10.1.00110** we conclude that yz is a free direction for P.

Case 10.1.10111: $R \in [1,0,1,1,1]$. The midpoints of RB and AY' represent the same class $\langle \frac{1}{2},0,\frac{1}{2},0,\frac{1}{2}\rangle \in \mathbb{Z}_{1/2}^5$. Using Lemma 3.3 for dual cell $\mathcal{D}(x)$ and the face with dual cell $AYY'A'$ we get that $\mathcal{D}(x)$ contains a point $R' \in [1,0,1,1,0]$. From **Case 10.1.10110** we conclude that yz is a free direction for P.

Case 10.1.01111: $R \in [0,1,1,1,1]$. The midpoints of RB and AZ' represent the same class $\langle 0,\frac{1}{2},\frac{1}{2},0,\frac{1}{2}\rangle \in \mathbb{Z}_{1/2}^5$. Using Lemma 3.3 for dual cell $\mathcal{D}(x)$ and the face with dual cell $AZZ'A'$ we get that $\mathcal{D}(x)$ contains a point $R' \in [0,1,1,1,0]$. From **Case 10.1.01110** we conclude that yz is a free direction for P.

Case 10.1.1111*: $R \in [1,1,1,1,0]$ or $R \in [1,1,1,1,1]$. This case is similar to **Case 9.1.1111*** of Lemma 9.1. □

Lemma 10.2. *If a triangular face xyz of P with prismatic dual cell has exactly two edges xz and yz with dual cells equivalent to pyramids over triangular prism, then P has a free direction.*

Proof. Suppose P does not have a free direction, then $\mathcal{D}(x) = BAXYZA'X'Y'Z'$ according to Lemma 10.1. The proof generally repeats the proof of Lemma 9.2 with minor changes. It might be useful to use Figure 4 to track dual cells of faces and edges we use in the arguments.

Parallelogram $XYY'X'$ is a dual cell of a face of the tiling $\mathcal{T}_P$. Let G be the 3-dimensional face of $\mathcal{T}_P$ such that $\mathcal{D}(G) = XYY'X'$. In particular, triangle $F = xyz$ is a face of G. Let H_{xy} be the face of G adjacent to F by xy. The dual cell of H_{xy} contains the points X, Y, Y' and X' and is contained in $\mathcal{D}(xy) = AXYZA'X'Y'Z'$, hence $\mathcal{D}(H_{xy}) = AXYA'X'Y'$ as this is the only 3-cell within $\mathcal{D}(xy)$ that contains $XYY'X'$ other than $\mathcal{D}(F)$. Since $\mathcal{D}(H_{xy})$

is equivalent to triangular prism, then H_{xy} is a triangle due to Lemma 7.1 unless P has a free direction.

Let $H_{xy} = xyt$ for some point t. We look on the dual cell of the edge xt. This dual cell contains $AXYA'X'Y'$, the dual cell of xyt, and is contained in $BAXYZA'X'Y'Z'$, the dual cell of x obtained in Lemma 10.1. The dual cell $\mathcal{D}(xt)$ does not contain Z (or Z') as in that case the copy of P centered at Z (or Z') would contain all three vertices of xyt but not the triangle xyt itself. Hence, $\mathcal{D}(xt)$ contains B and $\mathcal{D}(xt) = BAXYA'X'Y'$.

The dual cells $\mathcal{D}(xz) = BXYZX'Y'Z'$ and $D(xt) = BAXYA'X'Y'$ intersect by dual 3-cell $BXYX'Y'$ and therefore the edges xz and xt belong to two-dimensional face H_{xz} of G with the dual cell $\mathcal{D}(H_{xz}) = BXYB'X'Y'$.

We can use Lemma 10.1 to find the dual cell $\mathcal{D}(y)$ as similarly to x, y is a vertex of xyz incident to edges having two non-equivalent dual cells. Thus, $\mathcal{D}(y) = CAXYZA'X'Y'Z'$. Similar arguments that we presented for finding $\mathcal{D}(xt)$ show that $\mathcal{D}(yt) = CAXYA'X'Y'$.

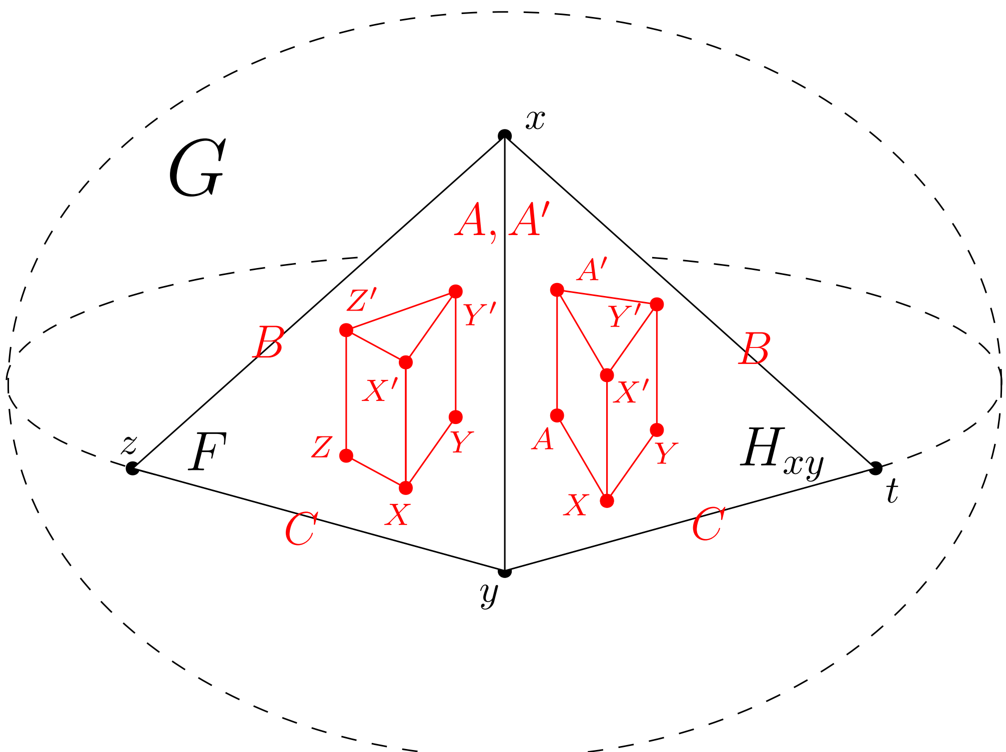

FIGURE 4. An illustration for the proof of Lemma 10.2. The face G with $\mathcal{D}(G) = XYY'X$ and its triangular faces xyz and xyt with prismatic dual cells. We put dual cells of two-dimensional faces inside corresponding triangles and show only additional points corresponding to edges. Dual cells are shown in red.

Similarly to edges xz and xt, the dual cells $\mathcal{D}(yz) = CXYZX'Y'Z'$ and $\mathcal{D}(yt) = CAXYA'X'Y'$ intersect by dual 3-cell $CXYX'Y'$ and therefore yz and yt belong to a two-dimensional face H_{yz} of G with $\mathcal{D}(H_{yz}) = CXYX'Y'$. Two 2-dimensional faces H_{yz} and H_{xz} have two vertices z and t in common, hence zt is an edge of both and $H_{xz} = xzt$ and

$H_{yz} = yzt$. This means that the face G is a tetrahedron $xyzt$ as we identified four triangular faces $F = xyz$, $H_{xy} = xyt$, $H_{xz} = xzt$, and $H_{yz} = yzt$ of G.

However the dual cell $\mathcal{D}(G)$ is $XYY'X'$, so G is a contact face and must be centrally symmetric. Hence G cannot be a tetrahedron. □

11. Pyramid-Pyramid-Pyramid case

In this section we assume that P does not have a dual 3-cell equivalent to a cube. Also, if F is a two-dimensional face of P with dual cell equivalent to a triangular prism, then F is a triangle and dual cells of all edges of F are pyramids over this prism. In all other cases P has a free direction and hence satisfies the Voronoi conjecture.

For such P we show that P admits a canonical scaling or P has a free direction. In both cases P satisfies the Voronoi conjecture. The main tool we use to establish canonical scaling for P is the gain function, see Definition 2.12. We extend the notion of gain function for two facet vectors within a non-triangular dual 2-cell.

Definition 11.1. Let KL and LM be two facet vectors of a dual cell $KLMN$ equivalent to a parallelogram. If O is point such that $OKLMN$ is a dual 3-cell, then we define

$$\gamma(KL, LM) := \gamma(KL, OL) \cdot \gamma(OL, LM).$$

This definition can also be extended to a sequence of facet vectors with each pair of consequent vectors within one dual 2-cell.

We note that this definition gives a way (may be ambiguous) to define the gain function for each pair of appropriate facet vectors as every dual 2-face equivalent to a parallelogram belongs to a pyramidal dual 3-face as all parallelograms in a prismatic dual cell (dual cell of a triangle xyz) belong to pyramid subcells that are faces of pyramids over triangular prism (dual cells of edges xy, xz, and yz).

We also note that this definition may give multiple values for the gain function $\gamma(KL, LM)$ if $KLMN$ is a subcell for two or more pyramidal dual 3-cells. We say that $KLMN$ is *coherent* dual cell if $\gamma(KL, LM)$ does not depend on the choice of O for the cell $OKLMN$ equivalent to a pyramid over parallelogram.

Lemma 11.2. *If all dual 2-cells of $\mathcal{T}_P$ equivalent to parallelograms are coherent, then the Voronoi conjecture is true for P.*

Proof. First we claim that the value of gain function γ is 1 on every cycle of facet vectors of $\mathcal{T}_P$. It is enough to show that for cycles within single dual 3-cell of $\mathcal{T}_P$. For dual 3-cells equivalent to a tetrahedron or an octahedron we refer to [18]. For a dual 3-cell equivalent to a pyramid, the cycle around its apex has gain function 1 due to [18] while all other cycles can be reduced to a multiples of the cycle around apex using Definition 11.1 and a trivial property that if KLM is a triangular dual 2-face, then $\gamma(KL, LM, KM, KL) = 1$.

The last case is a prismatic dual cell as $\mathcal{T}_P$ does not have cubical dual 3-cells. We fix one prismatic dual cell $XYZX'Y'Z'$ and show that all cycles within this cell have gain function 1. All the cycles within this cell are generated by cycles around vertices of the prism, so it is enough to show that

$$\gamma(XX', XY, YZ, XX') = 1.$$

If P does not have a free direction, then $XYZX'Y'Z'$ belongs to a 4-cell $AXYZX'Y'Z'$ equivalent to a pyramid over triangular prism. We use pyramids and tetrahedra within this 4-cell to extract the value of $\gamma(XX', XY, YZ, XX')$. From the dual 3-cell $AXYY'X'$ we know that $\gamma(XX', XY) = \gamma(XX', XA, XY)$. From the tetrahedral dual 3-cell $AXYZ$ we know that $\gamma(XA, XY, YZ) = \gamma(XA, AZ, YZ)$. And from the pyramidal dual 3-cell $AYZZ'Y'$ we know that $\gamma(AZ, YZ, XX') = \gamma(AZ, XX')$ because XX', YY' and ZZ' represent equivalent facet vectors. Combining these equalities together we get

$$\gamma(XX', XY, YZ, XX') = \gamma(XX', XA, XY, YZ, XX') = \\ = \gamma(XX', XA, AZ, YZ, XX') = \gamma(XX', XA, AZ, XX').$$

The last quantity is 1 because $XX' - XA - AZ - XX'$ is a cycle within the pyramidal dual 3-cell $AXZZ'X'$

Once the gain function γ has value 1 on every cycle of facet vectors of $\mathcal{T}_P$, then $\mathcal{T}_P$ admits a canonical scaling. In this case we fix a facet $F \in \mathcal{T}_P^4$ (the set of all facets of $\mathcal{T}_P$) and set $s(F) := 1$ where $s : \mathcal{T}_P^4 \longrightarrow \mathbb{R}_+$ is the canonical scaling we construct. For a facet $G \in \mathcal{T}_P^4$ we choose any path $F = F_0, F_1, \dots, F_k = G$ such that F_i and F_j share a face of codimension 2 and define

$$s(G) = \gamma(F_0, F_1, \dots, F_k).$$

It is easy to see that if all parallellograms are coherent then s is indeed a canonical scaling for $\mathcal{T}_P$ and hence P satisfies the Voronoi conjecture.

We also refer to [18] for more details on connection between gain function and canonical scaling. □

It remains to prove that all dual 2-cells equivalent to parallelograms are coherent.

Lemma 11.3. *If G is a contact 3-dimensional face of $\mathcal{T}_P$, then $\mathcal{D}(G)$ is coherent or P has a free direction.*

Proof. We consider only the case when P does not have a free direction.

Suppose that G_1 and G_2 are two two-dimensional faces of $\mathcal{T}_P$ incident to G such that $\mathcal{D}(G_1)$ and $\mathcal{D}(G_2)$ are equivalent to pyramids over parallelograms. We need to show that G_1 and G_2 give rise to the same value of gain function between two facet vectors of $\mathcal{D}(G)$ using Definition 11.1.

We note that no two two-dimensional faces of G adjacent by an edge can both have dual cells equivalent to triangular prisms because the common edge of these two faces has dual cell equivalent to a pyramid over triangular prism (this is the case in this Section). However a pyramid over triangular prism has only one face equivalent to triangular prism.

G is a 3-dimensional polytope and G_1 and G_2 are two-dimensional faces of G. We connect a vertex of G_1 with a vertex of G_2 by a path of edges of G. For every edge of this path, at least one of two incident two-dimensional faces of G has dual cell equivalent to a pyramid, so it is enough to show that if G_1 and G_2 share a vertex, then they give rise to the same value of gain function between two facet vectors of $\mathcal{D}(G)$.

If G_1 and G_2 share an edge e, then e does not belong to a two-dimensional face of $\mathcal{T}_P$ with dual 3-cell equivalent to a triangular prism. Indeed, in that case the dual cell $\mathcal{D}(e)$ would be a pyramid over triangular prism, but a pyramid over triangular prism does not have two pyramidal faces with a common base, and such two faces must be dual cells of G_1 and G_2. Thus, the two-dimensional faces that contain e have tetrahedral, octahedral, or pyramidal dual 3-cells. Then e is a locally "Ordine" edge meaning that the dual cell $\mathcal{D}(e)$ does not contain cubical or prismatic dual 3-cells as subcells. In that case the parallelogram dual cell of G has the same gain function within cells corresponding to faces incident to e, see [40, Sec. 7], in particular for the faces G_1 and G_2.

If $G_1 \cap G_2$ is a vertex of G, then there is a cycle of two-dimensional faces of G around this vertex, so there are two non-intersecting paths of two-dimensional faces of G from G_1 to G_2, both with a common vertex $G_1 \cap G_2$. If one of these paths does not contain two-dimensional faces with prismatic dual cells, then the faces G_1 and G_2 give rise to the same value of gain function between two facet vectors of $\mathcal{D}(G)$ as this value does not change if we travel along the path around $G_1 \cap G_2$ using only two-dimensional faces with pyramidal dual 3-cells. Suppose that there is a face with prismatic dual cell on each path. It means that there are two triangular faces H_1 and H_2 of G that share a vertex $V = G_1 \cap G_2$ such that both $\mathcal{D}(H_1)$ and $\mathcal{D}(H_2)$ are equivalent to triangular prisms, see Figure 5.

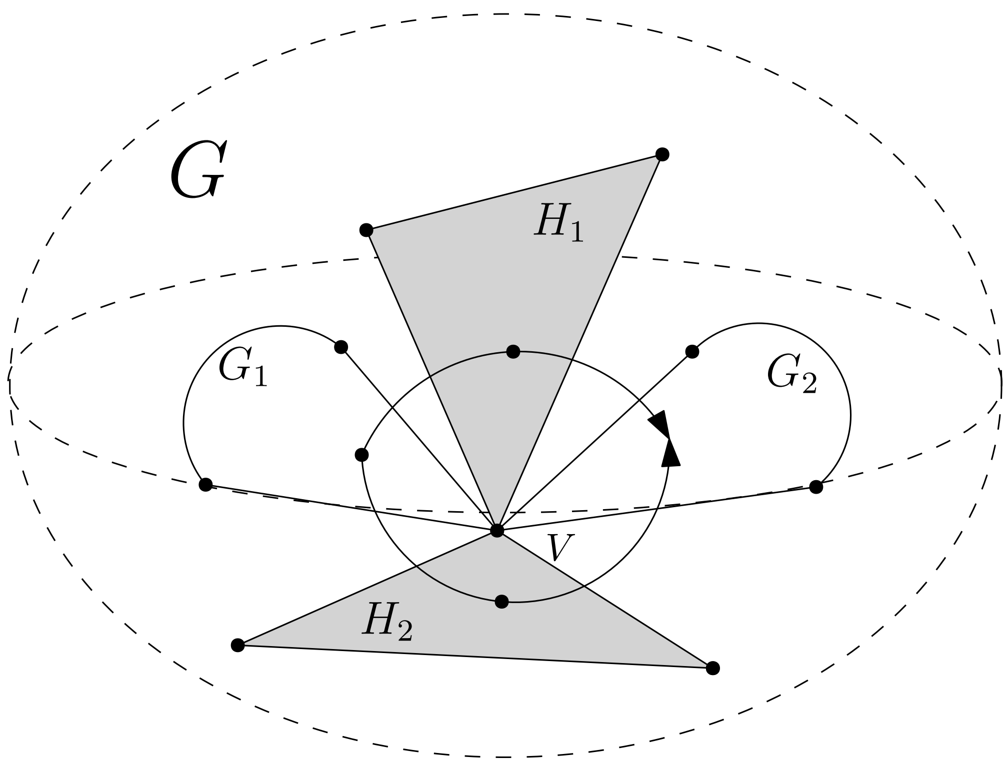

FIGURE 5. An illustration for the proof of Lemma 11.3. The face G with $\mathcal{D}(G) = XYY'X$ and its faces G_1 and G_2 with a common vertex V. Two paths from G_1 to G_2 around V pass through triangular faces H_1 and H_2 with prismatic dual cells.

Let $XYY'X'$ be the dual cell of G and let $XYZX'Y'Z'$ and $XYTX'Y'U$ be dual cells of H_1 and H_2. We can assume that $\overrightarrow{XX'} = \overrightarrow{YY'} = \overrightarrow{ZZ'}$ but $\overrightarrow{TU}$ may be equal to either $\overrightarrow{XX'}$ or $\overrightarrow{XY}$.

We look on the parity class of the point T. The three-dimensional subspace of $\mathbb{Z}_p^5$ spanned by $XYZX'Y'Z'$ contains parity classes of X, Y, Z, $X+Y+Z$, X', Y', Z', and $X'+Y'+Z'$. None of these classes can be the class of T due to Lemmas 2.10 and 3.6 as vertices of $XYZX'Y'Z'$ and T are in the dual cell of V. Thus, there is an edge e_A of H_1 with dual cell $AXYZX'Y'Z'$ such that A and T are in one affine subspace of $\mathbb{Z}_p^5$ parallel to the plane spanned by $XYZX'Y'Z'$.

Let π_A be the 4-dimensional subspace of $\mathbb{Z}^5_{1/2}$ spanned by the set of midpoints of $\mathcal{D}(e_A) = AXYZX'Y'Z'$. The set of midpoints of the pyramid $AXYZX'Y'Z'$ contains 14 classes in π_A. 8 of these classes are in the set of midpoints of $XYZX'Y'Z'$ and the remaining 6 classes correspond to midpoints of facet vectors AX, AY, AZ, AX', AY', and AZ'. Among midpoints of the dual cell $XYTX'Y'U$, the midpoints of TX, TY, TX', and TY' contain two classes that correspond to contact faces of codimension 2 (parallellogram subcells of $XYTX'Y'U$ that contain T). 4 midpoints of TX, TY, TX', and TY' together with 6 midpoints of AX, AY, AZ, AX', AY', and AZ' give 8 classes (two classes are repeated twice) that form an affine subspace of π_A parallel to the space spanned by 8 midpoints of $XYZX'Y'Z'$.

Now we can use Lemma 3.7 for the edge e_A and four-dimensional subspace π_A. Among 16 classes in π_A 14 are the classes of midpoints of $\mathcal{D}(e_A)$ and two correspond to contact faces of codimension 2 of $XYTX'Y'U$. Thus, e_A is a free direction for P in that case. □

Corollary 11.4. *If for every face F of P with prismatic dual 3-cell, dual cells of all edges of F are equivalent to pyramids over triangular prisms, then P satisfies the Voronoi conjecture.*

Proof. If all dual 2-cells of $\mathcal{T}_P$ equivalent to parallelogram are coherent, then P satisfies the Voronoi conjecture due to Lemma 11.2. Otherwise, there is a contact 3-dimensional face of P with incoherent dual 2-cell and P has a free direction due to Lemma 11.3. In this case P satisfies the Voronoi conjecture as well. □

The proof of Lemma 11.3 above and the general approach for parallelohedra without dual 3-cells equivalent to prisms or cubes in Theorem 4.1 rely on the proof of Ordine [40, Sec. 7]. The most complicated part of the proof of Ordine and the only part that involves computer computations using `PORTA` software is Case 4 in [40, Subsection 7.6]. In this particular case Ordine shows that there is no dual 4-cell (with all dual 3-cells equivalent to tetrahedra, octahedra, or pyramids) with incoherent parallelograms forming a family $\mathcal{R}$ such that

- each two parallelograms in $\mathcal{R}$ intersect over a vertex;
- each vertex of a parallelogram in $\mathcal{R}$ belongs to at least one other parallelogram in $\mathcal{R}$.

In five-dimensional case these computations can be avoided.

Particularly, if e is an edge of five-dimensional parallelohedron P with dual 4-cell that contains a family of incoherent parallelograms satisfying the conditions above, then the first condition implies that $\mathcal{D}(e)$ contains two parallelograms $ABCD$ and $AXYZ$. For a certain choice of coordinate system in $\mathbb{Z}^5_p$, the points A, B, C, D, X, Y and Z belong to the following parity classes

$$\begin{array}{ll} A \in [0,0,0,0,0], & \\ B \in [1,0,0,0,0], & X \in [0,0,1,0,0], \\ C \in [0,1,0,0,0], & Y \in [0,0,0,1,0], \\ D \in [1,1,0,0,0], & Z \in [0,0,1,1,0]. \end{array}$$

In that case, the set of midpoints $M_{\mathcal{D}(e)}$ contains all points from the 4-dimensional space $x_5 = 0$ of $\mathbb{Z}^5_{1/2}$ and e is a free direction of P according to Lemma 3.7

12. Concluding remarks

In this section we explain why our approach cannot be carried out in higher dimensions without significant improvement. Our approach relies on two results that seem to require additional elaboration in order to be used in dimensions 6 and beyond.

The first result is the classification of five-dimensional Dirichlet-Voronoi parallelohedra from [10] and verification of the combinatorial condition from [18] done in [11] for every five-dimensional Dirichlet-Voronoi parallelohedra. While the verification is computationally simple for a given parallelohedron, the full classification in $\mathbb{R}^6$ and beyond looks unreachable at this moment. Particularly, the paper [44] reports about more than 250 000 types of Delone triangulations (and consequently, primitive parallelohedra) in $\mathbb{R}^6$; a more recent paper [1] reports about more than 500 000 000 types of primitive parallelohedra in $\mathbb{R}^6$. Both computations were terminated before finding all triangulations/parallelohedra and both suggest that the total number of parallelohedra in $\mathbb{R}^6$, both primitive and not, is too large for computational study without additional insight.

The second result is the classification of dual 3-cells by Delone [5]. In five-dimensional case, dual 3-cells originate from two-dimensional faces that have a fairly simple structure that allowed us to prove many properties in Sections 7 through 11. In higher dimension, we would need to deal either with three-dimensional faces of parallelohedra with additional co-dimension in the spaces Λ_p and $\Lambda_{1/2}$, or with dual 4-cells. However at this point there is no complete classification of dual 4-cells and, in particular, the question on dimension of affine space spanned by vertices of a dual 4-cell is still open.

As a final remark, we would like to formulate two well-known conjectures on dual cells. These conjectures are still open and having a counterexample for each of them will immediately give a counterexample to the Voronoi conjecture.

Conjecture (Dimension conjecture)**.** *For every dual k-cell, the dimension of its affine span is equal to k.*

This conjecture is proved for $k \leq 3$ as all dual 3-cells are known. A stronger version of this conjecture imposes additional structure coming from Delone tilings.

Conjecture. *For every dual k-cell D there is a k-dimensional lattice Λ such that there is a k-dimensional cell in the Delone tessellation of Λ equivalent to D.*

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Alexey Garber
The University of Texas Rio Grande Valley, Brownsville, TX, USA
E-mail address: alexeygarber@gmail.com

Alexander Magazinov
Higher School of Economics, Moscow, Russia
E-mail address: magazinov-al@yandex.ru