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THE QUANTIZATION OF THE STANDARD TRIADIC CANTOR DISTRIBUTION

MRINAL KANTI ROYCHOWDHURY

ABSTRACT. The quantization scheme in probability theory deals with finding a best approximation of a given probability distribution by a probability distribution that is supported on finitely many points. For a given $k \geq 2$, let $\{S_j : 1 \leq j \leq k\}$ be a set of k contractive similarity mappings such that $S_j(x) = \frac{1}{2^{k-1}}x + \frac{2(j-1)}{2^{k-1}}$ for all $x \in \mathbb{R}$, and let $P = \frac{1}{k} \sum_{j=1}^k P \circ S_j^{-1}$. Then, P is a unique Borel probability measure on \mathbb{R} such that P has support the Cantor set generated by the similarity mappings S_j for $1 \leq j \leq k$. In this paper, for the probability measure P , when $k = 3$, we investigate the optimal sets of n -means and the n th quantization errors for all $n \geq 2$. We further show that the quantization coefficient does not exist though the quantization dimension exists.

1. INTRODUCTION

One of the main mathematical aims of the quantization problem is to study the error in the approximation of a given probability measure with a probability measure of finite support. We refer to [GL1, GL3, GL4, GL5, P] for more theoretical results, and [P1, P2] for promising applications of quantization theory. One may see [GG, GN, Z] for its deep background in information theory and engineering technology. Let \mathbb{R}^d denote the d -dimensional Euclidean space, $\|\cdot\|$ denote the Euclidean norm on \mathbb{R}^d for any $d \geq 1$, and $n \in \mathbb{N}$. For a finite set $\alpha \subset \mathbb{R}^d$, the number $\int \min_{a \in \alpha} \|x - a\|^2 dP(x)$ is often referred to as the *cost* or *distortion error* for α , and is denoted by $V(P; \alpha)$. Then, the n th quantization error, denoted by $V_n := V_n(P)$, is defined by

$$V_n = \inf \left\{ V(P; \alpha) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n \right\}.$$

Such a set α for which the infimum occurs and contains no more than n points is called an *optimal set of n -means*, and is denoted by $\alpha_n := \alpha_n(P)$. It is known that for a continuous probability measure an optimal set of n -means always has exactly n elements (see [GL1]). To see some work in the direction of optimal sets of n -means, one is referred to [CR, DR1, GL2, RR, R1–R5]. The number

$$D(P) := \lim_{n \rightarrow \infty} \frac{2 \log n}{-\log V_n(P)},$$

if it exists, is called the *quantization dimension* of P and is denoted by $D(P)$. For any $s \in (0, +\infty)$, the number

$$\lim_{n \rightarrow \infty} n^{\frac{2}{s}} V_n(P),$$

if it exists, is called the *s -dimensional quantization coefficient* for P . Given a finite subset $\alpha \subset \mathbb{R}^d$, the Voronoi region generated by $a \in \alpha$ is the set of all elements in \mathbb{R}^d which are closer to a than to any other element in α . Let us now state the following proposition (see [GG, GL1]).

Proposition 1.1. *Let α be an optimal set of n -means, $a \in \alpha$, and $M(a|\alpha)$ be the Voronoi region generated by $a \in \alpha$, i.e., $M(a|\alpha) = \{x \in \mathbb{R}^d : \|x - a\| = \min_{b \in \alpha} \|x - b\|\}$. Then, for every $a \in \alpha$, (i) $P(M(a|\alpha)) > 0$, (ii) $P(\partial M(a|\alpha)) = 0$, (iii) $a = E(X : X \in M(a|\alpha))$.*

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From the above proposition, we can say that if α is an optimal set of n -means for P , then each $a \in \alpha$ is the conditional expectation of the random variable X given that X takes values in the Voronoi region of a . Sometimes, we also refer to such an $a \in \alpha$ as the centroid of its own Voronoi region. In this regard, interested readers can see [DFG, DR2, R1].

Let $k \geq 2$ be a positive integer, and let $\{S_j : 1 \leq j \leq k\}$ be a set of contractive similarity mapping such that $S_j(x) = \frac{1}{2^{k-1}}x + \frac{2(j-1)}{2^{k-1}}$ for all $x \in \mathbb{R}$, and let $P = \frac{1}{k} \sum_{j=1}^k P \circ S_j^{-1}$. Then, P is a unique Borel probability measure on \mathbb{R} , and P has support the Cantor set C generated by the similarity mappings S_j for $1 \leq j \leq k$, and C satisfies the invariance equality $C = \bigcup_{j=1}^k S_j(C)$ (see [F,H]). The Cantor set C generated by the k similarity mappings is called the k -adic Cantor set, more specifically the *standard k -adic Cantor set*, and the probability measure P is called the k -adic Cantor distribution, more specifically the *standard k -adic Cantor distribution*. If $k = 2$, then we have two similarity mappings given by $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{3}$ for all $x \in \mathbb{R}$, and then the probability measure P is given by $P = \frac{1}{2}P \circ S_1^{-1} + \frac{1}{2}P \circ S_2^{-1}$, which has support the classical Cantor set C satisfying $C = S_1(C) \cup S_2(C)$. For this dyadic Cantor distribution, in [GL2], Graf and Luschgy determined the optimal sets of n -means and the n th quantization errors for all $n \geq 2$. They also showed that the quantization dimension $D(P)$ of P exists, and equals the Hausdorff dimension of the invariant set C , but the quantization coefficient does not exist.

By a word σ of length n , where $n \geq 1$, over the alphabet $\{1, 2, 3\}$, it is meant that $\sigma := \sigma_1\sigma_2 \cdots \sigma_n$, where $\sigma_j \in \{1, 2, 3\}$ for all $1 \leq j \leq n$. By $\{1, 2, 3\}^n$, we denote the set of all words over the alphabet $\{1, 2, 3\}$ of some finite length $n \geq 0$. Notice that the empty word \emptyset has length zero. For $\sigma := \sigma_1\sigma_2 \cdots \sigma_n \in \{1, 2, 3\}^n$, by S_σ it is meant that $S_\sigma := S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}$. For the empty word \emptyset , by S_\emptyset it is meant the identity mapping on \mathbb{R} . Write

$$\{1, 2, 3\}^* := \bigcup_{n=0}^{\infty} \{1, 2, 3\}^n,$$

i.e., $\{1, 2, 3\}^*$ denotes the set of all words over the alphabet $\{1, 2, 3\}$ including the empty word \emptyset . Let X be a random variable with probability distribution P . For words $\beta, \gamma, \dots, \delta$ in $\{1, 2, 3\}^*$, by $a(\beta, \gamma, \dots, \delta)$ we mean the conditional expectation of the random variable X given $J_\beta \cup J_\gamma \cup \dots \cup J_\delta$, i.e.,

$$a(\beta, \gamma, \dots, \delta) = E(X|X \in J_\beta \cup J_\gamma \cup \dots \cup J_\delta) = \frac{1}{P(J_\beta \cup \dots \cup J_\delta)} \int_{J_\beta \cup \dots \cup J_\delta} x dP(x).$$

Definition 1.2. For $n \in \mathbb{N}$ with $n \geq 3$ let $\ell(n)$ be the unique natural number with $3^{\ell(n)} \leq n < 3^{\ell(n)+1}$. Write $\alpha_2 := \{a(1, 21), a(22, 23, 3)\}$ and $\alpha_3 := \{a(1), a(2), a(3)\}$. For $n \geq 3$, define $\alpha_n := \alpha_n(I)$ as follows:

$$\alpha_n(I) = \begin{cases} \{a(\omega) : \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \cup \bigcup_{\omega \in I} S_\omega(\alpha_2) & \text{if } 3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}, \\ \{S_\omega(\alpha_2) : \omega \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \cup \bigcup_{\omega \in I} S_\omega(\alpha_3) & \text{if } 2 \cdot 3^{\ell(n)} < n < 3^{\ell(n)+1}, \end{cases}$$

where $I \subset \{1, 2, 3\}^{\ell(n)}$ with $\text{card}(I) = n - 3^{\ell(n)}$ if $3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}$; and $\text{card}(I) = n - 2 \cdot 3^{\ell(n)}$ if $2 \cdot 3^{\ell(n)} < n < 3^{\ell(n)+1}$.

In this paper, in Section 3 we show that for all $n \geq 2$, the sets α_n given by Definition 1.2 form the optimal sets of n -means for the standard triadic Cantor distribution P . In Section 4, we show that the quantization coefficient does not exist though the quantization dimension $D(P)$ exists. Notice that the probability measure P is symmetric about the point $\frac{1}{2}$, i.e., if two intervals of equal lengths are equidistant from the point $\frac{1}{2}$, then they have the same probability. Thus, it seems, which is true in the case of dyadic Cantor distribution (see [GL2]), that if the closed interval $[0, 1]$ is partitioned in the middle, then the conditional expectations of the left half $[0, \frac{1}{2}]$, and the right half $[\frac{1}{2}, 1]$ will form the optimal set of two-means. In Proposition 2.6, we show that it is not true in the case of triadic Cantor distribution. The result in this paper

extends the well-known result for the dyadic Cantor distribution given by Graf-Luschgy, and we are grateful to say that the work in this paper was motivated by their work (see [GL2]).

2. PRELIMINARIES

Let S_j for $1 \leq j \leq 3$ be the contractive similarity mappings on \mathbb{R} given by $S_j(x) = \frac{1}{5}x + \frac{2}{5}(j-1)$ for all $x \in \mathbb{R}$. For $\sigma := \sigma_1\sigma_2 \cdots \sigma_n \in \{1, 2, 3\}^n$, set $J_\sigma := S_\sigma([0, 1])$, where $S_\sigma := S_{\sigma_1} \circ \cdots \circ S_{\sigma_n}$. For the empty word \emptyset , write $J := J_\emptyset = S_\emptyset([0, 1]) = [0, 1]$. Then, the set $C := \bigcap_{n \in \mathbb{N}} \bigcup_{\sigma \in \{1, 2, 3\}^n} J_\sigma$ is known as the *Cantor set* generated by the mappings S_j , and equals the support of the probability measure P given by $P = \sum_{j=1}^3 \frac{1}{3} P \circ S_j^{-1}$. For $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \{1, 2, 3\}^*$, $n \geq 0$, write $p_\sigma := \frac{1}{3^n}$ and $s_\sigma := \frac{1}{5^n}$.

Let us now give the following lemmas. The proofs are similar to the similar lemmas in [GL2].

Lemma 2.1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}^+$ be Borel measurable and $k \in \mathbb{N}$. Then,*

$$\int f dP = \sum_{\sigma \in \{1, 2, 3\}^k} \frac{1}{3^k} \int (f \circ S_\sigma)(x) dP(x).$$

Lemma 2.2. *Let X be a random variable with probability distribution P . Then, $E(X) = \frac{1}{2}$ and $V := V(X) = \frac{1}{9}$, and for any $x_0 \in \mathbb{R}$, $\int (x - x_0)^2 dP(x) = V + (x_0 - \frac{1}{2})^2$.*

From Lemma 2.1 and Lemma 2.2 the following corollary is true.

Corollary 2.3. *Let $\sigma \in \{1, 2, 3\}^*$ and $x_0 \in \mathbb{R}$. Then,*

$$(1) \quad \int_{J_\sigma} (x - x_0)^2 dP(x) = p_\sigma \left(s_\sigma^2 V + \left(S_\sigma\left(\frac{1}{2}\right) - x_0 \right)^2 \right).$$

Remark 2.4. From the above lemma it follows that the optimal set of one-mean is the expected value and the corresponding quantization error is the variance V of the random variable X . For $\sigma \in \{1, 2, 3\}^n$, $n \geq 1$, we have $a(\sigma) = E(x : X \in J_\sigma) = S_\sigma(\frac{1}{2})$.

Proposition 2.5. *Let $\alpha_n := \alpha_n(I)$ be the set given by Definition 1.2. If $3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}$, then the number of such sets is $3^{\ell(n)} C_{n-3^{\ell(n)}}$, and the corresponding distortion error is given by*

$$V(P; \alpha_n(I)) = \frac{1}{75^{\ell(n)}} \left((2 \cdot 3^{\ell(n)} - n)V + (n - 3^{\ell(n)})V(P; \alpha_2) \right).$$

If $2 \cdot 3^{\ell(n)} < n < 3^{\ell(n)+1}$, then the number of such sets is $3^{\ell(n)} C_{n-2 \cdot 3^{\ell(n)}}$, and the corresponding distortion error is given by

$$V(P; \alpha_n(I)) = \frac{1}{75^{\ell(n)}} \left((3^{\ell(n)+1} - n)V(P; \alpha_2) + (n - 2 \cdot 3^{\ell(n)})V(P; \alpha_3) \right).$$

Proof. If $3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}$, then the subset I can be chosen in $3^{\ell(n)} C_{n-3^{\ell(n)}}$ different ways, and so, the number of such sets is given by $3^{\ell(n)} C_{n-3^{\ell(n)}}$, and the corresponding distortion error is obtained as

$$\begin{aligned} V(P; \alpha_n(I)) &= \sum_{\sigma \in \{1, 2, 3\}^{\ell(n)} \setminus I} \int_{J_\sigma} (x - a(\sigma))^2 dP + \sum_{\sigma \in I} \int_{J_\sigma} \min_{a \in S_\sigma(\alpha_2)} (x - a)^2 dP \\ &= \frac{1}{3^{\ell(n)}} \frac{1}{25^{\ell(n)}} \left(\sum_{\sigma \in \{1, 2, 3\}^{\ell(n)} \setminus I} V + \sum_{\sigma \in I} V(P; \alpha_2) \right) = \frac{1}{75^{\ell(n)}} \left((2 \cdot 3^{\ell(n)} - n)V + (n - 3^{\ell(n)})V(P; \alpha_2) \right). \end{aligned}$$

Similarly, the other part of the proposition can be derived. Thus, the proof of the proposition is complete. \square

The following proposition is helpful to find the optimal set of two-means given in Lemma 3.1. It also shows that though the triadic Cantor distribution is uniform and symmetric about the point $\frac{1}{2}$, the set consisting of the conditional expectations of the left half $[0, \frac{1}{2}]$, and the right half $[\frac{1}{2}, 1]$ does not form an optimal set of two-means.

Proposition 2.6. *Let $a_1 := E(X : X \in [0, \frac{1}{2}])$, and $a_2 := E(X : X \in [\frac{1}{2}, 1])$. Then, the set $\gamma := \{a_1, a_2\}$ does not form an optimal set of two-means for P .*

Proof. By the hypothesis, we have

$$a_1 = E(X : X \in [0, \frac{1}{2}]) = E\left(X : X \in J_1 \cup J_{21} \cup J_{221} \cup \dots\right), \text{ and}$$

$$a_2 = E(X : X \in [\frac{1}{2}, 1]) = E\left(X : X \in J_3 \cup J_{23} \cup J_{223} \cup \dots\right),$$

yielding

$$a_1 = 2 \sum_{n=1}^{\infty} \frac{1}{3^n} \frac{1}{2} \frac{5^n - 4}{5^n} = \frac{3}{14}, \text{ and } a_2 = 2 \sum_{n=1}^{\infty} \frac{1}{3^n} \frac{1}{2} \frac{5^n + 4}{5^n} = \frac{11}{14}.$$

and the corresponding distortion error is given by

$$V(P; \gamma) = 2 \int_{J_1 \cup J_{21} \cup J_{221} \cup J_{2221} \dots} \left(x - \frac{3}{14}\right)^2 dP$$

implying

$$V(P; \gamma) = 2 \left(\sum_{n=1}^{\infty} \frac{1}{75^n} V + \sum_{n=1}^{\infty} \frac{1}{3^n} \left(\frac{1}{2} \frac{5^n - 4}{5^n} - \frac{3}{14} \right)^2 \right) = \frac{13}{441}.$$

Let us now consider the set $\beta := \{a(1, 21), a(22, 23, 3)\}$. Since $\frac{11}{25} = S_{21}(1) < \frac{1}{2}(a(1, 21) + a(22, 23, 3)) = \frac{117}{250} < S_{22}(0) = \frac{12}{25}$, the distortion error due to the set β is given by

$$V(P; \beta) = \int_{J_1 \cup J_{21}} (x - a(1, 21))^2 dP + \int_{J_{22} \cup J_{23} \cup J_3} (x - a(22, 23, 3))^2 dP = \frac{821}{28125}.$$

Since $V(P; \gamma) = \frac{13}{441} > \frac{821}{28125} = V(P; \beta)$, the set γ does not form an optimal set of two-means yielding the proposition. \square

3. OPTIMAL SETS OF n -MEANS AND THE n TH QUANTIZATION ERRORS FOR ALL $n \geq 2$

Let α_n be the set given by Definition 1.2. In this section, we show that for all $n \geq 2$, the sets α_n form the optimal sets of n -means for the standard triadic Cantor distribution P . To calculate the quantization error we will frequently use the formula given by (1).

Lemma 3.1. *The set $\alpha_2 := \{a(1, 21), a(22, 23, 3)\}$ forms an optimal set of two-means, and the corresponding quantization error is given by $V_2 = \frac{821}{28125}$.*

Proof. Consider the set β of two points given by $\beta := \{a(1, 21), a(22, 23, 3)\}$. The distortion error due to the set β is given by

$$\int \min_{b \in \beta} (x - b)^2 dP = \int_{J_1 \cup J_{21}} (x - a(1, 21))^2 dP + \int_{J_{22} \cup J_{23} \cup J_3} (x - a(22, 23, 3))^2 dP = \frac{821}{28125}.$$

Since V_2 is the quantization error for two-means, we have $V_2 \leq \frac{821}{28125}$. Let $\alpha := \{a, b\}$ be an optimal set of two-means. Since the optimal quantizers are the expected values of their own Voronoi regions, we have $0 < a < b < 1$. By Proposition 2.6, we see that the boundary of the Voronoi regions can not pass through the midpoint $\frac{1}{2}$. Thus, without any loss of generality we assume that the boundary of the Voronoi regions, i.e., the point $\frac{1}{2}(a + b)$ lies to the left of the midpoint $\frac{1}{2}$, i.e., $\frac{1}{2}(a + b) < \frac{1}{2}$. We now show that the Voronoi region of a contains points from J_2 . For the sake of contradiction, assume that the Voronoi region of a does not contain any point from J_2 . Then,

$$V_2 \geq \int_{J_2 \cup J_3} (x - a(2, 3))^2 dP = \frac{4}{135} > V_2,$$

which leads to a contradiction. Hence, we can assume that the Voronoi region of a contains points from J_2 , i.e., $S_2(0) = \frac{2}{5} < \frac{1}{2}(a+b) < \frac{1}{2}$. Assume that $S_2(0) = \frac{2}{5} < \frac{1}{2}(a+b) < S_{2112}(0)$. Then, writing $A_1 := J_{2112} \cup J_{2113} \cup J_{212} \cup J_{213} \cup J_{22} \cup J_{23} \cup J_3$, we have

$$V_2 \geq \int_{J_1} (x - a(1))^2 dP + \int_{A_1} (x - a(2112, 2113, 212, 213, 22, 23, 3))^2 dP = \frac{452551888}{15092578125} > V_2,$$

which leads to a contradiction. Hence, $S_{2112}(0) < \frac{1}{2}(a+b) < \frac{1}{2}$. Assume that $S_{2112}(0) < \frac{1}{2}(a+b) < S_{2113}(0)$. Then, writing $A_2 := J_{2113} \cup J_{212} \cup J_{213} \cup J_{22} \cup J_{23} \cup J_3$, we have

$$V_2 \geq \int_{J_1 \cup J_{2111}} (x - a(1, 2111))^2 dP + \int_{A_2} (x - a(2113, 212, 213, 22, 23, 3))^2 dP = \frac{775266524}{25913671875} > V_2,$$

which gives a contradiction. Hence, $S_{2113}(0) < \frac{1}{2}(a+b) < \frac{1}{2}$. Assume that $S_{2113}(0) < \frac{1}{2}(a+b) < S_{212}(0)$. Then, writing $A_3 := J_{212} \cup J_{213} \cup J_{22} \cup J_{23} \cup J_3$, we have

$$V_2 \geq \int_{J_1 \cup J_{2111} \cup J_{2112}} (x - a(1, 2111, 2112))^2 dP + \int_{A_3} (x - a(212, 213, 22, 23, 3))^2 dP = \frac{4180404512}{140389453125}$$

yielding $V_2 > V_2$, which is a contradiction. Hence, $S_{212}(0) < \frac{1}{2}(a+b) < \frac{1}{2}$. Assume that $S_{212}(0) < \frac{1}{2}(a+b) < S_{2122}(0)$. Then, writing $A_4 := J_{2122} \cup J_{2123} \cup J_{213} \cup J_{22} \cup J_{23} \cup J_3$, we have

$$V_2 \geq \int_{J_1 \cup J_{211}} (x - a(1, 211))^2 dP + \int_{A_4} (x - a(2122, 2123, 213, 22, 23, 3))^2 dP = \frac{210852992}{7119140625} > V_2,$$

which is a contradiction. Hence, $S_{2122}(0) < \frac{1}{2}(a+b) < \frac{1}{2}$. Assume that $S_{2122}(0) < \frac{1}{2}(a+b) < S_{2123}(0)$. Then, writing $A_5 := J_{2123} \cup J_{213} \cup J_{22} \cup J_{23} \cup J_3$, we have

$$V_2 \geq \int_{J_1 \cup J_{211} \cup J_{2121}} (x - a(1, 211, 2121))^2 dP + \int_{A_5} (x - a(2123, 213, 22, 23, 3))^2 dP = \frac{12733641776}{432558984375}$$

yielding $V_2 > V_2$, which is a contradiction. Hence, $S_{2123}(0) < \frac{1}{2}(a+b) < \frac{1}{2}$. Assume that $S_{2123}(0) < \frac{1}{2}(a+b) < S_{213}(0)$. Writing $A_6 = J_1 \cup J_{211} \cup J_{2121} \cup J_{2122}$, and $A_7 = J_{213} \cup J_{22} \cup J_{23} \cup J_3$, we have

$$V_2 \geq \int_{A_6} (x - a(1, 211, 2121, 2122))^2 dP + \int_{A_7} (x - a(213, 22, 23, 3))^2 dP = \frac{332546}{11390625} > V_2,$$

which give a contradiction. Hence, $S_{213}(0) < \frac{1}{2}(a+b) < \frac{1}{2}$. Assume that $S_{213}(0) < \frac{1}{2}(a+b) < S_{21313}(0)$. Then, writing $A_8 = J_1 \cup J_{211} \cup J_{212}$, $a_8 = E(X : X \in A_8)$, $A_9 = J_{21313} \cup J_{2132} \cup J_{2133} \cup J_{22} \cup J_{23} \cup J_3$, and $a_9 = E(X : X \in A_9)$, we have

$$V_2 \geq \int_{A_8} (x - a_8)^2 dP + \int_{A_9} (x - a_9)^2 dP = \frac{489226058779}{16680146484375} > V_2,$$

which leads to a contradiction. Hence, $S_{21313}(0) < \frac{1}{2}(a+b) < \frac{1}{2}$. Assume that $S_{21313}(0) < \frac{1}{2}(a+b) < S_{2132}(0)$. Then, writing $A_{10} = J_1 \cup J_{211} \cup J_{212} \cup J_{21311} \cup J_{21312}$, $a_{10} = E(X : X \in A_{10})$, $A_{11} = J_{2132} \cup J_{2133} \cup J_{22} \cup J_{23} \cup J_3$, and $a_{11} = E(X : X \in A_{11})$, we have

$$V_2 \geq \int_{A_{10}} (x - a_{10})^2 dP + \int_{A_{11}} (x - a_{11})^2 dP = \frac{2996202402374}{101383681640625} > V_2,$$

which leads to a contradiction. Hence, $S_{2132}(0) < \frac{1}{2}(a+b) < \frac{1}{2}$.

Assume that $S_{2132}(0) < \frac{1}{2}(a+b) < S_{2133}(0)$. Partition the interval $[S_{2132}(0), S_{2133}(0)]$ into the following three subintervals:

$$[S_{21321}(0), S_{21322}(0)], [S_{21322}(0), S_{21323}(0)], [S_{21323}(0), S_{2133}(0)].$$

Then, $\frac{1}{2}(a+b)$ belongs to one of the above three subintervals. First, assume that $\frac{1}{2}(a+b) \in [S_{21321}(0), S_{21322}(0)]$. Then, as before we can show that the distortion error is larger than $0.0291911 \geq V_2$, which leads to a contradiction. To show that for $\frac{1}{2}(a+b) \in [S_{21321}(0), S_{21322}(0)]$,

the distortion error is larger, we may need to partition the subinterval $[S_{21321}(0), S_{21322}(0)]$ into the following three sub-subintervals:

$$[S_{213211}(0), S_{213212}(0)], [S_{213212}(0), S_{213213}(0)], [S_{213213}(0), S_{21322}(0)],$$

and then we can separately consider the cases as follows:

$$\frac{1}{2}(a+b) \in [S_{213211}(0), S_{213212}(0)], [S_{213212}(0), S_{213213}(0)], \text{ or } [S_{213213}(0), S_{21322}(0)].$$

If needed, we can further partition the above sub-subintervals to check that the distortion is larger. Similarly, we can show that if $\frac{1}{2}(a+b)$ belongs to either $[S_{21322}(0), S_{21323}(0)]$, or $[S_{21323}(0), S_{2133}(0)]$, then the contradiction arises. Therefore, $S_{21313}(0) < \frac{1}{2}(a+b) < S_{2132}(0)$ can not happen. Using the similar arguments, we can show that neither $S_{2132}(0) < \frac{1}{2}(a+b) < S_{2133}(0)$, nor $S_{2133}(0) < \frac{1}{2}(a+b) < S_{2133}(1)$ can happen. Hence, we can assume that $S_{21}(1) < \frac{1}{2}(a+b) < \frac{1}{2}$.

Assume that $S_{22222}(0) < \frac{1}{2}(a+b) < \frac{1}{2}$. Then, writing $A_{12} = J_1 \cup J_{21} \cup J_{221} \cup J_{2221} \cup J_{22221}$, and $a_{12} = E(X : X \in A_{12})$, and by Proposition 2.6, noting the fact that $\int_{[\frac{1}{2}, 1]} (x - E(X : X \in [\frac{1}{2}, 1]))^2 dP = \frac{13}{882}$, we have

$$V_2 \geq \int_{A_{12}} (x - a_{12})^2 dP + \frac{13}{882} = \frac{61346429393}{2093027343750} > V_2,$$

which gives a contradiction. Hence, $S_{21}(1) < \frac{1}{2}(a+b) < S_{22222}(0)$. Assume that $S_{222212}(0) < \frac{1}{2}(a+b) < S_{22222}(0)$. Then, writing $A_{13} = J_1 \cup J_{21} \cup J_{221} \cup J_{2221} \cup J_{222211}$, $a_{13} = E(X : X \in A_{13})$, $A_{14} = J_{22222} \cup J_{22223} \cup J_{2223} \cup J_{223} \cup J_{23} \cup J_3$, and $a_{14} = E(X : X \in A_{14})$, we have

$$V_2 \geq \int_{A_{13}} (x - a_{13})^2 dP + \int_{A_{14}} (x - a_{14})^2 dP = \frac{1031786636229481}{35273384033203125} > V_2,$$

which is a contradiction. Hence, $S_{21}(1) < \frac{1}{2}(a+b) < S_{222212}(0)$. Proceeding in this way, and using the similar technique of partitioning the intervals into subintervals, as mentioned in the previous paragraph, we can show that $S_{22}(0) < \frac{1}{2}(a+b) < S_{222212}$ cannot happen. Hence, $S_{21}(1) < \frac{1}{2}(a+b) < S_{22}(0)$. Thus, the set $\alpha_2 := \{a(1, 21), a(22, 23, 3)\}$ forms an optimal set of two-means, and the corresponding quantization error is given by $V_2 = \frac{821}{28125}$. Hence, the proof of the lemma is complete. \square

Corollary 3.2. Let α_2 be an optimal set of two-means. Then, for $1 \leq j \leq 3$, we have $\int_{J_j} \min_{a \in S_j(\alpha_2)} (x - a)^2 dP = \frac{1}{75} V_2$.

Proof. We have

$$\begin{aligned} \int_{J_j} \min_{a \in S_j(\alpha_2)} (x - a)^2 dP &= \frac{1}{3} \int_{J_j} \min_{a \in S_j(\alpha_2)} (x - a)^2 d(P \circ S_j^{-1}) = \frac{1}{3} \int \min_{a \in S_j(\alpha_2)} (S_j(x) - a)^2 dP \\ &= \frac{1}{3} \int \min_{a \in \alpha_2} (S_j(x) - S_j(a))^2 dP = \frac{1}{75} \int \min_{a \in \alpha_2} (x - a)^2 dP = \frac{1}{75} V_2. \end{aligned}$$

Thus, the proof of the corollary is complete. \square

Lemma 3.3. The set $\alpha_3 := \{S_1(\frac{1}{2}), S_2(\frac{1}{2}), S_3(\frac{1}{2})\}$ forms an optimal set of three-means, and the corresponding quantization error is given by $V_3 = \frac{1}{225}$.

Proof. Let β be a set of three points such that $\beta := \{S_j(\frac{1}{2}) : j = 1, 2, 3\}$. Then,

$$\int \min_{b \in \beta} (x - b)^2 dP = \sum_{j=1}^3 \int_{J_j} (x - S_j(\frac{1}{2}))^2 dP = \frac{1}{225}.$$

Since V_3 is the quantization error for three-means, we have $V_3 \leq \frac{1}{225}$. Let $\alpha := \{a_1, a_2, a_3\}$ be an optimal set of three-means. Since the elements in an optimal set are the centroids of their

own Voronoi regions, without any loss of generality, we can assume that $0 < a_1 < a_2 < a_3 < 1$. We now prove that $\alpha \cap J_j \neq \emptyset$ for all $1 \leq j \leq 3$. Suppose that $\alpha \cap J_1 = \emptyset$. Then,

$$V_3 \geq \int_{J_1} \left(x - \frac{1}{5}\right)^2 dP = \frac{13}{2700} > V_3,$$

which is a contradiction. So, we can assume that $\alpha \cap J_1 \neq \emptyset$. Similarly, $\alpha \cap J_3 \neq \emptyset$. We now that that $\alpha \cap J_2 \neq \emptyset$. Suppose that $\alpha \cap J_2 = \emptyset$. Then, either $a_2 < \frac{2}{5}$, or $a_2 > \frac{3}{5}$. Assume that $a_2 < \frac{2}{5}$. Then, as $\frac{1}{2}(\frac{2}{5} + \frac{4}{5}) = \frac{3}{5}$, we have

$$V_3 \geq \int_{J_2} \left(x - \frac{2}{5}\right)^2 dP + \int_{J_3} \left(x - S_3\left(\frac{1}{2}\right)\right)^2 dP = \frac{17}{2700} > V_3,$$

which is a contradiction. Similarly, we can show that if $\frac{3}{5} < a_2$, then a contradiction arises. Thus, we can assume that $a_2 \in J_2$, i.e., $\alpha \cap J_2 \neq \emptyset$. Now, if the Voronoi region of a_1 contains points from J_2 , we have $\frac{1}{2}(a_1 + a_2) > \frac{2}{5}$ implying $a_2 > \frac{4}{5} - a_1 \geq \frac{4}{5} - \frac{1}{5} = \frac{3}{5}$, which is a contradiction as $a_2 \in J_2$. Thus, the Voronoi region of a_1 does not contain any point from J_2 . Similarly, the Voronoi region of a_2 does not contain any point from J_1 and J_3 . Likewise, the Voronoi region of a_3 does not contain any point from J_2 . Since the optimal quantizers are the centroids of their own Voronoi regions, we have $a_1 = S_1(\frac{1}{2})$, $a_2 = S_2(\frac{1}{2})$, and $a_3 = S_3(\frac{1}{2})$, and the corresponding quantization error is given by $V_3 = \frac{1}{225}$. Thus, the proof of the lemma is complete. \square

Remark 3.4. By Lemma 3.3, we see that the set $\{S_j(\frac{1}{2}) : 1 \leq j \leq 3\}$ forms an optimal set of three-means. Similarly, we can show that the sets $\{S_1(\frac{1}{2}), S_2(\frac{1}{2})\} \cup S_3(\alpha_2)$, $\{S_1(\frac{1}{2})\} \cup S_2(\alpha_2) \cup S_3(\alpha_2)$, $S_1(\alpha_2) \cup S_2(\alpha_2) \cup S_3(\alpha_2)$, $S_1(\alpha_3) \cup S_2(\alpha_2) \cup S_3(\alpha_2)$, and $S_1(\alpha_3) \cup S_2(\alpha_3) \cup S_3(\alpha_2)$ form optimal sets of n -means for $n = 4, 5, 6, 7, 8$, respectively. Due to technicality of the proofs, we do not show them in the paper.

Proposition 3.5. *Let α_n be an optimal set of n -means for any $n \geq 3$. Then, $\alpha_n \cap J_j \neq \emptyset$ for all $1 \leq j \leq 3$, and α_n does not contain any point from the open intervals $(\frac{1}{5}, \frac{2}{5})$ and $(\frac{3}{5}, \frac{4}{5})$. Moreover, the Voronoi region of any point in $\alpha_n \cap J_j$ does not contain any point from J_i , where $1 \leq i \neq j \leq 3$.*

Proof. Due to Lemma 3.3, and Remark 3.4, the proposition is true for $3 \leq n \leq 8$. Let us now prove that the proposition is true for $n \geq 9$. Let $\alpha_n := \{a_1, a_2, \dots, a_n\}$ be an optimal set of n -means for $n \geq 9$. Since the points in an optimal set are the centroids of their own Voronoi regions, without any loss of generality, we can assume that $0 < a_1 < a_2 < \dots < a_n < 1$. Consider the set of nine elements $\beta := \{S_\sigma(\frac{1}{2}) : \sigma \in \{1, 2, 3\}^2\}$. Then,

$$\int \min_{a \in \beta} (x - a)^2 dP = \sum_{\sigma \in \{1, 2, 3\}^2} \int_{J_\sigma} (x - a(\sigma))^2 dP = \frac{1}{25^2} V = \frac{1}{5625}.$$

Since V_n is the quantization error for n -means for $n \geq 9$, we have $V_n \leq V_9 \leq \frac{1}{5625}$. Suppose that $\frac{1}{5} < a_1$. Then,

$$V_n \geq \int_{J_1} \left(x - \frac{1}{5}\right)^2 dP = \frac{13}{2700} > V_n,$$

which is a contradiction. So, we can assume that $a_1 \leq \frac{1}{5}$. Similarly, $\frac{4}{5} \leq a_n$. Thus, $\alpha_n \cap J_1 \neq \emptyset$, and $\alpha_n \cap J_3 \neq \emptyset$. We now show that $\alpha_n \cap J_2 \neq \emptyset$. For the sake of contradiction, assume that $\alpha_n \cap J_2 = \emptyset$. Let $a_j := \max\{a_i : a_i < \frac{2}{5} \text{ for } 1 \leq i \leq n-1\}$. Then, $a_j < \frac{2}{5}$. As $\alpha_n \cap J_2 = \emptyset$, we

have $\frac{3}{5} < a_{j+1}$. Thus, using the symmetry and the formula given by (1), we have

$$\begin{aligned}
V_n &\geq 2 \sum_{n=2}^{\infty} \int_{J_{2^{n-1}}} (x - \frac{2}{5})^2 dP = 2 \sum_{n=2}^{\infty} \frac{1}{3^n} \left(\frac{1}{25^n} V + (S_{2^{n-1}}(\frac{1}{2}) - \frac{2}{5})^2 \right) \\
&= 2 \sum_{n=2}^{\infty} \left(\frac{1}{75^n} V + \frac{1}{3^n} \left((S_{2^{n-1}}(\frac{1}{2}))^2 - \frac{4}{5} S_{2^{n-1}}(\frac{1}{2}) + \frac{4}{25} \right) \right) \\
&= 2 \sum_{n=2}^{\infty} \frac{1}{75^n} V + 2 \sum_{n=2}^{\infty} \frac{1}{3^n} \left(\frac{5^n - 4}{5^{n-1} 10} \right)^2 - \frac{8}{5} \sum_{n=2}^{\infty} \frac{1}{3^n} \frac{5^n - 4}{5^{n-1} 10} + \frac{8}{25} \sum_{n=2}^{\infty} \frac{1}{3^n} \\
&= \frac{19}{18900} > V_n,
\end{aligned}$$

which gives a contradiction. Hence, we can conclude that $\alpha_n \cap J_2 \neq \emptyset$. Next, suppose that α_n contains a point from the open interval $(\frac{1}{5}, \frac{2}{5})$. Let $a_j := \max\{a_i : a_i < \frac{1}{5} \text{ for } 1 \leq i \leq n-2\}$. Then, $a_{j+1} \in (\frac{1}{5}, \frac{2}{5})$, and $a_{j+2} \in J_2$. The following cases can arise:

Case 1. $\frac{1}{5} < a_{j+1} \leq \frac{3}{10}$.

Then, $\frac{1}{2}(a_{j+1} + a_{j+2}) > \frac{2}{5}$ implying $a_{j+2} > \frac{4}{5} - a_{j+1} \geq \frac{4}{5} - \frac{3}{10} = \frac{1}{2}$ implying, as before,

$$\begin{aligned}
V_n &\geq \sum_{n=2}^{\infty} \int_{J_{2^{n-1}}} (x - \frac{1}{2})^2 dP \\
&= \sum_{n=2}^{\infty} \frac{1}{75^n} V + \sum_{n=2}^{\infty} \frac{1}{3^n} \left(\frac{5^n - 4}{5^{n-1} 10} \right)^2 - \sum_{n=2}^{\infty} \frac{1}{3^n} \frac{5^n - 4}{5^{n-1} 10} + \frac{1}{4} \sum_{n=2}^{\infty} \frac{1}{3^n} \\
&= \frac{1}{1350} > V_n,
\end{aligned}$$

which leads to a contradiction.

Case 2. $\frac{3}{10} \leq a_{j+1} < \frac{2}{5}$.

Then, $\frac{1}{2}(a_j + a_{j+1}) < \frac{1}{5}$ implying $a_j \leq \frac{2}{5} - a_{j+1} \leq \frac{2}{5} - \frac{3}{10} = \frac{1}{10}$. Then,

$$V_n \geq \int_{J_{13}} (x - \frac{1}{10})^2 dP = \frac{37}{50625} > V_n,$$

which yields a contradiction.

Thus, by Case 1 and Case 2, we can conclude that α_n does not contain any point from the open interval $(\frac{1}{5}, \frac{2}{5})$. Reflecting the situation with respect to the point $\frac{1}{2}$, we can conclude that α_n does not contain any point from the open interval $(\frac{3}{5}, \frac{4}{5})$ as well. To prove the last part of the proposition, we proceed as follows: $a_j = \max\{a_i : a_i < \frac{1}{5} \text{ for } 1 \leq i \leq n-2\}$. Then, a_j is the rightmost element in $\alpha_n \cap J_1$, and $a_{j+1} \in \alpha_n \cap J_2$. Suppose that the Voronoi region of a_j contains points from J_2 . Then, $\frac{1}{2}(a_j + a_{j+1}) > \frac{2}{5}$ implying $a_{j+1} > \frac{4}{5} - a_j \geq \frac{4}{5} - \frac{1}{5} = \frac{3}{5}$, which yields a contradiction as $a_{j+1} \in J_2$. Thus, the Voronoi region of any point in $\alpha_n \cap J_1$ does not contain any point J_2 , and so from J_3 as well. Similarly, we can prove that the Voronoi region of any point in $\alpha_n \cap J_2$ does not contain any point from J_1 and J_3 , and the Voronoi region of any point in $\alpha_n \cap J_3$ does not contain any point from J_1 and J_2 . Thus, the proof of the proposition is complete. \square

The following lemma is similar to Lemma 4.5 in [GL2].

Lemma 3.6. *Let $n \geq 3$, and let α_n be an optimal set of n -means such that $\alpha_n \cap J_j \neq \emptyset$ for all $1 \leq j \leq 3$, and α_n does not contain any point from the open intervals $(\frac{1}{5}, \frac{2}{5})$ and $(\frac{3}{5}, \frac{4}{5})$. Further assume that the Voronoi region of any point in $\alpha_n \cap J_j$ does not contain any point from J_i , where $1 \leq i \neq j \leq 3$. Set $\beta_j := \alpha_n \cap J_j$, and $n_j := \text{card}(\beta_j)$ for $1 \leq j \leq 3$. Then, $S_j^{-1}(\beta_j)$ is an optimal set of n_j -means, and $V_n = \frac{1}{75}(V_{n_1} + V_{n_2} + V_{n_3})$.*

Proof. By the hypothesis, $\beta_j \neq \emptyset$, for all $1 \leq j \leq 3$, and α_n does not contain any point from the open intervals $(\frac{1}{5}, \frac{2}{5})$ and $(\frac{3}{5}, \frac{4}{5})$. Hence, $\alpha_n = \bigcup_{j=1}^3 \beta_j$. Since α_n is an optimal set of n -means,

$$V_n = \sum_{j=1}^3 \int_{J_j} \min_{a \in \alpha_n} \|x - a\|^2 dP = \sum_{j=1}^3 \int_{J_j} \min_{a \in \beta_j} \|x - a\|^2 dP.$$

Now, using Lemma 2.1 we have

$$(2) \quad V_n = \frac{1}{75} \sum_{j=1}^3 \int \min_{a \in \beta_j} \|x - S_j^{-1}(a)\|^2 dP = \frac{1}{75} \sum_{j=1}^3 \int \min_{a \in S_j^{-1}(\beta_j)} \|x - a\|^2 dP.$$

If $S_1^{-1}(\beta_1)$ is not an optimal set of n_1 -means, then we can find a set $\gamma_1 \subset \mathbb{R}^2$ with $\text{card}(\gamma_1) = n_1$ such that

$$\int \min_{a \in \gamma_1} \|x - a\|^2 dP < \int \min_{a \in S_1^{-1}(\beta_1)} \|x - a\|^2 dP.$$

But, then $S_1(\gamma_1) \cup \beta_2 \cup \beta_3$ will be a set of cardinality n , and

$$\begin{aligned} & \int \min\{\|x - a\|^2 : a \in S_1(\gamma_1) \cup \beta_2 \cup \beta_3\} dP \\ &= \int_{J_1} \min_{a \in S_1(\gamma_1)} \|x - a\|^2 dP + \frac{1}{75} \sum_{j=2}^3 \int \min_{a \in S_j^{-1}(\beta_j)} \|x - a\|^2 dP \\ &= \frac{1}{75} \int \min_{a \in S_1(\gamma_1)} \|x - S_1^{-1}(a)\|^2 dP + \frac{1}{75} \sum_{j=2}^3 \int \min_{a \in S_j^{-1}(\beta_j)} \|x - a\|^2 dP \\ &= \frac{1}{75} \int \min_{a \in \gamma_1} \|x - a\|^2 dP + \frac{1}{75} \sum_{j=2}^3 \int \min_{a \in S_j^{-1}(\beta_j)} \|x - a\|^2 dP \\ &< \frac{1}{75} \int \min_{a \in S_1^{-1}(\beta_1)} \|x - a\|^2 dP + \frac{1}{75} \sum_{j=2}^3 \int \min_{a \in S_j^{-1}(\beta_j)} \|x - a\|^2 dP. \end{aligned}$$

Thus by (2), we have $\int \min\{\|x - a\|^2 : a \in S_1(\gamma_1) \cup \beta_2 \cup \beta_3\} dP < V_n$, which contradicts the fact that α_n is an optimal set of n -means, and so $S_1^{-1}(\beta_1)$ is an optimal set of n_1 -means. Similarly, one can show that $S_j^{-1}(\beta_j)$ are optimal sets of n_j -means for $j = 2, 3$. Thus, (2) implies that $V_n = \frac{1}{75} (V_{n_1} + V_{n_2} + V_{n_3})$. This completes the proof of the lemma. \square

Let us now state and prove the following theorem which gives the optimal sets of n -means for all $n \geq 3$.

Theorem 3.7. *Let P be the standard triadic Cantor distribution on \mathbb{R} with support the Cantor set C generated by the three contractive similarity mappings S_j for $j = 1, 2, 3$. Let $n \in \mathbb{N}$ with $n \geq 3$. Then, the set $\alpha_n := \alpha_n(I)$ given by Definition 1.2 forms an optimal set of n -means for P with the corresponding quantization error $V_n := V(P; \alpha_n(I))$, where $V(P; \alpha_n(I))$ is given by Proposition 2.5.*

Proof. We will proceed by induction on $\ell(n)$. If $\ell(n) = 1$, then the theorem is true by Remark 3.4. Let us assume that the theorem is true for all $\ell(n) < m$, where $m \in \mathbb{N}$ and $m \geq 2$. We now show that the theorem is true if $\ell(n) = m$. Let us first assume that $3^m \leq n \leq 2 \cdot 3^m$. Let α_n be an optimal set of n -means for P such that $3^m \leq n \leq 2 \cdot 3^m$. Let $\text{card}(\alpha_n \cap J_j) = n_j$ for $j = 1, 2, 3$. Then, by Lemma 3.6, we have

$$(3) \quad V_n = \frac{1}{75} (V_{n_1} + V_{n_2} + V_{n_3}).$$

Without any loss of generality, we can assume that $n_1 \geq n_2 \geq n_3$. Let $p, q, r \in \mathbb{N}$ be such that

$$(4) \quad 3^p \leq n_1 \leq 2 \cdot 3^p, \quad 3^q \leq n_2 \leq 2 \cdot 3^q, \quad \text{and} \quad 3^r \leq n_3 \leq 2 \cdot 3^r.$$

We will show that $p = q = r = m - 1$. Since $n_1 \geq n_2 \geq n_3$, we have $n_1 \geq 3^{m-1}$, and $n_3 \leq 2 \cdot 3^{m-1}$ implying $r \leq m - 1 \leq p$. If \tilde{V}_n is the distortion error due to the set $\{a(\sigma) : \sigma \in \{1, 2, 3\}^m \setminus I\} \cup (\cup_{\sigma \in I} S_\sigma(\alpha_2))$, by Proposition 2.5, we have

$$\tilde{V}_n = \frac{1}{75^m} ((2 \cdot 3^m - n)V + (n - 3^m)V_2).$$

Thus, by the induction hypothesis, (3) implies $\tilde{V}_n \geq V_n$ yielding

$$\begin{aligned} \frac{1}{75^m} ((2 \cdot 3^m - n)V + (n - 3^m)V_2) &\geq \frac{1}{75^{p+1}} ((2 \cdot 3^p - n_1)V + (n_1 - 3^p)V_2) \\ &+ \frac{1}{75^{q+1}} ((2 \cdot 3^q - n_2)V + (n_2 - 3^q)V_2) + \frac{1}{75^{r+1}} ((2 \cdot 3^r - n_3)V + (n_3 - 3^r)V_2) \end{aligned}$$

which upon simplification gives,

$$(5) \quad 3 \left((2V - V_2) - \frac{n}{3^m}(V - V_2) \right) \geq 25^{m-p-1} \left((2V - V_2) - \frac{n_1}{3^p}(V - V_2) \right) \\ + 25^{m-q-1} \left((2V - V_2) - \frac{n_2}{3^q}(V - V_2) \right) + 25^{m-r-1} \left((2V - V_2) - \frac{n_3}{3^r}(V - V_2) \right).$$

Hence, using the bounds of $\frac{n}{3^m}$, $\frac{n_1}{3^p}$, $\frac{n_2}{3^q}$, and $\frac{n_3}{3^r}$, i.e., putting $\frac{n}{3^m} = 1$, and $\frac{n_1}{3^p} = \frac{n_2}{3^q} = \frac{n_3}{3^r} = 2$, from the above inequality, we obtain

$$(6) \quad 11.419 = \frac{3V}{V_2} \geq 25^{m-p-1} + 25^{m-q-1} + 25^{m-r-1}.$$

Recall that $r \leq m - 1 \leq p$. Moreover, $p > m$ is not possible. Thus, from (6), to obtain the values of p, q , and r , we proceed as follows:

(i) If $p = r = m - 1$, then (6) implies that $q = m - 1$.

(ii) If $p = m - 1$ and $r = m - 2$, then $11.419 \geq 1 + 25^{m-q-1} + 25$, which gives a contradiction, in fact, a contradiction arises for $p = m - 1$ and any $r < m - 1$.

(iii) If $p = m$ and $r = m - 1$, then (6) implies that $q \geq m - 1$. Notice that if $q = m$, then $n > n_1 + n_2 \geq 3^m + 3^m = 2 \cdot 3^m$, which is a contradiction. Also, $q > m$ is not possible. So, $q = m - 1$ is the only choice, but then also a contradiction arises as shown below: For $p = m, q = m - 1$, and $r = m - 1$, (5) implies

$$3 \left((2V - V_2) - \frac{n}{3^m}(V - V_2) \right) \geq (2V - V_2) \left(\frac{1}{25} + 2 \right) - (V - V_2) \left(\frac{1}{25} \frac{n_1}{3^m} + \frac{n_2}{3^{m-1}} + \frac{n_3}{3^{m-1}} \right),$$

i.e.,

$$(2V - V_2) \frac{24}{25} \geq (V - V_2) \left(\frac{n}{3^{m-1}} - \frac{n_2}{3^{m-1}} - \frac{n_3}{3^{m-1}} - \frac{1}{25} \frac{n_1}{3^m} \right),$$

which upon simplification yields,

$$\frac{n_1}{3^m} \leq \frac{2V - V_2}{V - V_2} \frac{12}{37} = \frac{5429}{7104} < 1,$$

which is a contradiction because $3^p \leq n_1 \leq 2 \cdot 3^p$, and $p = m$.

(iv) If $p = m$ and $r = m - 2$, then (6) implies $11.419 \geq \frac{1}{25} + 25^{m-q-1} + 25$, which is not true. In fact, a contradiction arises for $p = m$ and any $r < m - 1$.

Hence, we can conclude that $p = q = r = m - 1$. Since by Lemma 3.6, for $S_j^{-1}(\alpha_n \cap J_j)$ is an optimal set of n_j means where $3^{m-1} \leq n_j \leq 2 \cdot 3^{m-1}$, we have

$$S_j^{-1}(\alpha_n \cap J_j) = \{a(\sigma) : \sigma \in \{1, 2, 3\}^{m-1} \setminus I_j\} \cup (\cup_{\sigma \in I_j} S_\sigma(\alpha_2)),$$

where $I_j \subseteq \{1, 2, 3\}^{m-1}$ with $\text{card}(I_j) = n_j - 3^{m-1}$ for $1 \leq j \leq 3$. Hence,

$$\alpha_n := \alpha_n(I) = \bigcup_{j=1}^3 S_j^{-1}(\alpha_n \cap J_j) = \{a(\sigma) : \sigma \in \{1, 2, 3\}^{\ell(n)} \setminus I\} \cup (\cup_{\sigma \in I} S_\sigma(\alpha_2)),$$

where $I \subseteq \{1, 2, 3\}^m$ with $\text{card}(I) = n - 3^m$, is an optimal set of n -means. The corresponding quantization error is

$$V_n = \frac{1}{75^m} ((2 \cdot 3^m - n)V + (n - 3^m)V_2) = V(P; \alpha_n(I)),$$

where $V(P; \alpha_n(I))$ is given by Proposition 2.5. Thus, the theorem is true if $3^m \leq n \leq 2 \cdot 3^m$. Similarly, we can prove that the theorem is true if $2 \cdot 3^m < n < 3^{m+1}$. Hence, by the induction principle, the proof of the theorem is complete. \square

Remark 3.8. In Theorem 3.7, if $n = 3^{\ell(n)}$, then I is the empty set implying $\alpha_n := \alpha_n(I) = \{a(\sigma) : \sigma \in \{1, 2, 3\}^{\ell(n)}\}$, and the corresponding quantization error is given by $V_n = \frac{1}{25^{\ell(n)}}V$.

4. QUANTIZATION DIMENSION AND QUANTIZATION COEFFICIENT

Since the Cantor set C under investigation satisfies the strong separation condition, with each S_j having contracting factor of $\frac{1}{5}$, the Hausdorff dimension of the Cantor set is equal to the similarity dimension. Hence, from the equation $3(\frac{1}{5})^\beta = 1$, we have $\dim_{\text{H}}(C) = \beta = \frac{\log 3}{\log 5}$. By Theorem 14.17 in [GL1], the quantization dimension $D(P)$ exists and is equal to β . In this section, we show that β dimensional quantization coefficient for P does not exist.

Lemma 4.1. Let $f : [1, 2] \rightarrow \mathbb{R}$ be a function defined by

$$f(x) = x^{\frac{2}{\beta}} ((2V - V_2) - x(V - V_2)) = \frac{1}{28125} (5429 - 2304x)x^{2/\beta}.$$

Then,

$$(i) \ 1 < \frac{5429}{1152(\beta+2)} < 2, \text{ and}$$

$$(ii) \ f(\frac{5429}{1152(\beta+2)}) > f(2) > f(1), \text{ and } f([1, 2]) = [f(1), f(\frac{5429}{1152(\beta+2)})].$$

Proof. Notice that f is a continuous function on the closed interval $[1, 2]$, and

$$f'(x) = \frac{x^{\frac{2}{\beta}-1}(10858 - 2304(\beta+2)x)}{28125\beta}.$$

$f'(x) = 0$ implies that $x = \frac{5429}{1152(\beta+2)}$. Moreover, $f'(1) = -\frac{2(1152\beta-3125)}{28125\beta} > 0$, and $f'(2) = \frac{4^{1/\beta}(821-2304\beta)}{28125\beta} < 0$. Thus, f is maximum at $x = \frac{5429}{1152(\beta+2)}$. Again, $f(2) = \frac{821}{28125}4^{1/\beta} > \frac{1}{9} = f(1)$. Hence, $1 < \frac{5429}{1152(\beta+2)} < 2$, and $f(\frac{5429}{1152(\beta+2)}) > f(2) > f(1)$, and $f([1, 2]) = [f(1), f(\frac{5429}{1152(\beta+2)})]$. Thus, the proof of the lemma is complete. \square

Theorem 4.2. The β -dimensional quantization coefficient does not exist.

Proof. Let $M = \max\{f(x) : 1 \leq x \leq 2\}$. Then, by Lemma 4.1, we have $M = \frac{5429}{1152(\beta+2)}$. Let $(n_k)_{k \in \mathbb{N}}$ be a subsequence of the set of natural numbers such that $3^{\ell(n_k)} \leq n_k < 2 \cdot 3^{\ell(n_k)}$. The assertion of the theorem will follow if we show that the set of accumulation points of the sequence $(n_k^{\frac{2}{\beta}} V_{n_k})_{k \geq 1}$ is $[f(1), f(M)]$. Let $y \in [f(1), f(M)]$, then $y = f(x)$ for some $x \in [1, 2]$. Set $n_{k_\ell} = \lfloor x3^\ell \rfloor$, where $\lfloor x3^\ell \rfloor$ denotes the greatest integer less than or equal to $x3^\ell$. Then, $n_{k_\ell} < n_{k_{\ell+1}}$ and $\ell(n_{k_\ell}) = \ell$, where by $\ell(n_{k_\ell}) = \ell$ it is meant that $3^\ell \leq n_{k_\ell} < 2 \cdot 3^\ell$. Notice that then there exists $x_{k_\ell} \in [1, 2]$ such that $n_{k_\ell} = x_{k_\ell}3^\ell$. Recall that $3^{\frac{1}{\beta}} = 5$, and if $3^{\ell(n)} \leq n \leq 2 \cdot 3^{\ell(n)}$, then by Theorem 3.7, we have

$$V_n = \frac{1}{75^{\ell(n)}} ((2 \cdot 3^{\ell(n)} - n)V + (n - 3^{\ell(n)})V_2).$$

Thus, putting the values of n_{k_ℓ} and $V_{n_{k_\ell}}$, we obtain

$$n_{k_\ell}^{\frac{2}{\beta}} V_{n_{k_\ell}} = n_{k_\ell}^{\frac{2}{\beta}} \frac{1}{75^\ell} ((2 \cdot 3^\ell - n_{k_\ell})V + (n_{k_\ell} - 3^\ell)V_2)$$

yielding

$$(7) \quad n_{k_\ell}^{\frac{2}{\beta}} V_{n_{k_\ell}} = x_{k_\ell}^{\frac{2}{\beta}} ((2V - V_2) - x_{k_\ell}(V - V_2)) = f(x_{k_\ell}).$$

Again, $x_{k_\ell} 3^\ell \leq x 3^\ell < x_{k_\ell} 3^\ell + 1$, which implies $x - \frac{1}{3^\ell} < x_{k_\ell} \leq x$, and so, $\lim_{\ell \rightarrow \infty} x_{k_\ell} = x$. Since f is continuous, we have

$$\lim_{\ell \rightarrow \infty} n_{k_\ell}^{\frac{2}{\beta}} V_{n_{k_\ell}} = f(x) = y,$$

which yields the fact that y is an accumulation point of the subsequence $(n_k^{\frac{2}{\beta}} V_{n_k})_{k \geq 1}$ whenever $y \in [f(1), f(M)]$. To prove the converse, let y be an accumulation point of the sequence $(n_k^{\frac{2}{\beta}} V_{n_k})_{k \geq 1}$. Then, there exists a subsequence $(n_{k_i}^{\frac{2}{\beta}} V_{n_{k_i}})_{i \geq 1}$ of $(n_k^{\frac{2}{\beta}} V_{n_k})_{k \geq 1}$ such that $\lim_{i \rightarrow \infty} n_{k_i}^{\frac{2}{\beta}} V_{n_{k_i}} = y$. Set $\ell_{k_i} = \ell(n_{k_i})$ and $x_{k_i} = \frac{n_{k_i}}{3^{\ell_{k_i}}}$. Then, $x_{k_i} \in [1, 2]$, and as shown in (7), we have

$$n_{k_i}^{\frac{2}{\beta}} V_{n_{k_i}} = f(x_{k_i}).$$

Let $(x_{k_{i_j}})_{j \geq 1}$ be a convergent subsequence of $(x_{k_i})_{i \geq 1}$, then we obtain

$$y = \lim_{i \rightarrow \infty} n_{k_i}^{\frac{2}{\beta}} V_{n_{k_i}} = \lim_{j \rightarrow \infty} n_{k_{i_j}}^{\frac{2}{\beta}} V_{n_{k_{i_j}}} = \lim_{j \rightarrow \infty} f(x_{k_{i_j}}) \in [f(1), f(M)].$$

Thus, the set of accumulation points of the sequence $(n_k^{\frac{2}{\beta}} V_{n_k})_{k \geq 1}$ is $[f(1), f(M)]$, i.e., the β -dimensional quantization coefficient for P does not exist. Hence, the proof of the theorem is complete. \square

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