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Duffing-van der Pol-type nonlinear oscillator system

Jing Cui
University of Texas-Pan American

DUFFING–VAN DER POL–TYPE NONLINEAR OSCILLATOR SYSTEM

A Thesis

by

JING CUI

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COMMITTEE MEMBERS

Zhaosheng Feng
Chair of Committee

Andras Balogh
Committee Member

Paul Bracken
Committee Member

Bao-Feng Feng
Committee Member

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ABSTRACT

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In this thesis, we are concerned with the nonlinear Duffing–van der Pol–type oscillator system by means of the Lie symmetry reduction method. This system has physical relevance as a simple model in certain flow-induced structural vibration problems, which includes the van der Pol oscillator and the damped Duffing oscillator etc as particular cases. By applying the Lie symmetry analysis, we find two nontrivial infinitesimal generators, and use them to construct canonical variables. Through the inverse analysis, some dynamical properties of the nonlinear system under certain parametric conditions are presented. Comparison with the existing results by the Prelle–Singer procedure is provided.

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CHAPTER I

INTRODUCTION

Many models of dynamical problems appearing as differential equations arise in physical, chemical and biological phenomena [2, 11]. Problems in dynamics have fascinated applied mathematicians, physicists and engineers for a long time. Over the past few decades applications in solid and structural mechanics as well as fluid mechanics have appeared, and there is now widespread interest in the engineering and applied science communities in nonlinear oscillators, strange attractors, chaos, and dynamical systems theory. In spite of the great elegance and simplicity of such equations, the solutions of specific problems proved remarkably difficult [18]. Finding innovative methods to analyze and solve these equations has been an interesting subject in the field of differential equations and dynamical systems [21, 25, 26]. For many nonlinear problems, it is not always possible and sometimes not even advantageous to express exact solutions of nonlinear differential equations explicitly in terms of elementary functions, but it is possible to find elementary functions that are constant on solution curves, that is, elementary first integrals. These first integrals allow us to occasionally deduce properties that an explicit solution would not necessarily reveal. In the pioneering work [27], Prelle and Singer introduced a procedure to find the first integrals of first-order ordinary differential equations (ODEs) of the form $y = P(x,y)/Q(x,y)$, with both $P(x,y)$ and $Q(x,y)$ polynomials whose coefficients lie in the field of complex numbers $\mathbb{C}$. Duarte et al. [6] extended this procedure to second-order ODEs which is based on a conjecture that if the given second-order ODE has an elementary solution, then there exists at least one elementary first integral $I(x,y,y')$ whose derivatives are all rational functions of x, y and y'. Another powerful technique for seeking the first integrals of various differential equations is described in

[8, 9, 10, 11, 12, 13, 14, 15, 16, 20, 32] by means of the Lie reduction method. Recently, considerable attention has been received to various nonlinear oscillator systems for finding the first integrals and related dynamical properties [5, 18, 21, 29]. Special types of first integrals and dynamical behaviors are of fundamental importance to our understanding of physical, chemical and biological phenomena modelled differential equations [11].

In this paper, we consider the nonlinear oscillator system of the form

$$\ddot{y}+(\delta+\beta y^{m})\dot{y}-\mu y+\alpha y^{m+1}+\varphi y^{n}=0, \tag{1}$$

where the over-dot represents differentiation with respect to the independent variable x, and all coefficients δ, β, μ,α and φ are real constants with $\delta\beta\mu\alpha\varphi\neq 0$. In paper [11], we already studied the simple nonlinear oscillator system of the form

$$\ddot{y}+(\delta+\beta y^{m})\dot{y}-\mu y+\alpha y^{n}=0. \tag{2}$$

It is usually referred as to the Duffing–van der Pol–type oscillator, since in the choices $\alpha=0$ and $m=2$, the equation (2) reduces to the van der Pol oscillator

$$\ddot{y}+(\delta+\beta y^{2})\dot{y}-\mu y=0, \tag{3}$$

which was originally proposed by the Dutch electrical engineer and physicist Balthasar van der Pol in electrical circuits [30, 31]; and if $\beta=0$ and $n=3$ are taken, then the equation (2) becomes the damped Duffing equation [7, 18]

$$\ddot{y}+\delta\dot{y}-\mu y+\alpha y^{3}=0; \tag{4}$$

when we take $\beta = 0$ and $n = 2$, equation (2) becomes the damped Helmholtz oscillator [1, 28]

$$\ddot{y} + \delta\dot{y} - \mu y + \alpha y^2 = 0; \tag{5}$$

moreover, the choices $m = 2$ and $n = 3$ lead equation (2) to the standard form of the Duffing–van der Pol oscillator equation, whose autonomous version (force-free) takes the form

$$\ddot{y} + (\delta + \beta y^2)\dot{y} - \mu y + \alpha y^3 = 0. \tag{6}$$

Equation (6) arises in a model describing the propagation of voltage pulses along a neuronal axon and has recently received much attention from many authors. A large amount of literature exists on this equation; for details and applications, see [19, 22] and references therein. It is well known that there are great numbers of theoretical works to deal with equations (3)–(6) [16, 18, 19, 21], and applications of these four equations and the related systems can be seen in quite a few scientific areas [1, 4, 11, 17].

In the present paper, we study the nonlinear oscillator system (1), which is more general than equation (2) and also named Duffing–van der Pol–type oscillator, to obtain its first integrals under certain parametric conditions by applying the Lie symmetry method. The paper is organized as follows. In the next chapter, in order to make this paper well self-contained and present our results in a straightforward way, we summarize the Lie symmetry method for constructing the first integrals of second-order ODEs. In chapter III, we deduce the first integral of equation (2) by the Lie symmetry method. In chapter IV, we use the Lie symmetry method to derive the general first integral of the Duffing–van dor Pol–type oscillator system (1). In chapter V, we will give the exact solution of the Helmholtz oscillator (5) and discuss the first integrals of some other particular cases, such as the damped Duffing equation and the force-free Duffing–van der Pol oscillator system. In chapter VI, we present a brief conclusion.

CHAPTER II

PRELIMINARIES

Following references [3, 20, 24] and the Master's thesis of Guangyue Gao [15], I will give the brief introduction of Lie symmetry.

2.1 Symmetries of Planar Objects

In order to understand symmetries of differential equations, firstly we can consider the symmetries of planar objects. Roughly speaking, a symmetry of a geometrical object is a transformation whose action leaves the object apparently unchanged. For instance, consider the result of rotating an equilateral triangle anticlockwise about its center. After a rotation of $2\pi/3$, the triangle looks the same as it did before the rotation, so this transformation is a symmetry. Rotations of $4\pi/3$ and 2π are also symmetries of the equilateral triangle. In fact, rotating by 2π is equivalent to doing nothing, because each point is mapped to itself. The transformation mapping each point to itself is a symmetry of any geometrical object: it is called the trivial symmetry. In summary, a transformation is a symmetry if it satisfies the following:

(S1) The transformation preserves the structure,

(S2) The transformation is a diffeomorphism,

(S3) The transformation maps the object to itself.

Henceforth, we restrict attention to transformation satisfying (S1) and (S2). Such transformations are symmetries if they also satisfy (S3), which is called the symmetry condition.

2.2 Lie Symmetry for ODEs

For simplicity, we shall consider only ODEs of the form

$$y^{(n)} = \omega(x, y, y', \cdots, y^{(n-1)}), \qquad y^{(k)} \equiv \frac{d^k y}{dx^k}. \tag{7}$$

It is assumed that ω is a (locally) smooth function of all of its arguments. We begin by stating the symmetry condition and examining some of its consequences. A symmetry of equation (7) is a diffeomorphism that maps the set of solutions of the ODE to itself. Any diffeomorphism,

$$\Gamma : (x, y) \mapsto (\hat{x}, \hat{y}),$$

maps smooth planar curves to smooth planar curves. This action of Γ on the plane induces an action on the derivatives $y^{(k)}$, which is the mapping

$$\Gamma : (x, y, y', \cdots, y^{(n)}) \mapsto \left(\hat{x}, \hat{y}, \hat{y}', \cdots, \hat{y}^{(n)}\right), \qquad y^{(k)} = \frac{d^k \hat{y}}{d\hat{x}^k}, \qquad k = 1, \cdots, n.$$

This mapping is called the nth prolongation of Γ. The functions $\hat{y}^{(k)}$ are calculated recursively (using the chain rule) as

$$\hat{y}^{(k)} = \frac{d\hat{y}^{(k-1)}}{d\hat{x}} = \frac{D_x \hat{y}^{k-1}}{D_x \hat{x}}, \quad \hat{y}^{(0)} \equiv \hat{y}. \tag{8}$$

Here D_x is the total derivative with respect to x

$$D_x = \partial_x + y' \partial_y + y'' \partial_{y'} + \cdots.$$

The symmetry condition for ODE (7) is

$$\hat{y}^{(n)} = \omega(\hat{x}, \hat{y}, \hat{y}', \cdots, \hat{y}^{(n-1)}), \qquad \text{when equation (7) holds,} \tag{9}$$

where the functions $\hat{y}^{(k)}$ $(k = 1, 2, \cdots, n)$ are given by formula (8).

For almost all ODEs, the symmetry condition (9) is nonlinear. Lie symmetries are obtained by linearizing (9) about $\varepsilon = 0$. No such linearization is possible for discrete symmetries, which makes them hard to find. However, it is usually easy to find out whether or not a given diffeomorphism is a symmetry of a particular ODE. The trivial symmetry corresponding to $\varepsilon = 0$ leaves every point unchanged. Therefore, for ε sufficiently close to zero, the prolonged Lie symmetries are of the form

$$\begin{aligned} \hat{x} &= x + \varepsilon\xi + \mathbb{O}(\varepsilon^2), \\ \hat{y} &= y + \varepsilon\eta + \mathbb{O}(\varepsilon^2), \\ \hat{y}^{(k)} &= y^{(k)} + \varepsilon\eta^{(k)} + \mathbb{O}(\varepsilon^2), \ \ k \geq 1. \end{aligned} \tag{10}$$

[N.B. The superscript in $\eta^{(k)}$ is merely an index, it does not denote a derivative of η.] We substitute (10) into the symmetry condition (9); the $\mathbb{O}(\varepsilon)$ terms yield the linearized symmetry condition:

$$\eta^{(n)} = \xi\omega_x + \eta\omega_y + \eta^{(1)}\omega_{y'} + \cdots + \eta^{(n-1)}\omega_{y^{(n-1)}}, \qquad \text{when equation (7) holds,} \tag{11}$$

where the functions $\hat{y}^{(k)}$ and $\eta^{(k)}$ $(k = 1, 2, \cdots, n)$ can be derived recursively from formula (8). That is,

$$\begin{aligned} \hat{y}^{(1)} &= \frac{D_x\hat{y}}{D_x\hat{x}} = \frac{y' + \varepsilon D_x\eta + \mathbb{O}(\varepsilon^2)}{1 + \varepsilon D_x\xi + \mathbb{O}(\varepsilon^2)} \\ &= y' + \varepsilon(D_x\eta - y'D_x\xi) + \mathbb{O}(\varepsilon^2), \\ \hat{y}^{(k)} &= \frac{y^{(k)} + \varepsilon D_x\eta^{(k-1)} + \mathbb{O}(\varepsilon^2)}{1 + \varepsilon D_x\xi + \mathbb{O}(\varepsilon^2)}, \end{aligned}$$

where, from (10) we have

$$\eta^{(1)} = D_x\eta - y'D_x\xi, \tag{12}$$

$$\eta^{(k)}(x,y,y',\cdots,y^{(k)}) = D_x\eta^{(k-1)} - y^{(k)}D_x\xi. \tag{13}$$

The function ξ, η and $\eta^{(k)}$ can all be written in terms of the characteristic, $Q = \eta - y'\xi$, as follows:

$$\begin{aligned}
\xi &= -Qy', \\
\eta &= Q - y'Q_{y'}, \\
\eta^{(k)} &= D_x^kQ - y^{(k+1)}Q_{y'},\ k \geq 1.
\end{aligned}$$

In order to find the symmetry group G admitted by a differential equation with infinitesimal operator

$$X = \xi\partial_x + \eta\partial_y.$$

We introduce the prolonged infinitesimal generator

$$X^{(n)} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'} + \cdots + \eta^{(n)}\partial_{y^{(n)}}.$$

We can use the prolonged infinitesimal generator to write the linearized symmetry condition (11) in a compact form:

$$X^{(n)}\left(y^{(n)} - \omega(x,y,y',\cdots,y^{(n-1)})\right) = 0, \qquad \text{when equation (7) holds.}$$

2.3 Determining Equations for Lie Point Symmetries

Every symmetry that we have met so far is a diffeomorphism of the form

$$(\hat{x},\hat{y}) = (\hat{x}(x,y),\ \hat{y}(x,y)).$$

This type of diffeomorphism is called a point transformation; any point transformation that is also a symmetry is called a point symmetry. For now, we restrict attention to point symmetries.

To find the Lie point symmetries of an ODE (7), we must first calculate $\eta^{(k)}, k = 1, \cdots, n$. The functions ξ and η depend upon x and y only, and therefore (12) and (13) give the following results.

$$\begin{aligned}
\eta^{(1)} &= \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2, && (14)\\
\eta^{(2)} &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 \\
&\quad + \{\eta_y - 2\xi_x - 3\xi_y y'\}y'', && (15)\\
\eta^{(3)} &= \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y' + 3(\eta_{xyy} - \xi_{xxy})y'^2 + (\eta_{yyy} - 3\xi_{xyy})y'^3 \\
&\quad - \xi_{yyy}y'^4 + 3\{\eta_{xy} - \xi_{xx} + (\eta_{yy} - 3\xi_{xy})y' - 2\xi_{yy}y'^2\}y'' \\
&\quad - 3\xi_y y''^2 + \{\eta_y - 3\xi_x - 4\xi_y y'\}y'''.
\end{aligned}$$

The number of terms in $\eta^{(k)}$ increases exponentially with k, so computer algebra is recommended for the study of high-order ODEs.

So now we restrict our attention on second-order ODEs

$$y'' = F(x, y, y').$$

The linearized symmetry condition is obtained by substituting (14) and (15) into condition (11) and then replacing y'' by $F(x, y, y')$. This gives

$$\begin{aligned}
&\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + \{\eta_y - 2\xi_x - 3\xi_y y'\}F \\
= \; &\xi F_x + \eta F_y + \{\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2\}F_{y'}. && (16)
\end{aligned}$$

Although equation (16) looks complicated, in some cases it can be solved without much trouble. Both ξ and η are independent of y', and therefore (16) can be decomposed into a system of PDEs, which are the *determining equations* for the Lie point symmetries.

2.4 Lie Symmetry for PDEs

Point symmetries of PDEs [20, 24] are defined in much the same way as those of ODEs. For simplicity, let us start by considering PDEs with one dependent variable, u , and two independent variables, x and t. A point transformation is a diffeomorphism

$$\Gamma : (x,t,u) \mapsto (\hat{x}(x,t,u), \hat{t}(x,t,u), \hat{u}(x,t,u)).$$

This transformation maps the surface $u = u(x,t)$ to the following surface (which is parameterized by x and t):

$$\begin{aligned} \hat{x} &= \hat{x}(x,t,u(x,t)), \\ \hat{t} &= \hat{t}(x,t,u(x,t)), \\ \hat{u} &= \hat{u}(x,t,u(x,t)). \end{aligned} \tag{17}$$

To calculate the prolongation of a given transformation, we need to differentiate (17) with respect to each of the parameters x and t. To do this, we introduce the following *total derivatives*:

$$\begin{aligned} D_x &= \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{xt}\partial_{u_t} + \cdots, \\ D_t &= \partial_t + u_t\partial_u + u_{xt}\partial_{u_x} + u_{tt}\partial_{u_t} + \cdots. \end{aligned}$$

(Total derivatives treat the dependent variable u and its derivatives as functions of the independent variables.)

The first two equations of (17) may be inverted (locally) to give x and t in terms of $\hat{x}$ and $\hat{t}$, provided that the Jacobian is nonzero, that is,

$$H \equiv \begin{vmatrix} D_x\hat{x} & D_x\hat{t} \\ D_t\hat{x} & D_t\hat{t} \end{vmatrix} \neq 0 \qquad \text{when} \quad u = u(x,t). \tag{18}$$

If (18) is satisfied, then the last equation of (17) can be rewritten as

$$\hat{u} = \hat{u}(\hat{x}, \hat{t}). \tag{19}$$

Applying the chain rule to equation (19), we obtain

$$\begin{bmatrix} D_x\hat{u} \\ D_t\hat{u} \end{bmatrix} = \begin{bmatrix} D_x\hat{x} & D_x\hat{t} \\ D_t\hat{x} & D_t\hat{t} \end{bmatrix} \begin{bmatrix} \hat{u}_{\hat{x}} \\ \hat{u}_{\hat{t}} \end{bmatrix},$$

and therefore (by Cramer's rule)

$$\hat{u}_{\hat{x}} = \frac{1}{H} \begin{vmatrix} D_x\hat{u} & D_x\hat{t} \\ D_t\hat{u} & D_t\hat{t} \end{vmatrix}, \qquad \hat{u}_{\hat{t}} = \frac{1}{H} \begin{vmatrix} D_x\hat{x} & D_x\hat{u} \\ D_t\hat{x} & D_t\hat{u} \end{vmatrix}. \tag{20}$$

Higher-order prolongations are obtained recursively by repeating the above argument. If $\hat{u}_J$ is any derivative of $\hat{u}$ with respect to $\hat{x}$ and $\hat{t}$, then

$$\begin{aligned} \hat{u}_{J\hat{x}} \equiv \partial\hat{u}_J/\partial\hat{x} &= \frac{1}{H} \begin{vmatrix} D_x\hat{u}_J & D_x\hat{t} \\ D_t\hat{u}_J & D_t\hat{t} \end{vmatrix}, \\ \hat{u}_{J\hat{t}} \equiv \partial\hat{u}_J/\partial\hat{t} &= \frac{1}{H} \begin{vmatrix} D_x\hat{x} & D_x\hat{u}_J \\ D_t\hat{x} & D_t\hat{u}_J \end{vmatrix}. \end{aligned} \tag{21}$$

For example, the transformation is prolonged to second derivatives as follows:

$$\hat{u}_{\hat{x}\hat{x}} = \frac{1}{H} \begin{vmatrix} D_x\hat{u}_{\hat{x}} & D_x\hat{t} \\ D_t\hat{u}_{\hat{x}} & D_t\hat{t} \end{vmatrix}, \qquad \hat{u}_{\hat{t}\hat{t}} = \frac{1}{H} \begin{vmatrix} D_x\hat{x} & D_x\hat{u}_{\hat{t}} \\ D_t\hat{x} & D_t\hat{u}_{\hat{t}} \end{vmatrix},$$

$$\hat{u}_{\hat{x}\hat{t}} = \frac{1}{H} \begin{vmatrix} D_x\hat{u}_{\hat{t}} & D_x\hat{t} \\ D_t\hat{u}_{\hat{t}} & D_t\hat{t} \end{vmatrix} = \frac{1}{H} \begin{vmatrix} D_x\hat{x} & D_x\hat{u}_{\hat{x}} \\ D_t\hat{x} & D_t\hat{u}_{\hat{x}} \end{vmatrix}.$$

We are now in a position to define point symmetries of an nth order PDE:

$$\Delta(x,t,u,u_x,u_t,...) = 0. \tag{22}$$

For simplicity, we shall restrict attention to PDEs of the form

$$\Delta = u_\sigma - \omega(x,t,u,u_x,u_t,...) = 0, \tag{23}$$

where u_σ is one of the nth order derivatives of u and ω is independent of u_σ. (More generally, u_σ could be of order $k < n$ provided that ω does not depend upon u_σ or any derivatives of u_σ.)

The point transformation Γ is a point symmetry of equation (22) if

$$\Delta(\hat{x},\hat{t},\hat{u},\hat{u}_{\hat{x}},\hat{u}_{\hat{t}},...) = 0 \qquad \text{when equation (22) holds.} \tag{24}$$

Typically, the symmetry condition (24) is extremely complicated, so we shall not try to solve it directly. Nevertheless, it is quite easy to check whether or not a given point transformation is a symmetry of a particular PDE.

Generally speaking, we do not know *a priori* what form the point symmetries of a given PDE will take. However, it is usually possible to carry out a systematic search for one-parameter Lie groups of point symmetries. The technique is essentially the same as for ODEs. We seek point symmetries of the form

$$\begin{aligned}
\hat{x} &= x + \varepsilon\xi(x,t,u) + \mathbb{O}(\varepsilon^2), \\
\hat{t} &= t + \varepsilon\tau(x,t,u) + \mathbb{O}(\varepsilon^2), \\
\hat{u} &= u + \varepsilon\eta(x,t,u) + \mathbb{O}(\varepsilon^2).
\end{aligned} \tag{25}$$

Just as for Lie point transformations of the plane, each one-parameter (local) Lie group of point transformations is obtained by exponentiating its infinitesimal generator, which is

$$X = \xi \partial_x + \tau \partial_t + \eta \partial_u.$$

Equivalently, we can obtain $(\hat{x}, \hat{t}, \hat{u})$ by solving

$$\frac{d\hat{x}}{d\varepsilon} = \xi(\hat{x}, \hat{t}, \hat{u}), \qquad \frac{d\hat{t}}{d\varepsilon} = \tau(\hat{x}, \hat{t}, \hat{u}), \qquad \frac{d\hat{u}}{d\varepsilon} = \eta(\hat{x}, \hat{t}, \hat{u}),$$

subject to the initial conditions

$$(\hat{x}, \hat{t}, \hat{u}) \mid_{\varepsilon=0} = (x, t, u).$$

A surface $u = u(x,t)$ is mapped to itself by the group of transformations generated by X if

$$X(u - u(x,t)) = 0 \qquad \text{when} \quad u = u(x,t). \tag{26}$$

This condition can be expressed neatly by using the *characteristic* of the group, which is

$$Q = \eta - \xi u_x - \tau u_t.$$

From (26), the surface $u = u(x,t)$ is invariant provided that

$$Q = 0 \qquad \text{when} \quad u = u(x,t). \tag{27}$$

Equation (27) is called the *invariant surface condition*; it is central to some of the main techniques for finding exact solutions of PDEs.

The prolongation of the point transformation (25) to first derivatives is

$$\hat{u}_{\hat{x}} = u_x + \varepsilon \eta^x(x, t, u, u_x, u_t) + \mathbb{O}(\varepsilon^2),$$

$$\hat{u}_{\hat{t}} = u_t + \varepsilon\eta^t(x,t,u,u_x,u_t) + \mathbb{O}(\varepsilon^2),$$

where, from (20),

$$\begin{aligned}
\eta^x(x,t,u,u_x,u_t) &= D_x\eta - u_x D_x\xi - u_t D_x\tau, \\
\eta^t(x,t,u,u_x,u_t) &= D_t\eta - u_x D_t\xi - u_t D_t\tau.
\end{aligned}$$

The transformation is prolonged to higher-order derivatives recursively, using (21). Suppose that

$$\hat{u}_J = u_J + \varepsilon\eta^J + \mathbb{O}(\varepsilon^2), \qquad u_J \equiv \frac{\partial^{j_1+j_2}u}{\partial x^{j_1}\partial t^{j_2}}, \quad \hat{u}_J \equiv \frac{\partial^{j_1+j_2}\hat{u}}{\partial \hat{x}^{j_1}\partial \hat{t}^{j_2}}$$

for some numbers j_1 and j_2. Then (21) yields

$$\begin{aligned}
\hat{u}_{J\hat{x}} &= u_{Jx} + \varepsilon\eta^{Jx} + \mathbb{O}(\varepsilon^2), \\
\hat{u}_{J\hat{t}} &= u_{Jt} + \varepsilon\eta^{Jt} + \mathbb{O}(\varepsilon^2),
\end{aligned}$$

where

$$\begin{aligned}
\eta^{Jx} &= D_x\eta^J - u_{Jx}D_x\xi - u_{Jt}D_x\tau, \\
\eta^{Jt} &= D_t\eta^J - u_{Jx}D_t\xi - u_{Jt}D_t\tau. \qquad (28)
\end{aligned}$$

Alternatively, we can express the functions η^J in terms of the characteristic, for example,

$$\begin{aligned}
\eta^x &= D_xQ + \xi u_{xx} + \tau u_{xt}, \\
\eta^t &= D_tQ + \xi u_{xt} + \tau u_{tt}.
\end{aligned}$$

The higher-order terms are obtained by induction on j_1 and j_2:

$$\eta^J = D_J Q + \xi D_J u_x + \tau D_J u_t, \qquad D_J \equiv D_x^{j_1} D_t^{j_2}.$$

The infinitesimal generator is prolonged to derivatives by adding all terms of the form $\eta^J \partial_{u_J}$ up to the desired order. For example,

$$\begin{aligned} X^{(1)} &= \xi \partial_x + \tau \partial_t + \eta \partial_u + \eta^x \partial_{u_x} + \eta^t \partial_{u_t} = X + \eta^x \partial_{u_x} + \eta^t \partial_{u_t}, \\ X^{(2)} &= X^{(1)} + \eta^{xx} \partial_{u_{xx}} + \eta^{xt} \partial_{u_{xt}} + \eta^{tt} \partial_{u_{tt}}, \end{aligned}$$

From now on, we adopt the convention that the generator is prolonged as many times as is needed to describe the group's action on all variables. (We shall not usually refer explicitly to the order of prolongation.) To find the Lie point symmetries, we need explicit expressions for (28), such as

$$\begin{aligned} \eta^x &= \eta_x + (\eta_u - \xi_x) u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t, \\ \eta^t &= \eta_t - \xi_t u_x + (\eta_u - \tau_t) u_t - \xi_u u_x u_t - \tau_u u_t^2, \\ \eta^{xx} &= \eta_{xx} + (2\eta_{xu} - \xi_{xx}) u_x - \tau_{xx} u_t + (\eta_{uu} - 2\xi_{xu}) u_x^2 \\ &\quad -2\tau_{xu} u_x u_t - \xi_{uu} {u_x}^3 - \tau_{uu} u_x^2 u_t + (\eta_u - 2\xi_x) u_{xx} \\ &\quad -2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}, \\ \eta^{xt} &= \eta_{xt} + (\eta_{tu} - \xi_{xt}) u_x + (\eta_{xu} - \tau_{xt}) u_t - \xi_{tu} u_x^2 \\ &\quad +(\eta_{uu} - \xi_{xu} - \tau_{tu}) u_x u_t - \tau_{xu} u_t^2 - \xi_{uu} u_x^2 u_t - \tau_{uu} u_x u_t^2 \\ &\quad -\xi_t u_{xx} - \xi_u u_t u_{xx} + (\eta_u - \xi_x - \tau_t) u_{xt} - 2\xi_u u_x u_{xt} \\ &\quad -2\tau_u u_t u_{xt} - \tau_x u_{tt} - \tau_u u_x u_{tt}, \\ \eta^{tt} &= \eta_{tt} - \xi_{tt} u_x + (2\eta_{tu} - \tau_{tt}) u_t - 2\xi_{tu} u_x u_t \\ &\quad +(\eta_{uu} - 2\tau_{tu}) u_t^2 - \xi_{uu} u_x u_t^2 - \tau_{uu} u_t^3 - 2\xi_t u_{xt} \\ &\quad -2\xi_u u_t u_{xt} + (\eta_u - 2\tau_t) u_{tt} - \xi_u u_x u_{tt} - 3\tau_u u_t u_{tt}. \end{aligned}$$

Lie point symmetries are obtained by differentiating the symmetry condition (24) with respect to ε at $\varepsilon = 0$. We obtain the linearized symmetry condition

$$X\Delta = 0 \qquad \text{when} \quad \Delta = 0. \tag{29}$$

Restriction (23) enables us to eliminate u_σ from condition (29); then we split the remaining terms (according to their dependence on derivatives of u) to obtain a linear system of *determining equations* for ξ, τ, and η. The vector space $\mathscr{L}$ of all Lie point symmetry generators of a given PDE is a Lie algebra, although it may not be finite dimensional.

Let us consider the Lie group of symmetries of a given PDE

$$\Delta = 0. \tag{30}$$

For now, we restrict attention to scalar PDEs with two independent variables. Recall that a solution $u = u(x,t)$ is invariant under the group generated by

$$X = \xi \partial_x + \tau \partial_t + \eta \partial_u$$

if and only if the characteristic vanishes on the solution. In other words, every invariant solution satisfies the invariant surface condition

$$Q \equiv \eta - \xi u_x - \tau u_t = 0. \tag{31}$$

Usually (31) is much easier to solve than the original PDE. Having solved (31), we can find out which solutions also satisfy (30). For example, the characteristic

$$Q = -cu_x - u_t.$$

The travelling wave ansatz

$$u = F(x - ct)$$

is the general solution of the invariant surface condition $Q = 0$.

For now, suppose that ξ and τ are not both zero. Then the invariant surface condition is a first-order quasilinear PDE that can be solved by the method of characteristics. The characteristic equations are

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}. \tag{32}$$

If $r(x,t,u)$ and $v(x,t,u)$ are two functionally independent first integrals of (32), every invariant of the group is a function of r and v. Usually, it is convenient to let one invariant play the role of a dependent variable. Suppose (without lose of generality) that $v_u \neq 0$; then the general solution of the invariant surface condition is

$$v = F(r). \tag{33}$$

This solution is now substituted into the PDE (30) to determine the function F.

If r and v both depend on u, it is necessary to find out whether the PDE has any solutions of the form

$$r = c. \tag{34}$$

These are the only solutions of the invariant surface condition that are not (locally) of the form (33). If r is a function of the independent variables x and t only, then (34) cannot yield a solution $u = u(x,t)$.

CHAPTER III

SIMPLE DUFFING–VAN DER POL–TYPE OSCILLATOR

In this chapter, we will consider the simple Duffing–van der Pol–type oscillator (2) and we focus on the case where $n = m+1$.

3.1 Determining Equation System

Firstly, we write the oscillator equation (2) as following form:

$$\ddot{y} = -(\delta + \beta y^m)\dot{y} + \mu y - \alpha y^{m+1} = F(x,y,y'). \tag{35}$$

To investigate the integrability of this equation, the Lie theory of differential equations will be used [20]. However, it should be noted that the integrability of a differential equation can also be analyzed by means of Divisor theorem method [13]. Here we apply the Lie theory to study equation (35) because the Lie symmetry method not only provides us useful information about when the equation is integrable, but also enables us to simplify the problem to canonical variables which eases the integration of the equation in a more effective way without complicated calculations [1].

It can be seen in [20] that in order to find the symmetry group G admitted by a differential equation with infinitesimal operator

$$X = \eta(x,y)\frac{\partial}{\partial y} + \xi(x,y)\frac{\partial}{\partial x},$$

it is needed to find an infinitesimal operator $X^{(2)}$ such that

$$X^{(2)}(\ddot{y} + (\delta + \beta y^m)\dot{y} - \mu y + \alpha y^{m+1}) = 0. \tag{36}$$

The operator $X^{(2)}$ is

$$X^{(2)} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + A(x,y,\dot{y})\frac{\partial}{\partial \dot{y}} + B(x,y,\dot{y},\ddot{y})\frac{\partial}{\partial \ddot{y}},$$

where $A(x,y,\dot{y})$ and $B(x,y,\dot{y},\ddot{y})$ are defined as follows [1]:

$$\begin{aligned} A(x,y,\dot{y}) &= \eta_x + \dot{y}(\eta_y - \xi_x) - \dot{y}^2\xi_y, \\ B(x,y,\dot{y},\ddot{y}) &= \eta_{xx} + \dot{y}(2\eta_{xy} - \xi_{xx}) + \dot{y}^2(\eta_{yy} - 2\xi_{xy}) \\ &\quad -\dot{y}^3\xi_{yy} + \ddot{y}(\eta_y - 2\xi_x - 3\dot{y}\xi_y). \end{aligned}$$

All $\xi(x,y)$ and $\eta(x,y)$ that verify equation (36) generate infinitesimal operators $X^{(2)}$ as in equation (36) which comprise the symmetries of the differential equation. Also, it is known that the order of an ordinary differential equation can be reduced by one, if the equation admits an infinitesimal generator. Thus, the Duffing–van der Pol–type oscillator (35) will be integrated only if we can find two linearly independent infinitesimal operators generated by $\xi(x,y)$ and $\eta(x,y)$ [1].

Following the procedure to determine the symmetries of a differential equation mentioned in the former section, we have

$$\begin{aligned} &\eta_{xx} + (2\eta_{xy} - \xi_{xx})y'_x + (\eta_{yy} - 2\xi_{xy})(y'_x)^2 - \xi_{yy}(y'_x)^3 \\ = \;& (2\xi_x - \eta_y + 3\xi_y y'_x)F + \xi F_x + \eta F_y + [\eta_x + (\eta_y - \xi_x)y'_x - \xi_y(y'_x)^2]F_{y'_x}, \end{aligned} \tag{37}$$

which is a second-order partial differential equation for two unknown functions $\xi(x,y)$ and $\eta(x,y)$. Although equation (37) looks complicated, it is commonly easy to solve. Since the unknown functions do not depend on the derivative y', after setting the coefficients of the powers $(y')^i$ ($i = 0, 1, 2, 3$) in (37) to zero, one can get the determining equations which can be split and represented as the system [11]

$$
\begin{aligned}
[y']^0 &: \eta_{xx} = (\mu\eta - \delta\eta_x) + (2\mu\xi_x - \mu\eta_y)y \\
&\quad -((m+1)\alpha\eta + \beta\eta_x)\,y^m + (\eta_y - 2\xi_x)\alpha y^{m+1}, && (38)\\
[y']^1 &: 2\eta_{xy} - \xi_{xx} = -\delta\xi_x + 3\mu\xi_y y - m\beta\eta y^{m-1} - \beta\xi_x y^m - 3\alpha\xi_y y^{m+1}, && (39)\\
[y']^2 &: \eta_{yy} - 2\xi_{xy} = -2\delta\xi_y - 2\beta\xi_y y^m, && (40)\\
[y']^3 &: \xi_{yy} = 0. && (41)
\end{aligned}
$$

From the equation (41), it is obvious that

$$\xi = a(x)y + b(x). \tag{42}$$

and apply this result (42) in equation (40) implies that

$$\eta = a'(x)y^2 - \delta a(x)y^2 - \frac{2\beta a(x)}{(m+1)(m+2)}y^{m+2} + c(x)y + d(x),\ m \neq -1,\ m \neq -2, \tag{43}$$

where $a(x), b(x), c(x), d(x)$ are functions of x need to be determined. Plugging results (42) and (43) into equation (38), we obtain a polynomial of y with degree $2m+2$ which is zero if and only if the coefficient of each variable is set to zero. Then we have

$$
\begin{aligned}
&[y^{2m+2}] : \beta^2 a' - \alpha\beta a = 0,\\
&[y^{m+2}] : \frac{2\beta\mu a}{m+2} + \frac{2\beta\delta a'}{(m+1)(m+2)} + \frac{2\beta a''}{(m+1)(m+2)}\\
&\qquad -\alpha(m+1)a' + \alpha(m-1)a\delta - \beta a'' + \beta\delta a' = 0,\\
&[y^{m+1}] : \alpha c m + c'\beta + 2\alpha b' = 0,\\
&[y^m] : (m+1)\alpha d + d'\beta = 0, \qquad\qquad (44)\\
&[y^2] : a''' - a'\mu - \delta^2 a' - \delta a\mu = 0,
\end{aligned}
$$

$$[y^1]: c'' + \delta c' - 2b'\mu = 0,$$

$$[y^0]: d'' - d\mu + \delta d' = 0.$$

Similarly, if both results (42) and (43) are substituted into equation (39), we obtain a polynomial of y with degree $2m+1$ which is zero if and only if the following equations are verified

$$\begin{aligned}
&[y^{2m+1}]: \frac{2ma\beta^2}{(m+1)(m+2)} = 0,\\
&[y^{m+1}]: -(m+1)a'\beta + ma\delta\beta - 3\alpha a = -\frac{4\beta}{m+1}a',\\
&[y^m]: -cm\beta - \beta b' = 0,\\
&[y^{m-1}]: -md\beta = 0,\\
&[y^1]: 3a'' - 3\delta a' - 3a\mu = 0,\\
&[y^0]: 2c' - b'' = -\delta b'.
\end{aligned} \tag{45}$$

Here we restrict $\alpha \neq 0$ and $\beta \neq 0$. These above two equation systems (44) and (45) imply that $a(x) = 0$ and $d(x) = 0$. Then the determining system about $b(x)$ and $c(x)$ is reduced to

$$cm + b' = 0, \tag{46}$$

$$c'\beta + \alpha b' = 0, \tag{47}$$

$$2c' - b'' + \delta b' = 0, \tag{48}$$

$$c'' + \delta c' - 2b'\mu = 0. \tag{49}$$

From equations (46) and (47), $b(x)$ and $c(x)$ can be solved as

$$b = \frac{-c_0}{\alpha}\beta e^{\frac{\alpha m}{\beta}x} + b_0, \tag{50}$$

$$c = c_0 e^{\frac{\alpha m}{\beta}x}, \tag{51}$$

where b_0 and c_0 are arbitrary constants.

Here there are only two options. The first one is $c_0 \equiv 0$. In this case b_0 can be an arbitrary constant, and this means that

$$\xi = 1, \qquad \eta = 0.$$

Hence, one infinitesimal operator is obtained, namely $\chi_1 = \partial x$.

The second one is assuming $c_0 \neq 0$. Substituting equations (50) and (51) into equation (48), we obtain one parametric condition

$$m = \frac{\delta\beta}{\alpha} - 2. \tag{52}$$

Then substituting equations (50) and (51) into equation (49), another parametric condition is obtained as

$$\frac{\alpha^2 m}{\beta^2} = -\frac{\alpha\delta}{\beta} - 2\mu. \tag{53}$$

Because b_0 and c_0 are arbitrary constants, for our convenience, we may assume $b_0 = 0$ and $c_0 = 1$. Then we have

$$b = -\frac{1}{\alpha}\beta e^{\frac{\alpha m}{\beta}x}, \qquad c = e^{\frac{\alpha m}{\beta}x},$$

those are equivalent to

$$\xi = -\frac{1}{\alpha}\beta e^{\frac{\alpha m}{\beta}x}, \qquad \eta = e^{\frac{\alpha m}{\beta}x}y. \tag{54}$$

After using ξ and η, the second infinitesimal generator is found, namely

$$\chi_2 = -\frac{1}{\alpha}\beta e^{\frac{\alpha m}{\beta}x}\partial x + e^{\frac{\alpha m}{\beta}x}y\partial y.$$

Every infinitesimal generator is of the form:

$$\chi = c_1\chi_1 + c_2\chi_2,$$

where χ_1 is a homothety operator and χ_2 is a translation operator.

In conclusion, only under the parametric conditions (52) and (53), the oscillator is completely integrable. Otherwise, the oscillator is only partially integrable and there is no way to write down the solution in terms of elementary functions.

3.2 Reduction to Canonical Variables

We know that if an ordinary differential equation admits an infinitesimal generator, then there exists a pair of variables:

$$t = f(x,y), \qquad u = g(x,y),$$

called canonical variables, with f and $g(g \neq 0)$ being arbitrary particular solutions of the first-order linear partial differential equations [12]

$$\xi(x,y)\frac{\partial f}{\partial x} + \eta(x,y)\frac{\partial f}{\partial y} = \chi, \tag{55}$$

$$\xi(x,y)\frac{\partial g}{\partial x} + \eta(x,y)\frac{\partial g}{\partial y} = 0, \tag{56}$$

where χ is a nonzero constant and can be chosen arbitrarily. Suppose that the general solution of the characteristic equation

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)},$$

has the form $U(x,y) = C$, where C is arbitrary, then the general solutions of (55) and (56) can be expressed by

$$f(x,y) = \chi \int \frac{dx}{\xi^*(x,U)} + \Phi_1(U), \tag{57}$$

$$g(x,y) = \Phi_2(U), U = U(x,y), \tag{58}$$

where $\Phi_1(U)$ and $\Phi_2(U)$ are arbitrary functions, $\xi^*(x,U(x,y)) \equiv \xi(x,y)$, and U in the integral is regarded as a parameter later. Choosing $\chi = m$ in (55), by solving equations (57) and (58) under condition (54), we obtain a particular solution

$$f(x,y) = e^{-\frac{\alpha m}{\beta}x}, \qquad g(x,y) = ye^{\frac{\alpha}{\beta}x}. \tag{59}$$

Since $t = f(x,y)$ and $u = g(x,y)$, formula (59) is equivalent to the parametric form

$$x = -\frac{\beta}{\alpha m}\ln t, \qquad y = ut^{\frac{1}{m}}. \tag{60}$$

Using the nonlinear transformations (60), we have:

$$\frac{\partial y}{\partial x} = -\frac{\alpha m}{\beta}u_t' t^{\frac{m+1}{m}} - \frac{\alpha}{\beta}ut^{\frac{1}{m}}, \tag{61}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\alpha^2 m^2}{\beta^2}u_{tt}'' t^{\frac{2m+1}{m}} + \frac{\alpha^2 m(m+2)}{\beta^2}t^{\frac{m+1}{m}}u_t' + \frac{\alpha^2}{\beta^2}t^{\frac{1}{m}}u. \tag{62}$$

Substituting equations (61) and (62) into equation (35), we obtain

$$t^{\frac{2m+1}{m}}\left(\frac{\alpha^2 m^2}{\beta^2}u'' - \alpha m u' u^m\right) + t^{\frac{1}{m}}\left(\frac{\alpha^2}{\beta^2}u - \frac{\alpha\delta u}{\beta} - \mu u\right)$$
$$+ t^{\frac{m+1}{m}}\left(\frac{\alpha^2 m(m+2)}{\beta^2}u' - \frac{\delta\alpha m}{\beta}u' - \alpha u^{m+1} + \alpha u^{m+1}\right) = 0. \tag{63}$$

Under the parametric conditions (52) and (53), equation (63) is simplified to an autonomous equation

$$\frac{\alpha m}{\beta^2}u_{tt}'' = u_t' u^m,$$

which is easily integrated as

$$u_t' = \frac{\beta^2}{\alpha m(m+1)}u^{m+1} + I, \tag{64}$$

where I is an arbitrary constant. Utilizing the inverse transformations of (60) we have

$$\frac{\partial u}{\partial t} = -\frac{\beta}{\alpha m}y' e^{\frac{\alpha(m+1)}{\beta}x} - \frac{1}{m}ye^{\frac{\alpha(m+1)}{\beta}x}. \tag{65}$$

Substituting equations (65) into (64), under the parametric conditions (52) and (53), consequently we obtain the first integral to the oscillator system (2) when $n = m+1$ as follows

$$\left(y' + \frac{\alpha}{\beta}y + \frac{\beta}{m+1}y^{m+1}\right)e^{\frac{\alpha(m+1)}{\beta}x} = I. \tag{66}$$

CHAPTER IV

FIRST INTEGRALS FOR DUFFING–VAN DER POL–TYPE SYSTEM

In this chapter, we are going to find the first integral of a more general and complex Duffing–van der Pol–type oscillator system (1) by the method of Lie symmetry theory.

4.1 Condition of Integrability

Write the oscillator equation (1) as the following form

$$\ddot{y} = -(\delta + \beta y^m)\dot{y} + \mu y - \alpha y^{m+1} - \varphi y^n = F(x, y, y').$$

We will have

$$\begin{aligned} F_x &= 0 \\ F_y &= -\beta m y' y^{m-1} + \mu - \alpha(m+1)y^m - \varphi n y^{n-1} \\ F_{y'} &= -(\delta + \beta y^m) \end{aligned}$$

Following the procedure to get the determining system of a nonlinear second-order differential equation introduced in the preceding chapter, we can have the determining system of oscillator (1)

$$\begin{aligned} [y']^0 : \eta_{xx} = \eta[\mu - \alpha(m+1)y^m - \varphi n y^{n-1}] - \eta_x(\delta + \beta y^m) \\ +(2\xi_x - \eta_y)(\mu y - \alpha y^{m+1} - \varphi y^n), \end{aligned} \tag{67}$$

$$[y']^1 : 2\eta_{xy} - \xi_{xx} = -\xi_x(\delta + \beta y^m) + 3\xi_y(\mu y - \alpha y^{m+1} - \varphi y^n) - \eta \beta m y^{m-1}, \tag{68}$$

$$[y']^2 : \eta_{yy} - 2\xi_{xy} = -2\delta\xi_y - 2\beta\xi_y y^m, \tag{69}$$

$$[y']^3 : \xi_{yy} = 0. \tag{70}$$

The last equation (70) gives

$$\xi = a(x)y + b(x). \tag{71}$$

After substituting (71) to equation (69) yields

$$\eta = a'(x)y^2 - \delta a(x)y^2 - \frac{2\beta a(x)}{(m+1)(m+2)}y^{m+2} + c(x)y + d(x),\ m \neq -1,\ m \neq -2, \tag{72}$$

where $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are functions of x. Plugging equations (71) and (72) into equation (67), we obtain a polynomial of y with degree $n+m+1$ which is zero if and only if each variable coefficient is set to zero

$$\begin{aligned}
&[y^{n+m+1}]:\ \frac{2a\beta\varphi(n-m-2)}{(m+1)(m+2)} = 0,\\
&[y^{n+1}]:\ -2a\varphi\delta - \varphi n a' + \varphi n a\delta = 0,\\
&[y^{n}]:\ -2\varphi b' + c\varphi - c\varphi n = 0,\\
&[y^{n-1}]:\ \varphi n d = 0,\\
&[y^{2m+2}]:\ \beta^2 a' - \alpha\beta a = 0,\\
&[y^{m+2}]:\ \frac{2\beta\mu a}{m+2} + \frac{2\beta\delta a'}{(m+1)(m+2)} + \frac{2\beta a''}{(m+1)(m+2)} - \alpha(m+1)a'\\
&\qquad +\alpha(m-1)a\delta - \beta a'' + \beta\delta a' = 0,\\
&[y^{m+1}]:\ \alpha c m + c'\beta + 2\alpha b' = 0,\\
&[y^{m}]:\ (m+1)\alpha d + d'\beta = 0,\\
&[y^{2}]:\ a''' - a'\mu - \delta^2 a' - \delta a\mu = 0,\\
&[y^{1}]:\ c'' + \delta c' - 2b'\mu = 0,\\
&[y^{0}]:\ d'' - d\mu + \delta d' = 0.
\end{aligned} \tag{73}$$

Similarly, substituting (71) and (72) into equation (68), we obtain a polynomial of y with degree n which is zero if and only if the following equations are satisfied

$$
\begin{aligned}
&[y^n]:\ a\varphi = 0, \\
&[y^{2m+1}]:\ \frac{2am\beta^2}{(m+1)(m+2)} = 0, \\
&[y^{m+1}]:\ \frac{4\beta a'}{m+1} - (m+1)a'\beta + m\beta a\delta - 3a\alpha = 0, \\
&[y^m]:\ b'\beta + m\beta c = 0, \\
&[y^{m-1}]:\ m\beta d = 0, \\
&[y^1]:\ a'' - a'\delta - a\mu = 0, \\
&[y^0]:\ 2c' - b'' + b'\delta = 0.
\end{aligned}
\tag{74}
$$

In our study we require $\alpha \neq 0$, $\beta \neq 0$ and $\varphi \neq 0$. Analyzing the above two resultant equation systems (73) and (74), we deduce that $a(x) = 0$ and $d(x) = 0$, and the determining system for $b(x)$ and $c(x)$ is obtained as

$$
\begin{aligned}
&c'' + \delta c' - 2b'\mu = 0, \\
&\alpha cm + c'\beta + 2\alpha b' = 0, \\
&-2b' + c - cn = 0, \qquad (75) \\
&2c' - b'' + b'\delta = 0, \\
&b'\beta + m\beta c = 0, \qquad (76)
\end{aligned}
$$

From equations (75) and (76), we can have that n can only be $2m+1$; then the determining system should be reduced to

$$cm + b' = 0, \tag{77}$$

$$c'\beta + \alpha b' = 0, \tag{78}$$

$$2c' - b'' + \delta b' = 0, \tag{79}$$

$$c'' + \delta c' - 2b'\mu = 0, \tag{80}$$

Through analyzing equations (77) and (78), we solve for $b(x)$ and $c(x)$ as

$$b(x) = \frac{-c_0}{\alpha}\beta e^{\frac{\alpha m}{\beta}x} + b_0, \tag{81}$$

$$c(x) = c_0 e^{\frac{\alpha m}{\beta}x}, \tag{82}$$

where b_0 and c_0 are arbitrary constants. Similar with chapter III, when $c_0 \equiv 0$, we will get

$$\xi = 1, \qquad \eta = 0,$$

and one infinitesimal operator is

$$\chi_1 = \partial x.$$

Otherwise $c_0 \neq 0$, after we substitute equations (81) and (82) into equations (79) and (80), we can get the parametric conditions as

$$m = \frac{\delta\beta}{\alpha} - 2, \qquad \frac{\alpha^2 m}{\beta^2} = -\frac{\alpha\delta}{\beta} - 2\mu. \tag{83}$$

Because b_0 and c_0 are arbitrary constants, for simplicity, we will assume $b_0 = 0$ and $c_0 = 1$. Then we find

$$\xi = -\frac{1}{\alpha}\beta e^{\frac{\alpha m}{\beta}x}, \qquad \eta = e^{\frac{\alpha m}{\beta}x}y.$$

Using ξ and η, we obtain the second infinitesimal generator as follows

$$\chi_2 = -\frac{1}{\alpha}\beta e^{\frac{\alpha m}{\beta}x}\partial x + e^{\frac{\alpha m}{\beta}x}y\partial y.$$

4.2 Reduction to First Integral

Because equation (1) has the same two infinitesimal generators with equation (2), so we can substitute equations (61) and (62) into equation (1) directly, we obtain

$$
\begin{aligned}
& t^{\frac{2m+1}{m}}\left(\frac{\alpha^2 m^2}{\beta^2}u'' + \varphi u^{2m+1} - \alpha m u' u^m\right) \\
& + t^{\frac{m+1}{m}}\left(\frac{\alpha^2 m(m+2)}{\beta^2}u' - \frac{\delta \alpha m}{\beta}u' - \alpha u^{m+1} + \alpha u^{m+1}\right) \\
& + t^{\frac{1}{m}}\left(\frac{\alpha^2}{\beta^2}u - \frac{\alpha \delta u}{\beta} - \mu u\right) = 0.
\end{aligned} \tag{84}
$$

Under the parametric condition (83), equation (84) is simplified to

$$
\frac{\alpha^2 m^2}{\beta^2}u'' + \varphi u^{2m+1} - \alpha m u' u^m = 0. \tag{85}
$$

Rewrite equation (85) as

$$
\ddot{u} + f(u)\dot{u} + g(u) = 0, \tag{86}
$$

with

$$
f(u) = -\frac{\beta^2}{\alpha m}u^m, \qquad g(u) = \frac{\varphi \beta^2}{\alpha^2 m^2}u^{2m+1}. \tag{87}
$$

With using the well known transformation

$$
\dot{u} \equiv \frac{du}{dt} = \zeta(u), \tag{88}
$$

the equation (86) will be transferred into

$$
\zeta\zeta' + f(u)\zeta + g(u) = 0, \qquad ' \equiv \frac{d}{du}. \tag{89}
$$

Let

$$
\zeta = F(u)G(\omega(u)). \tag{90}
$$

Then we can have

$$\zeta' = F'G + F\frac{\partial G}{\partial \omega}\omega'. \tag{91}$$

Substituting equations (90) and (91) into equation (89), we obtain

$$\omega' = -\frac{F'}{F}\frac{G}{\frac{\partial G}{\partial \omega}} - \frac{f}{F}\frac{1}{\frac{\partial G}{\partial \omega}} - \frac{g}{F^2}\frac{1}{G\frac{\partial G}{\partial \omega}}, \tag{92}$$

where $\frac{F'}{F}$, $\frac{f}{F}$ and $\frac{g}{F^2}$ are functions with respect to the variable u; $\frac{G}{\frac{\partial G}{\partial \omega}}$, $\frac{1}{\frac{\partial G}{\partial \omega}}$ and $\frac{1}{G\frac{\partial G}{\partial \omega}}$ are functions with respect to the variable ω. So if we want to use the method of separation of variables to integrate the equation (92), the only choice is

$$\frac{F'}{F} = a\frac{f}{F} = b\frac{g}{F^2}, \tag{93}$$

where a and b are arbitrary constants. From condition (93) we can have

$$\left(\frac{g}{f}\right)' = \frac{a^2}{b}f, \tag{94}$$

where $\frac{a^2}{b}$ is an arbitrary constant because a and b are arbitrary.

Since from (87) we can have

$$\left(\frac{g}{f}\right)' = \lambda f, \tag{95}$$

with $\lambda = \frac{\varphi(m+1)}{\beta^2}$ is a constant. So equation (89) satisfies the condition (94), which is a special case of Liouville's condition. Assume

$$\zeta = \frac{g}{f}\omega, \tag{96}$$

then by equations (95) and (96), equation (89) becomes to

$$\omega' = -\frac{f^2}{g}\frac{\lambda\omega^2 + \omega + 1}{\omega}. \tag{97}$$

Separating dependence from ω and u, making further use of condition (95), and integrating on both sides of (97), gives

$$P(\omega) \equiv \int \frac{-\omega}{\lambda\omega^2+\omega+1}d\omega = \frac{1}{\lambda}ln\frac{g}{f}. \tag{98}$$

The integral $P(\omega)$ is given by

$$P(\omega) = \frac{1}{2\lambda}\left[-\frac{2}{2\lambda\omega+1} - ln(\lambda\omega^2+\omega+1)\right], \quad \Delta = 0, \tag{99}$$

$$P(\omega) = \frac{1}{2\lambda}\left[\frac{1}{\sqrt{\Delta}}ln\frac{2\lambda\omega+1-\sqrt{\Delta}}{2\lambda\omega+1+\sqrt{\Delta}} - ln(\lambda\omega^2+\omega+1)\right], \quad \Delta > 0, \tag{100}$$

$$P(\omega) = \frac{1}{2\lambda}\left[\frac{2}{\sqrt{-\Delta}}tan^{-1}\left(\frac{2\lambda\omega+1}{\sqrt{-\Delta}}\right) - ln(\lambda\omega^2+\omega+1)\right], \quad \Delta < 0, \tag{101}$$

where $\Delta = 1-4\lambda$. By equations (88) and (96), we have

$$\omega = \frac{f}{g}\dot{u} = -\frac{\alpha m}{\varphi}u^{-(m+1)}\dot{u}. \tag{102}$$

Case $\Delta = 0$ means $\lambda = \frac{1}{4}$. Substituting equations (99) and (102) into equation (98), we can obtain the first integral of equation (86) as

$$\frac{2ru^{m+1}}{2ru^{m+1}+\dot{u}} + ln(2ru^{m+1}+\dot{u}) = I,$$

where $r = -\frac{\varphi}{\alpha m}$, I is an arbitrary constant. Utilizing the transformation of equation (65), we have the first integral of system (1) as

$$\frac{2\varphi y^{m+1}}{2\varphi y^{m+1}+\beta y'+\alpha y} + \frac{\alpha(m+1)}{\beta}x + ln\left[-\frac{1}{\alpha m}(2\varphi y^{m+1}+\beta y'+\alpha y)\right] = I.$$

Case $\Delta > 0$ means $\lambda < \frac{1}{4}$. Substituting equations (100) and (102) into equation (98), we can have

$$\frac{1}{\sqrt{\Delta}}ln\frac{2\lambda\dot{u}+(1-\sqrt{\Delta})ru^{m+1}}{2\lambda\dot{u}+(1+\sqrt{\Delta})ru^{m+1}} - ln(r^2u^{2m+2}+ru^{m+1}\dot{u}+\lambda\dot{u}^2) = I$$

as the first integral of equation (86), where $r = -\frac{\varphi}{\alpha m}$, I is an arbitrary constant. Applying the equation (65), we obtain the first integral of the oscillator system (1) as

$$\frac{1}{\sqrt{\Delta}} ln \frac{2(m+1)(\beta y' + \alpha y) + (1-\sqrt{\Delta})\beta^2 y^{m+1}}{2(m+1)(\beta y' + \alpha y) + (1+\sqrt{\Delta})\beta^2 y^{m+1}} - \frac{2\alpha(m+1)}{\beta} x$$
$$-ln\left[\varphi^2 y^{2m+2} + \varphi y^{m+1}(\beta y' + \alpha y) + \frac{\varphi(m+1)}{\beta^2}(\beta y' + \alpha y)^2\right] = I.$$

Case $\Delta < 0$ means $\lambda > \frac{1}{4}$. Using equations (101) and (102) into equation (98), we can have the first integral of equation (86) as

$$\frac{2}{\sqrt{-\Delta}} tan^{-1}\left(\frac{ru^{m+1} + 2\lambda \dot{u}}{\sqrt{-\Delta} ru^{m+1}}\right) - ln(r^2 u^{2m+2} + ru^{m+1}\dot{u} + \lambda \dot{u}^2) = I,$$

where $r = -\frac{\varphi}{\alpha m}$, I is an arbitrary constant. Utilizing the equation (65), we get the first integral of the equation (1) as

$$\frac{2}{\sqrt{-\Delta}} tan^{-1}\left[\frac{2(m+1)(\beta y' + \alpha y) + \beta^2 y^{m+1}}{\sqrt{-\Delta}\beta^2 y^{m+1}}\right] - \frac{2\alpha(m+1)}{\beta} x$$
$$-ln\left[\varphi^2 y^{2m+2} + \varphi y^{m+1}(\beta y' + \alpha y) + \frac{\varphi(m+1)}{\beta^2}(\beta y' + \alpha y)^2\right] = I.$$

CHAPTER V

PARTICULAR CASES

Following the previous procedure, we can apply this Lie symmetry method to solve some special cases of Duffing–van der Pol–type system.

5.1 Helmholtz Oscillator

In this section we are going to study the Helmholtz oscillator equation

$$\ddot{y} + \delta \dot{y} - \mu y + \alpha y^2 = 0, \tag{103}$$

which is a nonlinear oscillator with a quadratic nonlinearity [1, 5]. Under the conditions $m = 1$ and β, $\varphi = 0$, from the equations (71) and (72), it is obvious that

$$\xi = a(x)y + b(x),$$

and

$$\eta = a'(x)y^2 - \delta a(x)y^2 + c(x)y + d(x),$$

where $a(x), b(x), c(x), d(x)$ are arbitrary functions. Following (73) and (74), we have

$$[y^3] : \ \alpha a' = 0,$$

$$[y^2] : \ a''' - a'\mu - a'\delta^2 - \mu a\delta + \alpha c + 2\alpha b' = 0,$$

$$[y^1] : \ 2\alpha d + c'' + \delta c' - 2b'\mu = 0,$$

$$[y^0] : \ d'' - d\mu + \delta d' = 0,$$

and

$$[y^2]:\ -3\alpha a=0,$$
$$[y^1]:\ 3a''-3\delta a'-3a\mu=0,$$
$$[y^0]:\ 2c'-b''=-\delta b',$$

respectively. Analyzing the above two resultant systems gives that $a(x)=0$, so we obtain a determining system about $b(x)$ and $c(x)$ as

$$c+2b'=0, \tag{104}$$
$$2c'-b''+\delta b'=0, \tag{105}$$
$$2\alpha d+c''+\delta c'-2b'\mu=0, \tag{106}$$
$$d''-d\mu+\delta d'=0. \tag{107}$$

Equations (104) and (105) indicate that

$$c(x)=c_0 e^{\frac{\delta}{5}x},$$
$$b(x)=-\frac{5}{2\delta}c_0 e^{\frac{\delta}{5}x}+b_0,$$

where b_0 and c_0 are constants. Substituting $b(x)$ and $c(x)$ into equation (106) we derive a parametric condition

$$d=-\frac{1}{2\alpha}\left(\frac{6}{25}\delta^2+\mu\right)c.$$

Similarly, substituting $d(x)$ and $c(x)$ into equation (107) we derive another parametric condition

$$-\frac{1}{2\alpha}c\left(\frac{6}{25}\delta^2+\mu\right)\left(\frac{6}{25}\delta^2-\mu\right)=0.$$

Assumption $c_0 = 0$ produces an infinitesimal operator $\chi_1 = \partial x$. If $c_0 \neq 0$, we separate our discussions into two cases:

Case 1. when $\mu = \frac{6}{25}\delta^2$, because b_0 and c_0 are arbitrary constants, for our convenience, we assume $b_0 = 0$ and $c_0 = 1$ that gives:

$$b = \frac{-5}{2\delta} e^{\frac{\delta}{5}x}, \quad c = e^{\frac{\delta}{5}x}, \quad d = \frac{-6}{25\alpha}\delta^2 e^{\frac{\delta}{5}x}.$$

Then, we obtain

$$\xi = \frac{-5}{2\delta} e^{\frac{\delta}{5}x}, \qquad \eta = e^{\frac{\delta}{5}x}\left(y - \frac{6\delta^2}{25\alpha}\right).$$

Therefore two infinitesimal generators are found, named

$$\chi_1 = \partial x, \qquad \chi_2 = \frac{-5}{2\delta} e^{\frac{\delta}{5}x}\partial x + e^{\frac{\delta}{5}x}\left(y - \frac{6\delta^2}{25\alpha}\right)\partial y.$$

Choosing $\chi = \frac{1}{2}$ in (55) and using (57) and (58), we obtain a particular solution of f and g as

$$f(x,y) = e^{-\frac{\delta}{5}x}, \qquad g(x,y) = e^{\frac{2\delta}{5}x}\left(y - \frac{6\delta^2}{25\alpha}\right). \tag{108}$$

Since $t = f(x,y)$ and $u = g(x,y)$, formula (108) is equivalent to the parametric form:

$$x = -\frac{5}{\delta}\ln t, \qquad y = ut^2 + \frac{6\delta^2}{25\alpha}. \tag{109}$$

By this nonlinear transformation, we have

$$\frac{\partial y}{\partial x} = -\frac{\delta}{5}u'_t t^3 - \frac{2\delta}{5}ut^2, \tag{110}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\delta^2}{25}u''_{tt}t^4 + \frac{\delta^2}{5}u'_t t^3 + \frac{4\delta^2}{25}ut^2. \tag{111}$$

Substituting equations (110) and (111) into equation (103), we obtain

$$\left(\frac{\delta^2}{25}u''+\alpha u^2\right)t^4+\left(\frac{6\delta^2}{25}u-\mu u\right)t^2+\frac{36\delta^4}{625\alpha}-\frac{6\delta^2}{25\alpha}\mu=0. \tag{112}$$

Under the parametric condition $\mu=\frac{6\delta^2}{25}$, equation (112) reduces to:

$$\frac{\delta^2}{25}u''_{tt}=-\alpha u^2,$$

which is easily integrated as

$$(u'_t)^2=-\frac{50\alpha}{3\delta^2}u^3+I. \tag{113}$$

Now we substitute the reverse transformation of (109)

$$\frac{\partial u}{\partial t}=\left(-\frac{5}{\delta}y'-2y+\frac{12\delta^2}{25\alpha}\right)e^{\frac{3\delta}{5}x},$$

into (113), we obtain the first integral of equation (103)

$$\left[\frac{24\delta^2}{25\alpha}y-8y^2+\frac{50\alpha}{3\delta^2}y^3+\frac{20}{\delta}yy'-\frac{24\delta}{5\alpha}y'+\frac{25}{\delta^2}(y')^2\right]e^{\frac{6\delta}{5}x}=I, \tag{114}$$

Case 2. If we choose $\mu=-\frac{6}{25}\delta^2$. Base on the same argument, we can obtain another first integral of equation (103) as

$$\left[\frac{25}{\delta^2}(y')^2+4y^2+\frac{20}{\delta}yy'+\frac{50\alpha}{3\delta^2}y^3\right]e^{\frac{6}{5}\delta x}=I, \tag{115}$$

It is notable that the corresponding results presented in [1, 5, 12, 13] agree well with formulas (114) and (115). Rewrite equation (115) to

$$\left[\left(y'+\frac{2\delta y}{5}\right)^2+\frac{2\alpha y^3}{3}\right]e^{\frac{6\delta x}{5}}=c,$$

where c is an arbitrary constant, because $c = \frac{\delta^2}{25} I$ and I is arbitrary. Then

$$y' = \pm\sqrt{ce^{-\frac{6\delta x}{5}} - \frac{2\alpha y^3}{3}} - \frac{2\delta y}{5}.$$

Firstly, we consider

$$y' = \frac{dy}{dx} = \sqrt{ce^{-\frac{6\delta x}{5}} - \frac{2\alpha y^3}{3}} - \frac{2\delta y}{5}. \tag{116}$$

From equation (116) we can have

$$\left(\sqrt{ce^{-\frac{6\delta x}{5}} - \frac{2\alpha y^3}{3}} - \frac{2\delta y}{5}\right) dx + (-1) dy = 0.$$

Let us assume

$$M(x,y) = \sqrt{ce^{-\frac{6\delta x}{5}} - \frac{2\alpha y^3}{3}} - \frac{2\delta y}{5}, \qquad N(x,y) = -1, \qquad \mu(x,y) = \frac{e^{-\frac{\delta x}{5}}}{\sqrt{ce^{-\frac{6\delta x}{5}} - \frac{2\alpha y^3}{3}}}.$$

which yields

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}.$$

Now we need to find $v(x,y)$, satisfies

$$\frac{\partial v}{\partial x} = \mu M = e^{-\frac{\delta x}{5}} - \frac{2\delta y}{5} e^{-\frac{\delta x}{5}} A^{-\frac{1}{2}}, \tag{117}$$

$$\frac{\partial v}{\partial y} = \mu N = -e^{-\frac{\delta x}{5}} A^{-\frac{1}{2}}. \tag{118}$$

where

$$A = ce^{-\frac{6\delta x}{5}} - \frac{2\alpha y^3}{3}.$$

Integrating equation (117) with respect to x we can have

$$v = -\frac{5}{\delta} e^{-\frac{\delta x}{5}} + \int -\frac{2\delta y}{5} e^{-\frac{\delta x}{5}} A^{-\frac{1}{2}} dx + \varphi(y). \tag{119}$$

Then from equations (118) and (119) we can obtain

$$\begin{aligned}\frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{5}{\delta}e^{-\frac{\delta x}{5}}\right)+\frac{\partial}{\partial y}\left(-\int\frac{2\delta y}{5}e^{-\frac{\delta x}{5}}A^{-\frac{1}{2}}dx\right)+\frac{\partial\varphi(y)}{\partial y}\\ &= -e^{-\frac{\delta x}{5}}A^{-\frac{1}{2}}.\end{aligned}$$

So

$$\frac{\partial\varphi(y)}{\partial y}=-e^{-\frac{\delta x}{5}}A^{-\frac{1}{2}}+\frac{\partial}{\partial y}\left(\int\frac{2\delta y}{5}e^{-\frac{\delta x}{5}}A^{-\frac{1}{2}}dx\right).$$

Then

$$\begin{aligned}\varphi(y) &= \int -e^{-\frac{\delta x}{5}}A^{-\frac{1}{2}}dy+\int\left[\frac{\partial}{\partial y}\left(\int\frac{2\delta y}{5}e^{-\frac{\delta x}{5}}A^{-\frac{1}{2}}dx\right)\right]dy\\ &= \int -e^{-\frac{\delta x}{5}}A^{-\frac{1}{2}}dy+\int\frac{2\delta y}{5}e^{-\frac{\delta x}{5}}A^{-\frac{1}{2}}dx. \qquad (120)\end{aligned}$$

Substituting (120) into equation (119) we can have

$$\begin{aligned}v &= -\frac{5}{\delta}e^{-\frac{\delta x}{5}}-\int\frac{2\delta y}{5}e^{-\frac{\delta x}{5}}A^{-\frac{1}{2}}dx-\int e^{-\frac{\delta x}{5}}A^{-\frac{1}{2}}dy+\int\frac{2\delta y}{5}e^{-\frac{\delta x}{5}}A^{-\frac{1}{2}}dx\\ &= -\frac{5}{\delta}e^{-\frac{\delta x}{5}}-\int e^{-\frac{\delta x}{5}}A^{-\frac{1}{2}}dy.\end{aligned}$$

Therefore the general solution of equation (116) is

$$-\frac{5}{\delta}e^{-\frac{\delta x}{5}}-\sqrt[3]{\frac{-3}{2\alpha}}e^{-\frac{\delta x}{5}}\int\frac{1}{\sqrt{u^3+ce^{-\frac{6\delta x}{5}}}}du=I_1, \qquad (121)$$

with $u=\sqrt[3]{\frac{-2\alpha}{3}}y$.

Case $c=0$. The equation (121) is changed to

$$-\frac{5}{\delta}e^{-\frac{\delta x}{5}}-\sqrt[3]{\frac{-3}{2\alpha}}e^{-\frac{\delta x}{5}}\int\frac{1}{\sqrt{u^3}}du=I_1, \qquad (122)$$

which is easy to solve. By solving equation (122) we can obtain the solution of equation (116) as

$$y = \frac{-6\delta^2}{25\alpha}(1 + c_1 e^{\frac{\delta x}{5}})^{-2},$$

where c_1 is an arbitrary constant because I_1 is arbitrary.

Case $c < 0$. Rewrite equation (121) to

$$-\frac{5}{\delta}e^{-\frac{\delta x}{5}} - \sqrt[3]{\frac{-3}{2\alpha}}e^{-\frac{\delta x}{5}} \int \frac{1}{\sqrt{u^3 - \eta^3}}du = I_1, \tag{123}$$

where $\eta = \sqrt[3]{-c}e^{-\frac{2\delta x}{5}} > 0$. Because $u^3 - \eta^3$ has three zeros including one positive real zero η and two complex zeros $-\frac{\eta}{2} \pm \frac{\sqrt{3}}{2}\eta i$, whose real parts are negative. So we can write $u^3 - \eta^3$ as $(u - z_1)\{(u + z_2)^2 + z_3^2\}$, where $z_1 = \eta$, $z_2 = \frac{\eta}{2}$ and $z_3 = \frac{\sqrt{3}}{2}\eta$ are all positive. Clearly, $u \geq z_1$ so that $u^3 - \eta^3 > 0$. Then [23]

$$\int \frac{1}{\sqrt{u^3 - \eta^3}}du = \sqrt{2}(p+q)\int \frac{dy}{\sqrt{[(y+p)^2 - (y-q)^2][(q+z_2)(y+p)^2 + (p-z_2)(y-q)^2]}},$$

with

$$p = \sqrt{\{(z_1 + z_2)^2 + z_3^2\}} - z_1, \qquad q = \sqrt{\{(z_1 + z_2)^2 + z_3^2\}} + z_1.$$

We now make the substitution

$$t = \frac{u-q}{u+p}, \tag{124}$$

where t increases monotonically for increasing u. This gives

$$\begin{aligned}
\int \frac{1}{\sqrt{u^3 - \eta^3}}du &= \sqrt{\frac{2}{p - z_2}}\int \frac{dt}{\sqrt{\{(1-t^2)((\frac{q+z_2}{p-z_2})^2 + t^2)\}}}, \\
&= -\sqrt{\frac{2}{p+q}}cn^{-1}t, \\
&= -3^{-\frac{1}{4}}\eta^{-\frac{1}{2}}cn^{-1}t,
\end{aligned} \tag{125}$$

and the modulus being given by

$$k^2 = \frac{p - z_2}{p + q} = \frac{2 - \sqrt{3}}{4}.$$

Apply (125) to equation (123) we can have

$$t = cn[c_1 + \varepsilon e^{-\frac{\delta x}{5}}],$$

where c_1 is an arbitrary constant because $c_1 = \sqrt[3]{\frac{-2\alpha}{3}}\sqrt[4]{3}\sqrt[6]{-c}I_1$ and I_1 is arbitrary; $\varepsilon = \frac{5}{\delta}\sqrt[3]{\frac{-2\alpha}{3}}\sqrt[4]{3}\sqrt[6]{-c}$. Then solve equation (124) we can have

$$y = \frac{-3\delta^2}{50\alpha}\varepsilon^2\left(\frac{1}{\sqrt{3}} + \frac{1 + cn[c_1 + \varepsilon e^{-\frac{\delta x}{5}}, k]}{1 - cn[c_1 + \varepsilon e^{-\frac{\delta x}{5}}, k]}\right)e^{-\frac{2\delta x}{5}}.$$

Case $c > 0$. We write equation (121) as

$$-\frac{5}{\delta}e^{-\frac{\delta x}{5}} - \sqrt[3]{\frac{-3}{2\alpha}}e^{-\frac{\delta x}{5}}\int\frac{1}{\sqrt{u^3 + \eta^3}}du = I_1,$$

where $\eta = \sqrt[3]{c}e^{-\frac{2\delta x}{5}} > 0$. After we write $u^3 + \eta^3$ as $(u - z_1)\{(u + z_2)^2 + z_3^2\}$, we can have $z_1 = -\eta$, $z_2 = -\frac{\eta}{2}$ and $z_3 = \frac{\sqrt{3}}{2}\eta$. Thus by the same way in case $c < 0$, we only need to change values of z_1 and z_2, then we can have the solution of equation (116) is

$$y = \frac{-3\delta^2}{50\alpha}\varepsilon^2\left(-\frac{1}{\sqrt{3}} + \frac{1 + cn[c_1 + \varepsilon e^{-\frac{\delta x}{5}}, k']}{1 - cn[c_1 + \varepsilon e^{-\frac{\delta x}{5}}, k']}\right)e^{-\frac{2\delta x}{5}},$$

where $k'^2 = \frac{2+\sqrt{3}}{4}$, c_1 is an arbitrary constant and $\varepsilon = \frac{5}{\delta}\sqrt[3]{\frac{-2\alpha}{3}}\sqrt[4]{3}\sqrt[6]{c}$.

Similarly, we can get the solutions of

$$y' = \frac{dy}{dx} = -\sqrt{ce^{-\frac{6\delta x}{5}} - \frac{2\alpha y^3}{3}} - \frac{2\delta y}{5},$$

which are similar with the solutions of equation (116), the only difference is the sign of ε being reversed for each case of $c < 0$ and $c > 0$.

Thus the general solutions of equation (115) are

$$
\begin{aligned}
y &= \frac{-6\delta^2}{25\alpha}(1+c_1 e^{\frac{\delta x}{5}})^{-2}, \quad I=0,\\
y &= \frac{-3\delta^2}{50\alpha}c_2^2\left(\frac{1}{\sqrt{3}}+\frac{1+cn[c_1+c_2e^{-\frac{\delta x}{5}},k]}{1-cn[c_1+c_2e^{-\frac{\delta x}{5}},k]}\right)e^{-\frac{2\delta x}{5}}, \quad I<0,\\
y &= \frac{-3\delta^2}{50\alpha}c_2^2\left(-\frac{1}{\sqrt{3}}+\frac{1+cn[c_1+c_2e^{-\frac{\delta x}{5}},k']}{1-cn[c_1+c_2e^{-\frac{\delta x}{5}},k']}\right)e^{-\frac{2\delta x}{5}}, \quad I>0,
\end{aligned}
$$

where c_2 is an arbitrary constant because $c_2 = \pm\varepsilon$ and c is arbitrary in each case of $c > 0$ and $c < 0$. These solutions are same as that of paper [1].

Write equation (114) as

$$
\left\{\left[y'+\frac{2\delta}{5}\left(y-\frac{6\delta^2}{25\alpha}\right)\right]^2+\frac{2\alpha}{3}\left(y-\frac{6\delta^2}{25\alpha}\right)^3\right\}e^{\frac{6\delta x}{5}}=c,
$$

where c is an arbitrary constant. Then from the solution set of equation (115), we can easy to get the solution set of equation (114) is

$$
\begin{aligned}
y &= \frac{-6\delta^2}{25\alpha}(1+c_1 e^{\frac{\delta x}{5}})^{-2}+\frac{6\delta^2}{25\alpha}, \quad I=0,\\
y &= \frac{-3\delta^2}{50\alpha}c_2^2\left(\frac{1}{\sqrt{3}}+\frac{1+cn[c_1+c_2e^{-\frac{\delta x}{5}},k]}{1-cn[c_1+c_2e^{-\frac{\delta x}{5}},k]}\right)e^{-\frac{2\delta x}{5}}+\frac{6\delta^2}{25\alpha}, \quad I<0,\\
y &= \frac{-3\delta^2}{50\alpha}c_2^2\left(-\frac{1}{\sqrt{3}}+\frac{1+cn[c_1+c_2e^{-\frac{\delta x}{5}},k']}{1-cn[c_1+c_2e^{-\frac{\delta x}{5}},k']}\right)e^{-\frac{2\delta x}{5}}+\frac{6\delta^2}{25\alpha}, \quad I>0.
\end{aligned}
$$

5.2 Other Special Cases

1. Duffing-Type Oscillator

Assume that $\alpha \neq 0$ and $\beta, \varphi \equiv 0$, then equation (1) changes into the form [11]

$$
\ddot{y}+\delta\dot{y}-\mu y+\alpha y^{m+1}=0. \tag{126}
$$

We suppose that $m \neq 0$ and $m \neq -1$. By an analogous argument in Chapter IV, we obtain $a(x) = 0$ and $d(x) = 0$, as well as a determining system about $b(x)$ and $c(x)$ as follows

$$cm + 2b' = 0, \tag{127}$$

$$2c' - b'' + \delta b' = 0, \tag{128}$$

$$c'' + \delta c' - 2b'\mu = 0. \tag{129}$$

From equations (127) and (128), we find $b(x)$ and $c(x)$ be of the form

$$b = \frac{-c_0(m+4)}{2\delta} e^{\frac{\delta m}{m+4}x} + b_0, \tag{130}$$

$$c = c_0 e^{\frac{\delta m}{m+4}x}, \tag{131}$$

where b_0 and c_0 are arbitrary constants.

There are two options here too. The first one is $c_0 = 0$, which gives

$$\xi = 1, \qquad \eta = 0.$$

So, only one corresponding infinitesimal operator is generated as $\chi_1 = \partial x$. The other option, in order to get two symmetries, is to assume $c_0 \neq 0$. Substituting equations (130) and (131) into equation (129), we derive the parametric condition as

$$\mu = -\frac{2m+4}{(m+4)^2}\delta^2. \tag{132}$$

For simplicity, we choose $b_0 = 0$ and $c_0 = 1$, and then have

$$b = \frac{-(m+4)}{2\delta} e^{\frac{\delta m}{m+4}x}, \qquad c = e^{\frac{\delta m}{m+4}x}.$$

Combining this with $a(x) = 0$ and $d(x) = 0$, we deduce

$$\xi = \frac{-(m+4)}{2\delta} e^{\frac{\delta m}{m+4}x}, \qquad \eta = e^{\frac{\delta m}{m+4}x} y.$$

Thus, another infinitesimal generator is found, namely

$$\chi_2 = \frac{-(m+4)}{2\delta} e^{\frac{\delta m}{m+4}x} \partial x + e^{\frac{\delta m}{m+4}x} y \partial y.$$

That is, under the parametric condition (132), the oscillator system (126) is completely integrable.

Following the procedure (55)-(58) and choosing $\chi = \frac{m}{2}$ in (55), we obtain particular solutions for f and g as

$$f(x,y) = e^{-\frac{\delta m}{m+4}x}, \qquad g(x,y) = y e^{\frac{2\delta}{m+4}x}. \tag{133}$$

Since $t = f(x,y)$ and $u = g(x,y)$, formulas (133) are equivalent to the parametric form

$$x = -\frac{m+4}{\delta m} \ln t, \qquad y = u t^{\frac{2}{m}}, \tag{134}$$

which gives

$$\frac{\partial y}{\partial x} = -\frac{\delta m}{m+4} u_t' t^{\frac{m+2}{m}} - \frac{2\delta}{m+4} u t^{\frac{2}{m}}, \tag{135}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\delta^2 m^2}{(m+4)^2} u_{tt}'' t^{\frac{2(m+1)}{m}} + \frac{\delta^2 m}{(m+4)} t^{\frac{2+m}{m}} u_t' + \frac{4\delta^2}{(m+4)^2} t^{\frac{2}{m}} u. \tag{136}$$

Substituting equations (135) and (136) into equation (126), we obtain

$$\begin{aligned} &\left(\frac{4\delta^2}{(m+4)^2} u - \frac{2\delta^2}{m+4} u - \mu u \right) t^{\frac{2}{m}} \\ &+ \left(\frac{m(m+2)}{(m+4)^2} \delta^2 u_t' + \frac{2m}{(m+4)^2} \delta^2 u_t' - \frac{m\delta^2}{m+4} u_t' \right) t^{\frac{m+2}{m}} \\ &+ \left(\left(\frac{m\delta}{m+4} \right)^2 u_{tt}'' + \alpha u^{m+1} \right) t^{\frac{2m+2}{m}} = 0. \end{aligned} \tag{137}$$

Under the parametric condition (132), equation (137) changes into an autonomous equation as

$$\frac{m^2\delta^2}{(m+4)^2}u''_{tt} = -\alpha u^{m+1},$$

which is integrated as

$$(u'_t)^2 = -\frac{2(m+4)^2}{m^2\delta^2}\frac{\alpha}{m+2}u^{m+2} + I_2. \tag{138}$$

Using the inverse transformation of (134) yields

$$\frac{\partial u}{\partial t} = \left(-\frac{m+4}{\delta m}y' - \frac{2y}{m}\right)e^{\frac{(m+2)}{m+4}\delta x}. \tag{139}$$

Substituting equations (139) into (138), under the parametric condition (132), we obtain the first integral of equation (126) as

$$\left[\frac{(m+4)^2}{(\delta m)^2}(y')^2 + \frac{4}{m^2}y^2 + \frac{4(m+4)}{m^2\delta}yy' + \frac{2\alpha(m+4)^2}{m^2\delta^2(m+2)}y^{m+2}\right]e^{\frac{2\delta(m+2)}{m+4}x} = I_2. \tag{140}$$

2. Damped Duffing Equation

The choice $m = 2$ leads equation (126) to the damped Duffing equation [7, 10, 18]

$$\ddot{y} + \delta\dot{y} - \mu y + \alpha y^3 = 0. \tag{141}$$

Substituting $m = 2$ into solution (140), we can derive the first integral of equation (141) immediately

$$\left[\frac{9}{\delta^2}(y')^2 + y^2 + \frac{6}{\delta}yy' + \frac{9\alpha}{2\delta^2}y^4\right]e^{\frac{4}{3}\delta x} = I_2, \tag{142}$$

under the parametric condition $\mu = -\frac{2}{9}\delta^2$.

It is noted that the corresponding results on equation (141) in [29] are identical to our formula (142) when $\alpha = 1$.

3. Force-free Duffing–van der Pol Oscillator

We know that the choices $m = 2$ and $n = 3$ lead equation (2) to the standard form of the Duffing-van der Pol oscillator equation (6). In [5, 29], the first integral of the Duffing-van der Pol oscillator when $\alpha = 1$ is considered, namely

$$\ddot{y} + (\delta + \beta y^2)\dot{y} - \mu y + y^3 = 0. \tag{143}$$

Following the parametric conditions (52) and (53), that are,

$$4\alpha = \delta\beta, \qquad \frac{2\alpha^2}{\beta^2} = -\frac{\alpha\delta}{\beta} - 2\mu, \tag{144}$$

and using formula (66), we can obtain immediately that the Duffing-van der Pol equation (143) has one first integral of the form

$$\left[\dot{y} + \frac{1}{\beta}y + \frac{\beta}{3}y^3\right] e^{\frac{3x}{\beta}} = I_1. \tag{145}$$

It is remarkable that in view of our parametric condition (144) and formula (145), it shows that our parametric constraint (144) is weaker than the corresponding ones described in the literature [5, 29], and the first integral presented in [5, 29] is just a particular case of (145). The results on the first integral established in [16] by using the Prelle-Singer method agree well with (145).

CHAPTER VI

CONCLUSION

Finding first integrals (conservation laws) and exact solutions for various nonlinear differential equations has been an interesting subject in mathematical and physical communities. Since 1983, Prelle and Singer presented a deductive method for solving first–order ODEs that presents a solution in terms of elementary functions if such a solution exists. This technique has attracted many researchers from diverse groups and has been extended to autonomous systems of ODEs of higher dimensions for finding the first integrals and exact solutions under certain assumptions. From illustrative examples in these works, the obtained first integrals of autonomous systems are usually of rational or quasi-rational forms and searching for solution sets (S, R) usually involves complicated calculations. However, the generalization of this procedure to autonomous/nonautonomous systems of higher dimensions to find elementary first integrals in an effective manner is still an interesting and important subject.

In this work, we showed that under certain parametric conditions, some new first integrals of the Duffing–van der Pol oscillator system (1) could be established. To reach our goal, we first made a series of nonlinear transformations to simplify equation (1) to a simple form by means of Lie symmetry method, then we derived the first integral of the resultant equation. Through the inverse transformations we obtained the first integrals of the original oscillator equations. Finally, using the established first integral, we obtained exact solutions of equation (1) in the special parametric form.

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BIOGRAPHICAL SKETCH

Jing Cui, the daughter of Liangying Cui and Guirong Wang, was born in China in 1985. She received her bachelor degree in Computer Science and Technology from Xuzhou Institute of Technology, Xuzhou, Jiangsu, China in July of 2007. In August of 2009, she joined the Mathematical Master's Program at the University of Texas-Pan American, Edinburg, Texas. Her main research interests were in Differential Equations and Dynamic Systems. Her permanent mailing address is: No31. Xisixiang, Jinpu East Rd, Yunlong district, Xuzhou city, Jiangsu province, P.R.China, 221000.