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ON THE VORONOI CONJECTURE FOR COMBINATORIALLY VORONOI PARALLELOHEDRA IN DIMENSION 5*

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Abstract. In a recent paper, Garber, Gavrilyuk, and Magazinov [*Discrete Comput. Geom.*, 53 (2015), pp. 245–260] proposed a sufficient combinatorial condition for a parallelohedron to be affinely Voronoi. We show that this condition holds for all 5-dimensional Voronoi parallelohedra. Consequently, the Voronoi conjecture in $\mathbb{R}^5$ holds if and only if every 5-dimensional parallelohedron is combinatorially Voronoi. Here, by saying that a parallelohedron P is *combinatorially Voronoi*, we mean that P is combinatorially equivalent to a Dirichlet–Voronoi polytope for some lattice Λ, and this combinatorial equivalence is naturally translated into equivalence of the tiling by copies of P with the Voronoi tiling of Λ. We also propose a new condition which, if satisfied by a parallelohedron P, is sufficient to infer that P is affinely Voronoi. The condition is based on the new notion of the Venkov complex associated with a parallelohedron and cohomologies of this complex.

Key words. tiling, parallelohedron, Voronoi conjecture

AMS subject classifications. 52B20, 52B70, 52C07

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1. Introduction.

A convex d-dimensional polytope P is called a *parallelohedron* if it tiles $\mathbb{R}^d$ by translations only. The systematic study of parallelohedra goes back to works of Minkowski [19] and Voronoi [24]. Even before that, Fedorov determined all five combinatorial types of parallelohedra that exist in $\mathbb{R}^3$.

There are numerous papers that study tilings of $\mathbb{R}^d$ by convex polytopes under various restrictions, for example, on the number of different tiles that are allowed to be used, on possible isometries of the tiling and so on. We refer the reader to Chapters 3 and 64 of [15] in particular for a review on open questions and known results on this topic.

In early works on parallelohedra, the parallelohedra tilings were considered to be face-to-face only. A remarkable result by Venkov [23] and, independently, McMullen [18] asserts that if a convex polytope P tiles $\mathbb{R}^d$ with translations, then there is a face-to-face tiling using translations of P. All face-to-face tilings by translates of P are translationally equivalent, so we will write $\mathcal{T}(P)$ to denote the unique face-to-face tiling of $\mathbb{R}^d$ by translates of P such that $\mathcal{T}(P)$ contains P itself as one of its tiles.

Minkowski [19] established that all parallelohedra are centrally symmetric. Consequently, the centers of all tiles of $\mathcal{T}(P)$ form a lattice provided the origin $\mathbf{0}$ is the center of P. We denote this lattice by $\Lambda(P)$. Then the set of tiles of $\mathcal{T}(P)$ is exactly $\{P + t \,|\, t \in \Lambda(P)\}$.

The connection between parallelohedra and lattices lies in the core of one of the most famous conjectures in the theory of parallelohedra. The conjecture is stated by Voronoi [24] and connects classification of translational lattices with classification of parallelohedra.

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Conjecture. *For every parallelohedron P, there is a lattice Λ such that P is an affine image of the Dirichlet–Voronoi polytope for Λ.*

It is noteworthy that in [24] Voronoi developed a framework that can classify all lattice Dirichlet–Voronoi parallelohedra in $\mathbb{R}^d$. Voronoi's approach, known as *Voronoi reduction theory*, with some modifications, allows one to construct examples of lattices that give best-known bounds for classical lattice packing and covering problems; see [21] for details.

In the following, we call a d-dimensional parallelohedron *Voronoi* if it is a Dirichlet–Voronoi polytope for some d-dimensional lattice and *affinely Voronoi* if it is an affine image of a Voronoi parallelohedron. Thus, the Voronoi conjecture claims that every parallelohedron is affinely Voronoi.

The Voronoi conjecture has been proved for some remarkable families of parallelohedra (see [24, 25, 17, 9, 20, 12, 16]) but still remains open in general. The conjecture was also confirmed for all 3- and 4-dimensional parallelohedra; for $d \geq 6$ the conjecture remains largely open.

A proof of the Voronoi conjecture in $\mathbb{R}^5$ was recently announced in the preprint [14] by the last two authors of this paper; the proof significantly relies on Theorem 1.3 proved here. Some sources suggest that the paper of Engel [8] contains a proof of the Voronoi conjecture in $\mathbb{R}^5$. However, we have strong doubts that Engel's argument can be considered a rigorous proof, and we also refer the reader to the preprint [14] for more discussion on that matter. See also [5] for partial interpretation and discussion of Engel's results.

The classification of lattices and the corresponding Voronoi parallelohedra is also a classical topic in crystallography. It is easy to check that parallelograms and centrally symmetric hexagons are parallelohedra for $d = 2$ and that there are no other 2-dimensional parallelohedra. The full classification of parallelohedra of a given dimension d exists only for $d \leq 4$. The case $d = 3$ is due to Fedorov [10] as we mentioned before, and the case $d = 4$ is due to Delaunay [2] with a correction by Stogrin [22].

The classification of 5-dimensional Voronoi parallelohedra has been obtained in [5] by Dutour-Sikirić et al. The proof of the Voronoi conjecture in $\mathbb{R}^5$ from [14] turns this classification into a complete classification of 5-dimensional parallelohedra.

When speaking about classification of parallelohedra, it is necessary to specify the equivalence relation used to split parallelohedra in equivalence classes; usually, the combinatorial equivalence is used. We introduce the notion of equivalence in the following definition, where $\mathcal{F}(\mathcal{T}(P))$ denotes the poset of all faces of $\mathcal{T}(P)$ ordered by inclusion.

Definition 1.1. *Two d-dimensional parallelohedra, P and P', are equivalent if there is an isomorphism of face posets $f : \mathcal{F}(\mathcal{T}(P)) \to \mathcal{F}(\mathcal{T}(P'))$ and a linear isomorphism of lattices $f_* : \Lambda(P) \to \Lambda(P')$ such that $f(P + t) = P' + f_*(t)$ for every $t \in \Lambda(P)$. In other words, the isomorphism f satisfies the additional restriction: The naturally associated isomorphism of lattices, namely, the one obtained by restricting the action of f to d-dimensional tiles and then passing to their centers, is linear.*

Remark. Equivalence (in the sense of Definition 1.1) for Voronoi parallelohedra reduces to the notion of an *L-type*. More precisely, two Voronoi parallelohedra are equivalent if and only if they belong to the same L-type. The concept of L-types originates in the work of Voronoi [24]. See also [5, section 3] for a modern treatment.

Of course, if P and P' are equivalent as parallelohedra, then they are combinatorially equivalent as convex polytopes. In addition, the equivalence classes in the

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sense of Definition 1.1 also retain information about the lattice structure of the tiling. Particularly, the class of a parallelohedron P retains the information about central symmetries in the points of half-lattice $\frac{1}{2}\Lambda(P)$, i.e., the symmetries preserving the lattice $\Lambda(P)$.

We also note that the second condition of Definition 1.1 of induced linear isomorphism of lattices might be redundant—it is entirely possible that every isomorphism between face posets of two parallelohedra tilings induces a linear isomorphism of corresponding lattices. Moreover, it is plausible that any isomorphism of two face posets $\mathcal{F}(P)$ and $\mathcal{F}(P')$ of parallelohedra P and P' induces an isomorphism of face posets of corresponding tilings together with a linear isomorphism of the lattices $\Lambda(P)$ and $\Lambda(P')$. If this were true, a natural injective homomorphism $Aut(P) \to GL(\Lambda(P))$ would exist, where $Aut(P)$ is the group of combinatorial automorphisms of P. However, we do not have a proof that the lattice structure can be inferred from either the combinatorics of a single parallelohedron or the combinatorics of the entire tiling. Neither do we have an example of two parallelohedra that are combinatorially equivalent as convex polytopes but nonequivalent in the sense of Definition 1.1.

Definition 1.2. *We say that a parallelohedron P is combinatorially Voronoi if there is a Voronoi parallelohedron P' such that P and P' are equivalent in the sense of Definition* 1.1.

As we mentioned earlier, this equivalence implies the usual combinatorial equivalence as convex polytopes, but we are unaware if the converse is true; see the discussion above.

One of the methods of proving Voronoi conjecture for certain particular families of parallelohedra consists of proving existence of a *canonical scaling* for each parallelohedron from this family. This method was introduced by Voronoi [24] and is used in other papers, including [25, 20, 16]. For a modern treatment of the canonical scaling condition and other equivalent conditions, we refer the reader to [3].

Recently, Garber, Gavrilyuk, and Magazinov [12, Theorem 4.6] proposed a sufficient condition for a parallelohedron to be affinely Voronoi, which is an adapted version of the canonical scaling approach; see section 2 for details. In this paper we combine this condition (the *GGM condition*) with the complete classification of 5-dimensional Voronoi parallelohedra due to Dutour Sikirić et al. [5], proving the following main result.

Theorem 1.3. *A* 5*-dimensional parallelohedron is affinely Voronoi if and only if it is equivalent to a Voronoi parallelohedron in the sense of Definition* 1.1. *In other words, a* 5*-dimensional parallelohedron is affinely Voronoi if and only if it is combinatorially Voronoi.*

Theorem 1.3 essentially reduces the Voronoi conjecture in 5 dimensions to its weaker combinatorial version. This reduction is used in the preprint [14] to prove the Voronoi conjecture in $\mathbb{R}^5$.

We note that the GGM condition makes no sense in dimensions $d = 1, 2$, whereas for all 3- and 4-dimensional parallelohedra it holds by the results of [12] and [13], respectively.

Additionally, we propose yet another sufficient condition, Theorem 3.9, or the *Venkov complex condition*, also depending only on the equivalence class, that is sufficient for a parallelohedron to be affinely Voronoi. This condition generalizes both the GGM condition and the so-called 3*-irreducibility condition* from [20], meaning that

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every parallelohedron that satisfies either the GGM or the 3-irreducibility condition must also satisfy the Venkov complex condition.

The paper is organized as follows.

In section 2 we recall the key concepts and statements of the paper [12]. Particularly, we state the GGM condition explicitly.

In section 3 we introduce the *Venkov complex*, a 2-dimensional simplicial complex $Ven(P)$ associated with the combinatorics of P. This notion is used to formulate the Venkov complex condition for a parallelohedron to be affinely Voronoi.

In section 4 the GGM condition is reformulated in a discrete form, namely, in terms of the 1-skeleton of $Ven(P)$ (or, equivalently, the *red Venkov graph* $VG_r(P)$).

In section 5 we describe two independent algorithms to verify Theorem 1.3 and provide the details of their implementation. Both algorithms iterate through all 5-dimensional combinatorially Voronoi parallelohedra. The first one verifies the Venkov complex condition, whereas the second one verifies the GGM condition.

In section 6 we show that the Venkov complex condition generalizes both the GGM and Ordine's 3-irreducibility conditions.

Finally, in section 7 we provide a brief summary of results. We also propose two open questions in relation to possible counterexample to the Voronoi conjecture.

2. The π-surface of a parallelohedron.

In this section we introduce several key properties of parallelohedra that we use in the latter discussion. Most of these properties are discussed in [12], and we refer the reader to this paper for more details. For the sake of brevity, we limit ourselves only to short descriptions rather than detailed definitions.

2.1. Dual cells.

We fix a d-dimensional parallelohedron P. As before, let $\mathcal{T}(P)$ be the face-to-face tiling of $\mathbb{R}^d$ with translations of P. Each $(d-2)$-face of the polytope P is incident either exactly to three copies or exactly to four copies of P in the tiling $\mathcal{T}(P)$; see, for instance, [23] or [18] for details.

If a $(d-2)$-face F is incident exactly to three copies of P in $\mathcal{T}(P)$, then F is called *primitive*. Otherwise, F is called *nonprimitive*.

Let G be a face of $\mathcal{T}(P)$ of an arbitrary dimension $0 \le k \le d$. Denote by $D(G)$ the *dual* $(d-k)$*-cell* of G, which is, by definition, the set of centers of all translates of P in $\mathcal{T}(P)$ that contain G as a face. The poset of all dual cells for the faces of $\mathcal{T}(P)$ with ordering by inclusion is called the *dual complex* of $\mathcal{T}(P)$ and denoted by $\mathcal{D}(P)$. $\mathcal{D}(P)$ is dual to the poset of faces of $\mathcal{T}(P)$.

We emphasize that a priori $\mathcal{D}(P)$ is an abstract cell complex. However, if P is Voronoi, then there is a natural way to identify $\mathcal{D}(P)$ with the Delaunay tessellation of the lattice $\Lambda(P)$, which, in this particular case, makes $\mathcal{D}(P)$ a geometric cell complex.

We equip each dual cell $D(G)$ with a face structure by considering its subcells. This way, we are able to talk about the combinatorics of $D(G)$.

A classification of possible combinatorial types of dual 3-cells is available due to a classical result of Delaunay [2]. Namely, the dual cell $D(G)$ of a $(d-3)$-dimensional face G is combinatorially a tetrahedron, an octahedron (a cross-polytope), a quadrangular pyramid, a triangular prism, or a cube with combinatorics inherited from the convex hull $\operatorname{conv} D(G)$. Moreover, all quadrangular faces of $\operatorname{conv} D(G)$ are parallelograms.

For example, if a dual 3-cell is combinatorially a cube, Delaunay's result implies that its vertex set is the vertex set of some 3-dimensional parallelepiped Π, its dual 2-subcells are exactly the six 4-tuples of vertices spanning the facets of Π, and its

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dual 1-subcells are exactly the 12 pairs of vertices that span the edges of Π. A similar description is valid for the other four combinatorial types of dual 3-cells.

2.2. δ- and π-surfaces. Let P_δ, the *δ-surface of P*, be the manifold obtained from ∂P, the surface of P, by removing all closed nonprimitive $(d-2)$-faces. Also, let the *π-surface of P* be the quotient of P_δ obtained by identifying opposite points with respect to the central symmetry of P. We also equip both P_δ and P_π with a face structure inherited from that of P.

If F and G are two facets of P such that the intersection $F \cap G$ is a primitive $(d-2)$-face, then $F \cap G$ belongs to exactly three copies of P in $\mathcal{T}(P)$, namely, P, P_F, and P_G. Let $\mathbf{e}_{P,P_F}$, $\mathbf{e}_{P_G,P}$, and $\mathbf{e}_{P_F,P_G}$ be unit normal to common facets of these polytopes; then there is unique (up to nonzero multiplier) linear dependency

$$a\mathbf{e}_{P,P_F} + b\mathbf{e}_{P_G,P} + c\mathbf{e}_{P_F,P_G} = 0$$

of these vectors. In this case we define the *gain function* $g(F,G) := \frac{|b|}{|a|}$.

We can extend the definition of the gain function from one pair of facets to any *generic path* γ (continuous and piecewise linear) on P_δ. If γ visits facets $F_0, F_1, \dots, F_k$ in that order, we denote

$$\langle\gamma\rangle := [F_0, F_1, \dots, F_k]$$

and define

$$g(\gamma) := g(\langle\gamma\rangle) = g([F_0, F_1, \dots, F_k]) := \prod_{i=1}^{k} g(F_i, F_{i-1}).$$

The gain function of a generic path on P_π is, by definition, the gain function of any of its two lifts onto P_δ. The value does not depend on the particular lift since the lifts are centrally symmetric to each other, and the identity $g(F,F') = g(-F,-F')$ holds for any two facets F, F' of P adjacent by a primitive $(d-2)$-face.

The following criterion holds.

PROPOSITION 2.1 (see [12, Lemma 2.6 and Theorem 4.6]). *The following conditions are equivalent for a parallelohedron P:*

1. *P is affinely Voronoi.*
2. *For every generic path $\gamma : [0,1] \longrightarrow P_\pi$ which is closed, i.e., $\gamma(0) = \gamma(1)$, it holds that $g(\gamma) = 1$.*

2.3. Key cases of generic paths on π-surfaces. It the latter sections we use some particular cases of closed curves γ on P_π with $g(\gamma) = 1$.

LEMMA 2.2. *Let γ be a generic closed path on P_π. Assume that γ has a lift γ_δ onto P_δ satisfying any of the conditions (HB), (TC), or (O) below. Then $g(\gamma) = 1$.*

(HB) $\langle\gamma_\delta\rangle = [F_1, F_2, F_3, -F_1]$, *where the facets F_1, F_2, and F_3 are parallel to some primitive $(d-2)$-face G of P.*

(TC) $\langle\gamma_\delta\rangle = [F_1, \dots, F_k, F_1]$, *where all facets $F_1, \dots, F_k$ are distinct and share a common $(d-3)$-face G of P.*

(O) $\langle\gamma_\delta\rangle = [F_1, F_2, F_3, F_1]$, *where*

$$F_1 = P \cap (P + \mathbf{x}_2 - \mathbf{x}_3), \quad F_2 = P \cap (P + \mathbf{x}_1), \quad F_3 = P \cap (P + \mathbf{x}_2)$$

and $\{\mathbf{0}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_1 + \mathbf{x}_3 - \mathbf{x}_2\}$ is the vertex set of a pyramidal dual 3-cell $D(G)$ with apex $\mathbf{0}$.

Before we proceed with the proof, let us give a name to each type of closed paths mentioned in Lemma 2.2.

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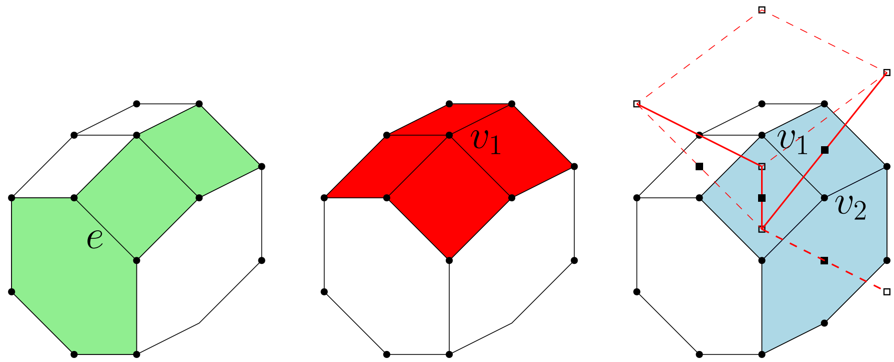

FIG. 1. *Three paths with trivial gain function. The leftmost green facets define a half-belt cycle generated by the edge e. The middle red facets define a trivially contractible cycle around the vertex v_1. The rightmost blue facets define an Ordine cycle because the dual cell of the same vertex v_1 is combinatorially pyramid and the solid red segments correspond to the pairs of neighbor facets in an Ordine circuit (we also can obtain this circuit as a trivially contractible circuit for the vertex v_2).*

DEFINITION 2.3. *In the notation of Lemma* 2.2:

1. *If the condition (HB) is satisfied, then γ is called a* half-belt circuit. *A belt of parallelohedron P is a collection of faces (codimensions* 1 *and* 2*) parallel to a single face of codimension* 2*; each belt contains four or six facets depending on primitivity of initial face of codimension* 2*. In the case of condition (HB), γ starts in one facet and ends in the center of the opposite facet staying within a single belt (of six facets) and using exactly one-half of this belt; see Figure* 1.
2. *If the condition (TC) is satisfied, then γ is called a* trivially contractible circuit. *In that case either $k = 3$ and $D(G)$ is a tetrahedral dual* 3*-cell or $k = 4$ and $D(G)$ is either octahedral dual* 3*-cell or pyramidal dual* 3*-cell with P representing its apex, so γ is in a neighborhood of G which is completely in P_π. In this case γ can be contracted on P_π to the trivial element of the corresponding fundamental group. In all other cases at least one edge incident to a vertex of $D(G)$ corresponds to nonprimitive $(d-2)$-face, and hence there is no circuit around the face G in a small neighborhood of G on P_π.*
3. *If the condition (O) is satisfied, then γ is called an* Ordine circuit. *The pyramidal dual* 3*-cells and combinatorics of corresponding parallelohedra tilings, particularly the connection between such circuits and existence of canonical scaling mentioned in the proof below, were studied by Ordine* [20]. *This connection explains our naming of these circuits.*

Remark. We warn the reader against a common misinterpretation of the definition above. A generic path on a δ- (or π-) surface does not directly correspond to a path on the dual complex. Instead, the correspondence is more intricate. More precisely, facets of P correspond by duality to edges of the dual complex incident to $\mathbf{0}$, while primitive $(d-2)$-faces correspond to dual triangles. Passing from one face to another on a δ-surface corresponds to passing from one dual edge to another via a common dual triangle. Consequently, the correct interpretation of a generic path on a δ-surface in terms of a dual complex is as follows: It is a sequence of dual triangles $[T_1, T_2, \dots, T_k]$ and dual edges $[E_0, E_1, \dots, E_k]$ such that each E_i is incident to $\mathbf{0}$ and each T_i is spanned by E_{i-1} and E_i. In particular, quadrangular dual 2-cells are disregarded altogether.

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Proof of Lemma 2.2. Consider each condition separately:

Case (HB). See [12, Lemma 4.5].

Case (TC). See [12, Lemma 3.6].

Case (O). This case follows from the existence of a local canonical scaling around a $(d-3)$-face G whose dual cell is combinatorially equivalent to a quadrangular pyramid. We provide a proof to make the argument self-contained.

Denote $\mathbf{x}_0 := \mathbf{0}$ and $\mathbf{x}_4 := \mathbf{x}_1 + \mathbf{x}_3 - \mathbf{x}_2$. Then

$$\{P + \mathbf{x}_i : i = 0, 1, \dots, 4\}$$

is the set of all parallelohedra of $\mathcal{T}(P)$ incident to G. By [12, Lemma 3.7], there exist affine functions $U_i : \mathbb{R}^d \to \mathbb{R}$, $i = 0, 1, \dots, 4$ such that if $P + \mathbf{x}_i$ and $P + \mathbf{x}_j$ share a common facet F_{ij}, then U_i and U_j coincide on the affine hull of F_{ij} and nowhere else. Define

$$\mathbf{a}_{ij} := \operatorname{grad} U_j - \operatorname{grad} U_i,$$

where $\operatorname{grad} U$ is the usual gradient vector of multivariate function U. Then the following identities hold:

$$g(F_1, F_2) = \frac{|\mathbf{a}_{01}|}{|\mathbf{a}_{14}|}, \quad g(F_2, F_3) = \frac{|\mathbf{a}_{02}|}{|\mathbf{a}_{01}|}, \quad g(F_3, F_1) = \frac{|\mathbf{a}_{23}|}{|\mathbf{a}_{02}|}. \tag{2.1}$$

Let us prove the first identity of (2.1). The $(d-2)$-face $F_1 \cap F_2$ is shared by exactly three parallelohedra of $\mathcal{T}(P)$, namely, P, $P + \mathbf{x}_1$, and $P + \mathbf{x}_2 - \mathbf{x}_3 = P + \mathbf{x}_1 - \mathbf{x}_4$. The facet F_2 is orthogonal to the vector $\mathbf{a}_{01}$. The facet F_1 is parallel to the facet $(P + \mathbf{x}_1) \cap (P + \mathbf{x}_4)$ and therefore is orthogonal to the vector $\mathbf{a}_{14}$. The facet $(P + \mathbf{x}_1) \cap (P + \mathbf{x}_1 - \mathbf{x}_4)$ is parallel to the facet $(P + \mathbf{x}_4) \cap P$ and is therefore orthogonal to the vector $\mathbf{a}_{04}$. But $\mathbf{a}_{01} + \mathbf{a}_{14} - \mathbf{a}_{04} = \mathbf{0}$; hence, indeed $g(F_1, F_2) = \frac{|\mathbf{a}_{01}|}{|\mathbf{a}_{14}|}$.

The proof of the third identity of (2.1) is obtained from that of the first identity by interchanging $\mathbf{x}_1$ with $\mathbf{x}_2$ and $\mathbf{x}_3$ with $\mathbf{x}_4$.

Concerning the second identity of (2.1), the $(d-2)$-face $F_2 \cap F_3$ is shared by parallelohedra P, $P + \mathbf{x}_1$, and $P + \mathbf{x}_2$. The normals to the faces F_2, F_3, and $(P + \mathbf{x}_1) \cap (P + \mathbf{x}_2)$ are, respectively, $\mathbf{a}_{01}$, $\mathbf{a}_{02}$, and $\mathbf{a}_{12}$. Since $\mathbf{a}_{01} + \mathbf{a}_{02} - \mathbf{a}_{12} = \mathbf{0}$, the second identity of (2.1) follows.

Finally, we have the identity

$$\mathbf{a}_{12} + \mathbf{a}_{23} + \mathbf{a}_{34} - \mathbf{a}_{14} = \mathbf{0}.$$

The vectors $\mathbf{a}_{12}$ and $\mathbf{a}_{23}$ span a 2-dimensional space, and $\mathbf{a}_{34}$ and $\mathbf{a}_{14}$ are collinear to $\mathbf{a}_{12}$ and $\mathbf{a}_{23}$, respectively. Hence, $\mathbf{a}_{12} = -\mathbf{a}_{34}$ and $\mathbf{a}_{23} = -\mathbf{a}_{14}$. In particular, $|\mathbf{a}_{23}| = |\mathbf{a}_{14}|$.

Expanding the definition of $g(\gamma)$ via (2.1) yields

$$g(\gamma) = g(F_1, F_2) g(F_2, F_3) g(F_3, F_1) = \frac{|\mathbf{a}_{01}|}{|\mathbf{a}_{14}|} \cdot \frac{|\mathbf{a}_{02}|}{|\mathbf{a}_{01}|} \cdot \frac{|\mathbf{a}_{23}|}{|\mathbf{a}_{02}|} = \frac{|\mathbf{a}_{23}|}{|\mathbf{a}_{14}|} = 1,$$

finishing the proof. ∎

To conclude this section, we reproduce the main result of [12]. See section 4 of this paper for further discussion of the approach. For brevity, we will call the condition of Theorem 2.4 *the GGM condition*.

THEOREM 2.4 (the GGM condition; see [12, Theorem 4.6]). *If the homology group $H_1(P_\pi, \mathbb{Q})$ is generated by half-belt cycles, then P is affinely Voronoi.*

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3. Simplicial complex approach. In this section we propose yet another sufficient condition for a parallelohedron to satisfy the Voronoi conjecture. Both GGM and Ordine conditions from [20] are special cases of our new condition, as we show later in section 6. We introduce Ordine conditions in section 6 as well.

We will introduce the notion of a *Venkov complex* $Ven(P)$ associated with a parallelohedron P. By definition, $Ven(P)$ will be a finite homogeneous 2-dimensional simplicial complex. The name is justified by the observation that the edge structure of $Ven(P)$ coincides with that of the *red Venkov graph* $VG_r(P)$. The graph $VG_r(P)$ may, however, have additional isolated vertices, and the number of isolated vertices is the number of 1-dimensional summands in the representation of P as a direct sum of irreducible parallelohedra, i.e., those that cannot be represented as the direct sum of parallelohedra of smaller dimension.

Let $\mathcal{A}$ be an arbitrary set (the *alphabet*) of *labels* and $T_m(\mathcal{A})$ be the set of all m-element subsets of $\mathcal{A}$. Every finite subset $X \subseteq T_m(\mathcal{A})$ defines a finite homogeneous $(m-1)$-dimensional simplicial complex $\mathcal{C}(X)$. Namely, the vertices of $\mathcal{C}(X)$ are in one-to-one correspondence with the set $\bigcup_{S\in X} S$ (i.e., the set of labels that are used at least once). The facets of $\mathcal{C}(X)$ are in one-to-one correspondence with elements of X so that each $S \in X$ corresponds to a facet with the vertex set labeled exactly by the elements of S. For our purposes we set $\mathcal{A} := \Lambda(P)/2\Lambda(P)$, i.e., the alphabet is the set of parity classes of the lattice $\Lambda(P)$. The element $x + 2\Lambda(P) \in \Lambda(P)/2\Lambda(P)$; i.e., the parity class of the lattice point x, will be denoted by $\bar{x}$.

We note that for two lattice points x, y, the parity classes $\overline{x+y}$ and $\overline{x-y}$ coincide. Moreover, if points x and y are connected with an edge in a dual cell, then the class $\overline{x+y} = \overline{x-y}$ can be seen as a representative of this edge as a vertex of the Venkov graph; see Definition 3.2 below.

It will be convenient to use a shorthand notation:

$$O(a,b,c,a',b',c') := \{\{a,b,c\},\{a',b',c'\},\{a',b,c\},\{a,b',c'\},\{a,b',c\},\\ \{a',b,c'\},\{a,b,c'\},\{a',b',c\}\}.$$

One can see that the complex $\mathcal{C}(O(a,b,c,a',b',c'))$ is combinatorially isomorphic to the surface of an octahedron with the pairs of opposite vertices labeled as $\{a,a'\}$, $\{b,b'\}$, and $\{c,c'\}$.

DEFINITION 3.1. *Let P be a parallelohedron of dimension $d \geq 4$. Let $\mathcal{D}^3(P)$ denote the set of all dual 3-cells of the tiling $\mathcal{T}(P)$. For each dual cell $D \in \mathcal{D}^3(P)$ define a set $X(D) \subseteq T_3(\mathcal{A})$, where $\mathcal{A} := \Lambda(P)/2\Lambda(P)$, as follows:*

1. *If D is a combinatorial tetrahedron and $V(D) = \{a,b,c,d\}$, set*
$$X(D) := O\left(\overline{a+b},\overline{a+c},\overline{a+d},\overline{c+d},\overline{b+d},\overline{b+c}\right).$$
2. *If D is a combinatorial pyramid, $V(D) = \{s,a,b,c,d\}$, and $a+c = b+d$, set*
$$X(D) := O\left(\overline{s+a},\overline{s+b},\overline{a+d},\overline{s+c},\overline{s+d},\overline{a+b}\right).$$
3. *If D is a combinatorial octahedron, $V(D) = \{a,b,c,d,e,f\}$, and $a+d = b+e = c+f$, set*
$$X(D) := O\left(\overline{a+b},\overline{a+c},\overline{b+c},\overline{a+e},\overline{a+f},\overline{b+f}\right).$$
4. *If D is a combinatorial prism, $V(D) = \{a,b,c,a',b',c'\}$, and $a-a' = b-b' = c-c'$, set*
$$X(D) := \{\{\overline{a+b},\overline{a+c},\overline{b+c}\}\}.$$

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5. *If D is a combinatorial cube, set $X(D) := \varnothing$.*
Write, finally,

$$X := \bigcup_{D \in \mathcal{D}^3(P)} X(D).$$

Then the simplicial complex $Ven(P) := \mathcal{C}(X)$ is called the Venkov complex *of P. The 2-dimensional faces of $Ven(P)$ are called the* Venkov triangles.

Remark. For every choice of D, the triangle xyz is included in $X(D)$ if and only if some edges of D represent the parity classes of x, y, and z and each pair of parity classes in $\{x, y, z\}$ is present as a pair of edges in at least one of the triangular subcells of D. For example, in case (3), triangle $\{\overline{a+b}, \overline{a+c}, \overline{b+f}\}$ is included in $D(X)$ because subcell abc contains edges corresponding to $\overline{a+b}$ and $\overline{a+c}$, subcell abf contains edges corresponding to $\overline{a+b}$ and $\overline{b+f}$, and subcell ace contains edges corresponding to $\overline{a+c}$ and $\overline{b+f} = \overline{c+e}$.

Remark. The cases (2)–(4) of dual 3-cells have certain linear relations between vertices. These relations force linear relations between parity classes as well.

For example, in case (2), the relation $a+c = b+d$ implies that $\overline{a+d} = \overline{b+c}$ and $\overline{a+b} = \overline{c+d}$, so the set $X(D)$ can be written in an equivalent way as

$$X(D) = O\left(\overline{s+a}, \overline{s+b}, \overline{b+c}, \overline{s+c}, \overline{s+d}, \overline{c+d}\right).$$

Similarly, in case (3), the relations $a+d = b+e = c+f$ between vertices of D imply the following relations between parity classes: $\overline{a+b} = \overline{d+e}$, $\overline{a+c} = \overline{d+f}$, $\overline{b+c} = \overline{e+f}$, $\overline{a+e} = \overline{b+d}$, $\overline{a+f} = \overline{c+d}$, $\overline{b+f} = \overline{c+e}$.

Finally, in case (4), the relations $a - a' = b - b' = c - c'$ imply $\overline{a+b} = \overline{a'+b'}$, $\overline{a+c} = \overline{a'+c'}$, $\overline{b+c} = \overline{b'+c'}$.

In the following, we may switch between equivalent parity classes without saying it explicitly.

For further simplicity, we identify the vertices of $Ven(P)$ with their labels.

Let us recall the definition of Venkov graphs.

DEFINITION 3.2 (the Venkov graph; see, for instance, [20]). *Let P be a parallelohedron. Set*

$$V := \{\{F, -F\} \mid F \text{ is a facet of } P\}.$$

In other words, V is the set of pairs of opposite facets of P. Let now $\{F, -F\}$ and $\{F', -F'\}$ be two distinct elements of V. We say that

- $\big\{\{F, -F\}, \{F', -F'\}\big\} \in E_b$ *if $F \cap F'$ is a nonprimitive $(d-2)$-face of $\mathcal{T}(P)$;*
- $\big\{\{F, -F\}, \{F', -F'\}\big\} \in E_r$ *if either $F \cap F'$ or $F \cap (-F')$ is a primitive $(d-2)$-face of $\mathcal{T}(P)$.*

Then $VG(P) := (V, E_b \cup E_r)$ is called the Venkov graph *of P, and $VG_r(P) := (V, E_r)$ (respectively, $VG_b(P) := (V, E_b)$) is the red (respectively,* blue*)* Venkov graph *of P.*

The next definition establishes a correspondence between the Venkov complex and the Venkov graph of a parallelohedron.

DEFINITION 3.3. *Given a parallelohedron P of dimension $d \geq 4$, let a map*

$$\varphi : vert(Ven(P)) \to vert(VG_r(P))$$

be defined as follows. For each $x \in vert(Ven(P))$ (and thus satisfying $x \in \Lambda(P)/2\Lambda(P)$) we set $\varphi(x) := \{F, -F\}$ if $F = P \cap (P + a)$, where x is the parity class of a.

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Remark. Each parity class of $\Lambda(P)$ (i.e., a coset in $\Lambda(P)/2\Lambda(P)$) either contains two opposite facet vectors of P or does not contain facet vectors. From the definition of $Ven(P)$ it follows immediately that each parity class $\xi \in vert(Ven(P))$ contains exactly two facet vectors, which, in turn, define the pair $(F, -F)$ uniquely.

LEMMA 3.4. *The map φ from Definition* 3.3 *has the following properties:*
1. *φ is injective.*
2. *φ induces a bijection between the edge sets of $VG_r(P)$ and $Ven(P)$.*

Proof. Assertion (1) holds since distinct parity classes of $\Lambda(P)$ determine distinct pairs of opposite facets.

Assertion (2) is proved by verifying the properties (a) and (b) below.

(a) If $\{x, y\}$ is an edge of $Ven(P)$, then there is a triangular cell D with two edges representing parity classes x and y. In that case $\{\varphi(x), \varphi(y)\}$ is an edge of $VG_r(P)$ because the face of $\mathcal{T}(P)$ associated with D is a primitive $(d-2)$-face.

(b) If $\{\{F, -F\}, \{F', -F'\}\}$ is an edge of $VG_r(P)$, then $\varphi^{-1}(\{F, -F\})$, and $\varphi^{-1}(\{F', -F'\})$ exist. Moreover, they are connected with an edge of $Ven(P)$. In order to verify this property, assume, with no loss of generality, that $F \cap F'$ is a primitive $(d-2)$-face of P. Then the property (b) is immediate by considering the set of triples $X(D(G))$, where G is an arbitrary $(d-3)$-subface of $F \cap F'$ and $D(G)$ is the dual 3-cell of G. ∎

By the following corollary, the Venkov complex is, in a sense, a 2-dimensional extension of the red Venkov graph, which justifies our terminology.

COROLLARY 3.5. *Let P be a parallelohedron of dimension $d \geq 4$. Then red Venkov graph $VG_r(P)$ can be obtained by adding a finite number (possibly zero) of isolated vertices to the* 1*-dimensional skeleton of $Ven(P)$.*

Remark. The number of additional isolated vertices in $VG_r(P)$ equals the number of 1-dimensional summands in the representation of P as a direct sum of irreducible parallelohedra; see [20].

There is a natural correspondence between closed paths on the π-surface and closed circuits on the 1-skeleton of $Ven(P)$, as explained by the lemma below.

LEMMA 3.6. *Let $x_1, x_2, \ldots, x_k \in \Lambda(P)/2\Lambda(P)$ be such that $x_1x_2, \ldots, x_kx_1$ is a closed path over the edges of $Ven(P)$. There exists a generic closed path γ on P_π whose lift γ_δ onto P_δ satisfies*

$$\langle\gamma_\delta\rangle = [F_1, F_2, \ldots, F_{k+1}], \quad \textit{where } F_1, F_{k+1} \in \varphi(x_1) \textit{ and } F_i \in \varphi(x_i) \textit{ for } i = 2, 3, \ldots, k.$$

Proof. Take $F_1 \in \varphi(x_1)$ arbitrarily. For convenience, set $x_{k+1} := x_1$. Then, consecutively for $i = 2, 3, \ldots, k+1$, we can choose $F_i \in \varphi(x_i)$ so that F_i is adjacent to F_{i-1} by a primitive $(d-2)$-face. In particular, we get $F_{k+1} \in \varphi(x_1)$, i.e., $F_{k+1} = \pm F_1$, and hence it is possible to construct γ_δ so that $\langle\gamma_\delta\rangle = [F_1, F_2, \ldots, F_{k+1}]$ so that the endpoints of γ_δ either coincide or are antipodal to each other. This means that γ, the image of γ_δ under the natural projection $P_\delta \to P_\pi$, is a closed path. ∎

Remark. It is convenient to assume, by definition, that a path on P_π that does not cross any $(d-2)$-face corresponds to an empty cycle in the Venkov graph (i.e., a cycle with no vertices and no edges). Then every generic closed path on P_π corresponds to a unique (not necessarily simple) cycle in the Venkov graph, and every cycle in the Venkov graph corresponds to a nonempty class of generic closed paths on P_π.

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Now we turn to the 2-dimensional structure of $Ven(P)$. Before we turn to the proof of the main result of this section, let us establish some important relations between the subcomplexes $X(D)$ and basic circuits in P_π (i.e., half-belt, trivially contractible, and Ordine).

LEMMA 3.7. *With correspondence understood in the sense of Lemma* 3.6, *the following assertions hold:*

(i) *Let D be a dual* 3*-cell combinatorially equivalent to a tetrahedron. Then four triangles of the octahedron $X(D) \subseteq Ven(P)$ correspond to half-belt cycles, while four other triangles correspond to trivially contractible cycles in P_π.*

(ii) *Let D be a dual* 3*-cell combinatorially equivalent to a quadrangular pyramid. Then four triangles of the octahedron $X(D) \subseteq Ven(P)$ correspond to half-belt cycles, four other triangles correspond to Ordine cycles, and one equator of $X(D)$ corresponds to a trivially contractible cycle in P_π.*

(iii) *Let D be a dual* 3*-cell combinatorially equivalent to an octahedron. Then four triangles of the octahedron $X(D) \subseteq Ven(P)$ (which have no pairwise common edges) correspond to half-belt cycles, and all three equators of $X(D)$ correspond to trivially contractible cycles in P_π.*

(iv) *Let D be a dual* 3*-cell combinatorially equivalent to a triangular prism. Then $X(D)$ is a triangle that corresponds to a half-belt cycle.*

In all assertions (i)–(iv) *"triangle" means a three-edge circuit around a triangular face of $Ven(P)$, and "equator" means a four-edge circuit around an equator of an octahedron.*

Proof. Let a vector $x \in \Lambda(P)$ be such that $P \cap (P + x)$ is a facet of both P and $P + x$. Then this facet will be denoted by $F(x)$.

We prove each assertion separately.

(i) Let $\{a, b, c, d\}$ be the vertices of D. Then a, b, and c span a triangular dual cell in the dual complex $\mathcal{D}(P)$. Consequently, the facets $F(\pm(a-b))$, $F(\pm(a-c))$, and $F(\pm(b-c))$ all exist and span a 6-belt of P. Hence, the triangle $\{\overline{a+b}, \overline{a+c}, \overline{b+c}\}$ corresponds to a half-belt cycle of P. A similar argument applies to triangles $\{\overline{a+b}, \overline{a+d}, \overline{b+d}\}$, $\{\overline{a+c}, \overline{a+d}, \overline{c+d}\}$, and $\{\overline{b+c}, \overline{b+d}, \overline{c+d}\}$.

Further, let G be the $(d-3)$-face of $T(P)$ that is dual to D. Then $G - a$ is a $(d-3)$-face of $T(P)$, and its dual 3-cell is a combinatorial tetrahedron spanned by $\{\mathbf{0}, b-a, c-a, d-a\}$. Since the dual of $G - a$ has $\mathbf{0}$ as one of its vertices, $G - a$ is a face of P. In addition, the facets that surround $G - a$ are $F(b-a)$, $F(c-a)$, and $F(d-a)$. These facets form a trivially contractible cycle that corresponds to the triangle $\{\overline{a+b}, \overline{a+c}, \overline{a+d}\}$. A similar argument applies to triangles $\{\overline{a+b}, \overline{b+c}, \overline{b+d}\}$, $\{\overline{a+c}, \overline{b+c}, \overline{c+d}\}$, and $\{\overline{a+d}, \overline{b+d}, \overline{c+d}\}$.

(ii) Let $\{s, a, b, c, d\}$ be the vertices of D, where s is the apex of the pyramid and $[a, b, c, d]$ is the natural cyclic order on the base. Then the triples $\{s, a, b\}$, $\{s, b, c\}$, $\{s, c, d\}$, and $\{s, a, d\}$ span triangular dual 2-cells in $\mathcal{D}(P)$. In a manner similar to the proof of assertion (i) we establish that the triangles $\{\overline{s+a}, \overline{s+b}, \overline{a+b}\}$, $\{\overline{s+b}, \overline{a+c}, \overline{b+c}\}$, $\{\overline{s+c}, \overline{s+d}, \overline{c+d}\}$, and $\{\overline{s+a}, \overline{s+d}, \overline{a+d}\}$ correspond to half-belt cycles. Since $a+b+c+d \in 2\Lambda$, we have $\{\overline{s+b}, \overline{a+c}, \overline{b+c}\} = \{\overline{s+b}, \overline{a+c}, \overline{a+d}\}$ and $\{\overline{s+c}, \overline{s+d}, \overline{c+d}\} = \{\overline{s+c}, \overline{s+d}, \overline{a+b}\}$. Consequently, all four triangles indeed belong to $X(D)$.

Further, let G be the $(d-3)$-face of $T(P)$ that is dual to D. Then $G - s$ is a $(d-3)$-face of $T(P)$, and its dual 3-cell is a combinatorial quadrangular pyramid spanned by $\{\mathbf{0}, a-s, b-s, c-s, d-s\}$. Since the dual of $G - s$ has $\mathbf{0}$ as one of its vertices, $G - s$ is a face of P. In addition, the facets that surround $G - s$ are $F(a-s)$,

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$F(b-s)$, $F(c-s)$, and $F(d-s)$ in that cyclic order. These facets form a trivially contractible cycle that corresponds to the equator $[\overline{s+a}, \overline{s+b}, \overline{s+c}, \overline{s+d}]$.

Finally, P corresponds to the apex $\mathbf{0}$ of the pyramidal dual 3-cell spanned by $\{\mathbf{0}, a-s, b-s, c-s, d-s\}$. Consequently, the following four cycles are Ordine by definition: $[F(a-s), F(b-s), F(a-d)]$, $[F(b-s), F(c-s), F(b-a)]$, $[F(c-s), F(d-s), F(d-a)]$ and $[F(d-s), F(a-s), F(a-b)]$. They correspond to the triangles $\{\overline{s+a}, \overline{s+b}, \overline{a+d}\}$, $\{\overline{s+b}, \overline{s+c}, \overline{a+b}\}$, $\{\overline{s+c}, \overline{s+d}, \overline{a+d}\}$, and $\{\overline{s+a}, \overline{s+d}, \overline{a+b}\}$, respectively.

(iii) Let $\{a, b, c, d, e, f\}$ be the vertices of D, where $a+d = b+e = c+f$. The four triples $\{a, b, c\}$, $\{a, b, f\}$, $\{a, c, e\}$, and $\{a, e, f\}$ span triangular dual 2-cells in $\mathcal{D}(P)$. In a manner similar to the proof of assertion (i) we establish that the triangles $\{\overline{a+b}, \overline{a+c}, \overline{b+c}\}$, $\{\overline{a+b}, \overline{a+f}, \overline{b+f}\}$, $\{\overline{a+c}, \overline{a+e}, \overline{c+e}\}$, and $\{\overline{a+e}, \overline{a+f}, \overline{e+f}\}$ correspond to half-belt cycles. Since $b+c+e+f \in 2\Lambda$, we have $\{\overline{a+c}, \overline{a+e}, \overline{c+e}\} = \{\overline{a+c}, \overline{a+e}, \overline{b+f}\}$ and $\{\overline{a+e}, \overline{a+f}, \overline{e+f}\} = \{\overline{a+e}, \overline{a+f}, \overline{b+c}\}$. Consequently, all four triangles indeed belong to $X(D)$.

Further, let G be the $(d-3)$-face of $T(P)$ that is dual to D. Then $G-a$ is a $(d-3)$-face of $T(P)$, and its dual 3-cell is a combinatorial octahedron spanned by $\{\mathbf{0}, b-a, c-a, d-a, e-a, f-a\}$. Since the dual 3-cell of $G-a$ has $\mathbf{0}$ as one of its vertices, $G-a$ is a face of P. In addition, the facets that surround $G-a$ are $F(b-a)$, $F(c-a)$, $F(e-a)$, and $F(f-a)$, in that cyclic order. These facets form a trivially contractible cycle that corresponds to the equator $[\overline{a+b}, \overline{a+c}, \overline{a+e}, \overline{a+f}]$. The correspondence between the other two equators of $X(D)$ and their respective trivially contractible cycles (namely, the ones around the $(d-3)$-faces $G-b$ and $G-c$) is shown in a similar way.

(iv) Let $\{a, b, c, a', b', c'\}$ be the vertices of D, where $a-a' = b-b' = c-c'$. The triple $\{a, b, c\}$ spans a triangular dual 2-cell in $\mathcal{D}(P)$. In a manner similar to the proof of assertion (i) we establish that the only triangle in $X(D)$, i.e., $\{\overline{a+b}, \overline{a+c}, \overline{b+c}\}$, corresponds to a half-belt cycle. ∎

We make use of Lemma 3.7 via the following application.

LEMMA 3.8. *Let a triangle T be a 2-dimensional face of $Ven(P)$. Let γ be a path in P_π corresponding to the three-edge circuit around T in the sense of Lemma* 3.6. *Then γ is either half-belt, trivially contractible, or Ordine, or the square of γ is a product of such cycles in the (singular) first homology group of P_π over the rational (or real) numbers.*

Proof. By the construction of the Venkov complex, there exists a dual 3-cell $D \in \mathcal{D}(P)$ such that $T \in X(D)$.

Since $X(D)$ is not empty, D cannot be a combinatorial cube.

If D is combinatorially equivalent to a tetrahedron, a quadrangular pyramid, or a triangular prism, then, by Lemma 3.7, T corresponds either to a half-belt, to a trivially contractible, or to an Ordine cycle.

If D is combinatorially equivalent to an octahedron, then T may correspond to a half-belt cycle. If this is not the case, then we may assume that $T = \{a+b, a+c, b+f\}$ in the notation of Definition 3.1 (3). This assumption does not lead to any loss of generality since the remaining cases are equivalent up to an appropriate permutation of a, b, c, d, e, f.

Whenever $[x_1, x_2, \ldots, x_k, x_1]$ ($x_i \in \Lambda(P)/2\Lambda(P)$) is a closed path over edges of $Ven(P)$, let $\gamma(x_1, x_2, \ldots, x_k)$ denote the closed generic path in P_π corresponding to $[x_1, x_2, \ldots, x_k, x_1]$ in the sense of Lemma 3.6. We also denote $x := \overline{a+b}$, $y = \overline{a+c}$, $z = \overline{b+f}$, $u = \overline{a+e}$, $v = \overline{a+f}$,$w = \overline{b+c}$.

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We claim that the following combination represents the zero class in the (singular) first homology of P_π:

$$2\gamma(x,y,z)-\gamma(x,y,w)-\gamma(y,z,u)-\gamma(z,x,v)\\ -\gamma(x,y,u,v)-\gamma(y,z,v,w)-\gamma(z,x,w,u)+\gamma(u,v,w).$$

Our claim follows from the fact that each primitive $(d-2)$-face is crossed equally many times in both directions by the combination above. But $\gamma(x,y,w)$, $\gamma(y,z,u)$, $\gamma(z,x,w)$, and $\gamma(u,v,w)$ are half-belt cycles by Lemma 3.7, and, finally, $\gamma(x,y,u,v)$, $\gamma(y,z,v,w)$, and $\gamma(z,x,w,u)$ are trivially contractible cycles according to the same Lemma 3.7. This gives the desired representation to the path $\gamma(x,y,z)$ that corresponds to the three-edge path around T. ∎

We are ready to proceed toward the main result of this section—a sufficient condition for a parallelohedron to be affinely Voronoi in terms of its Venkov complex. From now on we use the notation $C_k(K,R)$ (respectively, $C^k(K,R)$) for the spaces of chains (respectively, cochains) of a simplicial complex K with coefficients in a commutative ring R.

THEOREM 3.9 (the Venkov graph condition). *Let P be a parallelohedron of dimension $d \geq 4$. If the first cohomology group $H^1(Ven(P),\mathbb{R})$ is trivial, then P is affinely Voronoi.*

Proof. Assume that P is not affinely Voronoi. It suffices to construct a nontrivial cohomology class in $H^1(Ven(P),\mathbb{R})$. Equivalently, we will construct a cochain $c \in C^1(Ven(P),\mathbb{R})$ such that the coboundary operator δ vanishes on c, but at the same time c is not a coboundary itself (i.e., $c \neq \delta c'$ for any $c' \in C^0(Ven(P),\mathbb{R})$).

Let $x_1, x_2 \in vert(Ven(P))$ be such that $\{x_1,x_2\}$ is an edge of $Ven(P)$. For $i=1,2$ let $\{F_i,-F_i\}=\varphi(x_i)$. Then F_1 is adjacent either to F_2, or to $-F_2$ by a primitive $(d-2)$-face of P. We then set

$$\langle c,\overrightarrow{x_1x_2}\rangle := \ln g(F_1,F_2) \quad \text{or} \quad \langle c,\overrightarrow{x_1x_2}\rangle := \ln g(F_1,-F_2), \tag{3.1}$$

respectively.

Let us prove that c is a cocycle. Consider an arbitrary 2-dimensional face $\{x_1,x_2,x_3\}$ of $Ven(P)$. Applying Lemma 3.8 to $\{x_1,x_2,x_3\}$ yields a generic closed path γ on P_π. We have

$$\langle \delta c,\{x_1,x_2,x_3\}\rangle = \langle c,\partial\{x_1,x_2,x_3\}\rangle = \langle c,\overrightarrow{x_1x_2}\rangle + \langle c,\overrightarrow{x_2x_3}\rangle + \langle c,\overrightarrow{x_3x_1}\rangle = \ln g(\gamma) = 0,$$

where the last identity is a consequence of Lemma 2.2. Since δc vanishes on every 2-face of $Ven(P)$, c is indeed a cocycle.

In turn, c is not a coboundary. Indeed, by Proposition 2.1, there exists a closed path γ on P_π such that $g(\gamma)\neq 1$. Let γ_δ be a lift of γ onto P_δ. Suppose that $\langle\gamma_\delta\rangle = [F_1,F_2,\dots,F_k]$. Then

$$[\{F_1,-F_1\},\{F_2,-F_2\},\dots,\{F_k,-F_k\}=\{F_1,-F_1\}]$$

is a cycle in $VG_r(P)$. Therefore, $x_i := \varphi^{-1}(\{F_i,-F_i\})$ exists for each $i=1,2,\dots,k$, and $x_1 = x_k$. Consequently,

$$\sum_{i=1}^{k-1}\langle c,\overrightarrow{x_ix_{i+1}}\rangle = \sum_{i=1}^{k-1}(\langle c',x_{i+1}\rangle - \langle c',x_i\rangle) = 0.$$

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On the other hand,

$$\sum_{i=1}^{k-1} \langle c, \overrightarrow{x_i x_{i+1}} \rangle = \sum_{i=1}^{k-1} \ln g(F_i, F_{i+1}) = \ln g(\gamma) \neq 0,$$

from which we infer that c is not a coboundary. This concludes the proof. ∎

4. Graph approach. In this section we present a method to verify the GGM condition for a single parallelohedron P. This approach, using the group of cycles of the red Venkov graph $VG_r(P)$, was proposed in [13].

We recall that Lemma 3.6 establishes a natural correspondence between closed paths on P_π and (not necessarily) simple cycles in the Venkov graph (equivalently, the 1-skeleton of $Ven(P)$). Throughout this section the term *correspondence* will refer to the correspondence in the sense of Lemma 3.6, unless a different meaning is given explicitly.

DEFINITION 4.1. *A cycle $c \in C_1(Ven(P))$ is called a* combinatorial half-belt cycle *(respectively,* combinatorial trivially contractible cycle *or* combinatorial Ordine cycle*) if it corresponds to a half-belt (respectively, trivially contractible or Ordine) cycle γ on P_π.*

Now we are ready to reformulate the GGM condition in terms of the Venkov complex.

LEMMA 4.2. *The following assertions are equivalent:*

(i) *The group $H_1(P_\pi, \mathbb{Q})$ is generated by half-belt cycles.*

(ii) *The implication $(A_1(c) \wedge A_2(c)) \Rightarrow B(c)$ holds for all cochains $c \in C^1(Ven(P), \mathbb{Q})$, where*

$$\begin{aligned} A_1(c) &:= \big[\langle c, \gamma \rangle = 0 \text{ for everyhalf-belt cycle } \gamma \in C_1(Ven(P), \mathbb{Q})\big], \\ A_2(c) &:= \big[\langle c, \gamma \rangle = 0 \text{ for every trivially contractible cycle } \gamma \in C_1(Ven(P), \mathbb{Q})\big], \\ B(c) &:= \big[\langle c, \gamma \rangle = 0 \text{ for every } \gamma \in C_1(Ven(P), \mathbb{Q}) \text{ satisfying } \partial\gamma = 0\big]. \end{aligned}$$

Remark. The condition $B(c)$ is equivalent to the property that c is a coboundary.

Proof of Lemma 4.2. (i) $\Rightarrow$ (ii). Let P be a parallelohedron P satisfying (i). Consider an arbitrary cochain $c \in C^1(Ven(P), \mathbb{Q})$ for which both $A_1(c)$ and $A_2(c)$ are true. We claim that $B(c)$ is true as well.

Let $\gamma_\pi \subset P_\pi$ be a closed generic curve. By Lemma 3.6, γ_π corresponds to some cycle $\gamma \in C_1(Ven(P), \mathbb{Q})$. We then set

$$c^*(\gamma_\pi) := \langle c, \gamma \rangle.$$

Since $A_2(c)$ holds, the value of $c^*(\gamma_\pi)$ depends only on the homotopy type of γ_π. Therefore, c^* acts as a map $c^* : \pi_1(P_\pi) \to \mathbb{Q}$. By construction c^* is a homomorphism and vanishes on the commutator of $\pi_1(P_\pi)$, and its image lies in the field $\mathbb{Q}$ of characteristic zero. Consequently, the action of c^* on the group $H_1(P_\pi, \mathbb{Q})$ is also well defined. By the property $A_1(c)$, all half-belt cycles lie in the kernel of c^*. Using (i), we conclude that c^* acts on $H_1(P_\pi, \mathbb{Q})$ trivially, which is only possible if c is a coboundary. Hence, the implication (i) $\Rightarrow$ (ii) holds.

(ii) $\Rightarrow$ (i). Assume that (i) is false for P. Let G be the proper subgroup of $H_1(P_\pi, \mathbb{Q})$ generated by half-belt cycles. Let $\ell : H_1(P_\pi, \mathbb{Q}) \to \mathbb{Q}$ be a linear map such that $G \subseteq Ker\, \ell \subsetneq H_1(P_\pi, \mathbb{Q})$.

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Denote the set of all coboundary elements of $C_1(Ven(P), \mathbb{Q})$ as

$$\bar{C}_1(Ven(P), \mathbb{Q}) := \{\gamma \in C_1(Ven(P), \mathbb{Q}) : \partial\gamma = 0\}.$$

Let $\gamma \in \bar{C}_1(Ven(P), \mathbb{Q})$. Then there exists $n \in \mathbb{N}$ such that $n\gamma = \sum_{i=1}^{s} \gamma^i$, where each $\gamma^i = [x_1^i, x_2^i, \ldots, x_{k_i}^i = x_1^i]$ is a closed circuit on $Ven(P)$. For each γ^i there exists a corresponding closed generic path γ_π^i. Let h^i be the homology class of γ_π^i.

By setting

$$\langle \ell_*, \gamma \rangle := \frac{1}{n} \sum_{i=0}^{s} \langle \ell, h^i \rangle,$$

we obtain a linear map $\ell_* : \bar{C}_1(Ven(P), \mathbb{Q}) \to \mathbb{Q}$. Let $c \in C^1(Ven(P), \mathbb{Q})$ be an arbitrary continuation of ℓ_* onto $C_1(Ven(P), \mathbb{Q})$. Clearly, c satisfies conditions $A_1(c)$ and $A_2(c)$ but not $B(c)$. ∎

Condition (ii) of the previous lemma can be stated in terms of the group of cycles of the red Venkov graph $VG_r(P)$. Recall that, by Corollary 3.5, the edge structure of $VG_r(P)$ and the edge structure of the Venkov complex $Ven(P)$ are isomorphic.

Definition 4.3. *A cycle of $VG_r(P)$ that corresponds to either a half-belt or a trivially contractible cycle on P_π is called a* basic cycle. *The set of all basic cycles is denoted by $\mathcal{C}(P)$.*

Definition 4.4. *If we treat a finite (nondirected) graph G as a 1-dimensional simplicial complex, then the group $H_1(G, \mathbb{Q})$ is called* the group of cycles *of G.*

Remark. The group of cycles of G is a free abelian group (or a linear space over $\mathbb{Q}$) of rank $e - v + k$, where e is the number of edges, v is the number of vertices, and k is the number of connected components of G.

Then we can reformulate Lemma 4.2 in the following way.

Lemma 4.5. *The group $H_1(P_\pi, \mathbb{Q})$ is generated by half-belt cycles if and only if the group of cycles of the red Venkov graph of P is generated by $\mathcal{C}(P)$.*

Proof. Since condition (ii) of Lemma 4.2 does not use 2-dimensional simplices of $Ven(P)$, we can substitute $Ven(P)$ with the red Venkov graph in it. Then the implication $\big(A_1(c) \wedge A_2(c)\big) \Rightarrow B(c)$ means that the rank of the subgroup generated by $\mathcal{C}(P)$ is equal to the rank of the group of cycles of $VG_r(P)$, and $\mathcal{C}(P)$ generates the group of cycles. ∎

5. Computational results. This section describes the computer-assisted verification of Theorems 5.1 and 5.2 below. We note that both the GGM and the Venkov graph conditions are, in general, only sufficient conditions for the Voronoi conjecture. But whenever any of these conditions gets verified for *all* combinatorially Voronoi parallelohedra of a certain dimension d, this condition becomes necessary and sufficient for the Voronoi conjecture in $\mathbb{R}^d$. We show that in $\mathbb{R}^5$ this is the case for both conditions; essentially, we verify each of them for all 110 244 types of 5-dimensional Voronoi parallelohedra obtained in [5].

We note that the computer verification of Theorem 5.1 is redundant since, by Corollary 6.4 below, it immediately follows from the verification of Theorem 5.2. However, chronologically, we first verified Theorem 5.1, and the two verifications are independent other than using the same initial input. Moreover, both implementations can be used for parallelohedra in higher dimensions, and it may happen that the verification of GGM condition that uses the graph $VG_r(P)$ will be computationally

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more accessible than the Venkov graph condition despite the fact that the Venkov graph condition might hold for a wider class of parallelohedra. We summarize this discussion in section 7.

THEOREM 5.1. *Let a* 5*-dimensional parallelohedron* P *be equivalent to some* 5*-dimensional Voronoi parallelohedron. Then the cohomology group* $H^1(Ven(P), \mathbb{R})$ *is trivial.*

THEOREM 5.2. *Let a* 5*-dimensional parallelohedron* P *be equivalent to some* 5*-dimensional Voronoi parallelohedron. Then the GGM condition holds for* P*.*

Theorem 1.3 is an immediate corollary of each of the above theorems, as explained below.

Proof of Theorem 1.3. The "only if" part is straightforward.

The "if" part is a combination of either Theorems 5.1 and 3.9 or Theorems 5.2 and 2.4. ∎

It is sufficient to verify the conclusions of Theorems 5.1 and 5.2 for a single representative of each equivalence class of 5-dimensional Voronoi parallelohedra.

The list of representatives is available due to the algorithm of [5]. Each representative P_i $(1 \le i \le 110\,244)$ is presented in two ways:

1. As a cell of $\mathbf{0}$ in the Voronoi tessellation for $\mathbb{Z}^5$, where the metric is given by an explicit quadratic form Q_i, i.e., $\|a\|^2 = Q_i(a, a)$. Q_i is presented by its 5×5 matrix with integer entries.
2. As a convex hull of a set of vertices given explicitly by listing the coordinates. All coordinates are rational numbers. Additionally, every face of P_i is described by listing its vertices.

We note that the second representation can be obtained from the first one, though it may be computationally difficult. The second representation may give rise to multiple options for the quadratic forms Q_i for a single P_i (and, consequently, for multiple first representations) in case P_i is a direct sum of two parallelohedra of smaller dimensions. For example, the square with vertices $(\pm\frac{1}{2}, \pm\frac{1}{2})$ is the Voronoi cell for metrics defined by quadratic forms $ax^2 + by^2$ for all $a, b > 0$.

Our first goal is to compute the dual complex $\mathcal{D}(P_i)$, which is, in this case, a Delaunay tessellation $\mathcal{D}(\mathbb{Z}^5, Q_i)$ of $\mathbb{Z}^5$ defined by the form Q_i. The whole complex is infinite; however, it is invariant under the action of $\mathbb{Z}^5$ by translations, and the action has finitely many orbits. Providing a single representative from each orbit completely determines the complex $\mathcal{D}(P_i)$.

In order to reconstruct the Venkov complex $Ven(P_i)$, a single representative from each translational class of dual 3-cells is sufficient since $X(D) = X(D')$ when D and D' are translationally equivalent. Determining basic cycles of $VG_r(P_i)$ as in section 4 involves more dual 3-cells, namely, all that contain $\mathbf{0}$ as one of its vertices. Nevertheless, this set is also finite and can be easily reconstructed from the list of translational classes of dual 3-cells.

We use two different approaches. The direct approach uses the algorithm of [6], which is available in [4]. The second approach uses the following proposition.

PROPOSITION 5.3 (see, for instance, [20]). *The following assertions hold:*

1. *Let* G *be a face of parallelohedron* P*. Then*

$$D(G) = \{-v \mid v \in \Lambda(P) \text{ and } G + v \text{ is a face of } P\}.$$

2. D *is a dual* k*-cell of* $\mathcal{D}(P)$ *if and only if* $D = D(G) + v$*, where* G *is a* $(d-k)$*-face of* P *and* $v \in \Lambda(P)$*.*

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Using Proposition 5.3, one can compute $\mathcal{D}(P_i)$ from the vertex representation of P_i. All computations are performed over the field of rationals; therefore, we are not concerned about the issues with machine precision.

Now we address Theorems 5.1 and 5.2 separately.

For Theorem 5.1, we use $\mathcal{D}(P_i)$ to construct the simplicial complex $Ven(P_i)$. After that, we check the triviality of $H^1(Ven(P_i), \mathbb{R})$ by verifying the identity

$$\dim(\operatorname{Im}\delta_0) = \dim(\operatorname{Ker}\delta_1),$$

where δ_0 and δ_1 are restrictions of the coboundary operator δ to the spaces $C^0(Ven(P_i), \mathbb{R})$ and $C^1(Ven(P), \mathbb{R})$, respectively.

But $\dim(\operatorname{Im}\delta_0) = \operatorname{rank}(\delta_0)$ and $\dim(\operatorname{Ker}\delta_1) = f_1(Ven(P_i)) - \operatorname{rank}(\delta_1)$. Therefore, the condition of Theorem 3.9 is equivalent to the identity

$$\operatorname{rank}(\delta_0) + \operatorname{rank}(\delta_1) - f_1(Ven(P_i)) = 0. \tag{5.1}$$

The identity (5.1) is verified by running a `GAP` [11] program. We use the list of all 2-dimensional faces of $Ven(P_i)$ as the input, then compute the left-hand side of (5.1) and pass the result to the output. Most of the computation uses the `simpcomp` package [7], except for the rank function, which is in the core `GAP`.

The scripts that process the vertex representation of 5-dimensional parallelohedra into a `GAP` program are available on the Web page [26]. It takes on average about 1 second per 5-dimensional parallelohedron to convert the vertex representation into a list of dual 3-cells. Processing the dual cell data into a `GAP` program and running that program is significantly faster.

Similarly, for Theorem 5.2, the dual complex $\mathcal{D}(P_i)$ is used to construct the graph $VG_r(P_i)$ and to determine the half-belt and the trivially contractible cycles.

The group of cycles of $VG_r(P_i)$ has rank $e - v + k$, where e is the number of edges, v is the number of vertices, and k is the number of connected components. Therefore, by Lemma 4.5, it is sufficient to verify that the $\mathbb{Q}$-rank of the set $\mathcal{C}(P_i)$ equals $e - v + k$.

This approach is implemented using `SAGE` in [26] with the input data given as all dual cells incident to the origin; the running time is about 1–2 seconds per each 5-dimensional parallelohedron on an 8-year-old laptop. We note that the condition is only checked for parallelohedra with connected $VG_r(P_i)$, i.e., if $k = 1$, and for parallelohedra that have at least one nontetrahedral or nonoctahedral dual 3-cell. If $k \geq 2$, then the corresponding P_i is a direct sum of two parallelohedra of smaller dimensions, and the GGM condition is inherited from these summands. If dual 3-cells of P_i are tetrahedra and octahedra only, then $(P_i)_\delta$ is $\mathbb{S}^4$, and the GGM condition is trivially true for such P_i; the Voronoi conjecture for such parallelohedra is proved by Žitomirskiĭ (spelled in the source as Zitomirskij) in [25].

6. Relations between sufficient conditions. In this section we show that the cohomology condition generalizes both Ordine's 3-irreducibility condition [20] and the GGM condition. That is, we show that if a d-dimensional parallelohedron P satisfies either the GGM condition or the Ordine condition, then $H^1(Ven(P), \mathbb{R})$ is trivial; thus, P satisfies the cohomology condition as well.

First, we introduce the Ordine condition, which is also called the *3-irreducibility condition*. A d-dimensional parallelohedron P is 3-irreducible if no 3-cell of the dual complex $\mathcal{D}(P)$ is equivalent to a prism or a cube. This condition has been put forward by Ordine [20], who proved that every 3-irreducible parallelohedron of dimension at least 5 is affinely Voronoi.

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LEMMA 6.1. *Let P be a 3-irreducible parallelohedron of dimension $d \geq 5$. Then the group $H^1(Ven(P), \mathbb{R})$ is trivial.*

Proof. Let $c \in C^1(Ven(P), \mathbb{R})$ be a cocycle. We need to prove that c is a coboundary.

Let F and F' be two facets of P such that $F \cap F'$ is a primitive $(d-2)$-face. Then there is an edge between $\{F, -F\}$ and $\{F', -F'\}$ in $VG_r(P)$. Consequently, $\overrightarrow{xx'}$, where $x := \varphi^{-1}(\{F, -F\})$ and $x' := \varphi^{-1}(\{F', -F'\})$, is an oriented edge of $Ven(P)$.

Set

$$g(F, F') := \exp\left(c(\overrightarrow{xx'})\right).$$

One can check that plugging g into the argument of [20, Theorem 7] instead of the gain function for P is sufficient to produce an analogue of a canonical scaling. More precisely, there exists a function s mapping facets of P to positive real numbers such that the identities

$$\begin{aligned} s(F') &= s(F)g(F, F') \quad \text{whenever } F \cap F' \text{ is a primitive } (d-2)\text{-face}, \\ s(F) &= s(-F) \quad \text{for all facets } F \end{aligned}$$

are satisfied.

Letting $c'(x) := \ln s(F)$, where $F \in \varphi(x)$, yields $c = \delta c'$. Hence, c is indeed a coboundary. ∎

We proceed by considering the GGM condition. As in section 3, the term *Venkov triangle* will refer to any 2-dimensional face of the Venkov complex.

LEMMA 6.2. *The following assertions hold:*

(i) *Each combinatorial half-belt cycle is a boundary of a Venkov triangle.*

(ii) *Each combinatorial trivially contractible cycle is an integer combination of boundaries of Venkov triangles.*

Proof. Assertion (i). Let $[x_1, x_2, x_3, x_1]$ be a combinatorial half-belt cycle. Then one can choose two facets $F_1 \in \varphi(x_1)$ and $F_2 \in \varphi(x_2)$ so that $F_1 \cap F_2$ is a primitive $(d-2)$-face of P. Choose an arbitrary $(d-3)$-face $G \subset F_1 \cap F_2$. If D is the dual 3-cell of G, then $\{x_1, x_2, x_3\} \in X(D)$, finishing the proof of the assertion.

Assertion (ii). Let γ be a combinatorial trivially contractible cycle around a $(d-3)$-face G. Let D be the dual 3-cell of G. D cannot be combinatorially equivalent to a cube or to a triangular prism, as in these cases trivially contractible cycles do not exist. Hence, D has the combinatorcs of either a tetrahedron, an octahedron, or a quadrangular pyramid. Consequently, $X(D)$ is a combinatorial octahedron. In each case, by Lemma 3.7, γ is either a boundary of a Venkov triangle (when D is a tetrahedron) or an equator of $X(D)$ (when D is either a pyramid or an octahedron).

Since $X(D)$ is a simplicial sphere with all facets being Venkov triangles and since γ is a circuit on that sphere, we conclude that γ is indeed representable as an integer combination of boundaries of Venkov triangles. See also Figure 2 for an explicit representation of an equator of $X(D)$ as a sum of boundaries of Venkov triangles. ∎

LEMMA 6.3. *Let assertion* (ii) *of Lemma* 4.2 *hold. Then the group $H^1(Ven(P), \mathbb{R})$ is trivial.*

Proof. Note that the groups $H^1(Ven(P), \mathbb{R})$ and $H^1(Ven(P), \mathbb{Q})$ are either both trivial or both nontrivial. Therefore, for the rest of the proof we will be working over the field of rationals.

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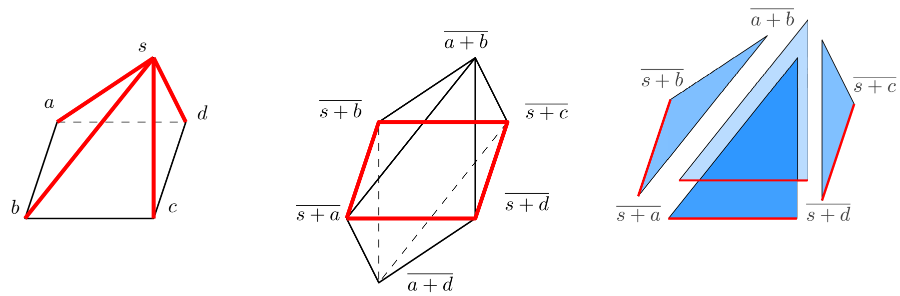

FIG. 2. *Left: The pyramidal dual cell D. The segments sa, sb, sc, and sd (highlighted) are dual to the facets of $\mathcal{T}(P)$ passed by a trivially contractible cycle. Center: The octahedron $X(D)$ and its equator (highlighted) corresponding to the trivially contractible cycle. Right: The equator is represented as a sum of boundaries of four Venkov triangles.*

Let $c \in C^1(Ven(P), \mathbb{R})$ be a 1-cocycle, i.e., $\delta c \equiv 0$. Equivalently, we have

$$\langle c, \partial\tau \rangle = \langle \delta c, \tau \rangle = 0$$

for every Venkov triangle τ.

By Lemma 6.2, every half-belt cycle and every trivially contractible cycle can be represented as a combination of boundaries of Venkov triangles. Hence, in the notation of Lemma 4.2, $A_1(c)$ and $A_2(c)$ hold. Therefore, $B(c)$ holds as well; i.e., c is a coboundary.

We thus conclude that every cocycle in $C^1(Ven(P), \mathbb{Q})$ is a coboundary. Hence, the group $H^1(Ven(P), \mathbb{Q})$ is trivial, and so is the group $H^1(Ven(P), \mathbb{R})$. ∎

This lemma immediately gives the following corollary.

COROLLARY 6.4. *If a parallelohedron P satisfies the GGM condition, then $H^1(Ven(P), \mathbb{R})$ is trivial.*

Proof. The GGM condition is exactly assertion (i) of Lemma 4.2. If it holds for P, then assertion (ii) of Lemma 4.2 also holds for P, and then $H^1(Ven(P), \mathbb{R})$ is trivial. ∎

Combining the results of this section we get the following.

THEOREM 6.5. *If a parallelohedron P satisfies the GGM condition, the* 3-*irreducibility condition, or both, then $H^1(Ven(P), \mathbb{R})$ is trivial.*

7. Concluding remarks.

There is a reasonable question why the approach presented in this paper is limited only to 5-dimensional Voronoi parallelohedra. While for each particular parallelohedron the computations involved in verification of Theorems 5.1 and 5.2 are plausible, the complete classification of d-dimensional Voronoi parallelohedra for $d \geq 6$ looks unreachable at this point without significant improvement or modification of the approach.

The first step of the enumeration presented in [5] is enumeration of all possible lattice Delaunay triangulations or, equivalently, primitive Voronoi parallelohedra in dimension 5. There were attempts to employ a similar enumeration of 6-dimensional Voronoi parallelohedra by Schürmann and Vallentin [21] and by Baburin and Engel [1]. Both implementations were not successful due to the enormous number of triangulations found and terminated for memory reasons. Particularly, the enumeration in [1] found about half a billion nonequivalent primitive Voronoi parallelohedra in dimension 6.

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We conclude the paper by posing an open problem. As noted in Theorem 6.5, if the first statement of this problem is true, then the second one is true as well.

Problem 7.1. *Determine whether the following statements are true or false:*
1. *The GGM condition holds for all Voronoi parallelohedra.*
2. *For every Voronoi parallelohedron P of dimension $d \geq 5$ the cohomology group $H^1(Ven(P), \mathbb{R})$ is trivial.*

If any of the statements of Problem 7.1 holds, then a hypothetical counterexample to the Voronoi conjecture should be nonequivalent to any Voronoi parallelohedron. Additionally, if the Voronoi conjecture is true, then it does not immediately imply that any of these two statements is true; in addition, these statements deal with partial combinatorics of P and $\mathcal{T}(P)$ only and do not take in account geometric properties of P or the corresponding tiling.

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