Number of bounded distance equivalence classes in hulls of repetitive Delone sets

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NUMBER OF BOUNDED DISTANCE EQUIVALENCE CLASSES IN HULLS OF REPETITIVE DELONE SETS

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ABSTRACT. Two Delone sets are bounded distance equivalent to each other if there is a bijection between them such that the distance of corresponding points is uniformly bounded. Bounded distance equivalence is an equivalence relation. We show that the hull of a repetitive Delone set with finite local complexity has either one equivalence class or uncountably many.

1. INTRODUCTION

Delone sets are central objects of study in the theory of aperiodic order and give rise to dynamical systems and topological objects with interesting properties [3]. A *Delone set* $\Lambda \subset \mathbb{R}^d$ is a set that is both *uniformly discrete* (that is, there is $r > 0$ such that each open ball of radius r contains at most one point of Λ) and *relatively dense* (that is, there is $R > 0$ such that each closed ball of radius R contains at least one point of Λ). A *point lattice* is the $\mathbb{Z}$-span $\langle v_1, \ldots, v_d \rangle_{\mathbb{Z}}$ of d linear independent vectors in $\mathbb{R}^d$. Each point lattice is a Delone set. A Delone set Λ in $\mathbb{R}^d$ is *d-periodic* if the set

$$P_\Lambda = \{t \in \mathbb{R}^d \mid t + \Lambda = \Lambda\}$$

of its period vectors is a point lattice. Of course for every point lattice Λ we have $\Lambda = P_\Lambda$, hence each point lattice is d-periodic. A Delone Λ set is *nonperiodic* if $P_\Lambda = \{0\}$.

Two Delone sets Λ, Λ' in $\mathbb{R}^d$ are called *bounded distance equivalent* ($\Lambda \overset{\text{bd}}{\sim} \Lambda'$) or *bde*, if there is a bijection $\varphi \colon \Lambda \to \Lambda'$ such that $|x - \varphi(x)|$ is uniformly bounded. It is easy to see that $\overset{\text{bd}}{\sim}$ is an equivalence relation for Delone sets.

In the 1990s several authors studied the question of whether a given Delone set in $\mathbb{R}^d$ is bounded distant equivalent to a point lattice [6, 7, 16]. In [7] it was shown that any two point lattices in $\mathbb{R}^d$ with the same density are bde. It is a simple consequence that any two d-periodic Delone sets in $\mathbb{R}^d$ with the same density are bde. This leads to considering aperiodic Delone sets, for instance the vertices of a Penrose tiling. There are two well studied classes of aperiodic Delone sets: cut-and-project sets and Delone sets arising from substitution tilings. For details and a precise definition of an aperiodic Delone set, see [3]. Recently, the question of whether an aperiodic Delone set is bde to a point lattice has gained some interest. For the bd-equivalence of cut-and-project sets to a point lattice see

[8, 10, 13, 14, 15]. The bd-equivalence of Delone sets from substitution tilings to a point lattice was studied in [1, 10, 11, 16, 23, 24]. Since the bd-equivalence of Delone sets to a point lattice is well understood, the focus now turns to the question of when two Delone sets are bde.

This is where hulls of nonperiodic Delone sets come into play. In the theory of aperiodic order the *hull* $\mathbb{X}_\Lambda$ of a given Delone set Λ is a central object of interest. The hull of Λ is the closure of the orbit of Λ under translations in the topology defined by the "big box" metric (aka the Gromov-Hausdorff topology).[1] Specifically, the distance between two (r, R)-Delone sets Λ and Λ' is the infimum over all $\varepsilon \in (0, 1)$ such that there exist $x, x' \in \mathbb{R}^d$, each of size $\varepsilon/2$ or less, such that $\Lambda - x$ and $\Lambda' - x'$ agree exactly on $B_{1/\varepsilon}$. If no such ε exists, then $d(\Lambda, \Lambda') = 1$.

The hull may be studied as a topological object [2, 17, 21], or as a dynamical system $(\mathbb{X}_\Lambda, \mathbb{R}^d)$ [3], where $\mathbb{R}^d$ stands for the action of translations by $x \in \mathbb{R}^d$.

This raises the question "How many bde classes does $\mathbb{X}_\Lambda$ contain?". A partial answer was given in [25]: If Λ comes from a substitution tiling meeting some technical conditions, then $\mathbb{X}_\Lambda$ consists of uncountably many bde classes. Here we provide a generalization that covers all Delone sets that are repetitive and have finite local complexity (see definitions below). This category includes the vast majority of the substitution tilings and cut-and-project sets that have been studied to date.

Two geometric properties of the Delone set Λ are closely associated with dynamical properties of $(\mathbb{X}_\Lambda, \mathbb{R}^d)$. A *patch* in a Delone set Λ is a set $\Lambda \cap K$ for some compact $K \subset \mathbb{R}^d$. A Delone set Λ has *finite local complexity* (FLC) if for any compact set $K \subset \mathbb{R}^d$ there are only finitely many different patches $(\Lambda - x) \cap K$ ($x \in \mathbb{R}^d$), up to translation. A Delone set Λ is called *repetitive* if for each compact set $K \subset \mathbb{R}^d$ such that $K \cap \Lambda$ is non-empty, the set

$$\{x \in \mathbb{R}^d \mid (\Lambda - x) \cap K = \Lambda \cap K\}$$

is a Delone set. The uniform density radius R of this Delone set is the *repetitivity radius* of the patch $\Lambda \cap K$. The following fact is the essence of the work of several authors, see [3, 21] for details.

Fact 1. *Let $\Lambda \subset \mathbb{R}^d$ be a Delone set. Then $\mathbb{X}_\Lambda$ is compact if and only if Λ has FLC. If Λ has FLC, then the dynamical system $(\mathbb{X}_\Lambda, \mathbb{R}^d)$ is minimal (that is, each orbit is dense) if and only if Λ is repetitive.*

Now we can state our main result.

Theorem 1.1. *Let Λ be a repetitive Delone set in $\mathbb{R}^d$ having FLC and such that the density of Λ exists. Let $\mathbb{X}_\Lambda$ be the hull of Λ. Then $\mathbb{X}_\Lambda$ either consists of a single bde class or $\mathbb{X}_\Lambda$ contains uncountably many bde classes.*

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[1]Some authors work with a different topology, called the "local rubber" or "Chabauty-Fell" topology. When the Delone set has finite local complexity, defined below, then the two topologies agree.

In fact we show that the number of bde classes is either 1 or $2^{\aleph_0}$, where $\aleph_0$ is the cardinality of $\mathbb{Z}$. It is clear that the number of bde classes can't be larger than $2^{\aleph_0}$ [12, 22], insofar as there are only $2^{\aleph_0}$ elements of $\mathbb{X}_\Lambda$, up to translation.

During the completion of this paper, Smilansky and Solomon released a preprint [22] in which a stronger version of this result is proven using the language of dynamical systems. Their theorem, which was developed completely independently from ours, does not require FLC and considers the orbit closure of Λ in the local rubber topology. They replace our assumption of repetitivity with one of minimality. They also do not require that the density of Λ exists. The cost of that generality is complexity, in that their proof is longer and considerably more technical than ours.

An interesting consequence of both results is the following: In [10] it was shown that a certain one-dimensional collection of Delone sets, namely, the set of cut-and-project sets using half of the window of the famous Fibonacci tiling [3], has at least two different bde classes. Theorem 1.1 then yields the following result.

Corollary 1.2. *The hull $\mathbb{X}_{HF}$ of the 'Half-Fibonacci' cut-and-project tiling contains uncountably many bde classes.*

2. Auxiliary results

Let us fix some more notation. In the sequel, let $\#M$ denote the cardinality of a (typically finite) set M. We denote the Euclidean norm of $x \in \mathbb{R}^d$ by $\|x\|$. The closed ball of radius r about x is denoted by $B_r(x)$. The d-dimensional Lebesgue measure of a set $A \subset \mathbb{R}^d$ is denoted by $\mu(A)$ and all sets in the paper are compact and measurable unless noted otherwise. Let $A^{+\varepsilon}$ denote the ε-tube of the boundary of A. That is,

$$A^{+\varepsilon} = \{x \in \mathbb{R}^d \mid d_2(x, \partial A) \le \varepsilon\},$$

where ∂A is the topological boundary of A and d_2 is the standard Euclidean distance in $\mathbb{R}^d$. Hence $\mu(A^{+1})$ is the Lebesgue measure of the set of all points whose distance to the boundary of A is one or less.

Any d-periodic Delone set has a well defined density, in the sense of "average number of points per unit volume". The same holds for cut-and-project sets and for Delone sets from primitive substitutions. In general, the definition of the density of a Delone set can be tricky. It is easy to construct Delone sets having no well defined density (for instance $-\mathbb{N} \cup 2\mathbb{N}$ in $\mathbb{R}$). There are even repetitive Delone sets without density, see [19, Thm. 5.1], or [9] for a simpler example.

In order to define the density of an arbitrary Delone sets we need van Hove sequences. A *van Hove sequence* is a sequence $(A_i)_i$ of compact subsets of $\mathbb{R}^d$ such that for all $\varepsilon > 0$

$$(1) \qquad\qquad \lim_{i \to \infty} \frac{\mu(A_i^{+\varepsilon})}{\mu(A_i)} = 0.$$

A Delone Λ set has density $\mathrm{dens}(\Lambda)$ if for all van Hove sequences $(A_i)_i$ the limits

$$\lim_{i\to\infty} \frac{\#(A_i \cap \Lambda)}{\mu(A_i)}$$

exist and are identical. In that case $\mathrm{dens}(\Lambda)$ is the value of these limits. An important tool in this context is the following result. In the context of this paper it translates as follows.

Theorem 2.1 ([18]). *Let Λ be a Delone set in $\mathbb{R}^d$ with density $\mathrm{dens}(\Lambda)$. Λ is bde to some lattice in $\mathbb{R}^d$ if and only if there is $c > 0$ such that, for all bounded measurable sets $E \subset \mathbb{R}^d$,*

$$\big|\#(\Lambda \cap E) - \mathrm{dens}(\Lambda)\mu(E)\big| \leq c\mu(E^{+1})$$

The proof of this result relies on the infinite version of the Hall Marriage Theorem [20]. The same arguments can be used not only to compare a Delone set Λ with $\alpha\mathbb{Z}^d$, but to compare two arbitrary Delone sets Λ, Λ' as well.

Theorem 2.2 ([11]). *Let Λ, Λ' be two Delone sets in $\mathbb{R}^d$. Suppose there is a van Hove sequence $(A_i)_i$ such that*

$$(2) \qquad \lim_{n\to\infty} \frac{|\#(\Lambda \cap A_i) - \#(\Lambda' \cap A_i)|}{\mu(A_i^{+1})} = \infty,$$

then $\Lambda \overset{\mathrm{bd}}{\not\sim} \Lambda'$.

In [11], the result above was formulated using sets A_i that are unions of lattice cubes of appropriate size. Since we deal with Delone sets we can "approximate" any van Hove sequence $(A_i)_i$ by an appropriate union of lattice cubes that are small enough that each cube contains either one or zero points of Λ. For more details see [12] or [11].

Example 2.3. As a test case for what follows we consider the set $L = \mathbb{Z} \cup \{1/2 + 2^n \mid n \in \mathbb{N}_0\}$ and compare L to the integers $\mathbb{Z}$. Both have density 1, but $\mathbb{Z} \overset{\mathrm{bd}}{\not\sim} L$. This can be seen by using Theorem 2.1 above, and observing for each $k > 0$ there are intervals $Q_i := [0, 2^i + 1]$ such that

$$\#(Q_i \cap \Lambda) - \#(Q_i \cap \mathbb{Z}) > i.$$

Note that L contains "rich" regions (i.e. regions where the number of points is larger than the expected number of points according to the density), but no "poor" regions.

3. Proof of Theorem 1.1

Note the difference between a patch $E \cap \Lambda$, which is a finite collection of points, and its support E. Also note that we have two different ways to move patches around: either by translating both its support and Λ to get $(E \cap \Lambda) - x = (E - x) \cap (\Lambda - x)$, or by translating just the support to get $(E - x) \cap \Lambda$. Both ways yield patches occurring in some $\Lambda' \in \mathbb{X}_\Lambda$ and we will use both.

The strategy of the proof of Theorem 1.1 is as follows. If Λ is not bde to a lattice, then there exist large regions that are "deviant" (meaning either very rich or very poor

in points of Λ compared to the expected number of points according to the density of Λ) and translates of those regions that are either "normal" (neither rich nor poor) or have a discrepancy of the opposite sign of the original region; we will refer to the latter sets as "normal" anyway. We will recursively define larger and larger regions P_i around the origin and find elements of $\mathbb{X}_\Lambda$ for which the regions P_i are either deviant or normal. In fact, for each infinite word in $\{D, N\}^{\mathbb{N}}$ we will construct an element of $\mathbb{X}_\Lambda$ where the form of each P_i corresponds to the i-th letter in the word. Any two such elements of $\mathbb{X}_\Lambda$ with words u, u' that differ in infinitely many letters are not bde to each other by Theorem 2.2. Since there are uncountably many sequences in $\{D, N\}^{\mathbb{N}}$, and since each tail-equivalence class is countable, there are uncountably many elements of $\mathbb{X}_\Lambda$, no two of which are bde.

We construct deviant and normal patches D_i' and N_i' around the origin recursively. Let ϱ be the density of Λ and suppose that D_{i-1}' and N_{i-1}' have already been constructed. Since Λ is not bde to a lattice, there exist a region D_i' such that the ratio

$$\frac{|\#(\Lambda \cap D_i') - \varrho\mu(D_i')|}{\mu(D_i^{+1})}$$

is arbitrarily large. In other words, for which D_i' is deviant. As shown below, there is also a "normal" translate N_i' of D_i' for which the discrepancy $\#(N_1' \cap \Lambda) - \varrho\mu(N_i')$ has the opposite sign as the discrepancy of D_i'. Without loss of generality, we can choose D_i' so large that there are copies of D_{i-1}' and N_{i-1}' near the center of D_i', and likewise near the center of N_i'. Note that the construction so far only involves looking for regions in the fixed Delone set Λ.

Next we construct Delone sets in $\mathbb{X}_\Lambda$ corresponding to each infinite word in $\{D, N\}^{\mathbb{N}}$. Let u be such a word. Place a copy of D_1' or N_1', centered at the origin, according to whether the first letter of u is D or N. Call this copy D_1 or N_1, and let $P_1 = D_1$ or N_1. Since both D_2' and N_2' contain copies of both D_1' and N_1' near their centers, we can extend P_1 to a copy D_2 or N_2 of D_2' or N_2', according to whether the second letter of u is D or N, and we can call this copy P_2. Repeat for each index $i \in \mathbb{N}$. Note that each P_i is a patch $N_i' - x_i$ or $D_i' - x_i$ in $\Lambda - x$. The union of the P_i's is a Delone set $\Lambda_u \in \mathbb{X}_\Lambda$. Specifically,

$$\Lambda_u = \bigcup_i P_i = \lim_{i \to \infty} (\Lambda - x_i).$$

The tricky point is that, if $u \neq u'$, then the i-th patch P_i of Λ_u is not perfectly aligned with the P_i of $\Lambda_{u'}$, so we cannot directly compare the two P_i's in Theorem 2.2. Instead, we must compare patches defined by small translates (within Λ_u) of supports of deviant patches D_i in Λ_u to normal patches N_i in $\Lambda_{u'}$, or vice-versa. The remainder of the proof is a series of estimates to show that suitable D_i''s and N_i''s exists, such that these small translates are still sufficiently deviant to apply Theorem 2.2.

The following result is standard, but for completeness we provide a sketch of the proof.

Lemma 3.1. *Let Λ be a Delone set in $\mathbb{R}^d$ with density ϱ and let E be a bounded measurable set. Then there exist vectors $x_1, x_2 \in \mathbb{R}^d$ such that*

$$\#((E + x_1) \cap \Lambda) \le \varrho\mu(E) \qquad and \qquad \#((E + x_2) \cap \Lambda) \ge \varrho\mu(E).$$

Sketch of proof: If we average $\#((E + x) \cap \Lambda)$ over all values of $x \in \mathbb{R}^d$ we must get $\varrho\mu(E)$. This means that for every $\varepsilon > 0$ there must a point x for which $\#((E+x) \cap \Lambda) \le \varrho\mu(E) + \varepsilon$. However, $\#((E + x) \cap \Lambda)$ is always an integer, and so for small enough ε cannot be strictly between $\varrho\mu(E)$ and $\varrho\mu(E) + \varepsilon$. Thus there must be an x_1 satisfying the first inequality. The second is similar. $\qquad\square$

The following result is then immediate and allows us to construct normal patches N_i' from deviant patches D_i'.

Lemma 3.2. *Let $\Lambda \subset \mathbb{R}^d$ be a Delone set with density ϱ such that Λ is not bde to any lattice in $\mathbb{R}^d$. Let E be a compact subset of $\mathbb{R}^d$ such that $\#(\Lambda \cap E) - \varrho\mu(E) \ge 0$ (resp. ≤ 0). Then there is a translation $E - x$ of E such that $\#(\Lambda \cap (E - x)) - \varrho\mu(E) \le 0$ (resp. ≥ 0).*

We also need the following elementary identity.

Lemma 3.3. *Let $r, \ell > 0$, then $(E^{+\ell})^{+r} \subseteq E^{+(\ell+r)}$.*

Proof. Let $z \in (E^{+\ell})^{+r}$. By definition there are $y \in \partial E$ and y' such that $\|z - y'\| \le r$ and $\|y' - y\| \le \ell$, hence

$$\|z - y\| \le \|z - y'\| + \|y' - y\| \le \ell + r,$$

hence $z \in E^{+(\ell+r)}$. $\qquad\square$

We also need some result on estimates on the minimal and the maximal number of points of a Delone set Λ within a region E. In the sequel we always assume that E is a bounded measurable set.

Lemma 3.4. *There exist constants η, η' such that for any Delone set Λ with parameters $r > 0$ and $R > 0$ and any bounded region $E \subset \mathbb{R}^d$,*

$$\#(E \cap \Lambda) \ge \frac{\eta}{R^d}(\mu(E) - \mu(E^{+R})) \quad and \quad \#(E \cap \Lambda) \le \frac{\eta'}{r^d}(\mu(E) + \mu(E^{+r}))$$

Here, η and η' depend only on r, R, and the dimension d.

Proof. Consider a periodic packing of $\mathbb{R}^d$ by balls of radius R. (For instance, we could tile $\mathbb{R}^d$ by cubes of side length $2R$ and place one ball in each cube.) Averaging over translates of this packing, the number of balls with center in $E - E^{+R}$ is the packing density (which is a constant η that depends on dimension divided by R^d) times the volume of $E - E^{+R}$. Therefore there exists a specific packing where the number of such balls is at least $\frac{\eta}{R^d}\mu(E - E^{+R})$, which in turn is at least $\frac{\eta}{R^d}(\mu(E) - \mu(E^{+R}))$. Each of these disjoint

balls is contained completely in E, and by the Delone property each contains at least one point of Λ. This establishes the first inequality.

The second inequality is similar, except that we use a covering by balls of radius r (e.g. by starting with a tiling of $\mathbb{R}^d$ by cubes of side length $2r/\sqrt{d}$ and using circumscribed balls) instead of a packing by balls of radius R. Each point in $E \cap \Lambda$ must lie in a ball whose center is either in E or in E^{+r}. However, each such ball can contain at most one point in Λ and the number of such balls is bounded by $\frac{\eta'}{r^n}(\mu(E \cup E^{+r}) \le \frac{\eta'}{r^n}(\mu(E) + \mu(E^{+r}))$. $\square$

Lemma 3.5. *For all $\ell \ge 1$, $\mu(E^{+\ell}) \le \ell^d \mu(E^{+1})$.*

Proof. First we prove $\mu((\frac{1}{\ell}E)^{+1}) \le \mu(E^{+1})$. Let $\varepsilon > 0$. Let $x_1, \ldots, x_n \in \partial(\frac{1}{\ell}E)$ be points such that the set of ε-balls centered at x_i covers $\partial(\frac{1}{\ell}E)$.

The balls of radii $1 + \varepsilon$ centered at x_i's cover $(\frac{1}{\ell}E)^{+1}$; let X be the union of these balls. Similarly, the balls of radii $1 + \varepsilon$ centered at ℓx_i's are contained in $E^{+(1+\varepsilon)}$; let Y be the union of these balls. Using a variant of the Kneser-Poulsen conjecture[2] for continuous contractions, see [4, 5], we get that $\mu(X) \le \mu(Y)$ and therefore

$$\mu((\frac{1}{\ell}E)^{+1}) \le \mu(X) \le \mu(Y) \le \mu(E^{+(1+\varepsilon)}).$$

Taking the limit as ε goes to 0, $\mu((\frac{1}{\ell}E)^{+1}) \le \mu(E^{+1})$.

Now we can prove the lemma. Scaling $E^{+\ell}$ down by ℓ^{-1} yields $(\frac{1}{\ell}E)^{+1}$. Hence

$$\mu(E^{+\ell}) = \ell^d \mu((\frac{1}{\ell}E)^{+1}) \le \ell^d \mu(E^{+1}).$$

$\square$

The next lemma ensures that, given a patch of $E \cap \Lambda$, every patch of Λ with translated support $E - x$ has "approximately" the same number of points, the difference in the number of points being governed by $\mu(E^{+1})$ and the length of the shift.

Lemma 3.6. *Let Λ be a Delone set with parameters $r > 0$ and $R > 0$. Let $\ell \ge r$. Then there is $q > 0$ such that, for every $x \in \mathbb{R}^d$ with $\|x\| \le \ell$ and for every region E,*

$$|\#((E+x) \cap \Lambda) - \#(E \cap \Lambda)| \le q\ell^d \mu(E^{+1}).$$

Here the constant q depends on the Delone set Λ and the dimension d.

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[2] Despite retaining its original name, this "conjecture" is actually a proven theorem.

Proof. Let $\ell \geq r$ and $\|x\| \leq \ell$.

$$|\#\big((E+x)\cap\Lambda\big) - \#\big(E\cap\Lambda\big)|$$

$$\leq \#\Big(\Big(\big((E+x)\setminus E\big)\cup\big(E\setminus(E+x)\big)\Big)\cap\Lambda\Big)$$

$$\leq \#\Big(\big((E+x)\setminus E\big)\cap\Lambda\Big) + \#\Big(\big(E\setminus(E+x)\big)\cap\Lambda\Big)$$

$$\overset{(x\leq\ell)}{\leq} \#\big((E+x)^{+\ell}\cap\Lambda\big) + \#\big(E^{+\ell}\cap\Lambda\big)$$

$$\overset{\text{(Lem. 3.4)}}{\leq} \frac{\eta'}{r^d}\Big(\mu\big((E+x)^{+\ell}\big) + \mu\big(((E+x)^{+\ell})^{+r}\big)\Big) + \frac{\eta'}{r^d}\Big(\mu\big(E^{+\ell}\big) + \mu\big((E^{+\ell})^{+r}\big)\Big)$$

$$\overset{\text{(Lem. 3.3)}}{\leq} 2\frac{\eta'}{r^d}\big(\mu(E^{+\ell}) + \mu(E^{+(\ell+r)})\big) \leq 2\frac{\eta'}{r^d}\big(\mu(E^{+\ell}) + \mu(E^{+(\ell+\ell)})\big)$$

$$\overset{\text{(Lem. 3.5)}}{\leq} 2\frac{\eta'}{r^d}(\ell^d + (\ell+\ell)^d)\mu(E^{+1}) \leq 2\frac{\eta'}{r^d}(1+2^d)\ell^d\mu(E^{+1}).$$

With $q = 2\frac{\eta'}{r^d}(1+2^d)$ and η' the constant from Lemma 3.4 the claim follows. □

The next lemma is the key to proving Theorem 1.1 and particularly to the construction of "deviant" patches of Λ.

Lemma 3.7. *Let $\Lambda \subset \mathbb{R}^d$ be a repetitive Delone set of FLC with $\operatorname{dens}(\Lambda) = \varrho$ such that Λ is not bde to any lattice in $\mathbb{R}^d$. Let $c > 0$ and $\ell > 0$ be given. Then there exists a bounded measurable set E such that for all $x \in \mathbb{R}^d$ with $\|x\| \leq \ell$ holds:*

$$|\#\big((E-x)\cap\Lambda\big) - \varrho\mu(E)| > c\mu(E^{+1})$$

Proof. Let q be as in Lemma 3.6. By Theorem 2.1 there is E such that

$$(3) \qquad\qquad |\#(E\cap\Lambda) - \varrho\mu(E)| > (c + q\ell^d)\mu(E^{+1}).$$

By Lemma 3.6 we have that $|\#(E\cap\Lambda) - \#((E-x)\cap\Lambda)| \leq q\ell^d\mu(E^{+1})$. Replacing $\#(E\cap\Lambda)$ by $\#((E-x)\cap\Lambda)$ in (3) changes the left hand side by less than $q\ell^d\mu(E^{+1})$. This yields the claim. □

Lemma 3.8. *Let Λ be a Delone set in $\mathbb{R}^d$ with $\operatorname{dens}(\Lambda) = \varrho$. Let $(E_i)_i$ be a sequence of bounded measurable subsets of $\mathbb{R}^d$ that violate the condition in Theorem 2.1. That is, let the sequence $(c_i)_i$ be such that $\lim_{i\to\infty} c_i = \infty$, and for each i*

$$(4) \qquad\qquad |\#(E_i\cap\Lambda) - \varrho\mu(E_i)| > c_i\mu(E_i^{+1}).$$

Then $(E_i)_i$ is a van Hove sequence.

Proof. Inequality (4) is equivalent to

$$(5) \qquad\qquad \frac{1}{c_i}\Big|\frac{\#(E_i\cap\Lambda)}{\mu(E_i)} - \varrho\Big| > \frac{\mu(E_i^{+1})}{\mu(E_i)}.$$

Using Lemma 3.4 we get

$$\frac{\eta}{R^d} - \frac{\mu(E_i^{+R})}{\mu(E_i)} \leq \frac{\#(E_i \cap \Lambda)}{\mu(E_i)} \leq \frac{\eta'}{r^d} + \frac{\mu(E_i^{+r})}{\mu(E_i)}.$$

From Lemma 3.5 we get upper bounds for $\mu(E_i^{+r})$ and $\mu(E_i^{+R})$ in terms of $\mu(E_i^{+1})$ (there is the trivial bound $\mu(E_i^{+1})$ if $r \leq 1$ or $R \leq 1$). Combining these estimates we get the following inequality for some positive constants α and β

$$\left| \frac{\#(E_i \cap \Lambda)}{\mu(E_i)} - \varrho \right| \leq \alpha + \beta \frac{\mu(E_i^{+1})}{\mu(E_i)}.$$

Once we plug this in inequality (5), we get

$$\frac{1}{c_i} \left(\alpha + \beta \frac{\mu(E_i^{+1})}{\mu(E_i)} \right) > \frac{\mu(E_i^{+1})}{\mu(E_i)}.$$

Considering the limits of both sides yields $\lim\limits_{i \to \infty} \frac{\mu(E^{+1})}{\mu(E_i)} = 0$. In order to show $\lim\limits_{i \to \infty} \frac{\mu(E^{+\varepsilon})}{\mu(E_i)} = 0$ for all $\varepsilon > 0$, we consider two cases. If $\varepsilon \leq 1$, then

$$\lim_{i \to \infty} \frac{\mu(E_i^{+\varepsilon})}{\mu(E_i)} \leq \lim_{i \to \infty} \frac{\mu(E_i^{+1})}{\mu(E_i)} = 0.$$

If $\varepsilon > 1$, then we use the estimate from Lemma 3.5 to get

$$\lim_{i \to \infty} \frac{\mu(E_i^{+\varepsilon})}{\mu(E_i)} \leq \lim_{i \to \infty} \frac{\varepsilon^d \mu(E_i^{+1})}{\mu(E_i)} = 0.$$

$\square$

Frequently a van Hove sequence $(E_i)_i$ is defined by requiring another property in addition to (1): namely, that the E_i exhaust the entire space. For instance, the sequence $(B_i(x_i))_i$, with $x_i = (i, 0, \cdots, 0)^T$, fulfills (1), but the (closure of the) union of the balls is only the half-space $\mathbb{R}^{\geq 0} \times \mathbb{R}^{d-1}$. The next result ensures that we do not need this further requirement in our context.

Proposition 3.9. *Let $(E_i)_i$ be a van Hove sequence. Then for each $R > 0$ there are $i \geq 0, t_i \in \mathbb{R}^d$ such that E_i contains a ball $B_R(t_i)$.*

Proof. Assume the contrary; that is, suppose there is $R > 0$ such that for all $t \in \mathbb{R}^d, i \geq 0$ holds: $B_R(t_i) \nsubseteq E_i$. This implies that for all t, i we have that the distance between t and ∂E_i is not greater than R. Consequently, for all i we have $E_i \setminus (E_i)^{+R} = \varnothing$. Since $E_i \neq \varnothing$ this implies that for all $i > 0$ holds $\mu(E_i) < \mu((E_i)^{+R})$, contradicting the van Hove property (1). $\square$

Now we can state the proof of our main result.

Proof of Theorem 1.1. Let Λ be a Delone set in $\mathbb{R}^d$ with $\mathrm{dens}(\Lambda) = \varrho$ such that Λ is not bde to any lattice in $\mathbb{R}^d$. We choose an infinite word $u = u_1 u_2 \cdots$ over the alphabet $\{D, N\}$, and $(c_i)_i$ such that $\lim_{i \to \infty} c_i = \infty$.

For $c > 0$, we call a bounded measurable set E *c-deviant* if

$$|\#(E \cap \Lambda) - \varrho\mu(E)| > c\mu(E^{+1}),$$

otherwise we call E *c-normal*.

Let ℓ_1 be arbitrary. By Theorem 2.1 there is a compact set D_1' that is c_1-deviant. By Lemma 3.2 there is set $N_1' := D_1' - y_1$ such that the *discrepancy* $\#(N_1' \cap \Lambda) - \varrho\mu(N_1')$ of N_1' is zero or has the opposite sign as the discrepancy of D_1'. (N_1' may or may not be c-normal for any prescribed c, but in all cases the discrepancy of N_1' differs greatly from that of D_1', which is what we actually need.) Pick a point $x_1 \in D_1'$. If the first letter u_1 of u is D then let $P_1 := (D_1' \cap \Lambda) - x_1 = (D_1' - x_1) \cap (\Lambda - x_1)$, which is a patch in $\Lambda - x_1$. Otherwise let $P_1 := (N_1' \cap \Lambda) + y_1 - x_1 = (N_1' + y_1 - x_1) \cap (\Lambda + y_1 - x_1)$, which is a patch in $\Lambda + y_1 - x_1$. The shape of support of P_1 is the same in both cases, but the underlying patch of the Delone set is different.

Choose ℓ_2 such that $\ell_2 > 2\max\{R_{rep}(D_1' \cap \Lambda), R_{rep}(N_1' \cap \Lambda)\}$, where $R_{rep}(P)$ denotes the repetitivity radius of the patch P (compare Section 1). By Lemma 3.7 there is D_2' such that $D_2' - x$ is c_2-deviant for all x with $\|x\| \le \ell_2$. By Lemma 3.2 there is y such that $N_2' = D_2' - y$ has the opposite discrepancy as D_2' (or this discrepancy is 0). Let $B_s(t_1)$ be the largest ball contained in D_2'.

By repetitivity, both $D_2' \cap \Lambda$ and $N_2' \cap \Lambda$ contain translates of both $D_1' \cap \Lambda$ and $N_1' \cap \Lambda$ within distance $\frac{\ell_2}{2}$ of the center t_1 of $B_s(t_1)$, say with the points $x_1 \in D_1'$ and $y_1 \in N_1'$ corresponding to x_{D2} and x_{N2} in D_2'. If $u_1 = D$ we take $x_2 = x_{D2}$ and if $u_1 = N$ we take $x_2 = x_{N2}$. Either way, define $D_2 = D_2' - x_2$, viewed as a pattern in $\Lambda - x_2$. That is, D_2 is a translate (of both support and Delone set) of D_2' with a copy of P_1 close to the center. We similarly create N_2 as a translate of N_2' that likewise extends P_1. Finally, we pick P_2 to be either D_2 or N_2, depending on whether u_2 is D or N.

The supports of the two possible choices of P_2 are not identical, since the relative positions of copies of P_1 in D_2' and N_2' are not the same. However, their supports differ by translation by less than ℓ_2. A translate of D_2 that has the same support as N_2 still has a discrepancy greater in magnitude than $c_2\mu(D_2^{+1})$, while the discrepancy of N_2 has a discrepancy of the opposite sign.

Now iterate: let ℓ_{i+1} be such that $\ell_{i+1} > 2\max\{r_{rep}(D_i' \cap \Lambda), r_{rep}(N_i' \cap \Lambda)\}$ and continue as above, finding regions D_{i+1}' and N_{i+1}' in Λ such that D_{i+1}' and all its translations by at most ℓ_{i+1} are c_{i+1}-deviant, N_{i+1}' has a discrepancy of the opposite sign as D_{i+1}', and such that both D_{i+1}' and N_{i+1}' contain copies of D_i' and N_i' within a distance $\ell_{i+1}/2$ of the center of a large sphere (as in Proposition 3.9). We then defined translates D_{i+1} and N_{i+1} of D_{i+1}' and N_{i+1}', viewed as regions in translates of Λ, such that these pattern extend P_i. Finally we pick P_{i+1} to be D_{i+1} or N_{i+1} depending on the $(i+1)$st letter of u.

For any word $u \in \{D, N\}^{\mathbb{N}}$, this procedure gives a nested sequence of patches $(P_i)_i$ (i.e. $P_i \subset P_{i+1}$). By Lemma 3.8 the supports of the patches P_i are a van Hove sequence. By Proposition 3.9 these van Hove sequences contain arbitrary large balls. Since we have chosen the balls $B_s(t_i)$ in each step to be the largest possible, their diameters s tend to infinity. By the closedness of $\mathbb{X}_\Lambda$ under translations we can assume without loss of generality that these balls are centered at 0. The union of the patches P_i is then a Delone set $\Lambda_u \in \mathbb{X}_\Lambda$. If $u, u' \in \{D, N\}^{\mathbb{N}}$ differ in infinitely many letters, then we look at the points of disagreement and compare N_i in one Delone set to a translate by less than ℓ_i of D_i in the other. Applying Theorem 2.2, we see that $\Lambda_u \overset{\text{bd}}{\not\sim} \Lambda_{u'}$ because the translate by at most ℓ_i of D_i is still c_i-deviant. There are $2^{\aleph_0}$ elements of $\{D, N\}^{\mathbb{N}}$ and only countably many elements of each bde class, so there are $2^{\aleph_0}$ distinct bde classes among the $\{\Lambda_u\}$ and there are at least $2^{\aleph_0}$ bde classes in $\mathbb{X}_\Lambda$.

However, Delone sets that differ only by a translation are bde and $\mathbb{X}_\Lambda$ consists of only $2^{\aleph_0}$ translational orbits, so $\mathbb{X}_\Lambda$ can have at most $2^{\aleph_0}$ bde classes. Thus the cardinality of the bde classes in $\mathbb{X}_\Lambda$ is exactly $2^{\aleph_0}$. □

Remark. In order to maintain a strict separation between our results and those of Smilansky and Solomon [22], we stated and proved Theorem 1.1 in the form in which we originally developed it, including the assumption that an overall density exists. However, it is extremely easy to drop that assumption. If there is no overall density, then there is an upper density ϱ_+ and a lower density ϱ_-. Instead of looking for regions that are rich or poor relative to "the" overall density, we can look instead for regions that are rich or poor relative to an arbitrary intermediate density $\varrho \in (\varrho_-, \varrho_+)$. For instance, we can look for Euclidean balls of large radius whose densities are greater than $(\varrho_+ + \varrho)/2$ or are less than $(\varrho_- + \varrho)/2$. The rest of the proof that there exist uncountably many bde classes proceeds exactly as before.

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