1-20-2021

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Abstract

In this article we give sufficient conditions for the generalized Dirac operator to obey the incomplete Huygens principle, as well as necessary and sufficient conditions to obey the Huygens principle by the Dirac operator in the curved spacetime of the Friedmann-Lemaître-Robertson-Walker models of cosmology.

0 Introduction

An interesting question in the physics of fundamental particles (fields) is the validity of the Huygens principle. In this article we find sufficient conditions for the incomplete Huygens principle for the generalized Dirac operator and necessary conditions for the Dirac operator in the curved spacetime of the Friedmann-Lemaître-Robertson-Walker (FLRW) models of cosmology. We use the definition of the Huygens principle due to Hadamard [25] as the absence of tails. Thus, the field equations satisfy the Huygens principle if and only if the solution has no tail, that is, solution depends on the source distributions on the past null cone of the field only and not on the sources inside the cone.

Fields of nonzero spin in a curved space have been studied from the mathematical and the physical (classical and quantum) point of view (see, e.g., [12, 19, 22] and references therein). It is known that, given Dirac equation in a curved four-dimensional spacetime (M, g_{ab}), the Huygens principle is generally violated by its solutions, due to the mass term in the equation and the curvature of spacetime [11]. The presence or absence of tails for waves has been established for some spacetime metrics g_{ab}, including constant curvature metrics [25]. In fact, the study of the Huygens principle has important applications to quantum field theory and cosmology, especially in inflationary theories of the early universe. The Huygens principle is a fundamental feature of the physics and the mathematics of propagation of waves. The fact that the support of the commutator or the anticommutator-distribution, respectively, lies on the null-cone if and only if the Huygens principle holds for the corresponding equation [16] shows significance of the Huygens principle for quantum field theory. The Huygens principle is also studied for gravitational waves in a curved background (see, e.g., [15]). In the analysis of partial differential equations the Huygens principle plays important role in the estimates of the Bahouri-Gérard concentration compactness method and Strichartz estimates (see, e.g., [14, 21]) and in study of blowup of solutions of nonlinear hyperbolic equations (see, e.g., [2]).

The spin-$\frac{1}{2}$ particle field in the Minkowski spacetime (see, e.g., [6, (20) Ch.1, Sec.6])

$$\left(i\gamma^0\partial_0 + i\gamma^k\partial_k - m\mathbb{I}_4\right)\Psi(x,t) = 0, \quad x \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \quad m \in \mathbb{C}, \tag{0.1}$$

satisfies the Huygens principle if and only if $m = 0$. Here and henceforth, the Einstein summation convention over repeated indexes is employed and also the speed of light c and Planck's constant h are set equal to unity. Even if we relax the Huygens principle by considering the Cauchy problem for (0.1) with the initial condition

$$\Psi(x,0) = (\Phi_0(x), \Phi_1(x), \Phi_2(x), \Phi_3(x))^T, \quad x \in \mathbb{R}^3,$$

that has only one non-vanishing component, say $\Phi_1(x) = \Phi_2(x) = \Phi_3(x) = 0$, the solution of the equation with $m \neq 0$ depends on the values of the function $\Phi_0(x)$ inside the past null cone of the field.

The metric tensor in the spatially flat FLRW spacetime in Cartesian coordinates is

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{pmatrix}, \quad \mu,\nu = 0,1,2,3.$$

We will focus on the de Sitter space with the scale factor $a(t) = e^{Ht}$ (see, e.g., [18]) that is modeling the expanding or contracting universe if $H > 0$ or $H < 0$, respectively. The Dirac equation in the de Sitter space is (see, e.g., [3])

$$\left(i\gamma^0\partial_0 + ie^{-Ht}\gamma^1\partial_1 + ie^{-Ht}\gamma^2\partial_2 + ie^{-Ht}\gamma^3\partial_3 + i\frac{3}{2}H\gamma^0 - m\mathbb{I}_4\right)\Psi = F, \tag{0.2}$$

where F is a source term, while the contravariant gamma matrices are (see, e.g., [6, p. 61])

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & -\mathbb{I}_2 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} \mathbb{O}_2 & \sigma^k \\ -\sigma^k & \mathbb{O}_2 \end{pmatrix}, \quad k = 1,2,3.$$

Here σ^k are Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\mathbb{I}_n$, $\mathbb{O}_n$ denote the $n \times n$ identity and zero matrices, respectively. We note here that for $H = 0$ and $F = 0$ the equation (0.2) coincides with (0.1).

The construction of a quantum field theory in curved spacetimes and the definition of a quantum vacuum demand a detailed investigation of the solutions of relativistic equations in curved backgrounds. (See, e.g., [5].) In [29] author presents the fundamental solutions and the solutions to the Cauchy problem via classical formulas for the wave equation in the Minkowski spacetime and the certain integral transform involving the Gauss's hypergeometric function in the kernel. One can regard that integral transform as an analytical mechanism that from the massless field in the Minkowski spacetime generates massive particle in the curved spacetime.

In the terms of the fundamental solutions the Huygens principle sustains that a delta-like impulse of field travels on a sharp front propagating along the light cone.

Even though nowadays, numerical solutions of differential equations are available, in some cases a deep understanding of properties of the solutions is possible only by the examination of the explicit formulas when they are known. This is the case with the Huygens principle. Some known results on the Huygens principle for the Dirac equation can be found in [9, 10, 11, 13, 17, 20, 24, 25].

Recall that a retarded fundamental solution (a retarded inverse) for the Dirac operator (0.2) is a matrix operator $\mathcal{E}^{ret} = \mathcal{E}^{ret}(x,t;x_0,t_0;m)$ that solves the equation

$$\left(i\gamma^0\partial_0 + ie^{-Ht}\gamma^\ell\partial_\ell + i\frac{3}{2}H\gamma^0 - m\mathbb{I}_4\right)\mathcal{E}(x,t;x_0,t_0;m) = \delta(x-x_0,t-t_0)\,\mathbb{I}_4, \quad (x,t,x_0,t_0) \in \mathbb{R}^8, \tag{0.3}$$

and with the support in the *causal future* $D_+(x_0,t_0)$ of the point $(x_0,t_0) \in \mathbb{R}^4$. The advanced fundamental solution (propagator) $\mathcal{E}^{adv} = \mathcal{E}^{adv}(x,t;x_0,t_0;\, m)$ solves the equation (0.3) and has a support in the *causal* *past* $D_-(x_0,t_0)$. The forward and backward light cones are defined as the boundaries of $D_+(x_0,t_0)$ and $D_-(x_0,t_0)$, respectively, where

$$D_\pm(x_0,t_0) := \left\{(x,t)\in\mathbb{R}^{3+1};\, |x-x_0|\le \pm(\phi(t)-\phi(t_0))\right\},$$

and $\phi(t):=(1-e^{-Ht})/H$ is a distance function. In fact, any intersection of $D_-(x_0,t_0)$ with the hyperplane $t=const<t_0$ determines the so-called *dependence domain* for the point (x_0,t_0), while the intersection of $D_+(x_0,t_0)$ with the hyperplane $t=const>t_0$ is the so-called *domain of influence* of the point (x_0,t_0). The Dirac equation (0.2) is non-invariant with respect to time inversion and its solutions have different properties in different directions of time.

Let $\mathcal{A}(x,\partial_x)$ be a differential operator $\mathcal{A}(x,\partial_x)=\sum_{|\alpha|\le p} a_\alpha(x)D_x^\alpha$ and the coefficients $a_\alpha(x)$ are C^∞-functions in $\mathbb{R}^3$, that is $a_\alpha\in C^\infty(\mathbb{R}^3)$. We consider defined in [29] the generalized Dirac equation

$$\left(i\gamma^0\partial_0+ie^{-Ht}\gamma^k A_k(x,\partial_x)+i\frac{3}{2}H\gamma^0-m\mathbb{I}_4\right)\Psi=0\,, \tag{0.4}$$

where $\mathcal{A}(x,\partial_x)$, $A_k(x,\partial_x)$, $k=1,2,3$, are the scalar operators with the property

$$\gamma^k A_k(x,\partial_x)\gamma^j A_j(x,\partial_x)=-\mathcal{A}(x,\partial_x)\mathbb{I}_4\,. \tag{0.5}$$

For $A_k(x,\partial_x)=\partial_{x_k}$, $k=1,2,3$, and $\mathcal{A}(x,\partial_x)=\Delta$ the equation (0.4) is the Dirac equation (0.2) without source term. Several examples of the generalized Dirac equation, including the equation for the motion of the charged spin-$\frac{1}{2}$ particle in a constant homogeneous magnetic field, one can find in [29]. Denote $\mathcal{E}^{we,\mathcal{A}}(x,t,D_x)$ the solution operator (fundamental solution) of the problem

$$\begin{cases} v_{tt}-\mathcal{A}(x,\partial_x)v=0, & x\in\mathbb{R}^3\,, \quad t\in[0,\infty)\,,\\ v(x,0)=\varphi(x), \quad v_t(x,0)=0\,, & x\in\mathbb{R}^3\,,\end{cases}$$

while by $\mathcal{E}^{we,\mathcal{A}}(x,t,y)$ we denote the Schwartz kernel of $\mathcal{E}^{we,\mathcal{A}}(x,t,D_x)$.

The so-called incomplete Huygens principle for the Klein-Gordon equation in the de Sitter spacetime was introduced in [27]. Here we define corresponding principle for the Dirac equation in the de Sitter spacetime.

Definition 0.1 *We say that the equation (0.4) obeys the incomplete Huygens principle with respect to the first 2-spinor initial data* Φ_0,Φ_1 *if the solution* $\Psi=(\Psi_0,\Psi_1,\Psi_2,\Psi_3)^t$ *with the second 2-spinor data* $\Phi_2=\Phi_3=0$ *vanishes at all points which cannot be reached from the support of initial data by a null geodesic.*

We say that the equation (0.4) obeys the incomplete Huygens principle with respect to the second 2-spinor initial data Φ_2,Φ_3 *if the solution* $\Psi=(\Psi_0,\Psi_1,\Psi_2,\Psi_3)^t$ *with the first 2-spinor data* $\Phi_0=\Phi_1=0$ *vanishes at all points which cannot be reached from the support of initial data by a null geodesic.*

It is evident that if the equation obeys the Huygens principle, then it obeys the incomplete Huygens principle. The main result of this paper is the following theorem.

Theorem 0.2 *Consider the generalized Dirac equation (0.4).*
(i) *If* $m=0$ *and the fundamental solution* $\mathcal{E}^{we,\mathcal{A}}(x,t,D_x)$ *with the differential operator* $\mathcal{A}(x,\partial_x)$ *obeys the Huygens principle, then the generalized Dirac equation (0.4) obeys it as well.*

(ii) *If* $m=iH$ *and the fundamental solution* $\mathcal{E}^{we,\mathcal{A}}(x,t,D_x)$ *with the differential operator* $\mathcal{A}(x,\partial_x)$ *obeys the Huygens principle, then the generalized Dirac equation (0.4) obeys the incomplete Huygens principle with respect to the first 2-spinor initial data* Φ_0,Φ_1.

(iii) *If* $m=-iH$ *and the fundamental solution* $\mathcal{E}^{we,\mathcal{A}}(x,t,D_x)$ *with the differential operator* $\mathcal{A}(x,\partial_x)$ *obeys the Huygens principle, then the generalized Dirac equation (0.4) obeys the incomplete Huygens principle with respect to the second 2-spinor initial data* Φ_2,Φ_3.

(iv) *The solution of the Dirac equation (0.2) with the mass* $m\in\mathbb{C}$ *in the de Sitter spacetime obeys the incomplete Huygens principle with respect to the first 2-spinor initial data* Φ_0,Φ_1 *if and only if* $m=0$ *or* $m=iH$.

(v) *The solution of the Dirac equation (0.2) with the mass $m \in \mathbb{C}$ in the de Sitter spacetime obeys the incomplete Huygens principle with respect to the second 2-spinor initial data Φ_2, Φ_3 if and only if $m = 0$ or $m = -iH$.*

(vi) *The solution of the Dirac equation (0.2) with the mass $m \in \mathbb{C}$ in the de Sitter spacetime obeys the Huygens principle if and only if $m = 0$ or $m = iH, -iH$.*

Remark 0.3 *We do not know if the generalized Dirac equation is necessarily produced by a spacetime metric.*

Remark 0.4 *The statement* (vi) *follows from Theorem by Wünsch [25]. In the present paper we give another proof based on the explicit formula for the solution.*

As it is proved by Wünsch [25] the system for symmetric spinor fields with the spin $s = \frac{1}{2}(n+1)$ $(n = 0, 1, 2, \ldots)$ in four dimensional spacetime satisfies the Huygens principle if and only if the spacetime has a constant curvature. The de Sitter spacetime has a constant curvature.

For the particle field satisfying the Dirac equation (0.2) in the de Sitter spacetime (in FLRW model) according to Theorem 0.2, there are three values of mass with which the equation obeys the Huygens principle. One of them $m = 0$, two others are $m = iH$ and $m = -iH$. The last two appear due to the curvature of the spacetime $R = -12H^2 \neq 0$, where the Hubble constant H is very small number around $2.2 \times 10^{-18} s^{-1}$. The particle with $m = 0$ in the standard textbooks refers as *neutrino* [6]. It is now known that there are three discrete neutrino masses with different tiny values less than 2.14×10^{-34} grams. It is very tempting to think that these three masses $m = 0, iH, -iH$ correspond to three neutrinos. But possible barrier can be the fact that the mass $m = \pm iH$ is an imaginary number. If m of the Dirac equation (0.2) is responsible for the gravitational interaction according to Newton's law, then it will be repulsive for two particles with the imaginary mass $m = iH$ and also between two particles with the imaginary mass $m = -iH$. On the other hand, it is commonly accepted that neutrino moves with speed of light. For interaction of neutrinos and gravitational fields see [8]. In particular, in [8] is considered the stress-energy tensor of the neutrino field, which must be inserted in the Einstein's equations, if the reaction of the neutrino field on the gravitational field is to be described.

It is possible that the Huygens principle affects the ability of particle to interact with other fields. The mass of a stationary electron has the value of about $9.1094 \times 10^{-28} g$. For the massive particle obeying the Huygens principle ($h \approx 6.6261 \times 10^{-27} cm^2 \cdot g \cdot s^{-1}$, $c \approx 2.99792458 \times 10^{10} cm \cdot s^{-1}$) in the physical units

$$|m| = \frac{Hh}{c^2} \approx 4.9 \times 10^{-65} g\,.$$

The duration of time when the factor $\exp\left(-i\, mc^2 t/h\right)$ of the kernel K_1 (1.7) of the integral representing the solution of the generalized Dirac equation with the mass $m = \pm iHh/c^2$ stays around unity is limited by

$$T \approx \frac{h}{c^2 m} \approx 10^{18} sec\,.$$

The age of the universe is $\approx 10^{18}$ sec. The only electrically neutral and long-lived particles in the Standard Model of particle physics are the neutrinos [7].

As experiments show, neutrinos have mass and they are a very natural Dark Matter candidate. The reasons why the known neutrinos cannot compose all of the observed Dark Matter are the smallness of their mass and the magnitude of their coupling to other particles. Despite their tiny masses, neutrinos are so numerous that their gravitational force can influence other matter in the universe. The weak interactions of neutrinos make it very difficult to study their properties. The observed neutrino flavour oscillations clearly indicate that *at least two neutrinos have non-vanishing mass* [7].

Since the neutrino does not respond directly to electromagnetic fields, if one intends to influence its trajectory one has to make use of gravitational fields. In other words, one has to consider the physics of a neutrino as a spinor in a curved metric.

From Theorem 0.2 by letting $H \to 0$ one can learn that the spin-$\frac{1}{2}$ particle field with mass $m \in \mathbb{C}$ in the Minkowski spacetime obeys the Huygens principle if and only if $m = 0$. On the other hand, the direct calculation of the limit of the kernel $K_1(r,t;M)$ and the corresponding operator is a challenging exercise although result is known (see, e.g., [29]).

This paper is organized as follows. In Section 1 we give the representation of the solution of the generalized Dirac equation in the de Sitter spacetime. Then, in Section 2, we prove the sufficiency part of Theorem 0.2. All remaining sections are devoted to the necessity part of Theorem 0.2.

1 Representation of the solution of the generalized Dirac equation in de Sitter spacetime

For $(x_0,t_0)\in\mathbb{R}^n\times\mathbb{R}$, $M\in\mathbb{C}$, $r=|x-x_0|/H$, we define the function

$$\begin{aligned} E(r,t;0,t_0;M) \quad &:= \quad 4^{-\frac{M}{H}} e^{M(t_0+t)}\left(\left(e^{-Ht_0}+e^{-Ht}\right)^2-(Hr)^2\right)^{\frac{M}{H}-\frac{1}{2}} \\ &\times F\left(\frac{1}{2}-\frac{M}{H},\frac{1}{2}-\frac{M}{H};1;\frac{\left(e^{-Ht}-e^{-Ht_0}\right)^2-(rH)^2}{\left(e^{-Ht}+e^{-Ht_0}\right)^2-(rH)^2}\right), \end{aligned} \tag{1.6}$$

where $(x,t)\in D_+(x_0,t_0)\cup D_-(x_0,t_0)$ and $F\big(a,b;c;\zeta\big)$ is the hypergeometric function. Denote

$$M_+=\frac{1}{2}H+im,\quad M_-=\frac{1}{2}H-im.$$

Let $e^{H\cdot}$ be the operator of multiplication by e^{Ht}. Theorem 0.2 [29] gives the representation formula for the solutions of the Cauchy problem. In order to formulate it we need the operator $\mathcal{G}(x,t,D_x;M)$ defined by

$$\mathcal{G}(x,t,D_x;M)[f]=2\int_0^t db\int_0^{\phi(t)-\phi(b)} E(r,t;0,b;M)\int_{\mathbb{R}^n}\mathcal{E}^w(x-y,r)f(y,b)\,dy\,dr,\quad f\in C_0^\infty(\mathbb{R}^{n+1}),$$

where $\mathcal{E}^w(x,r)$ is a fundamental solution of the Cauchy problem

$$\begin{cases} v_{tt}-\Delta v=0, & x\in\mathbb{R}^n,\quad t\in\mathbb{R}, \\ v(x,0)=\varphi(x), & v_t(x,0)=0,\quad x\in\mathbb{R}^n, \end{cases}$$

in the Minkowski spacetime, that is, for $n=3$ (see, e.g., [23])

$$\mathcal{E}^w(x,t):=\frac{1}{4\pi}\frac{\partial}{\partial t}\frac{1}{t}\delta(|x|-t).$$

The distribution $\delta(|x|-t)$ is defined by

$$\langle\delta(|\cdot|-t),\psi(\cdot)\rangle=\int_{|x|=t}\psi(x)\,dx\quad\text{for}\quad\psi\in C_0^\infty\left(\mathbb{R}^3\right).$$

We need also the kernel function

$$K_1(r,t;M)\quad:=\quad E(r,t;0,0;M) \tag{1.7}$$

and the operator $\mathcal{K}_1(x,t,D_x;M)$, which is defined as follows:

$$\mathcal{K}_1(x,t,D_x;M)\varphi(x)=2\int_0^{\phi(t)}K_1(s,t;M)\int_{\mathbb{R}^n}\mathcal{E}^w(x-y,s)\varphi(y)\,dy\,ds,\quad\varphi\in C_0^\infty(\mathbb{R}^n). \tag{1.8}$$

According to Theorem 0.2 [29], solution to the Cauchy problem

$$\begin{cases} \left(i\gamma^0\partial_0+ie^{-Ht}\gamma^k\partial_k+i\frac{3}{2}H\gamma^0-m\mathbb{I}_4\right)\Psi(x,t)=F(x,t), \\ \Psi(x,0)=\Phi(x), \end{cases}$$

$m \in \mathbb{C}$, is given by the following formula

$$\begin{aligned}\Psi(x,t) &= -e^{-Ht}\left(i\gamma^0\partial_0 + ie^{-Ht}\gamma^k\partial_k - i\frac{H}{2}\gamma^0 + m\mathbb{I}_4\right)\\ &\quad\times\left[\begin{pmatrix}\mathcal{G}(x,t,D_x;M_+)\mathbb{I}_2 & \mathbb{O}_2\\ \mathbb{O}_2 & \mathcal{G}(x,t,D_x;M_-)\mathbb{I}_2\end{pmatrix}[e^{H\cdot}F]\right.\\ &\quad\left.+i\gamma^0\begin{pmatrix}\mathcal{K}_1(x,t,D_x;M_+)\mathbb{I}_2 & \mathbb{O}_2\\ \mathbb{O}_2 & \mathcal{K}_1(x,t,D_x;M_-)\mathbb{I}_2\end{pmatrix}[\Phi]\right].\end{aligned}$$

In addition to the functions (1.6), (1.7) for $M \in \mathbb{C}$ we recall one more kernel function from [26, 28]

$$K_0(r,t;M) := -\left[\frac{\partial}{\partial b}E(r,t;0,b;M)\right]_{b=0}.$$

Then according to [28] the solution operator for the Cauchy problem for the scalar *generalized Klein-Gordon equation* in the de Sitter spacetime

$$\left(\partial_0^2 - e^{-2Ht}\mathcal{A}(x,\partial_x) - M^2\right)\psi = f, \quad \psi(x,0) = \varphi_0(x), \quad \psi_t(x,0) = \varphi_0(x),$$

where f is a source term, is given as follows

$$\psi(x,t) = \mathcal{G}(x,t,D_x;M)[f] + \mathcal{K}_0(x,t,D_x;M)[\varphi_0] + \mathcal{K}_1(x,t,D_x;M)[\varphi_1].$$

To describe the operators $\mathcal{G}, \mathcal{K}_0, \mathcal{K}_1$ we recall the results of Theorem 1.1 [28]. For $f \in C(\mathbb{R}^3 \times I)$, $I = [0,T]$, $0 < T \le \infty$, and $\varphi_0, \varphi_1 \in C(\mathbb{R}^3)$, let the function $v_f(x,t;b) \in C^{m,2,0}_{x,t,b}(\mathbb{R}^3 \times [0,(1-e^{-HT})/H] \times I)$ be the solution to the problem

$$\begin{cases} v_{tt} - \mathcal{A}(x,\partial_x)v = 0, & x \in \mathbb{R}^3, \quad t \in [0,(1-e^{-HT})/H],\\ v(x,0;b) = f(x,b), & v_t(x,0;b) = 0, \quad b \in I, \quad x \in \mathbb{R}^3,\end{cases} \tag{1.9}$$

and the function $v_\varphi(x,t) \in C^{m,2}_{x,t}(\Omega \times [0,(1-e^{-HT})/H])$ be the solution of the problem

$$\begin{cases} v_{tt} - \mathcal{A}(x,\partial_x)v = 0, & x \in \mathbb{R}^3, \quad t \in [0,(1-e^{-HT})/H],\\ v(x,0) = \varphi(x), & v_t(x,0) = 0, \quad x \in \mathbb{R}^3.\end{cases} \tag{1.10}$$

Then the function $u = u(x,t)$ defined by

$$\begin{aligned}u(x,t) &= 2\int_0^t db\int_0^{\phi(t)-\phi(b)} E(r,t;0,b;M)v_f(x,r;b)\,dr + e^{\frac{1}{2}Ht}v_{\varphi_0}(x,\phi(t))\\ &\quad + 2\int_0^{\phi(t)} K_0(s,t;M)v_{\varphi_0}(x,s)ds + 2\int_0^{\phi(t)} v_{\varphi_1}(x,s)K_1(s,t;M)ds, \quad x \in \mathbb{R}^3,\ t \in I,\end{aligned}$$

where $\phi(t) := (1-e^{-Ht})/H$, solves the problem

$$\begin{cases} u_{tt} - e^{-2Ht}\mathcal{A}(x,\partial_x)u - M^2u = f, & x \in \mathbb{R}^3,\ t \in I,\\ u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x), & x \in \mathbb{R}^3.\end{cases}$$

These representation formulas can be used for the functions in the Sobolev spaces as well. We stress here that the existence of the solutions in the problems (1.9) and (1.10) is assumed.

There exist two important examples of generalized Klein-Gordon equation. The first one has $\mathcal{A}(x,\partial_x) = \Delta$ and it is related to the problem written in the Cartesian coordinates. (See, e.g., [29]). The second one has the equation written in the spherical coordinates (r,θ,ϕ). In the FLRW spacetime with the line element

$$ds^2 = dt^2 - e^{2Ht}\left(\frac{1}{1-Kr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right)$$

the Klein-Gordon equation is

$$\partial_t^2\psi + 3H\partial_t\psi - e^{-2Ht}\mathcal{A}(x,\partial_x)\psi + m^2\psi = 0\,,$$

where

$$\mathcal{A}(x,\partial_x) \quad := \quad \frac{\sqrt{1-Kr^2}}{r^2}\frac{\partial}{\partial r}\left(r^2\sqrt{1-Kr^2}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial\psi}{\partial\phi}\right)$$

is the Laplace-Beltrami operator in the spatial variables, where $K = -1, 0$, or $+1$, for a hyperbolic, flat, or spherical spatial geometry, respectively. For more examples see references in [29].

Theorem 1.3 [29] allows us to write solution of the *generalized Dirac equation*. More exactly, assume that $\mathcal{A}(x,\partial_x)$, $A_k(x,\partial_x)$, $k = 1,2,3$, are the scalar operators with the properties (0.5). Then the solution to the Cauchy problem

$$\begin{cases}\left(i\gamma^0\partial_0 + ie^{-Ht}\gamma^k A_k(x,\partial_x) + i\frac{3}{2}H\gamma^0 - m\mathbb{I}_4\right)\Psi(x,t) = F(x,t),\\ \Psi(x,0) = \Phi(x)\end{cases}$$

is given as follows

$$\begin{aligned}\Psi(x,t) \quad = \quad & -e^{-Ht}\left(i\gamma^0\partial_0 + ie^{-Ht}\gamma^k A_k(x,\partial_x) - i\frac{H}{2}\gamma^0 + m\mathbb{I}_4\right)\\ & \times\left[\begin{pmatrix}\mathcal{G}(x,t,D_x;M_+)\mathbb{I}_2 & \mathbb{O}_2\\ \mathbb{O}_2 & \mathcal{G}(x,t,D_x;M_-)\mathbb{I}_2\end{pmatrix}[e^{H\cdot}F]\right.\\ & \left.+i\gamma^0\begin{pmatrix}\mathcal{K}_1(x,t,D_x;M_+)\mathbb{I}_2 & \mathbb{O}_2\\ \mathbb{O}_2 & \mathcal{K}_1(x,t,D_x;M_-)\mathbb{I}_2\end{pmatrix}[\Phi(x)]\right]\,. \end{aligned} \tag{1.11}$$

2 Proof of the sufficiency part

2.1 Some exceptional cases of operator $\mathcal{K}_1(x,t,D_x;M)$

Denote $V_\varphi(x,t)$ the solution of the problem

$$V_{tt} - \mathcal{A}(x,\partial_x)V = 0, \quad V(x,0) = 0, \quad V_t(x,0) = \varphi(x)\,, \tag{2.12}$$

then $V_\varphi(x,t)$ can related to the solution v_φ of (1.10) as follows

$$v_\varphi(x,t) = \frac{\partial}{\partial t}V_\varphi(x,t)\,.$$

The kernel $K_1(z,t;M)$ can be written in the explicit form:

$$\begin{aligned}K_1(r,t;M) \quad := \quad & 4^{-\frac{M}{H}}e^{Mt}\left(\left(1+e^{-Ht}\right)^2 - (Hr)^2\right)^{\frac{M}{H}-\frac{1}{2}}\\ & \times F\left(\frac{1}{2}-\frac{M}{H}, \frac{1}{2}-\frac{M}{H}; 1; \frac{\left(1-e^{-Ht}\right)^2-(rH)^2}{\left(1+e^{-Ht}\right)^2-(rH)^2}\right)\,.\end{aligned} \tag{2.13}$$

Further, according to (2.13) we obtain

$$\begin{aligned}K_1\left(r,t;-\frac{1}{2}H\right) \quad &= \quad K_1\left(r,t;\frac{1}{2}H\right) = \frac{1}{2}e^{\frac{1}{2}Ht}\,,\\ K_1\left(r,t;\frac{3}{2}H\right) \quad &= \quad \frac{1}{4}e^{-\frac{1}{2}Ht}\left(\left(1-H^2r^2\right)e^{2Ht}+1\right)\,.\end{aligned}$$

Consequently, by the definition (1.8) of the operator $\mathcal{K}_1$ we write

$$\mathcal{K}_1\left(x,t,D_x;\frac{1}{2}H\right)[\varphi(x)] = \mathcal{K}_1\left(x,t,D_x;-\frac{1}{2}H\right)[\varphi(x)] = e^{\frac{1}{2}Ht}V_\varphi(x,\phi(t)), \tag{2.14}$$

and

$$\begin{aligned} &\mathcal{K}_1\left(x,t,D_x;\frac{3}{2}H\right)[\varphi(x)] \\ &= 2\int_0^{\phi(t)} v_\varphi(x,s)\frac{1}{4}e^{-\frac{1}{2}Ht}\left(\left(1-H^2s^2\right)e^{2Ht}+1\right)ds \\ &= \frac{1}{2}e^{\frac{3}{2}Ht}\left(1+e^{-2Ht}\right)V_{\varphi_1}(x,\phi(t)) - H^2\frac{1}{2}e^{\frac{3}{2}Ht}\phi(t)^2V_\varphi(x,\phi(t)) + H^2e^{\frac{3}{2}Ht}\int_0^{\phi(t)}V_\varphi(x,s)sds\,. \end{aligned} \tag{2.15}$$

2.2 Sufficiency of $m=0,\pm iH$ for the incomplete Huygens principle

First we consider the case of $m=0$, then $M_+=M_-=M=\frac{1}{2}H$ and the kernel $K_1(s,t;M)$ that is given by (2.13). Consequently, for the solution of the generalized Dirac equation with $m=0$ and $F=0$ we obtain from (1.11)

$$\Psi(x,t) = e^{-Ht}\left(\partial_0\mathbb{I}_4 + e^{-Ht}\gamma^k\gamma^0\mathcal{A}_k(x,\partial_x) - \frac{H}{2}\mathbb{I}_4\right)e^{Ht/2}\begin{pmatrix} V_{\Phi_0}(x,\phi(t)) \\ V_{\Phi_1}(x,\phi(t)) \\ V_{\Phi_2}(x,\phi(t)) \\ V_{\Phi_3}(x,\phi(t)) \end{pmatrix}$$

and the Huygens principle is valid since by the assumption of the theorem it is valid for the problem (1.10).

Now we are going to prove that $m=\pm iH$ is sufficient for the incomplete Huygens principle for the generalized Dirac equation in the de Sitter spacetime and for the Huygens principle for the Dirac equation in the de Sitter spacetime. For $m=iH$ we have $M_+=-\frac{1}{2}H$ and $M_-=\frac{3}{2}H$. Then for the operators $\mathcal{K}_1\left(x,t,D_x;-\frac{1}{2}H\right)$ and $\mathcal{K}_1\left(x,t,D_x;\frac{3}{2}H\right)$ we have representation (2.14) and (2.15), respectively. Hence the following function

$$\Psi(x,t) = e^{-Ht}\left(\partial_0\mathbb{I}_4 + e^{-Ht}\gamma^k\gamma^0\mathcal{A}_k(x,\partial_x) - \frac{H}{2}\mathbb{I}_4 - im\gamma^0\right)\begin{pmatrix} \mathcal{K}_1(x,t,D_x;M_+)[\Phi_0(x)] \\ \mathcal{K}_1(x,t,D_x;M_+)[\Phi_1(x)] \\ \mathcal{K}_1(x,t,D_x;M_-)[\Phi_2(x)] \\ \mathcal{K}_1(x,t,D_x;M_-)[\Phi_3(x)] \end{pmatrix}$$

solves the generalized Dirac equation, where the matrices $\gamma^1\gamma^0$, $\gamma^2\gamma^0$,$\gamma^3\gamma^0$ can be written as follows

$$\gamma^1\gamma^0 = \begin{pmatrix} \mathbb{O}_2 & -\sigma_1 \\ -\sigma_1 & \mathbb{O}_2 \end{pmatrix}, \quad \gamma^2\gamma^0 = \begin{pmatrix} \mathbb{O}_2 & -\sigma_2 \\ -\sigma_2 & \mathbb{O}_2 \end{pmatrix}, \quad \gamma^3\gamma^0 = \begin{pmatrix} \mathbb{O}_2 & -\sigma_3 \\ -\sigma_3 & \mathbb{O}_2 \end{pmatrix}. \tag{2.16}$$

We split the initial function $\Phi(x)$ into two parts. First we consider the case of

$$\Phi_2(x) = \Phi_3(x) = 0\,,$$

then the solution $\Psi=\Psi(x,t)$ is as follows

$$\begin{aligned} \Psi(x,t) &= H^2e^{-Ht}\left(\partial_0\mathbb{I}_4 + e^{-Ht}\gamma^k\gamma^0\mathcal{A}_k(x,\partial_x) - \frac{H}{2}\mathbb{I}_4 - im\gamma^0\right)e^{Ht/2}\begin{pmatrix} V_{\Phi_0}(x,\phi(t)) \\ V_{\Phi_1}(x,\phi(t)) \\ 0 \\ 0 \end{pmatrix} \\ &= H^2e^{-Ht/2}\left(\partial_0\mathbb{I}_4 + e^{-Ht}\gamma^k\gamma^0\mathcal{A}_k(x,\partial_x) + H\mathbb{I}_4\right)\begin{pmatrix} V_{\Phi_0}(x,\phi(t)) \\ V_{\Phi_1}(x,\phi(t)) \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Thus, the sufficiency of $m = iH$ for the incomplete Huygens principle for the generalized Dirac equation in the de Sitter spacetime follows from the property of solution operator of the problem (1.10). It also proves partial validity of the incomplete Huygens principle for the Dirac equation in the de Sitter spacetime with respect to the first components of the initial function. On the other hand, for the second pair of components of the initial function, that is, for the case of

$$\Phi_0(x) = \Phi_1(x) = 0, \tag{2.17}$$

the solution is as follows

$$\begin{aligned}\Psi(x,t) &= e^{-Ht}\left[\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\partial_0 + e^{-Ht}\begin{pmatrix}0&0&0&-1\\0&0&-1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\mathcal{A}_1(x,\partial_x)\right.\\
&\quad +e^{-Ht}\begin{pmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}\mathcal{A}_2(x,\partial_x) + e^{-Ht}\begin{pmatrix}0&0&-1&0\\0&0&0&1\\-1&0&0&0\\0&1&0&0\end{pmatrix}\mathcal{A}_3(x,\partial_x)\\
&\quad \left.-\frac{H}{2}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} - im\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\right]\begin{pmatrix}0\\0\\\mathcal{K}_1(x,t,D_x;M_-)[\Phi_2(x)]\\\mathcal{K}_1(x,t,D_x;M_-)[\Phi_3(x)]\end{pmatrix}.\end{aligned}$$

For the components of the function $\Psi(x,t) = (\Psi_0(x,t), \Psi_1(x,t), \Psi_2(x,t), \Psi_3(x,t))^T$ we obtain

$$\begin{aligned}\Psi_0(x,t) &= e^{-Ht}\Big[-e^{-Ht}\mathcal{A}_1(x,\partial_x)\mathcal{K}_1(x,t,D_x;M_-)[\Phi_3(x)]\\
&\quad +e^{-Ht}i\mathcal{A}_2(x,\partial_x)\mathcal{K}_1(x,t,D_x;M_-)[\Phi_3(x)] - e^{-Ht}\mathcal{A}_3(x,\partial_x)\mathcal{K}_1(x,t,D_x;M_-)[\Phi_2(x)]\Big],\\
\Psi_1(x,t) &= e^{-Ht}\Big[-e^{-Ht}\mathcal{A}_1(x,\partial_x)\mathcal{K}_1(x,t,D_x;M_-)[\Phi_2(x)]\\
&\quad -ie^{-Ht}\mathcal{A}_2(x,\partial_x)\mathcal{K}_1(x,t,D_x;M_-)[\Phi_2(x)] + e^{-Ht}\mathcal{A}_3(x,\partial_x)\mathcal{K}_1(x,t,D_x;M_-)[\Phi_3(x)]\Big],\\
\Psi_2(x,t) &= e^{-Ht}\left(\partial_0 - \frac{H}{2} + im\right)\mathcal{K}_1(x,t,D_x;M_-)[\Phi_2(x)],\\
\Psi_3(x,t) &= e^{-Ht}\left(\partial_0 - \frac{H}{2} + im\right)\mathcal{K}_1(x,t,D_x;M_-)[\Phi_3(x)].\end{aligned}$$

It is easy to see that the tails of the functions $\Psi_k(x,t) = \left(\partial_0 - \frac{H}{2} + im\right)\mathcal{K}_1(x,t,D_x;M_-)[\Phi_k(x)]$ $(k = 2,3)$ are empty. Consider now the tails of the terms

$$\mathcal{A}_j(x,\partial_x)\mathcal{K}_1(x,t,D_x;M_-)[\Phi_k(x)], \quad j = 1,2,3, \quad k = 0,1\,. \tag{2.18}$$

Now we assume that the operator is the Dirac operator (0.2). Then according to (2.15)

$$\begin{aligned}\mathcal{A}_j(x,\partial_x)\mathcal{K}_1(x,t,D_x;M_-)[\Phi_k(x)] &= \frac{\partial}{\partial x_j}\left[\frac{1}{2}e^{\frac{3}{2}Ht}\left(1+e^{-2Ht}\right)V_{\Phi_k}(x,\phi(t))\right.\\
&\quad \left.-H^2\frac{1}{2}e^{\frac{3}{2}Ht}\phi(t)^2V_{\Phi_k}(x,\phi(t)) + H^2e^{\frac{3}{2}Ht}\int_0^{\phi(t)}V_{\Phi_k}(x,s)sds\right]\end{aligned}$$

and the only possible tail of function (2.18) is

$$H^2e^{\frac{3}{2}Ht}\frac{\partial}{\partial x_j}\int_0^{\phi(t)}sV_{\Phi_k}(x,s)ds\,.$$

We apply the Kirchhoff's formula and consider, for instance, the case of $j = 3$ in (2.18). Then up to an unimportant factor, the possible tail is:

$$\begin{aligned}
&\frac{\partial}{\partial x_3}\int_0^{\phi(t)} s^2 \iint_{|y|=1} \Phi_k(x+sy)dS_y = \frac{\partial}{\partial x_3}\iiint_{|y|\le\phi(t)} \Phi_k(x+y)d\,y_1\,dy_2\,dy_3 \\
= &\iiint_{|y|\le\phi(t)} \frac{\partial}{\partial x_3}\Phi_k(x+y)d\,y_1\,dy_2\,dy_3 = \iiint_{|y|\le\phi(t)} \frac{\partial}{\partial y_3}\Phi_k(x+y)d\,y_1\,dy_2\,dy_3 \\
= &\iint_{y_1^2+y_2^2\le\phi^2(t)} dy_1 dy_2 \int_{-\sqrt{\phi^2(t)-y_1^2-y_2^2}}^{\sqrt{\phi^2(t)-y_1^2-y_2^2}} \frac{\partial}{\partial y_3}\Phi_k(x_1+y_1, x_2+y_2, x_3+y_3)\,dy_3 \\
= &\iint_{y_1^2+y_2^2\le\phi^2(t)} \left\{\Phi_k\left(x_1+y_1, x_2+y_2, x_3+\sqrt{\phi^2(t)-y_1^2-y_1^2}\right)\right. \\
&\qquad\qquad \left. -\Phi_k\left(x_1+y_1, x_2+y_2, x_3-\sqrt{\phi^2(t)-y_1^2-y_1^2}\right)\right\} dy_1 dy_2\,.
\end{aligned}$$

For every $t > 0$ the points

$$\left(x_1+y_1, x_2+y_2, x_3 \pm \sqrt{\phi^2(t)-y_1^2-y_2^2}\right) \in \mathbb{R}^3, \quad \text{where } y_1^2+y_1^2 \le \phi^2(t),$$

belong to the sphere of the radius $\phi^2(t)$ in $\mathbb{R}^3$, that is, the domain of integration does not intersect the interior of the domain of dependence. Thus, the tail is empty. This completes the proof of the Huygens principle for the Dirac equation with $m = iH$ in the de Sitter spacetime.

For the case of $m = -iH$ we have $M_+ = \frac{3}{2}H$, $M_- = -\frac{1}{2}H$, and the proof is similar to the case of $m = iH$ that was discussed above. Therefore we skip the proof of the case with $m = -iH$. □

3 Necessity of $m = 0, \pm iH$ for the Huygens principle. The case of $m \neq i\frac{H}{2} + i\frac{H}{2}\ell, \quad \ell = 0, \pm1, \pm2, \ldots$

The proof of the necessity parts of the Theorem 0.2 is based on the large time asymptotic of the tail of solution. The initial data will be chosen radial having small support containing the origin. The tail is generated by the functions

$$\left(\frac{\partial}{\partial t} - \frac{H}{2} - im\right) K_1\left(r, t; \frac{1}{2}H + im\right), \quad \left(\frac{\partial}{\partial t} - \frac{H}{2} + im\right) K_1\left(r, t; \frac{1}{2}H - im\right).$$

where $K_1(r, t; M)$ (1.7) is the kernel of the integral representing the solution of the generalized Dirac equation. The non-huygensian part of the solution written via these functions contains some integrals over interior of the support of the initial function. This term simplifies only for the values of mass mentioned in the theorem and allows us to write solution via solutions of the wave equation without any non-huygensian operation.

We set $F = 0$ in the representation formula given by Theorem 0.2, then the solution is

$$\Psi(x,t) = e^{-Ht}\left(\partial_0\mathbb{I}_4 + e^{-Ht}\gamma^k\gamma^0\partial_k - \frac{H}{2}\mathbb{I}_4 - im\gamma^0\right)\begin{pmatrix} \mathcal{K}_1(x,t,D_x;M_+)[\Phi_0(x)] \\ \mathcal{K}_1(x,t,D_x;M_+)[\Phi_1(x)] \\ \mathcal{K}_1(x,t,D_x;M_-)[\Phi_2(x)] \\ \mathcal{K}_1(x,t,D_x;M_-)[\Phi_3(x)] \end{pmatrix},$$

where the matrices $\gamma^1\gamma^0$, $\gamma^2\gamma^0$,$\gamma^3\gamma^0$ are written in (2.16).

Consider the solution of the Cauchy problem with the radial function $\Phi_0(x) = \Phi_0(r)$, $\operatorname{supp}\Phi_0 \subset \{x \in \mathbb{R}^n; |x| \le \min\{1/2, \varepsilon/H\}\}$, $\varepsilon \in (0,1)$. If we choose the initial data

$$\Phi(x) = (\Phi_0(x), 0, 0, 0)^T, \tag{3.19}$$

then the solution $\Psi(x,t) = (\Psi_0, \Psi_1, \Psi_2, \Psi_3)^T$ is given by

$$\Psi(x,t) = e^{-Ht}\left(\partial_0 \mathbb{I}_4 + e^{-Ht}\gamma^a\gamma^0\partial_a - \frac{H}{2}\mathbb{I}_4 - im\gamma^0\right)(\mathcal{K}_1(x,t,D_x;M_+)[\Phi_0(x)],0,0,0)^T,$$

while its first component is as follows

$$\Psi_0(x,t) = e^{-Ht}\left(\partial_0 - \frac{H}{2} - im\right)\mathcal{K}_1(x,t,D_x;M_+)[\Phi_0(x)].$$

Now we are going to examine the first component of the vector-valued function $\Psi(x,t)$ for large time t. We calculate

$$\Psi_0(x,t) = e^{-Ht}\left(\frac{\partial}{\partial t} - \frac{H}{2} - im\right) 2\int_0^{\phi(t)} v_{\Phi_0}(x,s)K_1(s,t;M_+)ds.$$

We can rewrite $\Psi_0(x,t)$ in the terms of the function V_{Φ_0} defined in accordance to (2.12)

$$\Psi_0(x,t) = e^{-Ht}\left(\frac{\partial}{\partial t} - \frac{H}{2} - im\right) 2\int_0^{\phi(t)} \left(\frac{\partial}{\partial s}V_{\Phi_0}(x,s)\right)K_1(s,t;M_+)\,ds.$$

It follows,

$$\begin{aligned}\Psi_0(x,t) &= 2e^{-Ht}\left(\frac{\partial}{\partial t} - \frac{H}{2} - im\right)V_{\Phi_0}(x,\phi(t))K_1(\phi(t),t;M_+)\\ &\quad -2e^{-2Ht}V_{\Phi_0}(x,\phi(t))\left(\frac{\partial}{\partial s}K_1(s,t;M_+)\right)_{s=\phi(t)}\\ &\quad -2e^{-Ht}\int_0^{\phi(t)} V_{\Phi_0}(x,s)\left(\frac{\partial}{\partial t} - \frac{H}{2} - im\right)\frac{\partial}{\partial s}K_1(s,t;M_+)ds.\end{aligned}$$

In particular, since $x \in \mathbb{R}^3$, by the Kirchhoff's formula we have $V_{\Phi_0}(0,\tau) = \tau\Phi_0(\tau)$ and

$$V_{\Phi_0}(0,\phi(t)) = \phi(t)\Phi_0(\phi(t)) = \frac{1-e^{-Ht}}{H}\Phi_0\left(\frac{1-e^{-Ht}}{H}\right) = 0$$

for sufficiently large t, that is, if $1-e^{-Ht} > H\varepsilon$. Consequently, for large t we have

$$\begin{aligned}\Psi_0(0,t) &= -2e^{-Ht}\int_0^{\phi(t)} s\Phi_0(s)\left(\frac{\partial}{\partial t} - \frac{H}{2} - im\right)\frac{\partial}{\partial s}K_1(s,t;M_+)ds\\ &= 2e^{-Ht}\int_0^1 \left(\frac{\partial}{\partial s}s\Phi_0(s)\right)\left(\frac{\partial}{\partial t} - \frac{H}{2} - im\right)K_1(s,t;M_+)ds.\end{aligned}$$

Now we focus on the possible tail of the solution, that is, on the term generated by the integral. It is easy to see that

$$\begin{aligned}K_1(r,t;M_+) &= 2^{-1-\frac{2im}{H}}e^{\frac{1}{2}t(H+2im)}\left(\left(e^{-Ht}+1\right)^2 - H^2r^2\right)^{\frac{im}{H}}\\ &\quad \times F\left(-\frac{im}{H}, -\frac{im}{H}; 1; \frac{\left(-1+e^{-Ht}\right)^2 - H^2r^2}{\left(1+e^{-Ht}\right)^2 - H^2r^2}\right).\end{aligned}$$

Lemma 3.1 *For $m \in \mathbb{C}$ one has*

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \frac{H}{2} - im\right)K_1(r,t;M_+)\\ &= 2^{-\frac{2im}{H}}ime^{\frac{1}{2}t(2im-H)}\left(\left(1+e^{-Ht}\right)^2 - H^2r^2\right)^{\frac{im}{H}-2}\\ &\quad\times\left\{2\frac{im}{H}\left(1-e^{-2Ht}-H^2r^2\right)F\left(1-\frac{im}{H}, 1-\frac{im}{H}; 2; \frac{\left(1-e^{-Ht}\right)^2 - H^2r^2}{\left(1+e^{-Ht}\right)^2 - H^2r^2}\right)\right.\\ &\quad\left. -\left(1+e^{-Ht}\right)\left((1+e^{-Ht})^2 - H^2r^2\right)F\left(-\frac{im}{H}, -\frac{im}{H}; 1; \frac{\left(1-e^{-Ht}\right)^2 - H^2r^2}{\left(1+e^{-Ht}\right)^2 - H^2r^2}\right)\right\}.\end{aligned}$$

Proof. We skip the proof since this statement can be verified by straightforward calculations. □

Proposition 3.2 *Assume that* $m \in \mathbb{C}$ *and*

$$m \neq i\frac{H}{2} + i\frac{H}{2}\ell, \quad \ell = 0, \pm 1, \pm 2, \dots ,$$

then

$$\begin{aligned}
&2\frac{im}{H}\left(1-e^{-2Ht}-H^2r^2\right) F\left(1-\frac{im}{H}, 1-\frac{im}{H}; 2; \frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)\\
&-\left(e^{-Ht}+1\right)\left((1+e^{-Ht})^2-H^2r^2\right) F\left(-\frac{im}{H}, -\frac{im}{H}; 1; \frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)\\
= \quad & 2\frac{im}{H}4^{2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1-H^2r^2\right)^{1-2\frac{im}{H}} e^{-2imt}(1+R(m,H,r;t)),
\end{aligned}$$

where with large T *the remainder* $R(m,H,r;t)$ *can be estimated as follows*

$$|R(m,H,r;t)| \leq o(1) \quad \text{for all} \quad t \geq T \quad \textit{and} \quad 0 \leq r \leq \min\{1/2, 1/(2H)\}. \tag{3.20}$$

Proof. Consider the first term. In order to simplify notations we denote

$$\begin{aligned}
&A := Hr, \quad \tau := e^{-Ht},\\
&z := \frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2} = \frac{(1-\tau)^2-A^2}{(1+\tau)^2-A^2} = 1-\frac{4\tau}{1-H^2r^2}+\frac{8\tau^2}{\left(1-H^2r^2\right)^2}+O\left(\tau^3\right),\\
&1-z = \frac{4e^{-Ht}}{\left(1+e^{-Ht}\right)^2-H^2r^2} = \frac{4\tau}{(1+\tau)^2-A^2} = \frac{4\tau}{1-H^2r^2}-\frac{8\tau^2}{\left(1-H^2r^2\right)^2}+O\left(\tau^3\right).
\end{aligned}$$

There is a formula (A.1) in Appendix that ties together points $z=0$ and $z=1$ of the argument of the hypergeometric function. Hence, we have (A.1) for all $1/2 < z < 1$. For all $m \in \mathbb{C}$ such that

$$a = b = 1-\frac{im}{H}, \quad c = 2, \quad \frac{im}{H} \neq 0, \pm 1, \pm 2, \dots \quad \Longleftrightarrow \quad m \neq 0, \pm iH, \pm 2iH, \pm 3iH, \dots ,$$

according to (A.1) we can write

$$\begin{aligned}
F\left(1-\frac{im}{H}, 1-\frac{im}{H}; 2; z\right) = & \frac{\Gamma(2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2} F\left(1-\frac{im}{H}, 1-\frac{im}{H}; 1-2\frac{im}{H}; 1-z\right)\\
& +(1-z)^{2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2} F\left(1+\frac{im}{H}, 1+\frac{im}{H}; 1+2\frac{im}{H}; 1-z\right).
\end{aligned}$$

From $F(a,b;c;0) = 1$ it follows

$$\begin{aligned}
F\left(1-\frac{im}{H}, 1-\frac{im}{H}; 2; z\right) = & \frac{\Gamma(2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}\left(1+R_0(m,H,r;t)\right)\\
& +(1-z)^{2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1+R_1(m,H,r;t)\right),
\end{aligned}$$

where with large T the remainders $R_k(m,H,r;t)$, $k = 0,1$, can be estimated as follows

$$|R_k(m,H,r;t)| \leq o(1) \quad \text{for all} \quad t \geq T \quad \text{and} \quad 0 \leq r \leq 1/(2H).$$

For the second term of the proposition we note

$$a=b=-\frac{im}{H}\,,\quad c=1\,,\quad c-a-b=1+2\frac{im}{H}\neq 0,\pm 1,\pm 2,\dots\,, \tag{3.21}$$

$$\Longleftrightarrow m\neq i\frac{H}{2},\, i\frac{H}{2}\pm i\frac{H}{2},\, i\frac{H}{2}\pm 2i\frac{H}{2},\, i\frac{H}{2}\pm 3i\frac{H}{2},\dots\,, \tag{3.22}$$

and obtain

$$\begin{aligned} F\left(-\frac{im}{H},-\frac{im}{H};1;z\right) &= \frac{\Gamma(1+2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}F\left(-\frac{im}{H},-\frac{im}{H};-2\frac{im}{H};1-z\right)\\ &\quad +(1-z)^{1+2\frac{im}{H}}\frac{\Gamma(-1-2\frac{im}{H})}{[\Gamma(-\frac{im}{H})]^2}F\left(1+\frac{im}{H},1+\frac{im}{H};2+2\frac{im}{H};1-z\right). \end{aligned}$$

Assume that $\Re(im)\geq 0$. Then we have

$$F\left(-\frac{im}{H},-\frac{im}{H};1;z\right)=\frac{\Gamma(1+2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}+o(1)\quad\text{as}\quad z\nearrow 1.$$

It follows for $\Re(im)\geq 0$

$$\begin{aligned} & 2\frac{im}{H}\left(1-\tau^2-H^2r^2\right)F\left(1-\frac{im}{H},1-\frac{im}{H};2;z\right)\\ & -(\tau+1)\left((1+\tau)^2-H^2r^2\right)F\left(-\frac{im}{H},-\frac{im}{H};1;z\right)\\ =\; & 2\frac{im}{H}\left(1-\tau^2-H^2r^2\right)\left\{\frac{\Gamma(2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}+(1-z)^{2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}+o(1)\right\}\\ & -(\tau+1)\left((1+\tau)^2-H^2r^2\right)\left\{\frac{\Gamma(1+2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}+o(1)\right\}\\ =\; & 2\frac{im}{H}\left(1-\tau^2-H^2r^2\right)\left\{\frac{\Gamma(2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}+(1-z)^{2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}+o(1)\right\}\\ & -\left(1-H^2r^2\right)\left\{\frac{\Gamma(1+2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}+o(1)\right\}\\ =\; & 2\frac{im}{H}\left(1-\tau^2-H^2r^2\right)\left\{(1-z)^{2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}+o(1)\right\}. \end{aligned}$$

Thus,

$$\begin{aligned} & 2\frac{im}{H}\left(1-e^{-2Ht}-H^2r^2\right)F\left(1-\frac{im}{H},1-\frac{im}{H};2;z\right)\\ & -\left(e^{-Ht}+1\right)\left((1+e^{-Ht})^2-H^2r^2\right)F\left(-\frac{im}{H},-\frac{im}{H};1;z\right)\\ =\; & 2\frac{im}{H}\left(4e^{-Ht}\right)^{2\frac{im}{H}}\left(1-H^2r^2\right)^{1-2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}+o(1)\,,\quad\text{as}\quad t\to\infty\,, \end{aligned}$$

uniformly with respect to $r\in(0,\leq\min\{1/2,1/(2H)\})$. Next we assume that $\Re(im)<0$. Consider

$$\begin{aligned} F\left(1-\frac{im}{H},1-\frac{im}{H};2;z\right) &= \frac{\Gamma(2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}\left(1+R_0(m,H,r;t)\right)\\ &\quad +(1-z)^{2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1+R_1(m,H,r;t)\right). \end{aligned}$$

Since $\Re(im) < 0$, the principal term of the asymptotics is $(1-z)^{2\frac{im}{H}} \frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}$ and

$$F\left(1-\frac{im}{H},1-\frac{im}{H};2;z\right) = (1-z)^{2\frac{im}{H}} \frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1+R_1(m,H,r;t)\right).$$

Now for parameters a,b,c satisfying (3.21),(3.22) consider

$$\begin{aligned} F\left(-\frac{im}{H},-\frac{im}{H};1;z\right) &= \frac{\Gamma(1+2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}F\left(-\frac{im}{H},-\frac{im}{H};-2\frac{im}{H};1-z\right) \\ &\quad +(1-z)^{1+2\frac{im}{H}}\frac{\Gamma(-1-2\frac{im}{H})}{[\Gamma(-\frac{im}{H})]^2}F\left(1+\frac{im}{H},1+\frac{im}{H};2+2\frac{im}{H};1-z\right). \end{aligned}$$

If $1+2\frac{\Re(im)}{H}=\delta>0$, then the principal term is $\frac{\Gamma(1+2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}$ and we obtain

$$F\left(-\frac{im}{H},-\frac{im}{H};1;z\right) = \frac{\Gamma(1+2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}+O((1-z)^{\delta}) \quad \text{as} \quad z\nearrow 1.$$

If $1+2\frac{\Re(im)}{H}=\delta<0$ then the principal term is $(1-z)^{1+2\frac{im}{H}}\frac{\Gamma(-1-2\frac{im}{H})}{[\Gamma(-\frac{im}{H})]^2}$ and we obtain

$$F\left(-\frac{im}{H},-\frac{im}{H};1;z\right) = (1-z)^{1+2\frac{im}{H}}\frac{\Gamma(-1-2\frac{im}{H})}{[\Gamma(-\frac{im}{H})]^2}(1+O((1-z)^{-\delta})).$$

If $1+2\frac{\Re(im)}{H}=0$ then both terms are equivalent and we obtain

$$\begin{aligned} F\left(-\frac{im}{H},-\frac{im}{H};1;z\right) &= \frac{\Gamma(1+2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}F\left(-\frac{im}{H},-\frac{im}{H};-2\frac{im}{H};1-z\right) \\ &\quad +(1-z)^{1+2\frac{im}{H}}\frac{\Gamma(-1-2\frac{im}{H})}{[\Gamma(-\frac{im}{H})]^2}F\left(1+\frac{im}{H},1+\frac{im}{H};2+2\frac{im}{H};1-z\right) \\ &= \frac{\Gamma(1+2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}+(1-z)^{2i\frac{\Re(m)}{H}}\frac{\Gamma(-1-2\frac{im}{H})}{[\Gamma(-\frac{im}{H})]^2}+o(\tau). \end{aligned}$$

Thus, if $1+2\frac{\Re(im)}{H}=\delta>0$ and $\Re(im)<0 \Longleftrightarrow \Im(m)>0$, then

$$
\begin{aligned}
& 2\frac{im}{H}\left(1-e^{-2Ht}-H^2r^2\right)F\left(1-\frac{im}{H},1-\frac{im}{H};2;z\right) \\
& -\left(e^{-Ht}+1\right)\left((1+e^{-Ht})^2-H^2r^2\right)F\left(-\frac{im}{H},-\frac{im}{H};1;z\right) \\
= \; & 2\frac{im}{H}\left(1-e^{-2Ht}-H^2r^2\right)\left[(1-z)^{2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1+R_1(m,H,r;t)\right)\right] \\
& -\left(e^{-Ht}+1\right)\left((1+e^{-Ht})^2-H^2r^2\right)\left[\frac{\Gamma(1+2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}+O((1-z)^\delta)\right] \\
= \; & 2\frac{im}{H}\left(1-H^2r^2\right)\left[(1-z)^{2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1+R_1(m,H,r;t)\right)\right] \\
& -\left(1-H^2r^2\right)\left[\frac{\Gamma(1+2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}+O((1-z)^\delta)\right] \\
= \; & 2\frac{im}{H}\left(1-H^2r^2\right)(1-z)^{-2\frac{\Im(m)}{H}}(1-z)^{2\frac{i\Re(m)}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1+o(1)\right) \\
= \; & 2\frac{im}{H}\left(1-H^2r^2\right)\left(\frac{4\tau}{(1+\tau)^2-A^2}\right)^{-2\frac{\Im(m)}{H}+2\frac{i\Re(m)}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1+o(1)\right) \\
= \; & 4^{2\frac{im}{H}}2\frac{im}{H}\left(1-A^2\right)^{1-2\frac{im}{H}}\tau^{2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1+o(1)\right), \quad \text{as} \quad t\to\infty\,.
\end{aligned}
$$

Thus, if $1+2\frac{\Re(im)}{H}=\delta<0$ and $\Re(im)<0 \Longleftrightarrow \Im(m)>0$, then

$$
\begin{aligned}
& 2\frac{im}{H}\left(1-e^{-2Ht}-H^2r^2\right)F\left(1-\frac{im}{H},1-\frac{im}{H};2;z\right) \\
& -\left(e^{-Ht}+1\right)\left((1+e^{-Ht})^2-H^2r^2\right)F\left(-\frac{im}{H},-\frac{im}{H};1;z\right) \\
= \; & 2\frac{im}{H}\left(1-e^{-2Ht}-H^2r^2\right)\left[(1-z)^{2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1+R_1(m,H,r;t)\right)\right] \\
& -\left(e^{-Ht}+1\right)\left((1+e^{-Ht})^2-H^2r^2\right)\left[(1-z)^{1+2\frac{im}{H}}\frac{\Gamma(-1-2\frac{im}{H})}{[\Gamma(-\frac{im}{H})]^2}(1+O((1-z)))\right] \\
= \; & 2\frac{im}{H}\left(1-H^2r^2\right)(1-z)^{2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1+O((1-z))\right) \\
= \; & 4^{2\frac{im}{H}}2\frac{im}{H}\left(1-H^2r^2\right)^{1-2\frac{im}{H}}\tau^{2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1+O((1-z))\right)\,.
\end{aligned}
$$

Thus, if $1+2\frac{\Re(im)}{H}=0$ and $\Re(im)=-\frac{H}{2}<0 \Longleftrightarrow \Im(m)=\frac{H}{2}>0$, then

$$\begin{aligned}
& 2\frac{im}{H}\left(1-e^{-2Ht}-H^2r^2\right)F\left(1-\frac{im}{H},1-\frac{im}{H};2;z\right) \\
& -\left(e^{-Ht}+1\right)\left((1+e^{-Ht})^2-H^2r^2\right)F\left(-\frac{im}{H},-\frac{im}{H};1;z\right) \\
= & \ 2\frac{im}{H}\left(1-e^{-2Ht}-H^2r^2\right)\left[(1-z)^{2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1+R_1(m,H,r;t)\right)\right] \\
& -\left(e^{-Ht}+1\right)\left((1+e^{-Ht})^2-H^2r^2\right)\left[\frac{\Gamma(1+2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}+(1-z)^{2i\frac{\Re(m)}{H}}\frac{\Gamma(-1-2\frac{im}{H})}{[\Gamma(-\frac{im}{H})]^2}+o(\tau)\right] \\
= & \ 2\frac{im}{H}\left(1-e^{-2Ht}-H^2r^2\right)\left[(1-z)^{-1+2i\frac{\Im(im)}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1+R_1(m,H,r;t)\right)\right] \\
& -\left(e^{-Ht}+1\right)\left((1+e^{-Ht})^2-H^2r^2\right)\left[\frac{\Gamma(1+2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}+(1-z)^{2i\frac{\Re(m)}{H}}\frac{\Gamma(-1-2\frac{im}{H})}{[\Gamma(-\frac{im}{H})]^2}+o(\tau)\right] \\
= & \ 2\frac{im}{H}\left(1-H^2r^2\right)(1-z)^{2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1+o(\tau)\right)) \\
= & \ 4^{2\frac{im}{H}}2\frac{im}{H}\left(1-H^2r^2\right)^{1-2\frac{im}{H}}\tau^{2\frac{im}{H}}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1+O((1-z))\right)\,.
\end{aligned}$$

The proposition is proved. □

The necessity of $m\neq i\frac{H}{2}+i\frac{H}{2}\ell$, $\ell=0,\pm1,\pm2,\dots$ in Theorem 0.2. From Lemma 3.1 and Proposition 3.2 we have

$$\begin{aligned}
\Psi_0(0,t) = & \ 2e^{-Ht}\int_0^1\left(\frac{\partial}{\partial r}r\Phi_0(r)\right)\left[2^{-\frac{2im}{H}}ime^{\frac{1}{2}t(2im-H)}\left(\left(1+e^{-Ht}\right)^2-H^2r^2\right)^{\frac{im}{H}-2}\right. \\
& \times\left\{2\frac{im}{H}\left(1-e^{-2Ht}-H^2r^2\right)F\left(1-\frac{im}{H},1-\frac{im}{H};2;\frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)\right. \\
& \left.\left.-\left(e^{-Ht}+1\right)\left((1+e^{-Ht})^2-H^2r^2\right)F\left(-\frac{im}{H},-\frac{im}{H};1;\frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)\right\}\right]dr \\
= & \ 2e^{-Ht}\int_0^1\left(\frac{\partial}{\partial r}s\Phi_0(r)\right)\left[2^{-\frac{2im}{H}}ime^{\frac{1}{2}t(2im-H)}\left(\left(1+e^{-Ht}\right)^2-H^2r^2\right)^{\frac{im}{H}-2}\right. \\
& \left.\times\left\{2\frac{im}{H}4^{2\frac{im}{H}}e^{-2imt}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1-H^2r^2\right)^{1-2\frac{im}{H}}+R(m,H,r;t)\right\}\right]dr \\
= & \ 2e^{-Ht}\int_0^1\left(\frac{\partial}{\partial r}r\Phi_0(r)\right)\left[2^{-\frac{2im}{H}}ime^{\frac{1}{2}t(2im-H)}\left(\left(1+e^{-Ht}\right)^2-H^2r^2\right)^{\frac{im}{H}-2}\right. \\
& \left.\times2\frac{im}{H}4^{2\frac{im}{H}}e^{-2imt}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2}\left(1-H^2r^2\right)^{1-2\frac{im}{H}}+R(m,H,r;t)\right]dr \\
= & \ -e^{\frac{1}{2}t(-2im-3H)}H^{-1}m^2 2^{2\frac{im}{H}+2}\frac{\Gamma(-2\frac{im}{H})}{[\Gamma(1-\frac{im}{H})]^2} \\
& \times\int_0^1\left[\left(\frac{\partial}{\partial r}r\Phi_0(r)\right)\left(1-H^2r^2\right)^{-1-\frac{im}{H}}+R(m,H,r;t)\right]dr\,.
\end{aligned}$$

Next we choose Φ_0 such that

$$\int_0^1 \left(\frac{\partial}{\partial r} r\Phi_0(r)\right)\left(1-H^2r^2\right)^{-1-\frac{im}{H}}\,dr \neq 0\,.$$

The last equation shows that for $m \neq i\frac{H}{2}+i\frac{H}{2}\ell, \quad \ell=0,\pm1,\pm2,\ldots$, the value $\Psi_1(0,t)$ for large time depends on the values of the initial function inside of the characteristic conoid. This completes the proof of necessity of such values of m.

4 Necessity of $m=0,\pm iH$ for the Huygens principle. The case of $m=i\frac{H}{2}+i\frac{H}{2}\ell, \quad \ell=0,\pm1,\pm2,\ldots$, and $m\neq 0 \Longleftrightarrow \ell\neq -1$

We remind that after Section 3 to complete the proof it remains to consider the following values of mass: $m=i\frac{H}{2}+i\frac{H}{2}\ell, \quad \ell=0,\pm1,\pm2,\ldots$, and $m\neq 0 \Longleftrightarrow \ell\neq -1$. For all these values we have

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-\frac{H}{2}-im\right)K_1(r,t;M_+) &= \left(\frac{\partial}{\partial t}+\frac{H}{2}\ell\right)K_1\left(r,t;-\frac{H}{2}\ell\right)\\
&= H2^\ell(1+\ell)e^{-\frac{1}{2}H(\ell+2)t}\left(\left(1+e^{-Ht}\right)^2-H^2r^2\right)^{-\frac{1}{2}(\ell+5)}\\
&\quad\times\mathcal{F}\left(e^{-Ht},Hr;\ell;\frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right),
\end{aligned}$$

where the function $\mathcal{F}$ is defined as follows

$$\begin{aligned}
&\mathcal{F}\left(e^{-Ht},Hr;\ell;\frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)\\
:=\;& \left(1+e^{-Ht}\right)\left(\left(1+e^{-Ht}\right)^2-H^2r^2\right)F\left(\frac{\ell+1}{2},\frac{\ell+1}{2};1;\frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)\\
&+(\ell+1)\left(1-e^{-2Ht}-H^2r^2\right)F\left(\frac{\ell+3}{2},\frac{\ell+3}{2};2;\frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right).
\end{aligned}$$

We are going to study a large time asymptotics of $\mathcal{F}$. We cannot apply (A.1) since each term of the formula (A.1) has a pole when $c=a+b\pm k$, $(k=0,1,2,\ldots)$. We split all possible $m=i\frac{H}{2}+i\frac{H}{2}\ell, \quad \ell=0,\pm1,\pm2,\ldots$, $(m\neq 0 \Longleftrightarrow \ell\neq -1)$ into next seven sets:

1. $\ell=2k+1$, $k=-2,-3,-4,\ldots$,
2. $\ell=2k+1$, $k=1,2,3,\ldots$,
3. $\ell=2k$, $k=1,2,3,\ldots$,
4. $\ell=2k$, $k=-2,-3,-4,\ldots$,
5. $m=-i\frac{H}{2}$, that is $\ell=2k$, $k=-1$, $\ell=-2$,
6. $m=i\frac{H}{2}$, that is $\ell=2k$, $k=\ell=0$,
7. $m=iH$, $\ell=1$, $\ell=2k+1$, $k=0$.

5 The case of odd $\ell = 2k+1$, $k = 0, \pm 1, \pm 2, \ldots$

In order to complete the proof, it remains to consider the following values of mass:

$$m = i\frac{H}{2} + i\frac{H}{2}\ell, \quad \ell = 0, \pm 1, \pm 2, \ldots, \quad \ell \neq -1\,.$$

In this section consider the case of $\ell = 2k+1$, $k = 0, \pm 1, \pm 2, \ldots$, then

$$\begin{aligned}
&\mathcal{F}\left(e^{-Ht}, Hr; \ell; \frac{\left(1-e^{-Ht}\right)^2 - H^2r^2}{\left(1+e^{-Ht}\right)^2 - H^2r^2}\right) \\
= \quad & \left(1+e^{-Ht}\right)\left(\left(1+e^{-Ht}\right)^2 - H^2r^2\right) F\left(k+1, k+1; 1; \frac{\left(1-e^{-Ht}\right)^2 - H^2r^2}{\left(1+e^{-Ht}\right)^2 - H^2r^2}\right) \\
& +(2k+2)\left(1-e^{-2Ht} - H^2r^2\right) F\left(k+2, k+2; 2; \frac{\left(1-e^{-Ht}\right)^2 - H^2r^2}{\left(1+e^{-Ht}\right)^2 - H^2r^2}\right).
\end{aligned}$$

5.1 The case of negative odd $\ell = 2k+1$, $k = -2, -3, \ldots$

Denote $A := Hr \in [0, 1/2]$, $\tau = e^{-Ht}$, and $z := \frac{(1-\tau)^2 - A^2}{(1+\tau)^2 - A^2}$. Since

$$\begin{aligned}
F(-n, -n; 1; z) &= 1 + \sum_{j=1}^{n} \frac{\Gamma(n+1)^2}{\Gamma(j+1)^2\Gamma(n-j+1)^2} z^j\,, \\
F(-n, -n; 2; z) &= 1 + \sum_{j=1}^{n} \frac{\Gamma(n+1)^2}{\Gamma(j+2)\Gamma(j+1)\Gamma(n-j+1)^2} z^j\,,
\end{aligned}$$

we have with $-n = k+1$ and $-n = k+2$

$$\begin{aligned}
&\mathcal{F}\left(e^{-Ht}, Hr; \ell; \frac{\left(1-e^{-Ht}\right)^2 - H^2r^2}{\left(1+e^{-Ht}\right)^2 - H^2r^2}\right) \\
= \quad & (1+\tau)\left((1+\tau)^2 - A^2\right) F\left(k+1, k+1; 1; z\right) + (2k+2)\left(1-\tau^2 - A^2\right) F\left(k+2, k+2; 2; z\right) \\
= \quad & (1+\tau)\left((1+\tau)^2 - A^2\right)\left\{1 + \sum_{j=1}^{-k-1} \frac{\Gamma(-k)^2}{\Gamma(j+1)^2\Gamma(-k-j)^2}\left[\frac{(1-\tau)^2 - A^2}{(1+\tau)^2 - A^2}\right]^j\right\} \\
& +(2k+2)\left(1-\tau^2 - A^2\right)\left\{1 + \sum_{j=1}^{-k-2} \frac{\Gamma(-k-1)^2}{\Gamma(j+2)\Gamma(j+1)\Gamma(-k-j-1)^2}\left[\frac{(1-\tau)^2 - A^2}{(1+\tau)^2 - A^2}\right]^l\right\}.
\end{aligned}$$

For the function $\mathcal{F}\left(\tau, A; \ell; \frac{(1-\tau)^2 - A^2}{(1+\tau)^2 - A^2}\right)$ with these values of ℓ at $\tau = 0$ we have

$$\mathcal{F}\left(0, A; \ell; 1\right) = 0.$$

Then

$$\begin{aligned}
\partial_\tau \mathcal{F}\left(\tau, A; \ell; \frac{(1-\tau)^2 - A^2}{(1+\tau)^2 - A^2}\right) = \quad & \left(3(\tau+1)^2 - A^2\right) F\left(k+1, k+1; 1; \frac{(1-\tau)^2 - A^2}{(1+\tau)^2 - A^2}\right) \\
& + \frac{4(k+1)}{\tau^4 - 2\left(A^2+1\right)\tau^2 + \left(A^2-1\right)^2}\Bigg[(A-\tau+1)(A+\tau-1) \\
& \times \left(A^2 + 2\tau^3 + 3\tau^2 - 1\right) F\left(k+2, k+2; 2; \frac{(A-\tau+1)(A+\tau-1)}{(A-\tau-1)(A+\tau+1)}\right) \\
& -2(k+2)\left(A^2+\tau^2-1\right)^2 F\left(k+2, k+3; 2; \frac{(1-\tau)^2 - A^2}{(1+\tau)^2 - A^2}\right)\Bigg)\Bigg]
\end{aligned}$$

and

$$\partial_\tau \mathcal{F}\left(\tau, A; \ell; \frac{(1-\tau)^2 - A^2}{(1+\tau)^2 - A^2}\right)\Bigg|_{\tau=0} = -\frac{(-2k-2))!}{(2k+3)[(-k-1)!]^2}\left(A^2(2k+3)+1\right), \quad k=-3,-4,\dots.$$

If $k=-2$, then $\ell=-3$ and the function of Lemma 3.1 is simplified to the function independent of r that after substitution in $\Psi_0(0,t)$ creates a huygensian part and cannot be used to get a contradiction. In fact for $\ell=-3$ the mass is $m=-iH$, and the equation is huygensian. This explains why we exclude $\ell=-3$ from the further consideration.

Lemma 5.1 *For $n=-1,-2,\dots$ the polynomials $F(n+1,n+1;2;1-x)$ and $F(n,n;1;1-x)$ as $x \searrow 0$ satisfy*

$$\begin{aligned}
&F(n+1,n+1;2;1-x) = \frac{\Gamma(-2n)}{\Gamma(1-n)^2} + x\frac{1}{2}(n+1)^2\frac{2\Gamma(-2n-1)}{\Gamma(1-n)^2} + x^2 O(1), \quad n=-1,-2,\dots,\\
&F(n,n;1;1-x) = \frac{\Gamma(1-2n)}{\Gamma(1-n)^2} + xn^2\frac{\Gamma(-2n)}{\Gamma(1-n)^2} + x^2 O(1), \quad n=-1,-2,\dots,\\
&2nF(n+1,n+1;2;1) + F(n,n;1;1) = 0, \quad n=-1,-2,\dots,\\
&\frac{d}{dx}F(n+1,n+1;2;1-x)\Big|_{x=0} = -\frac{1}{2}(n+1)^2\frac{2\Gamma(-2n-1)}{\Gamma(1-n)^2},\\
&\frac{d}{dx}F(n,n;1;1-x)\Big|_{x=0} = -n^2\frac{\Gamma(-2n)}{\Gamma(1-n)^2},\\
&\frac{d}{dx}\left(2nF(n+1,n+1;2;1-x) + F(n,n;1;1-x)\right)\Big|_{x=0}\\
&= -\frac{n}{\Gamma(1-n)^2}(3n+2)\Gamma(-2n-1) = A(n) < 0, \quad n=-1,-2,\dots.
\end{aligned}$$

Hence,

$$\begin{aligned}
&2nF(n+1,n+1;2;1-x) + F(n,n;1;1-x)\\
=\;& x\frac{n}{\Gamma(1-n)^2}(3n+2)\Gamma(-2n-1) + x^2O(1), \quad x\searrow 0, \quad n=-1,-2,\dots.
\end{aligned}$$

Proof. For $n=-1,-2,\dots$ we have $c-a-b=-2n>0$ and the hypergeometric functions are polynomials. We apply (A.1) and obtain

$$F(n+1,n+1;2;1) = \frac{\Gamma(-2n)}{\Gamma(1-n)^2} \quad \text{if} \quad n<0, \quad \text{and} \quad F(n,n;1;1) = \frac{\Gamma(1-2n)}{\Gamma(1-n)^2} \quad \text{if} \quad n\le 0,$$

then

$$2nF(n+1,n+1;2;1) + F(n,n;1;1) = 2n\frac{\Gamma(-2n)}{\Gamma(1-n)^2} + \frac{\Gamma(1-2n)}{\Gamma(1-n)^2} = 0 \quad \text{if} \quad n<0.$$

Further according to [4, (7) Ch 2] and (A.1) we obtain

$$\begin{aligned}
\frac{d}{dx}F(n+1,n+1;2;1-x) &= -\frac{1}{2}(n+1)^2F(n+2,n+2;3;1-x),\\
\frac{d}{dx}F(n,n;1;1-x) &= -n^2F(n+1,n+1;2;1-x)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dx}F(n+1,n+1;2;1-x)\Big|_{x=0} &= -\frac{1}{2}(n+1)^2\frac{2\Gamma(-2n-1)}{\Gamma(1-n)^2}, \quad n=-1,-2,\dots,\\
\frac{d}{dx}F(n,n;1;1-x)\Big|_{x=0} &= -n^2\frac{\Gamma(-2n)}{\Gamma(1-n)^2}, \quad n=-1,-2,\dots.
\end{aligned}$$

Then for $n = -1, -2, -3, \ldots$

$$\begin{aligned} & \frac{d}{dx}\left(2nF(n+1,n+1;2;1-x)+F(n,n;1;1-x)\right)\Big|_{x=0} \\ = & \left(-2n\frac{1}{2}(n+1)^2F(n+2,n+2;3;1-x)-n^2F(n+1,n+1;2;1-x)\right)\Big|_{x=0}. \end{aligned}$$

For $3-2(n+2)>0$ and $2-2(n+1)>0$ (that is, $n=-1,-2,\ldots$) we obtain

$$\frac{d}{dx}\left(2nF(n+1,n+1;2;1-x)+F(n,n;1;1-x)\right)\Big|_{x=0} = -\frac{n}{\Gamma(1-n)^2}(3n+2)\Gamma(-2n-1)\,.$$

Lemma is proved. □

Corollary 5.2 *For every $k=-2,-3,\ldots$:*

$$\begin{aligned} & (1+\tau)\left((1+\tau)^2-A^2\right)F(k+1,k+1;1;z)+(2k+2)\left(1-\tau^2-A^2\right)F(k+2,k+2;2;z) \\ = & \ 2\frac{(k+1)}{\Gamma(-k)^2}\left[(A^2(-2k-3)+12k+19)\Gamma(-2k-3)+4\Gamma(-k)^2\right]\tau+\tau^2O(1), \end{aligned}$$

as $\tau \searrow 0$. Here for every $k=-2,-3,\ldots$ if A is such that

$$A<A(k)=\min\left\{\frac{1}{2},\left(\frac{1}{(2k+3)}\left[12k+19+4\frac{\Gamma(-k)^2}{\Gamma(-2k-3)}\right]\right)^{1/2}\right\}, \qquad (5.23)$$

then

$$(k+1)\left(\left[-A^2(2k+3)+12k+19\right]\frac{\Gamma(-2k-3)}{\Gamma(-k)^2}+4\right)>0\,.$$

Proof. First of all, we note that inequality

$$4\frac{\Gamma(-k)^2}{\Gamma(-2k-3)}\leq 2$$

implies

$$\left[-A^2(2k+3)+12k+19\right]\frac{\Gamma(-2k-3)}{\Gamma(-k)^2}+4<0,$$

for all A satisfying (5.23). Next we consider

$$\begin{aligned} & (1+\tau)\left((1+\tau)^2-A^2\right)F(k+1,k+1;1;z) \\ = & \left(\left(1-A^2\right)+\left(3-A^2\right)\tau+3\tau^2+\tau^3\right)\left[\frac{\Gamma(1-2(k+1))}{\Gamma(1-(k+1))^2}\right. \\ & \left.+\frac{4\tau}{(1+\tau)^2-A^2}(k+1)^2\frac{\Gamma(-2(k+1))}{\Gamma(1-(k+1))^2}+\left(\frac{4\tau}{(1+\tau)^2-A^2}\right)^2O(1)\right]. \end{aligned}$$

We can continue it as follows

$$\begin{aligned} = & \left(\left(1-A^2\right)+\left(3-A^2\right)\tau+3\tau^2+\tau^3\right) \\ & \times\left[\frac{\Gamma(-2k-1)}{\Gamma(-k)^2}+\left(-\frac{4\tau}{A^2-1}-\frac{8\tau^2}{(A^2-1)^2}-\frac{4\left(A^2+3\right)\tau^3}{(A^2-1)^3}+O\left(\tau^4\right)\right)(k+1)^2\frac{\Gamma(-2k-2)}{\Gamma(-k)^2}\right. \\ & \left.+\left(\frac{16\tau^2}{(A^2-1)^2}+\frac{64\tau^3}{(A^2-1)^3}+O\left(\tau^4\right)\right)\right] \end{aligned}$$

and, consequently,

$$\begin{aligned}
&(1+\tau)\left((1+\tau)^2-A^2\right)F(k+1,k+1;1;z)\\
=\;& -\frac{\left(A^2-1\right)\Gamma(-2k-1)}{\Gamma(-k)^2}+\tau\left(-\frac{\left(A^2+2k-1\right)\Gamma(-2k-1)}{\Gamma(-k)^2}\right)\\
&+\tau^2\left(\frac{(1-2k)\Gamma(-2k-1)}{\Gamma(-k)^2}-\frac{16}{A^2-1}\right)+\tau^3\left(\frac{\Gamma(-2k-1)}{\Gamma(-k)^2}-\frac{16\left(A^2+1\right)}{\left(A^2-1\right)^2}\right)+O\left(\tau^4\right).
\end{aligned}$$

For the term with $F(k+2,k+2;2;z)$ we obtain

$$\begin{aligned}
&(2k+2)\left(1-\tau^2-A^2\right)F(k+2,k+2;2;z)\\
=\;&(2k+2)\left(1-\tau^2-A^2\right)\left[\frac{\Gamma(-2k-2)}{\Gamma(-k)^2}+(1-z)\frac{1}{2}(k+2)^2\frac{2\Gamma(-2k-3)}{\Gamma(-k)^2}+(1-z)^2O(1)\right]\\
=\;&-\frac{2\left(\left(A^2-1\right)(k+1)\Gamma(-2k-2)\right)}{\Gamma(-k)^2}+\tau\left(8(k+1)\left(\frac{(k+2)^2\Gamma(-2k-3)}{\Gamma(-k)^2}+1\right)\right)\\
&+\tau^2\frac{2(k+1)}{A^2-1}\left(\frac{\left(A^2(2k+3)+8k^2+30k+29\right)\Gamma(-2k-3)}{\Gamma(-k)^2}+8\right)\\
&+\tau^3\frac{16\left(A^2+1\right)(k+1)}{\left(A^2-1\right)^2}\left(\frac{(k+2)^2\Gamma(-2k-3)}{\Gamma(-k)^2}+1\right)+O(\tau^4).
\end{aligned}$$

Hence,

$$\begin{aligned}
&(1+x)\left((1+\tau)^2-A^2\right)F(k+1,k+1;1;z)+(2k+2)\left(1-\tau^2-A^2\right)F(k+2,k+2;2;z)\\
=\;&2(k+1)\tau\left(\frac{\left(A^2(-(2k+3))+12k+19\right)\Gamma(-2k-3)}{\Gamma(-k)^2}+4\right)\\
&+\tau^2\frac{4}{A^2-1}\left(4k-\frac{(k+1)\left(A^2\left(2k^2+k-3\right)-k(6k+17)-13\right)\Gamma(-2k-3)}{\Gamma(-k)^2}\right)\\
&+\tau^3\frac{2}{\left(A^2-1\right)^2}\left(8\left(A^2+1\right)k+\frac{(k+1)\left(A^2+4k+7\right)\left(A^2(2k+3)+2k+5\right)\Gamma(-2k-3)}{\Gamma(-k)^2}\right)+O(\tau)^4.
\end{aligned}$$

Here the term with τ is

$$2\frac{(k+1)}{\Gamma(-k)^2}\left[(A^2(-2k-3)+12k+19)\Gamma(-2k-3)+4\Gamma(-k)^2\right]\tau$$

and for $k=-2$ it takes value $2\frac{(k+1)}{\Gamma(-k)^2}(A^2-1)\tau$. Corollary is proved. □

Now we can complete the proof of necessity part in Theorem 0.2 for the case of $\ell=2k+1$, $k=-3,-4,\ldots$.

Indeed, for these values of ℓ for the component Ψ_0 of the solution we have

$$\begin{aligned}\Psi_0(0,t) &= 2e^{-Ht}\int_0^1\left(\frac{\partial}{\partial s}s\Phi_0(s)\right)\Bigg\{H2^\ell(1+\ell)e^{-\frac{1}{2}H(\ell+2)t}\left(\left(1+e^{-Ht}\right)^2-H^2r^2\right)^{-\frac{1}{2}(\ell+5)}\\ &\quad\times\Bigg[\left(1+e^{-Ht}\right)\left(\left(1+e^{-Ht}\right)^2-H^2r^2\right)F\left(\frac{\ell+1}{2},\frac{\ell+1}{2};1;\frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)\\ &\quad+(\ell+1)\left(1-e^{-2Ht}-H^2r^2\right)F\left(\frac{\ell+3}{2},\frac{\ell+3}{2};2;\frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)\Bigg]\Bigg\}ds\\ &= e^{-2Ht}H2^{2k+2}(2+2k)^2e^{-\frac{1}{2}H(2k+3)t}\int_0^1\left(\frac{\partial}{\partial r}r\Phi_0(r)\right)\Bigg\{\left(\left(1+e^{-Ht}\right)^2-H^2r^2\right)^{-(k+3)}\\ &\quad\times\Bigg[\left(\frac{\left(H^2r^2(-2k-3)+12k+19\right)\Gamma(-2k-3)+4\Gamma(-k)^2}{\Gamma(-k)^2}\right)+e^{-Ht}O(1)\Bigg]\Bigg\}dr\,.\end{aligned}$$

Next for every given $k=-3,-4,\ldots$, $(\ell=2k+1)$ we choose the function Φ_0 such that

$$\int_0^1\left(\frac{\partial}{\partial r}r\Phi_0(r)\right)\left(1-H^2r^2\right)^{-(k+3)}\left[(H^2r^2(-2k-3)+12k+19)\Gamma(-2k-3)+4\Gamma(-k)^2\right]dr\neq 0 \qquad (5.24)$$

for all values of $Hr=A$ from Corollary 5.2. Here we note that only for $k=-2$ this integral is reduced to the expression that proportional to

$$\int_0^1\left(\frac{\partial}{\partial r}r\Phi_0(r)\right)dr=0.$$

For all values $k=-3,-4,\ldots$, the equation (5.24) shows that the value $\Psi_1(0,t)$ for large time depends on the values of initial function inside of the characteristic conoid. This completes the proof in this case.

5.2 The case of positive odd $\ell=2k+1$, $k=1,2,\ldots$

We skip the proof of the following simple lemma.

Lemma 5.3 *There are the following limits*

$$\begin{aligned}\lim_{x\searrow 0}x^{2n-1}F(n,n;1;1-x) &= F(1-n,1-n;1;1)=\frac{\Gamma(n-1)}{[\Gamma(n)]^2},\quad n=2,3,\ldots,\\ \lim_{x\searrow 0}x^{2n-2}F(n,n;2;1-x) &= F(2-n,2-n;2;1)=\frac{\Gamma(2n-2)}{[\Gamma(n)]^2},\quad n=2,3,\ldots.\end{aligned}$$

Corollary 5.4 *For $\ell=2k+1$, $k=1,2,\ldots$*

$$\begin{aligned}&\mathcal{F}\left(e^{-Ht},Hr;\ell;\frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)\\ &= (2k+2)\left(1-H^2r^2\right)^{3+2k}4^{-2k-2}e^{H(2k+2)t}\left\{\frac{\Gamma(2k+2)}{[\Gamma(k+2)]^2}+R(k,H,r;t)\right\},\quad k=1,2,\ldots,\end{aligned}$$

where with large T the remainder $R(k,H,r;t)$ can be estimated as follows

$$|R(k,H,r;t)|\leq o(1)\quad\text{for all}\quad t\geq T\quad\textit{and}\quad 0\leq r\leq 1/(2H)\,.$$

Proof. From Lemma 5.3, with $n = k+1 > 1$ and $n = k+2$ $(k > 0)$ we obtain

$$\begin{aligned}
& (1+\tau)\left((1+\tau)^2 - H^2r^2\right) F(k+1,k+1;1;z) + (2k+2)\left(1-\tau^2-A^2\right) F(k+2,k+2;2;z) \\
= \;& \left(1-A^2\right)(1-z)^{1-2n}\left[\frac{\Gamma(n-1)}{[\Gamma(n)]^2} + R_1(k,H,r;t)\right] \\
& + (2k+2)\left(1-A^2\right)(1-z)^{2-2n}\left[\frac{\Gamma(2n-2)}{[\Gamma(n)]^2} + R_2(k,H,r;t)\right] \\
= \;& \left(1-A^2\right)(1-z)^{-2k-1}\left[\frac{\Gamma(k)}{[\Gamma(k+1)]^2} + R_1(k,H,r;t)\right] \\
& + (2k+2)\left(1-A^2\right)(1-z)^{-2k-2}\left[\frac{\Gamma(2k+2)}{[\Gamma(k+2)]^2} + R_2(k,H,r;t)\right].
\end{aligned}$$

It follows

$$\begin{aligned}
& (1+\tau)\left((1+\tau)^2 - H^2r^2\right) F(k+1,k+1;1;z) + (2k+2)\left(1-\tau^2-A^2\right) F(k+2,k+2;2;z) \\
= \;& \left(1-A^2\right)^{2+2k} 4^{-2k-1} e^{H(2k+1)t}\left[\frac{\Gamma(k)}{[\Gamma(k+1)]^2} + R_1(k,H,r;t)\right] \\
& + (2k+2)\left(1-A^2\right)^{3+2k} 4^{-2k-2} e^{H(2k+2)t}\left[\frac{\Gamma(2k+2)}{[\Gamma(k+2)]^2} + R_2(k,H,r;t)\right] \\
= \;& (2k+2)\left(1-H^2r^2\right)^{3+2k} 4^{-2k-2} e^{H(2k+2)t}\left[\frac{\Gamma(2k+2)}{[\Gamma(k+2)]^2} + R(k,H,r;t)\right], \quad k = 1,2,\ldots.
\end{aligned}$$

Finally, since $k > 0$ we obtain the statement of the corollary. □

In order to complete the proof of (i) Theorem 0.2 for the case of $\ell = 2k+1$, $k = 1,2,3,\ldots$ we apply Corollary 5.4 and write

$$\begin{aligned}
\Psi_1(0,t) \;=\; & 2e^{-Ht}H2^\ell(1+\ell)e^{-\frac{1}{2}H(\ell+2)t}\int_0^1\left(\frac{\partial}{\partial s}s\Phi_0(s)\right)\left(\left(1+e^{-Ht}\right)^2 - H^2r^2\right)^{-\frac{1}{2}(\ell+5)} \\
& \times(2k+2)\left(1-H^2r^2\right)^{3+2k} 4^{-2k-2} e^{H(2k+2)t}\left[\frac{\Gamma(2k+2)}{[\Gamma(k+2)]^2} + R(k,H,r;t)\right] ds \\
=\; & 2^{-2k-2}H(2k+2)^2 e^{\frac{1}{2}H(2k-1)t}\int_0^1\left(\frac{\partial}{\partial r}r\Phi_0(r)\right)\left(\left(1+e^{-Ht}\right)^2 - H^2r^2\right)^{-(k+3)} \\
& \times\left(1-H^2r^2\right)^{3+2k}\left[\frac{\Gamma(2k+2)}{[\Gamma(k+2)]^2} + R(k,H,r;t)\right] dr\,.
\end{aligned}$$

For every given $k = 1,2,3,\ldots$ we choose Φ_0 such that

$$\int_0^1\left(\frac{\partial}{\partial r}r\Phi_0(r)\right)\left(1-H^2r^2\right)^{\frac{1}{2}(\ell-1)} dr = \int_0^1\left(\frac{\partial}{\partial r}r\Phi_0(r)\right)\left(1-H^2r^2\right)^k dr \neq 0.$$

The last equation shows that the value $\Psi_1(0,t)$ for large time depends on the values of initial function inside of the characteristic conoid. This completes the proof in this case.

6 Necessity of $m = 0, \pm iH$ for the Huygens principle. The case of even $\ell = 2k$, $k = 0, \pm1, \pm2, \ldots$

For the case of $\ell = 2k$, $k = 0,\pm1,\pm2,\ldots$ we have

$$\begin{aligned}
\mathcal{F}\left(e^{-Ht}, Hr; \ell; \frac{\left(1-e^{-Ht}\right)^2 - H^2r^2}{\left(1+e^{-Ht}\right)^2 - H^2r^2}\right) \;=\; & (1+\tau)\left((1+\tau)^2 - A^2\right) F\left(k+\frac{1}{2}, k+\frac{1}{2}; 1; z\right) \\
& + (2k+1)\left(1-\tau^2-A^2\right) F\left(k+\frac{3}{2}, k+\frac{3}{2}; 2; z\right).
\end{aligned}$$

6.1 The case of positive even $\ell = 2k$, $k = 1, 2, 3, \ldots$

If $k = 1, 2, 3, \ldots$ and $c - a - b = -2k$, then we apply (A.3) ([4, (14) Sec. 2.10]). Hence,

$$\begin{aligned}\lim_{z \nearrow 1} F\left(k+\frac{1}{2}, k+\frac{1}{2}; 1; z\right) &= \frac{\Gamma(2k)(1-z)^{-2k}}{[\Gamma(k+\frac{1}{2})]^2} \sum_{n=0}^{2k-1} \frac{[(\frac{1}{2}-k)_n]^2}{(1-2k)_n n!}(1-z)^n \\ &\quad + \frac{(-1)^{2k}}{[\Gamma(k+\frac{1}{2}-2k)]^2} \sum_{n=0}^{\infty} \frac{[(k+\frac{1}{2})_n]^2}{(n+2k)_n n!} \left[\bar{h}_n - \ln(1-z)\right](1-z)^n ,\end{aligned}$$

where $\bar{h}_n = \psi(1+n) + \psi(1+n+2k) - \psi(a+n) - \psi(b+n)$, implies

$$\begin{aligned}F\left(k+\frac{1}{2}, k+\frac{1}{2}; 1; z\right) &= \frac{\Gamma(2k)(1-z)^{-2k}}{[\Gamma(k+\frac{1}{2})]^2} + \frac{\Gamma(2k)(1-z)^{-2k}}{[\Gamma(k+\frac{1}{2})]^2} \sum_{n=1}^{2k-1} \frac{[(\frac{1}{2}-k)_n]^2}{(1-2k)_n n!}(1-z)^n \\ &\quad + \frac{(-1)^{2k}}{[\Gamma(k+\frac{1}{2}-2k)]^2} \sum_{n=0}^{\infty} \frac{[(k+\frac{1}{2})_n]^2}{(n+2k)_n n!} \left[\bar{h}_n - \ln(1-z)\right](1-z)^n \\ &= \frac{\Gamma(2k)(1-z)^{-2k}}{[\Gamma(k+\frac{1}{2})]^2} + (1-z)^{-2k+1}O(1) + |\ln(1-z)|O(1), \quad k = 1, 2, \ldots ,\end{aligned}$$

and

$$\begin{aligned}(1-z)^{2k} F\left(k+\frac{1}{2}, k+\frac{1}{2}; 1; z\right) &= \frac{\Gamma(2k)}{[\Gamma(k+\frac{1}{2})]^2} + (1-z)O(1), \quad k = 1, 2, \ldots , \\ \lim_{z \nearrow 1} (1-z)^{2k} F\left(k+\frac{1}{2}, k+\frac{1}{2}; 1; z\right) &= \frac{\Gamma(2k)}{[\Gamma(k+\frac{1}{2})]^2}, \quad k = 1, 2, \ldots .\end{aligned}$$

Further, according to (A.3) with $m = 2k+1$, for $k = 1, 2, \ldots$ we have

$$\begin{aligned}&F\left(k+\frac{3}{2}, k+\frac{3}{2}; 2; z\right) \\ &= \frac{\Gamma(2k+1)(1-z)^{-(2k+1)}}{[\Gamma(k+\frac{3}{2})]^2} \sum_{n=0}^{(2k+1)-1} \frac{[(\frac{1}{2}-k)_n]^2}{(-2k)_n n!}(1-z)^n \\ &\quad + \frac{(-1)^{(2k+1)}}{[\Gamma(\frac{1}{2}-k)]^2} \sum_{n=0}^{\infty} \frac{[(k+\frac{3}{2})_n]^2}{(n+2k+1)_n n!} \left[\bar{h}_n - \ln(1-z)\right](1-z)^n \\ &= \frac{\Gamma(2k+1)(1-z)^{-(2k+1)}}{[\Gamma(k+\frac{3}{2})]^2} + \frac{\Gamma(2k+1)(1-z)^{-(2k+1)}}{[\Gamma(k+\frac{3}{2})]^2} \sum_{n=1}^{(2k+1)-1} \frac{[(\frac{1}{2}-k)_n]^2}{(-2k)_n n!}(1-z)^n \\ &\quad + \frac{(-1)^{(2k+1)}}{[\Gamma(\frac{1}{2}-k)]^2} \sum_{n=0}^{\infty} \frac{[(k+\frac{3}{2})_n]^2}{(n+2k+1)_n n!} \left[\bar{h}_n - \ln(1-z)\right](1-z)^n \\ &= \frac{\Gamma(2k+1)(1-z)^{-(2k+1)}}{[\Gamma(k+\frac{3}{2})]^2} + \frac{\Gamma(2k+1)}{[\Gamma(k+\frac{3}{2})]^2}(1-z)^{-2k}O(1) + |\ln(1-z)|O(1) ,\end{aligned}$$

which implies

$$\begin{aligned}(1-z)^{(2k+1)} F\left(k+\frac{3}{2}, k+\frac{3}{2}; 2; z\right) &= \frac{\Gamma(2k+1)}{[\Gamma(k+\frac{3}{2})]^2} + (1-z)O(1) , \\ \lim_{z \nearrow 1} (1-z)^{(2k+1)} F\left(k+\frac{3}{2}, k+\frac{3}{2}; 2; z\right) &= \frac{\Gamma(2k+1)}{[\Gamma(k+\frac{3}{2})]^2} .\end{aligned}$$

Hence,

$$
\begin{aligned}
& (1+\tau)\left((1+\tau)^2-A^2\right) F\left(k+\frac{1}{2}, k+\frac{1}{2}; 1; \frac{(1-x)^2-A^2}{(1+x)^2-A^2}\right) \\
& +(2k+1)\left(1-\tau^2-A^2\right) F\left(k+\frac{3}{2}, k+\frac{3}{2}; 2; \frac{(1-x)^2-A^2}{(1+x)^2-A^2}\right) \\
= \; & (1+\tau)\left((1+\tau)^2-A^2\right)(1-z)^{-2k}\left\{\frac{\Gamma(2k)}{[\Gamma(k+\frac{1}{2})]^2}+(1-z)O(1)\right\} \\
& +(2k+1)\left(1-\tau^2-A^2\right)(1-z)^{-(2k+1)}\left\{\frac{\Gamma(2k+1)}{[\Gamma(k+\frac{3}{2})]^2}+(1-z)O(1)\right\} \\
= \; & (2k+1)\left(1-\tau^2-A^2\right)(1-z)^{-(2k+1)}\left\{\frac{\Gamma(2k+1)}{[\Gamma(k+\frac{3}{2})]^2}+(1-z)O(1)\right\}.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
& \lim_{\tau\to 0}(1-z)^{(2k+1)}\Bigg[(1+\tau)\left((1+\tau)^2-A^2\right) F\left(k+\frac{1}{2}, k+\frac{1}{2}; 1; z\right) \\
& +(2k+1)\left(1-\tau^2-A^2\right) F\left(k+\frac{3}{2}, k+\frac{3}{2}; 2; z\right)\Bigg] \\
= \; & (2k+1)\frac{\Gamma(2k+1)}{[\Gamma(k+\frac{3}{2})]^2}\left(1-A^2\right), \quad k=1,2,3,\ldots.
\end{aligned}
$$

In order to complete the proof of necessity part in Theorem 0.2 for $\ell = 2k$, $k = 1, 2, 3, \ldots$ we write

$$
\begin{aligned}
\Psi_1(0,t) = \; & 2e^{-Ht}\int_0^1 \left(\frac{\partial}{\partial r} r\Phi_0(r)\right)\Bigg\{H2^{\ell}(1+\ell)e^{-\frac{1}{2}H(\ell+2)t}\left(\left(1+e^{-Ht}\right)^2-H^2r^2\right)^{-\frac{1}{2}(\ell+5)} \\
& \times\Bigg\{\left(1+e^{-Ht}\right)\left(\left(1+e^{-Ht}\right)^2-H^2r^2\right) F\left(\frac{\ell+1}{2}, \frac{\ell+1}{2}; 1; \frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right) \\
& +(\ell+1)\left(1-e^{-2Ht}-H^2r^2\right) F\left(\frac{\ell+3}{2}, \frac{\ell+3}{2}; 2; \frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)\Bigg\}\Bigg\} dr \\
= \; & 2e^{-Ht}\int_0^1 \left(\frac{\partial}{\partial r} r\Phi_0(r)\right) H2^{\ell}(1+\ell)e^{-\frac{1}{2}H(\ell+2)t}\left(\left(1+e^{-Ht}\right)^2-H^2r^2\right)^{-\frac{1}{2}(\ell+5)} \\
& \times(2k+1)\left(1-\tau^2-A^2\right)\left(\frac{4e^{-Ht}}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)^{-(2k+1)}\left[\frac{\Gamma(2k+1)}{[\Gamma(k+\frac{3}{2})]^2}+o(1)\right] dr \\
= \; & H2^{-2k-1}e^{Ht(k-1)}(2k+1)^2\int_0^1 \left(\frac{\partial}{\partial r} r\Phi_0(r)\right)\left(\left(1+e^{-Ht}\right)^2-H^2r^2\right)^{\frac{1}{2}(2k-3)} \\
& \times\left(1-e^{-2Ht}-H^2r^2\right)\left[\frac{\Gamma(2k+1)}{[\Gamma(k+\frac{3}{2})]^2}+o(1)\right] dr\,.
\end{aligned}
$$

Next for every $\ell = 2k$, $k = 1, 2, 3, \ldots$ we can we choose a function Φ_0 such that

$$
\int_0^1 \left(\frac{\partial}{\partial r} r\Phi_0(r)\right)\left(1-H^2r^2\right)^{\frac{1}{2}(\ell-1)} dr = \int_0^1 \left(\frac{\partial}{\partial r} r\Phi_0(r)\right)\left(1-H^2r^2\right)^{\frac{1}{2}(2k-1)} dr \neq 0\,.
$$

The last equation shows that the value $\Psi_1(0,t)$ for large time depends on the values of initial function inside of the characteristic conoid. This completes the proof in this case.

6.2 The case of negative even $\ell = 2k$, $k = -2, -3, \dots$

For $\ell = 2k$, $k = -2, -3, \dots$ consider the function

$$\mathcal{F}\left(e^{-Ht}, Hr; \ell; \frac{\left(1-e^{-Ht}\right)^2 - H^2r^2}{\left(1+e^{-Ht}\right)^2 - H^2r^2}\right) = (1+\tau)\left((1+\tau)^2 - A^2\right) F\left(k+\frac{1}{2}, k+\frac{1}{2}; 1; \frac{(1-\tau)^2 - A^2}{(1+\tau)^2 - A^2}\right)$$
$$+(2k+1)\left(1-\tau^2 - A^2\right) F\left(k+\frac{3}{2}, k+\frac{3}{2}; 2; \frac{(1-\tau)^2 - A^2}{(1+\tau)^2 - A^2}\right).$$

Lemma 6.1 *The following formulas with $z \searrow 0$ hold*

$$F\left(k+\frac{1}{2}, k+\frac{1}{2}; 1; 1-z\right) = \frac{\Gamma(-2k)}{[\Gamma(\frac{1}{2}-k)]^2} - z\left(k+\frac{1}{2}\right)^2 \frac{\Gamma(-1-2k)}{[\Gamma(\frac{1}{2}-k)]^2} + O(z^2),$$
$$k = -1, -2, -3, \dots,$$
$$F\left(k+\frac{3}{2}, k+\frac{3}{2}; 2; 1-z\right) = \frac{\Gamma(-1-2k)}{[\Gamma(\frac{1}{2}-k)]^2} - z\left(k+\frac{3}{2}\right)^2 \frac{\Gamma(-2-2k)}{[\Gamma(\frac{1}{2}-k)]^2} + O(z^2),$$
$$k = -2, -3, \dots.$$

Proof. For the function $F\left(k+\frac{1}{2}, k+\frac{1}{2}; 1; z\right)$, since $c-a-b = -2k > 0$ and $c-b = \frac{1}{2}-k > 0$, we apply [4, (4) Sec. 2.1]

$$F\left(k+\frac{1}{2}, k+\frac{1}{2}; 1; 1\right) = \frac{\Gamma(1)\Gamma(-2k)}{[\Gamma(\frac{1}{2}-k)]^2}, \quad k = -1, -2, \dots.$$

For the function $F\left(k+\frac{3}{2}, k+\frac{3}{2}; 2; z\right)$, since $c-a-b = -1-2k > 0$ and $c-b = \frac{1}{2}-k > 0$ we apply [4, (4) Sec. 2.1]

$$F\left(k+\frac{3}{2}, k+\frac{3}{2}; 2; 1\right) = \frac{\Gamma(2)\Gamma(-1-2k)}{[\Gamma(\frac{1}{2}-k)]^2}, \quad k = -1, -2, \dots. \tag{6.25}$$

It follows

$$F\left(k+\frac{1}{2}, k+\frac{1}{2}; 1; 1\right) + (2k+1)F\left(k+\frac{3}{2}, k+\frac{3}{2}; 2; 1\right) = \frac{\Gamma(-2k)}{[\Gamma(\frac{1}{2}-k)]^2} + (2k+1)\frac{\Gamma(-1-2k)}{[\Gamma(\frac{1}{2}-k)]^2}$$
$$= 0, \quad k = -1, -2, \dots.$$

Then we look at the next term

$$\left[\frac{d}{dz}F\left(k+\frac{1}{2}, k+\frac{1}{2}; 1; z\right)\right]_{z=1} = -\left[\frac{d}{dz}F\left(k+\frac{1}{2}, k+\frac{1}{2}; 1; 1-z\right)\right]_{z=0}$$
$$= -\left[-\left(k+\frac{1}{2}\right)^2 F\left(k+\frac{3}{2}, k+\frac{3}{2}; 2; 1-z\right)\right]_{z=0}$$
$$= \left(k+\frac{1}{2}\right)^2 \frac{\Gamma(-1-2k)}{[\Gamma(\frac{1}{2}-k)]^2}, \quad k = -1, -2, \dots.$$

Also

$$\left[\frac{d}{dz}F\left(k+\frac{3}{2}, k+\frac{3}{2}; 2; z\right)\right]_{z=1} = -\left[\frac{d}{dz}F\left(k+\frac{3}{2}, k+\frac{3}{2}; 2; 1-z\right)\right]_{z=0}$$
$$= \frac{1}{2}\left(k+\frac{3}{2}\right)^2 F\left(k+\frac{5}{2}, k+\frac{5}{2}; 3; 1\right)$$
$$= \frac{1}{2}\left(k+\frac{3}{2}\right)^2 \frac{2\Gamma(-2-2k)}{[\Gamma(\frac{1}{2}-k)]^2}, \quad k = -2, -3, \dots.$$

It follows

$$\begin{aligned}
& F\left(k+\frac{1}{2},k+\frac{1}{2};1;1\right)+(2k+1)F\left(k+\frac{3}{2},k+\frac{3}{2};2;1\right)\\
= & \left(k+\frac{1}{2}\right)^2\frac{\Gamma(-1-2k)}{[\Gamma(\frac{1}{2}-k)]^2}+(2k+1)\frac{1}{2}\left(k+\frac{3}{2}\right)^2\frac{2\Gamma(-2-2k)}{[\Gamma(\frac{1}{2}-k)]^2}\\
= & \frac{(6k+7)\Gamma(-2k)}{8(k+1)[\Gamma(\frac{1}{2}-k)]^2}>0, \quad k=-2,-3,\dots .
\end{aligned}$$

The lemma is proved. □

The case of $k=-1$, that is, $\ell=-2$ and $m=-i\frac{H}{2}$ will be discussed in subsection 6.3.

Corollary 6.2 *For all $k=-2,-3,\dots$ and $0\le A<1/2$*

$$\begin{aligned}
& (1+\tau)\left((1+\tau)^2-A^2\right)F\left(k+\frac{1}{2},k+\frac{1}{2};1;\frac{(1-\tau)^2-A^2}{(1+\tau)^2-A^2}\right)\\
& +(2k+1)\left(1-\tau^2-A^2\right)F\left(k+\frac{3}{2},k+\frac{3}{2};2;\frac{(1-\tau)^2-A^2}{(1+\tau)^2-A^2}\right)\\
= & \frac{\left(-2A^2(k+1)+2k+1\right)\Gamma(-2k)}{2(k+1)\Gamma\left(\frac{1}{2}-k\right)^2}\tau+O\left(\tau^2\right),
\end{aligned}$$

where

$$\frac{\left(-2A^2(k+1)+2k+1\right)\Gamma(-2k)}{2(k+1)\Gamma\left(\frac{1}{2}-k\right)^2}>0, \quad k=-2,-3,\dots .$$

Proof. Indeed, according to Lemma 6.1

$$\begin{aligned}
& (1+\tau)\left((1+\tau)^2-A^2\right)F\left(k+\frac{1}{2},k+\frac{1}{2};1;\frac{(1-\tau)^2-A^2}{(1+\tau)^2-A^2}\right)\\
& +(2k+1)\left(1-\tau^2-A^2\right)F\left(k+\frac{3}{2},k+\frac{3}{2};2;\frac{(1-\tau)^2-A^2}{(1+\tau)^2-A^2}\right)\\
= & (1-A^2+(3-A^2)\tau+O(\tau^2))\\
& \times\left[\frac{\Gamma(-2k)}{[\Gamma(\frac{1}{2}-k)]^2}+\left(\frac{4\tau}{1-A^2}+\frac{8\tau^2}{(A^2-1)^2}+O\left(\tau^3\right)\right)\left(k+\frac{1}{2}\right)^2\frac{\Gamma(-1-2k)}{[\Gamma(\frac{1}{2}-k)]^2}+O(\tau^2)\right]\\
& +(2k+1)\left(1-A^2-x^2\right)\\
& \times\left[\frac{\Gamma(-1-2k)}{[\Gamma(\frac{1}{2}-k)]^2}-\left(\frac{4\tau}{1-A^2}+\frac{8\tau^2}{(A^2-1)^2}+O\left(x^3\right)\right)\left(k+\frac{3}{2}\right)^2\frac{\Gamma(-2-2k)}{[\Gamma(\frac{1}{2}-k)]^2}+O(z^2)\right]\\
= & \frac{\left(-2A^2(k+1)+2k+1\right)\Gamma(-2k)}{2(k+1)\Gamma\left(\frac{1}{2}-k\right)^2}\tau+O\left(\tau^2\right),
\end{aligned}$$

provided that $k=-2,-3,\dots$ and $0\le A^2<1/4$. The corollary is proved. □

To complete the proof of (i) Theorem 0.2 for the case of $\ell=2k$, $k=-2,-3,\dots$ we write

$$\begin{aligned}
\Psi_0(0,t) = & H2^{2k+1}(1+2k)e^{-H(k+3)t}\int_0^1\left(\frac{\partial}{\partial s}s\Phi_0(s)\right)\left(\left(1+e^{-Ht}\right)^2-H^2r^2\right)^{-\frac{1}{2}(2k+5)}\\
& \times\left[\frac{\left(-2H^2r^2(k+1)+2k+1\right)\Gamma(-2k)}{2(k+1)\Gamma\left(\frac{1}{2}-k\right)^2}+e^{-Ht}O\left(1\right)\right]ds\,.
\end{aligned}$$

Next for every $k=-2,-3,\dots$ we can chose choose Φ_0 such that

$$\int_0^1 \left(\frac{\partial}{\partial r} r\Phi_0(r)\right)\left(1-H^2r^2\right)^{-\frac{1}{2}(2k+5)}\left(-2H^2r^2(k+1)+2k+1\right)dr \neq 0\,.$$

The last equation shows that the value $\Psi_0(0,t)$ for large time depends on the values of initial function inside of the characteristic conoid. This completes the proof in this case. □

6.3 The case of $m=-i\frac{H}{2}$, that is $\ell=2k$, $k=-1$, $\ell=-2$

For $\ell=2k$, $k=-1$, consider

$$F\left(k+\frac{3}{2},k+\frac{3}{2};2;1-z\right) \;=\; F\left(\frac{1}{2},\frac{1}{2};2;1-z\right).$$

We have $c-a-b=1$ and $c-a=\frac{3}{2}$ and apply (6.25):

$$F\left(k+\frac{3}{2},k+\frac{3}{2};2;1\right) \;=\; F\left(\frac{1}{2},\frac{1}{2};2;1\right)=\frac{1}{[\Gamma(\frac{3}{2})]^2}=\frac{4}{\pi}\,,$$

while

$$\frac{d}{dz}F\left(k+\frac{3}{2},k+\frac{3}{2};2;z\right) \;=\; \frac{1}{8}F\left(\frac{3}{2},\frac{3}{2};3;z\right).$$

Then we use (A.2) with $m=0$, that is,

$$\begin{aligned}
F\left(\frac{3}{2},\frac{3}{2};3;z\right) &= \frac{\Gamma(3)}{[\Gamma(\frac{3}{2})]^2}\left[2\psi(1)-2\psi\left(\frac{3}{2}\right)-\ln(1-z)\right]\\
&\quad+\frac{\Gamma(3)}{[\Gamma(\frac{3}{2})]^2}\sum_{n=1}^{\infty}\frac{[(\frac{3}{2})_n]^2}{(n!)^2}\left[2\psi(n+1)-2\psi\left(\frac{3}{2}+n\right)-\ln(1-z)\right](1-z)^n\\
&= \frac{8}{\pi}\left[2\psi(1)-2\psi\left(\frac{3}{2}\right)-\ln(1-z)\right]+O(1)(1-z)\ln(1-z)\,,\\
&\qquad |\arg(1-z)|<\pi,\quad |1-z|<1\,.
\end{aligned}$$

Hence

$$\frac{d}{dz}F\left(k+\frac{3}{2},k+\frac{3}{2};2;z\right) \;=\; \frac{1}{\pi}\left[2\psi(1)-2\psi\left(\frac{3}{2}\right)-\ln(1-z)\right]+O(1)(1-z)\ln(1-z)\,.$$

It follows for $k=-1$

$$\begin{aligned}
&(\tau+1)\left((\tau+1)^2-A^2\right)F\left(k+\frac{1}{2},k+\frac{1}{2};1;\frac{(1-\tau)^2-A^2}{(\tau+1)^2-A^2}\right)\\
&+(2k+1)\left(-A^2-\tau+1\right)F\left(k+\frac{3}{2},k+\frac{3}{2};2;\frac{(1-\tau)^2-A^2}{(\tau+1)^2-A^2}\right)\\
=\;&(\tau+1)\left((\tau+1)^2-A^2\right)F\left(-\frac{1}{2},-\frac{1}{2};1;\frac{(1-\tau)^2-A^2}{(\tau+1)^2-A^2}\right)\\
&-\left(-A^2-\tau+1\right)F\left(\frac{1}{2},\frac{1}{2};2;\frac{(1-\tau)^2-A^2}{(\tau+1)^2-A^2}\right).
\end{aligned}$$

According to Lemma 6.1 for $0 \le A < \frac{1}{2}$ we obtain

$$
\begin{aligned}
& (\tau+1)\left((x+1)^2-A^2\right) F\left(-\frac{1}{2},-\frac{1}{2};1;\frac{(1-\tau)^2-A^2}{(x+1)^2-A^2}\right) \\
& -\left(-A^2-\tau+1\right) F\left(\frac{1}{2},\frac{1}{2};2;\frac{(1-\tau)^2-A^2}{(\tau+1)^2-A^2}\right) \\
= \; & (\tau+1)\left((\tau+1)^2-A^2\right)\left(\frac{\Gamma(-2k)}{[\Gamma(\frac{1}{2}-k)]^2} - z\left(k+\frac{1}{2}\right)^2 \frac{\Gamma(-1-2k)}{[\Gamma(\frac{1}{2}-k)]^2} + O(z^2)\right) \\
& -\left(1-A^2-\tau\right)\left(\frac{4}{\pi} + (-z)\left\{\frac{1}{\pi}\left[2\psi(1)-2\psi(\frac{3}{2})-\ln(z)\right] + O(1)(z)\ln(z)\right\}\right) \\
= \; & (\tau+1)\left((\tau+1)^2-A^2\right)\left(\frac{4}{\pi} - \left(\frac{4x}{(1+\tau)^2-A^2}\right)\left(-\frac{1}{2}\right)^2\frac{4}{\pi} + O(z^2)\right) \\
& -\left(1-A^2-\tau\right)\left(\frac{4}{\pi} - \left(\frac{4x}{(1+x)^2-A^2}\right)\left\{\frac{1}{\pi}\left[2\psi(1)-2\psi(\frac{3}{2})-\ln(z)\right] + O(1)(z)\ln(z)\right\}\right) \\
= \; & -\tau\frac{4}{\pi}\left(H^2r^2 - \ln\left(4-4H^2r^2\right) + \ln(\tau) + 1\right) + O\left(\tau^2\right) \\
= \; & -\tau\frac{4}{\pi}\left(\ln(\tau) + O(1)\right) + O\left(\tau^2\right) \quad \text{as} \quad t\to\infty \quad \text{and} \quad \tau = e^{-Ht} \to 0.
\end{aligned}
$$

The proof of (i) Theorem 0.2 for the case of $m = -iH/2$, $\ell = -2$, $k = -1$. We have

$$
\begin{aligned}
\Psi_1(0,t) \;=\; & 2^2 e^{-Ht}\int_0^1 \left(\frac{\partial}{\partial r} r\Phi_0(r)\right)\left\{-\frac{H}{8\left(\left(e^{-Ht}+1\right)^2 - H^2r^2\right)^{3/2}}\right. \\
& \times\left[\left(e^{-Ht}+1\right)\left(\left(e^{-Ht}+1\right)^2 - H^2r^2\right) F\left(-\frac{1}{2},-\frac{1}{2};1;\frac{\left(-1+e^{-Ht}\right)^2 - H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)\right. \\
& \left.\left. + \left(H^2r^2 + e^{-2Ht} - 1\right) F\left(\frac{1}{2},\frac{1}{2};2;\frac{\left(-1+e^{-Ht}\right)^2 - H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)\right]\right\} dr \\
= \; & 2^2 e^{-Ht}\int_0^1 \left(\frac{\partial}{\partial r} r\Phi_0(r)\right)\left\{-\frac{H}{8\left(\left(e^{-Ht}+1\right)^2 - H^2r^2\right)^{3/2}}\right. \\
& \left.\times\left[-\frac{4e^{-Ht}\left(H^2r^2 - \log\left(4-4H^2r^2\right) - Ht + 1\right)}{\pi} + e^{-2Ht}O\left(1\right)\right]\right\} dr\,.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\Psi_1(0,t) \;=\; & e^{-2Ht}H\int_0^1 \left(\frac{\partial}{\partial r} r\Phi_0(r)\right)\frac{1}{2\left(\left(e^{-Ht}+1\right)^2 - H^2r^2\right)^{3/2}} \\
& \times\left[\frac{4}{\pi}\left(H^2r^2 - \log\left(4-4H^2r^2\right) - Ht + 1\right) + e^{-Ht}O\left(1\right)\right] dr\,.
\end{aligned}
$$

Next we choose Φ_0 such that

$$
\int_0^1 \left(\frac{\partial}{\partial r} r\Phi_0(r)\right)\frac{1}{\left(1-H^2r^2\right)^{3/2}}\left[\frac{4}{\pi}\left(H^2r^2 - \log\left(4-4H^2r^2\right) - Ht + 1\right)\right] dr \neq 0\,.
$$

The last equation shows that the value $\Psi_1(0,t)$ for large time depends on the values of initial function inside of the characteristic conoid. This completes the proof in this case.

7 Necessity of $m = 0, \pm iH$ for the Huygens principle. Case of $m = i\frac{H}{2}$, $\ell = 0$

For $m = i\frac{H}{2}$ we have $M_+ = 0$ and

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \frac{H}{2} - im\right) K_1(r,t;M_+)\\
= \;& He^{-Ht}\left(\left(1+e^{-Ht}\right)^2 - H^2r^2\right)^{-\frac{5}{2}}\\
&\times\Bigg\{ \left(1+e^{-Ht}\right)\left(\left(1+e^{-Ht}\right)^2 - H^2r^2\right) F\left(\frac{1}{2},\frac{1}{2};1;\frac{\left(1-e^{-Ht}\right)^2 - H^2r^2}{\left(1+e^{-Ht}\right)^2 - H^2r^2}\right)\\
&+\left(1-e^{-2Ht} - H^2r^2\right) F\left(\frac{3}{2},\frac{3}{2};2;\frac{\left(1-e^{-Ht}\right)^2 - H^2r^2}{\left(1+e^{-Ht}\right)^2 - H^2r^2}\right)\Bigg\}.
\end{aligned}$$

We are going to study the asymptotics of the function

$$(1+x)\left((1+x)^2 - H^2r^2\right) F\left(\frac{1}{2},\frac{1}{2};1;\frac{(1-x)^2 - A^2}{(1+x)^2 - A^2}\right) + \left(1-x^2-A^2\right) F\left(\frac{3}{2},\frac{3}{2};2;\frac{(1-x)^2 - A^2}{(1+x)^2 - A^2}\right),$$

where $\tau = e^{-Ht}$ and $A = Hr$ as $\tau \to 0$. By (A.2) with $a = b = 1/2$ and $m = 0$ we obtain

$$\begin{aligned}
F\left(\frac{1}{2},\frac{1}{2};1;1-z\right) &= \frac{1}{\pi}\left[2\psi(n+1) - 2\psi\left(\frac{1}{2}\right) - \ln(z)\right]\\
&\quad+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(n!)^2}\left[2\psi(n+1) - \psi(a+n) - \psi(b+n) - \ln(z)\right]z^n\\
&= \frac{1}{\pi}\left[2\psi(n+1) - 2\psi\left(\frac{1}{2}\right) - \ln(z)\right] + z\ln(z)O(1) \quad \text{as} \quad z \searrow 0\,.
\end{aligned}$$

For the second hypergeometric function, since $c - a - b = -1$, we apply (A.3) with $m = 1$:

$$\begin{aligned}
F\left(\frac{3}{2},\frac{3}{2};2;1-z\right) &= \frac{4z^{-1}}{\pi} - \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{[(\frac{3}{2})_n]^2}{(n+1)_n n!}\left[\bar{h}_n - \ln(z)\right]z^n\\
&= \frac{4}{\pi}z^{-1} - O(1)\ln(z) \quad \text{as} \quad z \searrow 0\,,
\end{aligned}$$

where $\bar{h}_n = \psi(1+n) + \psi(2+n) - \psi(a+n) - \psi(b+n)$. Hence

$$\begin{aligned}
&(1+\tau)\left((1+\tau)^2 - A^2\right) F\left(\frac{1}{2},\frac{1}{2};1;\frac{(1-\tau)^2 - A^2}{(1+\tau)^2 - A^2}\right)\\
&+\left(1-\tau^2-A^2\right) F\left(\frac{3}{2},\frac{3}{2};2;\frac{(1-\tau)^2 - A^2}{(1+\tau)^2 - A^2}\right)\\
= \;& (1+\tau)\left((1+\tau)^2 - A^2\right)\left(\frac{1}{\pi}\left[2\psi(n+1) - 2\psi(\tfrac{1}{2}) - \ln(z)\right] + z\ln(z)O(1)\right)\\
&+\left(1-\tau^2-A^2\right)\left(\frac{4z^{-1}}{\pi} - O(1)\ln(z)\right)\\
= \;& \left(1-A^2\right)\left(\frac{4}{\pi}\left(\frac{4e^{-Ht}}{1-A^2}\right)^{-1} - O(1)\ln\left(\frac{4e^{-Ht}}{1-A^2}\right)\right)\\
= \;& \left(1-A^2\right)\left(\frac{1}{\pi}\left(1-A^2\right)e^{Ht} - O(1)t\right)\\
= \;& \left(1-A^2\right)^2\frac{1}{\pi}e^{Ht} - O(1)t
\end{aligned}$$

and, consequently,

$$\left(\frac{\partial}{\partial t} - \frac{H}{2} - im\right) K_1(r,t;M_+) \quad = \quad \frac{H^3}{\pi\sqrt{1-H^2r^2}} + e^{-2Ht}O\left(1\right) .$$

The proof of necessity part of Theorem 0.2 for the case of $m = iH/2$. For the solution for large time we obtain

$$\begin{aligned}\Psi_1(0,t) &= 2e^{-Ht}\int_0^1 \left(\frac{\partial}{\partial r} r\Phi_0(r)\right)\left(\frac{\partial}{\partial t}\right) K_1(r,t;0)\, dr \\ &= 2e^{-Ht}\int_0^1 \left(\frac{\partial}{\partial r} r\Phi_0(r)\right)\left(\frac{H}{\pi\sqrt{1-H^2r^2}} + e^{-2Ht}O\left(1\right)\right) dr .\end{aligned}$$

Next we choose Φ_0 such that

$$\int_0^1 \left(\frac{\partial}{\partial r} r\Phi_0(r)\right) \frac{1}{\sqrt{1-H^2r^2}} dr \neq 0 .$$

The last equation shows that the value $\Psi_1(0,t)$ for large time depends on the values of initial function inside of the characteristic conoid. This completes the proof in this case.

8 Necessity of $m = \pm iH$ for the incomplete Huygens principle

In order to prove necessity of $m = iH$ for the incomplete Huygens principle with respect to the first 2-spinor initial data Φ_0, Φ_1 for the Dirac equation (0.2), we set $m \neq 0$, $m \neq iH$ and chose initial data (3.19). Then we can repeat all arguments used for the cases of all possible values of mass except ones used in Sections 6, 7. This completes the proof of necessity part in Theorem 0.2 for $m = iH$.

In order to prove necessity of $m = -iH$ for the incomplete Huygens principle with respect to the second 2-spinor initial data Φ_2, Φ_3 for the Dirac equation (0.2), we set $m \neq 0$, $m \neq -iH$ and choose initial data (2.17) as in subsection 2.2. We set $F = 0$ and $M_- = \frac{H}{2} - im \neq -\frac{H}{2}$, then the kernel function is

$$\begin{aligned}K_1(r,t;M_-) &= 2^{-1+\frac{2im}{H}} e^{\frac{1}{2}t(H-2im)} \left(\left(e^{-Ht}+1\right)^2 - H^2r^2\right)^{-\frac{im}{H}} \\ &\quad \times F\left(\frac{im}{H}, \frac{im}{H}; 1; \frac{\left(-1+e^{-Ht}\right)^2 - H^2r^2}{\left(1+e^{-Ht}\right)^2 - H^2r^2}\right) ,\end{aligned}$$

while the operator $\mathcal{K}_1(x,t,D_x;M_-)$ is defined by (1.8). The solution to the Cauchy problem for the Dirac equation is the function

$$\Psi(x,t) \quad = \quad e^{-Ht}\left(\partial_0 \mathbb{I}_4 + e^{-Ht}\gamma^k\gamma^0\partial_k - \frac{H}{2}\mathbb{I}_4 - im\gamma^0\right)\begin{pmatrix} 0 \\ 0 \\ \mathcal{K}_1(x,t,D_x;M_-)[\Phi_2(x)] \\ 0 \end{pmatrix} ,$$

with the components (see (2.16))

$$\begin{aligned}\Psi_0(x,t) &= -e^{-2Ht}\partial_3\mathcal{K}_1(x,t,D_x;M_-)[\Phi_2(x)] , \\ \Psi_1(x,t) &= e^{-2Ht}\left(-\partial_1 - i\partial_2\right)\mathcal{K}_1(x,t,D_x;M_-)[\Phi_2(x)] , \\ \Psi_2(x,t) &= e^{-Ht}\left(\partial_0 - \frac{H}{2} + im\right)\mathcal{K}_1(x,t,D_x;M_-)[\Phi_2(x)] , \\ \Psi_3(x,t) &= 0 .\end{aligned}$$

We calculate

$$\Psi_2(x,t) \quad = \quad e^{-Ht}\left(\frac{\partial}{\partial t} - \frac{H}{2} + im\right) 2\int_0^{\phi(t)} v_{\Phi_2}(x,s)K_1(s,t;M_-)ds .$$

It can be rewritten in the terms of the function V_{Φ_2} defined in accordance to (2.12), that is,

$$\Psi_2(x,t) = e^{-Ht}\left(\frac{\partial}{\partial t}-\frac{H}{2}+im\right)2\int_0^{\phi(t)}\left(\frac{\partial}{\partial s}V_{\Phi_2}(x,s)\right)K_1(s,t;M_-)ds\,.$$

It follows,

$$\begin{aligned}\Psi_2(x,t) &= e^{-Ht}\left(\frac{\partial}{\partial t}-\frac{H}{2}+im\right)2\Big(V_{\Phi_0}(x,\phi(t))K_1(\phi(t),t;M_-)-V_{\Phi_2}(x,0)K_1(0,t;M_-)\Big)\\ &\quad -e^{-Ht}\left(\frac{\partial}{\partial t}-\frac{H}{2}+im\right)2\int_0^{\phi(t)}V_{\Phi_0}(x,s)\frac{\partial}{\partial s}K_1(s,t;M_-)ds\\ &= 2e^{-Ht}\left(\frac{\partial}{\partial t}-\frac{H}{2}+im\right)V_{\Phi_2}(x,\phi(t))K_1(\phi(t),t;M_-)\\ &\quad -2e^{-2Ht}V_{\Phi_2}(x,\phi(t))\left(\frac{\partial}{\partial s}K_1(s,t;M_-)\right)_{s=\phi(t)}\\ &\quad -2e^{-Ht}\int_0^{\phi(t)}V_{\Phi_2}(x,s)\left(\frac{\partial}{\partial t}-\frac{H}{2}+im\right)\frac{\partial}{\partial s}K_1(s,t;M_-)ds\,.\end{aligned}$$

In particular, since $x\in\mathbb{R}^3$, by the Kirchhoff's formula we have

$$V_{\Phi_2}(0,\phi(t))=\phi(t)\Phi_2(\phi(t))=\frac{1-e^{-Ht}}{H}\Phi_2\left(\frac{1-e^{-Ht}}{H}\right)=0$$

for sufficiently large t, that is, if $1-e^{-Ht}>H\varepsilon$. Consequently, for large t we have

$$\begin{aligned}\Psi_2(0,t) &= -2e^{-Ht}\int_0^{\phi(t)}s\Phi_0(s)\left(\frac{\partial}{\partial t}-\frac{H}{2}+im\right)\frac{\partial}{\partial s}K_1(s,t;M_-)ds\\ &= -2e^{-Ht}\int_0^{1}s\Phi_0(s)\left(\frac{\partial}{\partial t}-\frac{H}{2}+im\right)\frac{\partial}{\partial s}K_1(s,t;M_-)ds\\ &= 2e^{-Ht}\int_0^{1}\left(\frac{\partial}{\partial s}s\Phi_0(s)\right)\left(\frac{\partial}{\partial t}-\frac{H}{2}+im\right)K_1(s,t;M_-)ds\,.\end{aligned}$$

Now we focus on the tail of the solution, that is on the term generated by the integral. To discuss the last term we apply Lemma 3.1 when m is replaced with $-m$. Further, according to Proposition 3.2 if $m\in\mathbb{C}$ and

$$m\neq i\frac{H}{2}+i\frac{H}{2}\ell,\quad \ell=0,\pm1,\pm2,\dots,$$

then

$$\begin{aligned}&2\frac{im}{H}\left(1-e^{-2Ht}-H^2r^2\right)F\left(1+\frac{im}{H},1+\frac{im}{H};2;\frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)\\ &+\left(1+e^{-Ht}\right)\left((1+e^{-Ht})^2-H^2r^2\right)F\left(\frac{im}{H},\frac{im}{H};1;\frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)\\ =\;&-2\frac{im}{H}4^{-2\frac{im}{H}}e^{2imt}\frac{\Gamma(2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}\left(1-H^2r^2\right)^{1+2\frac{im}{H}}+R(m,H,r;t)\,,\end{aligned}$$

where with large T the remainder $R(m,H,r;t)$ can be estimated by (3.20). Further,

$$\begin{aligned}\Psi_2(0,t) &= 2e^{-Ht}\int_0^1\left(\frac{\partial}{\partial r}r\Phi_0(r)\right)\Bigg[2^{\frac{2im}{H}}ime^{\frac{1}{2}t(-H-2im)}\left(\left(1+e^{-Ht}\right)^2-H^2r^2\right)^{-\frac{im}{H}-2}\\ &\quad\times\Bigg\{2\frac{im}{H}\left(1-e^{-2Ht}-H^2r^2\right)F\left(1+\frac{im}{H},1+\frac{im}{H};2;\frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)\\ &\quad+\left(1+e^{-Ht}\right)\left((1+e^{-Ht})^2-H^2r^2\right)F\left(\frac{im}{H},\frac{im}{H};1;\frac{\left(1-e^{-Ht}\right)^2-H^2r^2}{\left(1+e^{-Ht}\right)^2-H^2r^2}\right)\Bigg\}\Bigg]dr\\ &= 2e^{-Ht}2^{\frac{2im}{H}}ime^{\frac{1}{2}t(-H-2im)}\int_0^1\left(\frac{\partial}{\partial r}r\Phi_0(r)\right)\left(\left(1+e^{-Ht}\right)^2-H^2r^2\right)^{-\frac{im}{H}-2}\\ &\quad\times\Bigg\{-2\frac{im}{H}4^{-2\frac{im}{H}}e^{2imt}\frac{\Gamma(2\frac{im}{H})}{[\Gamma(1+\frac{im}{H})]^2}\left(1-H^2r^2\right)^{1+2\frac{im}{H}}+R(m,H,r;t)\Bigg\}dr\,.\end{aligned}$$

Since $-1+\frac{im}{H}\neq 0$ we can choose the radial function $\Phi_0\in C^\infty(B_\varepsilon(0))$ such that

$$\int_0^1\left(\frac{\partial}{\partial r}r\Phi_0(r)\right)\left(1-H^2r^2\right)^{-1+\frac{im}{H}}dr\neq 0\,.$$

The last equation shows that the value $\Psi_2(0,t)$ for large time depends on the values of initial function inside of the characteristic conoid. This completes the proof in this case. □

9 Conclusions

The purpose of this paper was to examine the Huygens principle for generalized Dirac operator in the de Sitter spacetime. The generalized Dirac operator was firstly introduced in [29] and besides the mathematical importance for the theory of partial differential equations, it, in particular, includes the equation for the motion of the charged spin-$\frac{1}{2}$ particle in a constant homogeneous magnetic field. The last problem has been studied in physical literature (see [29] and the references therein). In the present paper, we introduced a novel definition of the so-called incomplete Huygens principle for this operator and comprehensively examined it. The incomplete Huygens principle was introduced in [27] for the scalar fields satisfying the Klein-Gordon equation in the de Sitter spacetime. In [27] it was shown that an existence of two scalar fields obeying the incomplete Huygens principle in the de Sitter spacetime implies that the spacetime is four-dimensional. The usual splitting of 4-spinors into 2-spinors allowed us to verify the Huygens principle for each 2-spinor separately. Mathematical analysis of the explicit formulas for the solutions, which is summarized in the main result of the present paper, Theorem 0.2, led to three exceptional values for the mass of the field: $m=0$, $m=-iHh/c^2$, and $m=iHh/c^2$, where H is the Hubble constant, c is the speed of light, and h is Planck's constant. It is remarkable that the duration of time when the factor $\exp\left(-i\,mc^2t/h\right)$ of the kernel of the integral representing the solution of the generalized Dirac equation with the mass $m=\pm iHh/c^2$ stays around unity, is limited by $\approx 10^{18}$ sec, which coincides with the age of the universe. A short paragraph written in the paper about neutrinos and imaginary mass touched on the possible physical interpretation in the framework of the Standard Model of particles physics. A more comprehensive discussion of the physical interpretation of the imaginary mass appearing as a result of mathematical analysis of the generalized Dirac equation in the de Sitter spacetime including, for instance, tachyons, was out of the scope of the present paper.

A Appendix

A.1 Some properties of hypergeometric function

There is a formula (See 15.3.6 of Ch.15[1] and [4, Sec.2.3.1].) that ties together points $z = 0$ and $z = 1$:

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b-c+1;1-z) \tag{A.1}$$
$$+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a,c-b;c-a-b+1;1-z)\,,$$

where $|\arg(1-z)| < \pi, \quad |1-z| < 1$, and $c-a-b \neq \pm 1, \pm 2, \ldots$.

A.2 Some properties of hypergeometric function. Case of zeros

We use (12) of [4, Sec. 2.10]:

$$F(a,b;a+b+m;z)\frac{1}{\Gamma(a+b+m)} = \frac{\Gamma(m)}{\Gamma(a+m)\Gamma(b+m)}\sum_{n=0}^{m-1}\frac{(a)_n(b)_n}{(1-m)_n n!}(1-z)^n \tag{A.2}$$
$$+\frac{(1-z)^m(-1)^m}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a+m)_n(b+m)_n}{(n+m)!n!}\left[h_n'' - \ln(1-z)\right](1-z)^n\,,$$

where $-\pi < \arg(1-z) < x, \quad a, b, \neq 0, -1, 2, \ldots,$

$$h_n'' = \psi(n+1) + \psi(n+m+1) - \psi(a+n+m) - \psi(b+n+m),$$

and $\sum_n^{m-1}$ is to be interpreted as zero when $m = 0$. The function $\psi(z)$ is the logarithmic derivative of the gamma function: $\psi(z) = \frac{d\ln\Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)}$.

A.3 Some properties of hypergeometric function. Case of poles

If $k = 1, 2, 3, \ldots$ since $c - a - b = -2k$ then we apply [4, (14) Sec. 2.10]:

$$F(a,b,a+b-m;z)\frac{1}{\Gamma(a+b-m)} = \frac{\Gamma(m)(1-z)^{-m}}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{m-1}\frac{(a-m)_n(b-m)_n}{(1-m)_n n!}(1-z)^n \tag{A.3}$$
$$+\frac{(-1)^m}{\Gamma(a-m)\Gamma(b-m)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(n+m)_n n!}\left[\bar{h}_n - \ln(1-z)\right](1-z)^n\,,$$

where $-\pi < \arg(1-z) < \pi, \quad a, b, \neq 0, -1, -2, \ldots$

$$\bar{h}_n = \psi(1+n) + \psi(1+n+m) - \psi(a+n) - \psi(b+n)\,,$$

and $\sum_{n=0}^{m-1}$ is set zero if $m = 0$.

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