12-17-2020

Ramanujan type congruences for quotients of level 7 Klein forms

Timothy Huber
The University of Texas Rio Grande Valley, timothy.huber@utrgv.edu

Dongxi Ye

Recommended Citation

Huber, T., & Ye, D. (2020). Ramanujan type congruences for quotients of level 7 Klein forms. Journal of Number Theory. https://doi.org/10.1016/j.jnt.2020.11.003

RAMANUJAN TYPE CONGRUENCES FOR QUOTIENTS OF LEVEL 7 KLEIN FORMS

TIMOTHY HUBER AND DONGXI YE

Abstract. Klein forms are used to construct generators for the graded algebra of modular forms of level 7. Dissection formulas for the series imply Ramanujan type congruences modulo powers of 7 for a family of generating functions that subsume the counting function for 7-core partitions. The broad class of arithmetic functions considered here enumerate colored partitions by weights determined by parts modulo 7. The method is a prototype for similar analysis of modular forms of level 7 and at other prime levels. As an example of the utility of the dissection method, the paper concludes with a derivation of novel congruences for the number of representations by $x^2+xy+2y^2$ in exactly k ways.

1. Introduction

In [8], Garvan, Kim and Stanton study arithmetic properties of the number $a_t(n)$ of partitions of n which are t-cores (see [13] for definitions) for $t \in \{5, 7, 11\}$. In particular, they derive the following congruences modulo powers of 5,7,11 for $a_t(n)$,

$$a_t(t^k n - \delta_t) \equiv 0 \pmod{t^k}, \quad \delta_t = (t^2-1)/24. \tag{1.1}$$

These are called Ramanujan type congruences because of their analogy to congruences for the classical partition function [19], and motivate a number of studies on Ramanujan type congruences for t-core partitions and other relevant counting functions (see, e.g., [2, 3, 4, 5, 6]). On the other hand, by the definition of $a_t(n)$ and simple manipulations of infinite products, the generating function may be written

$$\sum_{n=0}^{\infty} a_7(n) q^n = \frac{(q^7;q^7)_\infty^7}{(q;q)_\infty} = \frac{(q^7;q^7)_\infty^6}{(q,q^6;q^7)_\infty (q^2,q^5;q^7)_\infty (q^3,q^4;q^7)_\infty}, \tag{1.2}$$

where

$$(a;q)_n = \prod_{j=0}^{\infty}(1-aq^j) \qquad \text{and} \qquad (a_1,\ldots,a_k;q)_\infty = \prod_{j=1}^{k}(a_j;q)_\infty.$$

The rightmost infinite product representation expresses $a_7(n)$ as the enumeration of partitions of n such that parts congruent to $0 \pmod 7$ are 6-colored, distinct, and contribute ± 1 to the enumeration, according to the number of parts congruent to 0 modulo 7, while parts congruent to $i, 7-i \pmod 7$ appear with exactly one color for each of $i \in \{1,2,3\}$. This is a special case of the following general counting function considered in the present work.

Definition 1.1. *For $(a_0,a_1,a_2,a_3) \in \mathbb{Z}^4$, the colored-weighted partition $P_{a_0,a_1,a_2,a_3}(n)$ associated to (a_0,a_1,a_2,a_3) denotes the enumeration of partitions λ of n such that:*

- *If $a_i \leq 0$, parts congruent to $i, 7-i \pmod 7$ appear with $|a_i|$ colors;*

2010 *Mathematics Subject Classification.* 11F03; 11F11; 11P83; 11P84; 05A17.

Key words and phrases. Colored-weighted partition; Ramanujan type congruence; t-core partition.

The second author Dongxi Ye is supported by the Natural Science Foundation of China (grant No.11901586), the Natural Science Foundation of Guangdong Province (grant No.2019A1515011323) and the Sun Yat-sen University Research Grant for Youth Scholars (grant No.19lgpy244).

- *If $a_i > 0$, parts congruent to $i, 7-i \pmod 7$ are a_i-colored, distinct, and contribute $(-1)^{\ell_i(\lambda)}$ to the enumeration, where $\ell_i(\lambda)$ is the number of parts congruent to $i, 7-i \pmod 7$.*

The generating function for $P_{a_0,a_1,a_2,a_3}(n)$ has the infinite product representation

$$\sum_{n=0}^{\infty} P_{a_0,a_1,a_2,a_3}(n)q^n = (q^7;q^7)_\infty^{a_0}(q,q^6;q^7)_\infty^{a_1}(q^2,q^5;q^7)_\infty^{a_2}(q^3,q^4;q^7)_\infty^{a_3}. \tag{1.3}$$

The observations (1.2) and (1.3), together with identity (1.1) of Garvan et al. motivate an investigation of Ramanujan type congruences for $P_{6,a_1,a_2,a_3}(n)$ with $a_i \in \mathbb{Z}$ and $a_1 + a_2 + a_3 = -3$. Setting $q = e^{2\pi i\tau}$ for $\tau \in \mathbb{H}$ and viewing $\sum_{n=0}^{\infty} P_{6,a_1,a_2,a_3}(n)q^n$ as a function of τ, properties of Klein forms may be used to show that a certain class of these generating functions differ from a modular form of weight 3 for $\Gamma(7)$ by some power of q. This motivates an analysis from the context of modular forms. In this work, we provide a unified derivation of Ramanujan type congruences modulo powers of 7 for a family of colored-weighted partitions $P_{a_0,a_1,a_2,a_3}(n)$ with $a_0 = 4, 6$ and $a_1 + a_2 + a_3 = -2, -3$ that includes the 7-core partition function $a_7(n) = P_{6,-1,-1,-1}(n)$. We use Klein forms to construct generators for the graded algebra of holomorphic modular forms for $\Gamma(7)$ and prove a wealth of congruences. Since Klein forms are building blocks for more general modular forms, the analysis serves as a prototype for more general constructions at the higher weights and levels considered in [12].

Theorem 1.2. *Let $P_{a_0,a_1,a_2,a_3}(n)$ be defined as in Definition 1.1. Then the following Ramanujan type congruences hold.*

$$P_{6,1,0,-4}(7^k n - 3) \equiv 0 \pmod{7^{2k}}, \tag{1.4}$$
$$P_{6,-2,2,-3}(7^k n - 2) \equiv 0 \pmod{7^{k}}, \tag{1.5}$$
$$P_{6,-1,-1,-1}(7^k n - 2) \equiv 0 \pmod{7^{k}}, \tag{1.6}$$
$$P_{6,0,-4,1}(7^k n - 2) \equiv 0 \pmod{7^{2k}}, \tag{1.7}$$
$$P_{6,2,-3,-2}(7^k n - 3) \equiv 0 \pmod{7^{k}}, \tag{1.8}$$
$$P_{6,-4,1,0}(7^k n - 1) \equiv 0 \pmod{7^{2k}}, \tag{1.9}$$
$$P_{6,-2,-5,4}(7^k n - 1) \equiv 0 \pmod{7^{k}}, \tag{1.10}$$
$$P_{6,-3,-2,2}(7^k n - 1) \equiv 0 \pmod{7^{k}}, \tag{1.11}$$
$$P_{6,-5,4,-2}(7^k n - 1) \equiv 0 \pmod{7^{k}}, \tag{1.12}$$
$$P_{6,4,-2,-5}(7^k n - 4) \equiv 0 \pmod{7^{k}}. \tag{1.13}$$

In particular, $P_{6,-1,-1,-1}(n)$ is the 7-core partition function $a_7(n)$.

In addition to the congruence classes modulo powers of 7 in Theorem 1.2, we also establish the following associated Ramanujan type congruences.

Theorem 1.3. *Let $P_{a_0,a_1,a_2,a_3}(n)$ be defined as in Definition 1.1, and let $r \in \{3,5,6\}$, i.e., r is quadratic nonresidue modulo 7. Then the following Ramanujan type congruences hold.*

$$P_{6,1,0,-4}(7^{k+1} n + 7^k \cdot 5 - 3) \equiv 0 \pmod{7^{2k+1}}, \tag{1.14}$$
$$P_{6,-2,2,-3}(7^{k+1} n + 7^k \cdot r - 2) \equiv 0 \pmod{7^{2k}}, \tag{1.15}$$
$$P_{6,-1,-1,-1}(7^{k+1} n + 7^k \cdot r - 2) \equiv 0 \pmod{7^{2k}}, \tag{1.16}$$
$$P_{6,0,-4,1}(7^{k+1} n + 7^k \cdot 3 - 2) \equiv 0 \pmod{7^{2k+1}}, \tag{1.17}$$
$$P_{6,2,-3,-2}(7^{k+1} n + 7^k \cdot r - 3) \equiv 0 \pmod{7^{2k}}, \tag{1.18}$$

$$P_{6,-4,1,0}(7^{k+1}n + 7^k \cdot 6 - 1) \equiv 0 \pmod{7^{2k+1}}, \tag{1.19}$$
$$P_{6,-2,-5,4}(7^{k+1}n + 7^k \cdot 3 - 1) \equiv 0 \pmod{7^{2k}}, \tag{1.20}$$
$$P_{6,-2,-5,4}(7^{k+1}n + 7^k \cdot 5 - 1) \equiv 0 \pmod{7^{2k}}, \tag{1.21}$$
$$P_{6,-2,-5,4}(7^{k+1}n + 7^k \cdot 6 - 1) \equiv 0 \pmod{7^{2k+1}}, \tag{1.22}$$
$$P_{6,-3,-2,2}(7^{k+1}n + 7^k \cdot r - 1) \equiv 0 \pmod{7^{2k}}, \tag{1.23}$$
$$P_{6,-5,4,-2}(7^{k+1}n + 7^k \cdot 3 - 1) \equiv 0 \pmod{7^{2k}}, \tag{1.24}$$
$$P_{6,-5,4,-2}(7^{k+1}n + 7^k \cdot 5 - 1) \equiv 0 \pmod{7^{2k+1}}, \tag{1.25}$$
$$P_{6,-5,4,-2}(7^{k+1}n + 7^k \cdot 6 - 1) \equiv 0 \pmod{7^{2k}}, \tag{1.26}$$
$$P_{6,4,-2,-5}(7^{k+1}n + 7^k \cdot 3 - 4) \equiv 0 \pmod{7^{2k+1}}, \tag{1.27}$$
$$P_{6,4,-2,-5}(7^{k+1}n + 7^k \cdot 5 - 4) \equiv 0 \pmod{7^{2k}}, \tag{1.28}$$
$$P_{6,4,-2,-5}(7^{k+1}n + 7^k \cdot 6 - 4) \equiv 0 \pmod{7^{2k}}. \tag{1.29}$$

In particular, specializing (1.16) *to* $k = 1$ *recovers* [8, Corollary 1(3)–(5) $\pmod{7^2}$].

Remark 1.4. *Similar results also exist for the coefficients whose indices are quadratic residue classes modulo 7. However, these correspond primarily to congruences modulo* 7^k *instead of* 7^{2k}*.*

The arithmetic functions in the last two theorems are coefficients of a subset of the series $P_{a_0,a_1,a_2,a_3}(n)$ defined by (1.3) consisting of weight 3 modular forms for $\Gamma_1(7)$ spanned by components of a vector eigenform for the linear operator defined for $f = f(\tau) = \sum_{n=0}^{\infty} a(n)q^n$ by

$$U_p(f) = \sum_{n=0}^{\infty} a(pn)q^n.$$

This vector eigenform is constructed from a set of generators for the graded algebra of holomorphic modular forms for $\Gamma_1(7)$ and includes 10 of the modular forms of weight 3 that arise as quotients of a certain class of Klein forms. Our method provides a straightforward and general means to investigate dissections of products arising in this way, including t-core and other colored-weighted partition functions. Motivation for the method is based on a similar approach for modular forms of level 5 in [10] where the following modular forms of weight 1/5 are introduced:

$$A(\tau) = q^{1/5}(q;q)_\infty^{-3/5} \sum_{n=-\infty}^{\infty} (-1)^n q^{(5n^2-3n)/2},$$

$$B(\tau) = (q;q)_\infty^{-3/5} \sum_{n=-\infty}^{\infty} (-1)^n q^{(5n^2-n)/2}.$$

Critical ingredients for applying dissection operators in [10] are the formulas [10, Theorem 3.3]

$$A^5(\tau/5) = A^5 - 3A^4B + 4A^3B^2 - 2A^2B^3 + AB^4, \tag{1.30}$$
$$B^5(\tau/5) = B^5 + 3B^4A + 4B^3A^2 + 2B^2A^3 + BA^4. \tag{1.31}$$

The import of (1.30)–(1.31) is that the left sides (when $\tau/5$ is replaced with τ) form a basis for the weight one forms for $\Gamma_1(5)$, while the collection of functions on the right sides form a basis for weight one forms for $\Gamma(5)$. Moreover the full set of quotients on each side, respectively, generate the graded algebra over $\mathbb{C}$ of holomorphic modular forms over the aforementioned subgroups [11]. The series on the right sides of (1.30)–(1.31) serve as dissections of the series on the left since each term is of the form $q^{j/5}\sum_{n=0}^{\infty} a(n)q^n$ for a distinct reduced value of j modulo 5. This allows the dissection of powers, quotients, and linear combinations of the terms on the left by replacing τ by $\tau/5$, applying (1.30)–(1.31) and collecting the contribution in the resultant from a given congruence class modulo 5. Explicit formulas for applying U_5 to a vector of basis elements for $M_k(\Gamma_1(5))$ are given in [10, Corollary 3.6]. In particular for the basis of weight two forms for $\Gamma_1(5)$, $W = \begin{pmatrix} A^{10} & A^5B^5 & A^{10}\end{pmatrix}^T$,

$$U_5(W) = \begin{pmatrix} 1 & 22 & 0 \\ 0 & 5 & 0 \\ 0 & -22 & 1 \end{pmatrix} W.$$

Thus $A^5B^5 = q\sum_{n=0}^{\infty} a_5(n)q^n$ is an eigenfunction for the operator U_5 with eigenvalue 5, and (1.1) follows for $t = 5$ by iterating this operator. Extending these results to level 7 subgroups requires recognition of (1.30)–(1.31) in terms of parameters that translate between levels. In particular,

$$A^5(\tau) = \frac{K_{1/5}^2}{K_{2/5}^3}(5\tau), \quad B^5(\tau) = \frac{K_{2/5}^2}{K_{1/5}^3}(5\tau), \tag{1.32}$$

where $K_r := K_{(r,0)}$ are Klein forms defined for $q_z = e^{2\pi i(Q_1\tau+Q_2)}$, and $(Q_1, Q_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ by

$$K_{(Q_1,Q_2)}(\tau) = e^{\pi i Q_2(Q_1-1)} q^{\frac{1}{2}Q_1(Q_1-1)}(1-q_z)\prod_{n=1}^{\infty}(1-q_zq^n)(1-q_z^{-1}q^n)(1-q^n)^{-2}.$$

Quotients of Klein forms of level 7 analogous to (1.32) are defined in Lemma 2.2. A counterpart of the dissection identities (1.30)–(1.31) at level 7 is given by Lemma 2.3. The lemma is applied in the same way its counterpart is used in [10] to iteratively dissect modular forms of level 7, including the generating function for the enumeration of 7-cores $a_7(n)$. Additionally, in Section 2, we use properties of Klein forms to formulate generators for the graded algebra of holomorphic modular forms for $\Gamma_1(7)$ and $\Gamma(7)$. These are key ingredients in our derivations of Ramanujan type congruences for counting functions associated to modular forms of level 7. In Section 3, we state and prove two propositions formulating explicit representations for iterated dissections modulo 7. In Section 4 we turn our attention to deriving congruences for coefficients of a modular form counting the number of representations by the binary quadratic form $x^2 + xy + 2y^2$.

2. Notation and Preliminaries

In this section, we define certain notation and state preliminary results on the graded algebra over $\mathbb{C}$ for the holomorphic modular forms for $\Gamma(7)$. In the remainder of the present work, modular forms are referred to holomorphic modular forms. The following lemma allows us to verify that the reciprocal of the Klein forms $K_{1/7}(7\tau), K_{2/7}(7\tau), K_{3/7}(7\tau)$, are modular forms for $\Gamma(7)$. We subsequently use these properties to construct modular forms of weight 1 for $\Gamma(7)$ and $\Gamma_1(7)$.

Lemma 2.1 ([17])**.** *Let $K_{(Q_1,Q_2)}(\tau)$ be defined as above. Then the following assertions hold.*

(1) For $(Q_1, Q_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ and $(s_1, s_2) \in \mathbb{Z}^2$, one has

$$K_{(-Q_1,-Q_2)}(\tau) = -K_{(Q_1,Q_2)}(\tau)$$
$$K_{(Q_1,Q_2)+(s_1,s_2)}(\tau) = (-1)^{s_1s_2+s_1+s_2}e^{-\pi i(s_1Q_2-s_2Q_1)}K_{(Q_1,Q_2)}(\tau).$$

(2) For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$K_{(Q_1,Q_2)}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{-1}K_{(Q_1a+Q_2c,Q_1b+Q_2d)}(\tau).$$

(3) The order of vanishing of $K_{(Q_1,Q_2)}(\tau)$ at the cusp $i\infty$ is given by

$$\operatorname{ord}_\infty(K_{(Q_1,Q_2)}) = \frac{1}{2}\langle Q_1\rangle\left(\langle Q_1\rangle - 1\right),$$

where $\langle r\rangle = r - \lfloor r\rfloor$ denotes the fractional part of r.

For $r = 1, 3, 5$, define $T_r = T_r(\tau)$ by

$$T_r(\tau) = \frac{1}{K_{(\frac{7-r}{14},0)}(7\tau)}. \tag{2.1}$$

Then T_r are holomorphic modular forms of weight 1 for $\Gamma(7)$ (cf.[7]). Combinatorial interpretations for quotients of the Klein forms may be more readily deduced from the representation

$$T_r(\tau) = \exp\left(\frac{\pi i r}{14}\right)\eta^3(7\tau)\cdot\theta^{-1}\begin{bmatrix} r/7 \\ 1\end{bmatrix}(7\tau).$$

where $\eta(\tau)$ is the Dedekind eta function

$$\eta(\tau) = q^{1/24}(q;q)_\infty,$$

and the theta constants of level N and odd index $0 < r < N$ are given by

$$\theta\begin{bmatrix} r/N \\ 1\end{bmatrix}(N\tau) = \exp\left(\frac{\pi i r}{2N}\right) q^{r^2/(8N)}(q^{(N-r)/2};q^N)_\infty(q^{(N+r)/2};q^N)_\infty(q^N;q^N)_\infty.$$

For k a positive integer, denote by $\mathcal{M}_k(\Gamma)$ the vector space of modular forms for a congruence subgroup Γ. The next lemma formulates explicit bases for $\mathcal{M}_1(\Gamma(7))$ and $\mathcal{M}_1(\Gamma_1(7))$ grouped by congruence class of the order of vanishing. The quotients corresponding to the congruence class 0 generate the graded algebra over $\mathbb{C}$ for $\mathcal{M}_k(\Gamma_1((7))$, while the full set generate that for $\mathcal{M}_k(\Gamma(7))$.

Lemma 2.2. *Define the following subscripted vectors:*

$$\vec{v}_0 = \left\langle \frac{T_1^2}{T_3}, \frac{T_3^2}{T_5}, \frac{T_5^2}{T_1}\right\rangle_0, \quad \vec{v}_1 = \left\langle \frac{T_5^2}{T_3}, \frac{T_3T_1}{T_5}\right\rangle_1, \quad \vec{v}_2 = \left\langle \frac{T_1^2}{T_5}, \frac{T_3T_5}{T_1}\right\rangle_2, \tag{2.2}$$

$$\vec{v}_3 = \langle T_5\rangle_3, \quad \vec{v}_4 = \left\langle \frac{T_3^2}{T_1}, \frac{T_5T_1}{T_3}\right\rangle_4, \quad \vec{v}_5 = \langle T_3\rangle_5, \quad \vec{v}_6 = \langle T_1\rangle_6. \tag{2.3}$$

(1) Components of (2.2)–(2.3) form a basis for $\mathcal{M}_1(\Gamma(7))$.
(2) Components of $\vec{v}_0$ form a basis for $\mathcal{M}_1(\Gamma_1(7))$.
(3) Components of $\vec{v}_j$ have the form

$$q^{j/7}\sum_{n=0}^{\infty} a(n)q^n, \qquad a(n)\in\mathbb{Z}.$$

(4) Components of $\vec{v}_0$ generate the graded algebra of modular forms for $\Gamma_1(7)$, and the components (2.2)–(2.3) generate the graded algebra of modular forms for $\Gamma(7)$.

Proof. The transformation formulas for Parts (1) and (2) follow from (2.1) and Lemma 2.1 (2). A detailed proof of the modular properties of more general quotients subsuming the cases considered here appear in [7]. The required holomorphicity follows from (2.1) and Lemma 2.1 (2) and (3). The Fourier expansions $T_r(\tau)$ may be used to deduce the components of $\vec{v}_j$ are linearly independent over $\mathbb{C}$. Finally, Parts (1) and (2) follow from the coincidence of the number of components of $\vec{v}_j$ (resp. $\vec{v}_0$) and the dimension of $\mathcal{M}_1(\Gamma(7))$ (resp. $\mathcal{M}_1(\Gamma_1(7))$). Part (3) follows directly from the definition of $T_r(\tau)$. Part (4) follows from Part (1) and [14]. □

The next lemma is the foundation for subsequent results and is an analogue of (1.30)–(1.31).

Lemma 2.3. *Let $\vec{v}_j$ be defined as in Lemma 2.2, and let $(b_1, \ldots, b_k)_j = \vec{v}_j(b_1, \ldots, b_k)^T$. Then*

$$\frac{T_1^2}{T_3}(\tau/7) = (1,0,0)_0 + (1,1)_1 + (-2,0)_2 + (-1)_3 + (-1,3)_4 + (2)_5 + (-3)_6, \tag{2.4}$$

$$\frac{T_3^2}{T_5}(\tau/7) = (0,1,0)_0 + (1,-3)_1 + (1,-1)_2 + (2)_3 + (-2,0)_4 + (3)_5 + (-1)_6, \tag{2.5}$$

$$\frac{T_5^2}{T_1}(\tau/7) = (0,0,1)_0 + (2,0)_1 + (1,3)_2 + (3)_3 + (1,1)_4 + (1)_5 + (2)_6, \tag{2.6}$$

$$\frac{T_3T_1^3}{T_5^3}(\tau/7) = (1,-1,0)_0 + (0,4)_1 + (-3,1)_2 + (-3)_3 + (1,3)_4 + (-1)_5 + (-2)_6, \tag{2.7}$$

$$\frac{T_1T_5^3}{T_3^3}(\tau/7) = (1,0,1)_0 + (3,1)_1 + (-1,3)_2 + (2)_3 + (0,4)_4 + (3)_5 + (-1)_6, \tag{2.8}$$

$$\frac{T_5T_3^3}{T_1^3}(\tau/7) = (0,-1,1)_0 + (1,3)_1 + (0,4)_2 + (1)_3 + (3,1)_4 + (-2)_5 + (3)_6. \tag{2.9}$$

Proof. The left side of (2.4)–(2.6) are the vector components of $\vec{v}_0$ in Lemma 2.2 evaluated at $\tau/7$. It is straightforward to check that if $g(\tau)$ is a modular form for $\Gamma_0(7)$, then $g(\tau/7)$ is a modular form of the same weight for $\Gamma(7)$. Therefore, by Lemma 2.2, the left sides of (2.4)–(2.6) are elements of $\mathcal{M}_1(\Gamma(7))$. Hence, each expression on the left of (2.4)–(2.6) may be expressed as a linear combination of the basis elements from (2.2)–(2.3). The decomposition formulas listed in (2.7)–(2.9) follow similarly from the fact that the quotients $T_5T_3^3/T_1^3$, $T_3T_1^3/T_5^3$, $T_1T_5^3/T_3^3$ constitute an alternative basis for $\mathcal{M}_1(\Gamma_1(7))$. This can be verified using a procedure similar to that in the proof of Lemma 2.2. $\square$

The series on the left sides of each equality in Lemma 2.3 comprise a complete set of quotients of Klein forms of the form $T_5^{-a_1}T_3^{-a_2}T_1^{-a_3}$ in $\mathcal{M}_1(\Gamma_1(7))$. Moreover, the constituents of $\vec{v}_i$ in Lemma 2.2 are determined by the Fourier expansion of $T_5^{-a_1}T_3^{-a_2}T_1^{-a_3}$. To see this, note that by Lemma 2.1,

$$\begin{aligned}
\mathrm{Div}_{X_1(7)}(T_5) &= \frac{3}{7}[1/7] + \frac{5}{7}[2/7] + \frac{6}{7}[3/7],\\
\mathrm{Div}_{X_1(7)}(T_3) &= \frac{5}{7}[1/7] + \frac{6}{7}[2/7] + \frac{3}{7}[3/7],\\
\mathrm{Div}_{X_1(7)}(T_1) &= \frac{6}{7}[1/7] + \frac{3}{7}[2/7] + \frac{5}{7}[3/7].
\end{aligned}$$

Then for

$$T_5^{-a_1}T_3^{-a_2}T_1^{-a_3} = \sum_{n=\frac{-3a_1-5a_2-6a_3}{7}}^{\infty} P_{6,a_1,a_2,a_3}\left(n - \frac{-3a_1-5a_2-6a_3}{7}\right) q^n$$

to be a modular form for $\Gamma_1(7)$ of weight k, one must have that

$$(\mathrm{mod}\ 7)\ 0 \equiv -3a_1 - 5a_2 - 6a_3 \geq 0, \tag{2.10}$$

$$(\mathrm{mod}\ 7)\ 0 \equiv -5a_1 - 6a_2 - 3a_3 \geq 0, \tag{2.11}$$

$$(\mathrm{mod}\ 7)\ 0 \equiv -6a_1 - 3a_2 - 5a_3 \geq 0, \tag{2.12}$$

$$a_1 + a_2 + a_3 = -k. \tag{2.13}$$

For $k = 1$, there are exactly 6 solutions to the system of congruences. These correspond to the exponents appearing in the quotients on the left side of each identity in Lemma 2.3. A choice of 3 solutions corresponding to linearly independent series $T_5^{-a_1}T_3^{-a_2}T_1^{-a_3}$ determine $\vec{v}_0$ in Lemma 2.2.

To determine exponents so that $T_5^{-a_1}T_3^{-a_2}T_1^{-a_3} \in \mathcal{M}_1(\Gamma(7))$, congruence (2.10) is altered so that

$$(2.14) \qquad \pmod 7\ 1 \equiv -3a_1 - 5a_2 - 6a_3 \geq 0, \quad -5a_1 - 6a_2 - 3a_3 \geq 0, \quad -6a_1 - 3a_2 - 5a_3 \geq 0,$$

and congruences (2.11)–(2.12) are omitted. The resulting solutions are $(-2,1,0)$, $(1,-1,-1)$, $(-1,-2,2)$. Any two of the corresponding series $T_5^{-a_1}T_3^{-a_2}T_1^{-a_3}$ are linearly independent. The choice of the first two solutions determines $\vec{v}_1$ in Lemma 2.2. The constituents of $\vec{v}_i$ for $0 \leq i \leq 6$ are similarly established. For $k = 2$, there are precisely 15 integral solutions to the original system (2.10)–(2.13) given by

$$(-6,6,-2),\ (-5,3,0),\ (-4,0,2),\ (-3,-3,4),\ (-3,4,-3),\ (-2,-6,6),\ (-2,1,-1)^*$$
$$(-1,-2,1)^*,\ (0,-5,3),\ (0,2,-4),\ (1,-1,-2)^*,\ (2,-4,0),\ (3,0,-5),\ (4,-3,-3),\ (6,-2,-6).$$

For $k = 3$, the 28 integral solutions to to (2.10)–(2.13) are

$$(-9,9,-3),\ (-8,6,-1),\ (-7,3,1),\ (-6,0,3),\ (-5,-3,5),\ (-4,-6,7),\ (-3,-9,9),$$
$$(-5,4,-2)^*,\ (-4,1,0)^*,\ (-3,-2,2)^*,\ (-2,-5,4)^*,\ (-1,-8,6),\ (-3,5,-5), (-2,2,-3)^*,$$
$$(-1,-1,-1)^*,\ (0,-4,1)^*,\ (1,-7,3),\ (0,3,-6),\ (1,0,-4)^*,\ (2,-3,-2)^*,\ (3,-6,0),$$
$$(3,1,-7),\ (4,-2,-5)^*,\ (5,-5,-3),\ (6,-1,-8),\ (7,-4,-6),\ (9,-3,9), (-6,7,-4).$$

Starred (*) values of solutions to (2.10)–(2.13) for weights $k = 2, 3$ above refer to instances where $T_5^{-a_1}T_3^{-a_2}T_1^{-a_3}$ is an element of a subspace of of $M_k(\Gamma_1(7))$ corresponding to a vector eigenform for U_7 whose matrix of eigenvalues are integral multiples of 7. These determine the subscripts for colored-weighted partition functions $P_{2k,a_1,a_2,a_3}(n)$ appearing in Theorem 1.2 and Theorem 2.4 (see below). To describe how such subspaces are developed, note that for $k \geq 2$, Lemmas 2.2 and 2.3 permit the construction of a vector-valued eigenfunction for U_7 from a basis for $\mathcal{M}_k(\Gamma_1(7))$. For example, if we denote the quotients of the vector $\vec{v}_0$ from Lemma 2.2 by x, y, z, respectively, the components of $\begin{pmatrix} x^2 & y^2 & z^2 & xz & yz \end{pmatrix}^T = G$ form a basis for $\mathcal{M}_2(\Gamma_1(7))$. By Lemma 2.3 (cf. the proof of Lemma 3.1),

$$U_7(G) = \begin{pmatrix} 1 & 0 & 0 & -26 & 16 \\ 0 & 1 & 0 & 10 & -26 \\ 0 & 0 & 1 & 16 & 10 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix} G.$$

The last two rows of the eigenvalue matrix correspond to eigenforms for U_7 with eigenvalue 7. Therefore, to derive congruences modulo powers of 7, we apply U_7 to the subspace of $\mathcal{M}_2(\Gamma_1(7))$ spanned by the last two components of G. In particular, we may iteratively apply the U_7 operator to products of the weight one generators in the span of the components of G to obtain

$$(2.15) \qquad U_7^k(xy) = 7^k \begin{pmatrix} 1 & -1 \end{pmatrix} R, \quad U_7^k(xz) = 7^k \begin{pmatrix} 1 & 0 \end{pmatrix} R, \quad U_7^k(yz) = 7^k \begin{pmatrix} 0 & 1 \end{pmatrix} R,$$

where $R = \begin{pmatrix} xz & yz \end{pmatrix}^T$ and Klein's relation (3.4) was used to derive the leftmost equation of (2.15). This provides Fourier expansions witnessing the congruences in Theorem 2.4.

Theorem 2.4. *Let $P_{a_0,a_1,a_2,a_3}(n)$ be defined as in Definition 1.1. Then the following Ramanujan type congruences hold.*

$$P_{4,1,-1,-2}(7^k n - 2) \equiv 0 \pmod{7^k},$$
$$P_{4,-2,1,-1}(7^k n - 1) \equiv 0 \pmod{7^k},$$
$$P_{4,-1,-2,1}(7^k n - 1) \equiv 0 \pmod{7^k}.$$

In the next section, we prove the main theorems of the paper by constructing a corresponding subspace and eigenform for $\mathcal{M}_3(\Gamma_1(7))$ with an eigenvalue matrix that is a multiple of 7.

3. Proofs of Theorems 1.2 and 1.3

In this section, we prove Theorems 1.2 and 1.3 by establishing Propositions 3.2 and 3.3 below. One of the main tools is the formulation of an eigenvector for the U_7-operator whose components span the subpace of modular forms containing the ensuing dissections. If the quotients of the vector $\vec{v}_0$ from Lemma 2.2 are denoted x, y, z, respectively, then an ordered basis for $\mathcal{M}_3(\Gamma_1(7))$ is given by $\begin{pmatrix} x^3 & y^3 & z^3 & x^2z & y^2z & xz^2 & yz^2 \end{pmatrix}$. Lemma 2.3 may be used to conclude that only the last 4 elements listed in the basis are mapped under U_7 to a linear combination of elements in the basis in which the coefficients are integer multiples of 7. This determines the composition of the vector eigenform V for U_7 in Lemma 3.1.

Lemma 3.1. *Write* $\frac{T_1^2}{T_3}, \frac{T_3^2}{T_5}, \frac{T_5^2}{T_1}$ *as* x, y, z*, respectively, and let* $V = V(\tau) = \begin{pmatrix} x^2z & y^2z & xz^2 & yz^2 \end{pmatrix}^T$. *Then one has that*

$$U_7(V) = 7 \begin{pmatrix} 6 & 1 & -6 & 7 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 1 & -1 & 6 & 0 \end{pmatrix} V. \tag{3.1}$$

In particular, $A = PDP^{-1}$, *with*

$$P = \begin{pmatrix} -1 & 7 & -6 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}, \tag{3.2}$$

and thus,

$$A^{k-1} = \begin{pmatrix} \frac{(-1)^{k-1}+7^k}{8} & \frac{(-1)^k+7^{k-1}}{8} & \frac{3((-1)^{k-1}-7^{k-1})}{4} & \frac{7((-1)^k+7^{k-1})}{8} \\ 0 & 7^{k-1} & 0 & 0 \\ 0 & 0 & 7^{k-1} & 0 \\ \frac{(-1)^k+7^{k-1}}{8} & \frac{(-1)^{k-1}-7^{k-1}}{8} & \frac{3((-1)^k+7^{k-1})}{4} & \frac{7(-1)^{k-1}+7^{k-1}}{8} \end{pmatrix}.$$

Proof. Write $f(\tau) = x^2z$. Since $f(\tau)$ is a modular form of weight 3 for $\Gamma_1(7)$ by Lemma 2.2, then $f(\tau/7)$ is a modular form of weight 3 for $\Gamma(7)$, and thus by Lemma 2.3, one can write $f(\tau/7)$ as the product of the square of the right side of (2.4) and the right side of (2.6). Terms contributing integral exponents of q are

$$T_1^a T_3^b T_5^c, \qquad 6a + 5b + 3c \equiv 0 \pmod 7. \tag{3.3}$$

Extracting terms on both sides of the expansion of $f(\tau/7)$ of the form q^n for integral n and reducing using Klein's relation [15, 16]

$$\frac{T_1^2T_3}{T_5} + \frac{T_3^2T_5}{T_1} - \frac{T_1T_5^2}{T_3} = 0, \tag{3.4}$$

one deduces that

$$U_7(x^2z) = 7 \begin{pmatrix} 6 & 1 & -6 & 7 \end{pmatrix} V.$$

Identity (3.1) follows from a similar process applied to the remaining vector components. □

Any modular form $T_5^{-a_1}T_3^{-a_2}T_1^{-a_3}$ mapped by U_7 into the subspace of $\mathcal{M}_3(\Gamma_1(7))$ spanned by the components of V may be iteratively dissected by applying Lemma 3.1 to $T_5^{-a_1}T_3^{-a_2}T_1^{-a_3}$. The matrix formulation from Lemma 3.1 permits an explicit representation for dissections of the 10 quotients of Klein forms in the span of the components of the vector-valued eigenfunction V.

Proposition 3.2. *Let $P_{a_0,a_1,a_2,a_3}(n)$ be defined as in Definition 1.1 and denote x, y, z, V as in Lemma 3.1. Then the following identities hold.*

$$\sum_{n=1}^{\infty} P_{6,1,0,-4}(7^k n-3)q^n = 7^{2k}\begin{pmatrix}1 & 0 & -1 & 1\end{pmatrix}V, \tag{3.5}$$

$$\sum_{n=1}^{\infty} P_{6,-2,2,-3}(7^k n-2)q^n = 7^{k}\begin{pmatrix}\frac{(-1)^k+7^{k+1}}{8} & \frac{(-1)^{k-1}+7^k}{8} & \frac{3((-1)^k-7^k)}{4} & \frac{7(-1)^{k-1}+7^{k+1}}{8}\end{pmatrix}V, \tag{3.6}$$

$$\sum_{n=1}^{\infty} P_{6,-1,-1,-1}(7^k n-2)q^n = 7^{k}\begin{pmatrix}\frac{(-1)^k-7^k}{8} & \frac{(-1)^{k-1}+7^k}{8} & \frac{3(-1)^k+7^k}{4} & \frac{7(-1)^{k-1}-7^k}{8}\end{pmatrix}V, \tag{3.7}$$

$$\sum_{n=1}^{\infty} P_{6,0,-4,1}(7^k n-2)q^n = 7^{2k}\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}V, \tag{3.8}$$

$$\sum_{n=1}^{\infty} P_{6,2,-3,-2}(7^k n-3)q^n = 7^{k}\begin{pmatrix}\frac{(-1)^k-7^k}{8} & \frac{(-1)^{k-1}-7^{k+1}}{8} & \frac{3(-1)^k+7^k}{4} & \frac{7(-1)^{k-1}-7^k}{8}\end{pmatrix}V, \tag{3.9}$$

$$\sum_{n=1}^{\infty} P_{6,-4,1,0}(7^k n-1)q^n = 7^{2k}\begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}V, \tag{3.10}$$

$$\sum_{n=1}^{\infty} P_{6,-2,-5,4}(7^k n-1)q^n = 7^{k}\begin{pmatrix}\frac{(-1)^{k-1}+7^k}{8} & \frac{(-1)^k-9\cdot 7^k}{8} & \frac{3((-1)^{k-1}+7^k)}{4} & \frac{7(-1)^k+7^k}{8}\end{pmatrix}V, \tag{3.11}$$

$$\sum_{n=1}^{\infty} P_{6,-3,-2,2}(7^k n-1)q^n = 7^{k}\begin{pmatrix}\frac{(-1)^{k-1}+7^k}{8} & \frac{(-1)^k-7^k}{8} & \frac{3((-1)^{k-1}+7^k)}{4} & \frac{7(-1)^k+7^k}{8}\end{pmatrix}V, \tag{3.12}$$

$$\sum_{n=1}^{\infty} P_{6,-5,4,-2}(7^k n-1)q^n = 7^{k}\begin{pmatrix}\frac{(-1)^k+7^{k+1}}{8} & \frac{(-1)^{k-1}+7^k}{8} & \frac{(3(-1)^k+7^k)}{4} & \frac{7(-1)^{k-1}+7^{k+1}}{8}\end{pmatrix}V, \tag{3.13}$$

$$\sum_{n=1}^{\infty} P_{6,4,-2,-5}(7^k n-4)q^n = 7^{k}\begin{pmatrix}\frac{(-1)^{k-1}+9\cdot 7^k}{8} & \frac{(-1)^k+7^{k+1}}{8} & \frac{3(-1)^{k-1}-5\cdot 7^k}{4} & \frac{7(-1)^k+9\cdot 7^k}{8}\end{pmatrix}V. \tag{3.14}$$

Proof. Consider the weight 3 modular form $f(\tau) = T_1^2 T_3^3 T_5^{-2} = xy^2$ for $\Gamma_1(7)$. This is the generating function of $P_{6,2,-3,-2}(n)$ up a factor of q^3. To prove the congruences modulo 7 for $P_{6,2,-3,-2}(n)$, we perform a 7-dissection on $f(\tau)$ by applying the method in the proof of Lemma 3.1. By Lemma 2.3, $f(\tau/7)$ is a product of the right side of (2.4) and the square of the right side of (2.5). Extracting terms on both sides of the expansion of $f(\tau/7)$ of the form q^n for integral n using (3.3) and reducing by applying Klein's relation (3.4) results in

$$\sum_{n=1}^{\infty} P_{6,2,-3,-2}(7n-3)q^n = 7(-x^2z - 6y^2z + xz^2). \tag{3.15}$$

We can reapply Lemma 2.3 to repeat the dissection process. By employing (3.15) and iterating the dissection procedure an additional $k-1$ times through (3.1), we find

$$\sum_{n=1}^{\infty} P_{6,2,-3,-2}(7^k n-3)q^n = 7^k\begin{pmatrix}-1 & -6 & 1 & 0\end{pmatrix}A^{k-1}V,$$

where A is defined as in Lemma 3.1, and the identity follows. Proofs of the remaining identities follow along similar lines, with the caveat that for some of the values of a_1, a_2, a_3, dissection of the generating function for $P_{6,a_1,a_2,a_3}(n)$ must be accomplished through the last three identitites of Lemma 2.3. □

Proof of Theorem 1.2. These follow immediately from Proposition 3.2 and the fact that the Fourier coefficients of the components of V are all integral. $\square$

The last proposition was derived by iteratively applying the U_7 operator. Each iteration of U_7 may be followed by an iteration of $U_{7,r}(f)$, for particular r, to formulate additional congruences, where

$$U_{7,r}(f) = \sum_{n=0}^{\infty} a(7n+r) q^{\frac{7n+r}{7}}$$

for $f = f(\tau) = \sum_{n=0}^{\infty} a(n) q^{n/7}$. In the next proposition, explicit representations are given for dissections corresponding to the above congruences. For a fixed r, independent of k, the congruences in the last Theorem are witnessed by the same expression cup to a multiple of the modulus.

Proposition 3.3. *Let $P_{a_0,a_1,a_2,a_3}(n)$ be defined as in Definition 1.1. Let*

$$U_3 = \begin{pmatrix} \frac{T_1^4 T_5}{T_3^2} & \frac{T_3^5 T_5}{T_1^3} & \frac{T_3^2 T_5^2}{T_1} & \frac{T_5^5}{T_1^2} & \frac{T_5^6}{T_3^3} \end{pmatrix}^T, \quad U_5 = \begin{pmatrix} \frac{T_3^6}{T_1^3} & \frac{T_3^3 T_5}{T_1} & \frac{T_1^3 T_5^3}{T_3^3} & \frac{T_3 T_5^4}{T_1^2} & \frac{T_5^5}{T_3^2} \end{pmatrix}^T,$$

$$U_6 = \begin{pmatrix} \frac{T_1^5}{T_3^2} & \frac{T_3^5}{T_1^2} & \frac{T_3^3 T_5^3}{T_1^3} & \frac{T_5^4}{T_1} & \frac{T_1 T_5^5}{T_3^3} \end{pmatrix}^T.$$

Then the following identities hold.

$$\sum_{n=0}^{\infty} P_{6,1,0,-4}(7^{k+1}n + 7^k \cdot 3 - 3) q^{(7n+3)/7} = 7^{2k} \begin{pmatrix} 25 & -9 & 45 & 41 & -40 \end{pmatrix} U_3, \tag{3.16}$$

$$\sum_{n=1}^{\infty} P_{6,1,0,-4}(7^{k+1}n + 7^k \cdot 5 - 3) q^{(7n+5)/7} = 7^{2k+1} \begin{pmatrix} 0 & -3 & -5 & -9 & 9 \end{pmatrix} U_5, \tag{3.17}$$

$$\sum_{n=1}^{\infty} P_{6,1,0,-4}(7^{k+1}n + 7^k \cdot 6 - 3) q^{(7n+6)/7} = 7^{2k} \begin{pmatrix} 9 & -1 & -40 & 99 & -55 \end{pmatrix} U_6, \tag{3.18}$$

$$\sum_{n=1}^{\infty} P_{6,-2,2,-3}(7^{k+1}n + 7^k \cdot 3 - 2) q^{(7n+3)/7} = 7^{2k} \begin{pmatrix} 22 & -4 & 41 & 33 & -31 \end{pmatrix} U_3, \tag{3.19}$$

$$\sum_{n=1}^{\infty} P_{6,-2,2,-3}(7^{k+1}n + 7^k \cdot 5 - 2) q^{(7n+5)/7} = 7^{2k} \begin{pmatrix} 1 & -12 & -30 & -54 & 57 \end{pmatrix} U_5, \tag{3.20}$$

$$\sum_{n=1}^{\infty} P_{6,-2,2,-3}(7^{k+1}n + 7^k \cdot 6 - 2) q^{(7n+6)/7} = 7^{2k} \begin{pmatrix} 8 & -4 & -34 & 88 & -45 \end{pmatrix} U_6, \tag{3.21}$$

$$\sum_{n=1}^{\infty} P_{6,-1,-1,-1}(7^{k+1}n + 7^k \cdot 3 - 2) q^{(7n+3)/7} = 7^{2k} \begin{pmatrix} -3 & 5 & -4 & -8 & 9 \end{pmatrix} U_3, \tag{3.22}$$

$$\sum_{n=1}^{\infty} P_{6,-1,-1,-1}(7^{k+1}n + 7^k \cdot 5 - 2) q^{(7n+5)/7} = 7^{2k} \begin{pmatrix} 1 & 9 & 5 & 9 & -6 \end{pmatrix} U_5, \tag{3.23}$$

$$\sum_{n=1}^{\infty} P_{6,-1,-1,-1}(7^{k+1}n + 7^k \cdot 6 - 2) q^{(7n+6)/7} = 7^{2k} \begin{pmatrix} -1 & -3 & 6 & -11 & 10 \end{pmatrix} U_6, \tag{3.24}$$

$$\sum_{n=1}^{\infty} P_{6,0,-4,1}(7^{k+1}n + 7^k \cdot 3 - 2) q^{(7n+3)/7} = 7^{2k+1} \begin{pmatrix} 0 & 5 & -1 & 1 & -1 \end{pmatrix} U_3, \tag{3.25}$$

$$\sum_{n=1}^{\infty} P_{6,0,-4,1}(7^{k+1}n + 7^k \cdot 5 - 2) q^{(7n+5)/7} = 7^{2k} \begin{pmatrix} 9 & 46 & -4 & 11 & -12 \end{pmatrix} U_5, \tag{3.26}$$

$$\sum_{n=1}^{\infty} P_{6,0,-4,1}(7^{k+1}n+7^k\cdot 6-2)q^{(7n+6)/7} = 7^{2k}\begin{pmatrix}1 & -25 & 15 & -10 & 4\end{pmatrix} U_6, \tag{3.27}$$

$$\sum_{n=1}^{\infty} P_{6,2,-3,-2}(7^{k+1}n+7^k\cdot 3-3)q^{(7n+3)/7} = 7^{2k}\begin{pmatrix}-3 & -30 & 3 & -15 & 16\end{pmatrix} U_3, \tag{3.28}$$

$$\sum_{n=1}^{\infty} P_{6,2,-3,-2}(7^{k+1}n+7^k\cdot 5-3)q^{(7n+5)/7} = 7^{2k}\begin{pmatrix}-8 & -37 & 9 & -2 & 6\end{pmatrix} U_5, \tag{3.29}$$

$$\sum_{n=1}^{\infty} P_{6,2,-3,-2}(7^{k+1}n+7^k\cdot 6-3)q^{(7n+6)/7} = 7^{2k}\begin{pmatrix}-2 & 22 & -9 & -1 & 6\end{pmatrix} U_6, \tag{3.30}$$

$$\sum_{n=1}^{\infty} P_{6,-4,1,0}(7^{k+1}n+7^k\cdot 3-1)q^{(7n+3)/7} = 7^{2k}\begin{pmatrix}1 & -4 & 20 & -30 & 39\end{pmatrix} U_3, \tag{3.31}$$

$$\sum_{n=1}^{\infty} P_{6,-4,1,0}(7^{k+1}n+7^k\cdot 5-1)q^{(7n+5)/7} = 7^{2k}\begin{pmatrix}-1 & 5 & 9 & -2 & 27\end{pmatrix} U_5, \tag{3.32}$$

$$\sum_{n=1}^{\infty} P_{6,-4,1,0}(7^{k+1}n+7^k\cdot 6-1)q^{(7n+6)/7} = 7^{2k+1}\begin{pmatrix}0 & 0 & -1 & 3 & 3\end{pmatrix} U_6, \tag{3.33}$$

$$\sum_{n=1}^{\infty} P_{6,-2,-5,4}(7^{k+1}n+7^k\cdot 3-1)q^{(7n+3)/7} = 7^{2k}\begin{pmatrix}4 & -44 & 31 & -29 & 37\end{pmatrix} U_3, \tag{3.34}$$

$$\sum_{n=1}^{\infty} P_{6,-2,-5,4}(7^{k+1}n+7^k\cdot 5-1)q^{(7n+5)/7} = 7^{2k}\begin{pmatrix}-11 & -50 & 8 & -22 & 45\end{pmatrix} U_5, \tag{3.35}$$

$$\sum_{n=1}^{\infty} P_{6,-2,-5,4}(7^{k+1}n+7^k\cdot 6-1)q^{(7n+6)/7} = 7^{2k+1}\begin{pmatrix}0 & 4 & -4 & 6 & 1\end{pmatrix} U_6, \tag{3.36}$$

$$\sum_{n=1}^{\infty} P_{6,-3,-2,2}(7^{k+1}n+7^k\cdot 3-1)q^{(7n+3)/7} = 7^{2k}\begin{pmatrix}4 & -9 & 24 & -22 & 30\end{pmatrix} U_3, \tag{3.37}$$

$$\sum_{n=1}^{\infty} P_{6,-3,-2,2}(7^{k+1}n+7^k\cdot 5-1)q^{(7n+5)/7} = 7^{2k}\begin{pmatrix}-2 & -4 & 4 & -11 & 33\end{pmatrix} U_5, \tag{3.38}$$

$$\sum_{n=1}^{\infty} P_{6,-3,-2,2}(7^{k+1}n+7^k\cdot 6-1)q^{(7n+6)/7} = 7^{2k}\begin{pmatrix}1 & 3 & -13 & 32 & 11\end{pmatrix} U_6, \tag{3.39}$$

$$\sum_{n=1}^{\infty} P_{6,-5,4,-2}(7^{k+1}n+7^k\cdot 3-1)q^{(7n+3)/7} = 7^{2k}\begin{pmatrix}23 & -8 & 61 & 3 & 8\end{pmatrix} U_3, \tag{3.40}$$

$$\sum_{n=1}^{\infty} P_{6,-5,4,-2}(7^{k+1}n+7^k\cdot 5-1)q^{(7n+5)/7} = 7^{2k+1}\begin{pmatrix}0 & -1 & -3 & -8 & 12\end{pmatrix} U_5, \tag{3.41}$$

$$\sum_{n=1}^{\infty} P_{6,-5,4,-2}(7^{k+1}n+7^k\cdot 6-1)q^{(7n+6)/7} = 7^{2k}\begin{pmatrix}8 & -4 & -41 & 109 & -24\end{pmatrix} U_6, \tag{3.42}$$

$$\sum_{n=1}^{\infty} P_{6,4,-2,-5}(7^{k+1}n+7^k\cdot 3-4)q^{(7n+3)/7} = 7^{2k+1}\begin{pmatrix}4 & 3 & 6 & 8 & -8\end{pmatrix} U_3, \tag{3.43}$$

$$\sum_{n=1}^{\infty} P_{6,4,-2,-5}(7^{k+1}n + 7^k \cdot 5 - 4)q^{(7n+5)/7} = 7^{2k} \begin{pmatrix} 8 & 16 & -44 & -61 & 57 \end{pmatrix} U_5, \tag{3.44}$$

$$\sum_{n=1}^{\infty} P_{6,4,-2,-5}(7^{k+1}n + 7^k \cdot 6 - 4)q^{(7n+6)/7} = 7^{2k} \begin{pmatrix} 11 & -23 & -31 & 100 & -61 \end{pmatrix} U_6. \tag{3.45}$$

Proof. Define

$$W_3 = \begin{pmatrix} 22 & -4 & 41 & 33 & -31 \\ 0 & 35 & -7 & 7 & -7 \\ 1 & -4 & 20 & -30 & 39 \\ 4 & -9 & 24 & -22 & 30 \end{pmatrix}, \quad W_5 = \begin{pmatrix} 1 & -12 & -30 & -54 & 57 \\ 9 & 46 & -4 & 11 & -12 \\ -1 & 5 & 9 & -2 & 27 \\ -2 & -4 & 4 & -11 & 33 \end{pmatrix},$$

$$W_6 = \begin{pmatrix} 8 & -4 & -34 & 88 & -45 \\ 1 & -25 & 15 & -10 & 4 \\ 0 & 0 & -7 & 21 & 21 \\ 1 & 3 & -13 & 32 & 11 \end{pmatrix}.$$

Lemma 2.3 and Klein's relation may be used to show that for $r = 3, 5, 6$,

$$U_{7,r}(V) = W_r U_r. \tag{3.46}$$

As shown in the proof of Proposition 3.2, one has that

$$\sum_{n=1}^{\infty} P_{6,2,-3,-2}(7n-3)q^n = 7 \begin{pmatrix} -1 & -6 & 0 & 1 \end{pmatrix} V. \tag{3.47}$$

We now use the Jordan normal form (3.2) for the square matrix A in Equation (3.1) to apply the U_7-operator $k-1$ times to (3.47) followed by $U_{7,r}$ through Equation (3.46). This yields

$$\sum_{n=0}^{\infty} P_{6,2,-3,-2}(7^{k+1}n + r \cdot 7^k - 3)q^{(7n+r)/7} = 7^k \begin{pmatrix} -1 & -6 & 0 & 1 \end{pmatrix} A^{k-1} W_r U_r = 7^{2k} Q_r U_r^T,$$

where

$$Q_3 = \begin{pmatrix} -3 & -30 & 3 & -15 & 16 \end{pmatrix}, \quad Q_5 = \begin{pmatrix} -8 & -37 & 9 & -2 & 6 \end{pmatrix},$$

$$Q_6 = \begin{pmatrix} -2 & 22 & -9 & -1 & 6 \end{pmatrix}.$$

$\square$

Proof of Theorem 1.3. These follow immediately from Proposition 3.3 and the fact that the Fourier coefficients of the components of U_r are all integral. $\square$

Corollary 3.4. *Let $P_{a_0,a_1,a_2,a_3}(n)$ be defined as in Definition 1.1. Then the following identities hold.*

$$P_{6,1,0,-4}(7^2 n + 18) = 7^2 P_{6,1,0,-4}(7n), \tag{3.48}$$

$$P_{6,1,0,-4}(7^2 n + 32) = 7^2 P_{6,1,0,-4}(7n+2), \tag{3.49}$$

$$P_{6,1,0,-4}(7^2 n + 39) = 7^2 P_{6,1,0,-4}(7n+3), \tag{3.50}$$

$$P_{6,-2,2,-3}(7^2 n + 19) = 7^2 P_{6,-2,2,-3}(7n+1), \tag{3.51}$$

$$P_{6,-2,2,-3}(7^2 n + 33) = 7^2 P_{6,-2,2,-3}(7n+3), \tag{3.52}$$

$$P_{6,-2,2,-3}(7^2 n + 40) = 7^2 P_{6,-2,2,-3}(7n+4), \tag{3.53}$$

$$P_{6,-1,-1,-1}(7^2 n + 19) = 7^2 P_{6,-1,-1,-1}(7n+1), \tag{3.54}$$

$$P_{6,-1,-1,-1}(7^2n+33) = 7^2 P_{6,-1,-1,-1}(7n+3), \tag{3.55}$$
$$P_{6,-1,-1,-1}(7^2n+40) = 7^2 P_{6,-1,-1,-1}(7n+4), \tag{3.56}$$
$$P_{6,0,-4,1}(7^2n+19) = 7^2 P_{6,0,-4,1}(7n+1), \tag{3.57}$$
$$P_{6,0,-4,1}(7^2n+33) = 7^2 P_{6,0,-4,1}(7n+3), \tag{3.58}$$
$$P_{6,0,-4,1}(7^2n+40) = 7^2 P_{6,0,-4,1}(7n+4), \tag{3.59}$$
$$P_{6,2,-3,-2}(7^2n+18) = 7^2 P_{6,2,-3,-2}(7n), \tag{3.60}$$
$$P_{6,2,-3,-2}(7^2n+32) = 7^2 P_{6,2,-3,-2}(7n+2), \tag{3.61}$$
$$P_{6,2,-3,-2}(7^2n+39) = 7^2 P_{6,2,-3,-2}(7n+3), \tag{3.62}$$
$$P_{6,-4,1,0}(7^2n+20) = 7^2 P_{6,-4,1,0}(7n+2), \tag{3.63}$$
$$P_{6,-4,1,0}(7^2n+34) = 7^2 P_{6,-4,1,0}(7n+4), \tag{3.64}$$
$$P_{6,-4,1,0}(7^2n+41) = 7^2 P_{6,-4,1,0}(7n+5), \tag{3.65}$$
$$P_{6,-2,-5,4}(7^2n+20) = 7^2 P_{6,-2,-5,4}(7n+2), \tag{3.66}$$
$$P_{6,-2,-5,4}(7^2n+34) = 7^2 P_{6,-2,-5,4}(7n+4), \tag{3.67}$$
$$P_{6,-2,-5,4}(7^2n+41) = 7^2 P_{6,-2,-5,4}(7n+5), \tag{3.68}$$
$$P_{6,-3,-2,2}(7^2n+20) = 7^2 P_{6,-3,-2,2}(7n+2), \tag{3.69}$$
$$P_{6,-3,-2,2}(7^2n+34) = 7^2 P_{6,-3,-2,2}(7n+4), \tag{3.70}$$
$$P_{6,-3,-2,2}(7^2n+41) = 7^2 P_{6,-3,-2,2}(7n+5), \tag{3.71}$$
$$P_{6,-5,4,-2}(7^2n+20) = 7^2 P_{6,-5,4,-2}(7n+2), \tag{3.72}$$
$$P_{6,-5,4,-2}(7^2n+34) = 7^2 P_{6,-5,4,-2}(7n+4), \tag{3.73}$$
$$P_{6,-5,4,-2}(7^2n+41) = 7^2 P_{6,-5,4,-2}(7n+5), \tag{3.74}$$
$$P_{6,4,-2,-5}(7^2n+17) = 7^2 P_{6,4,-2,-5}(7n-1), \tag{3.75}$$
$$P_{6,4,-2,-5}(7^2n+31) = 7^2 P_{6,4,-2,-5}(7n+1), \tag{3.76}$$
$$P_{6,4,-2,-5}(7^2n+38) = 7^2 P_{6,4,-2,-5}(7n+2). \tag{3.77}$$

In particular, (3.54)–(3.56) *are exactly* [8, Corollary 1 (3)–(5)].

4. Congruences for the Number of Representations by $x^2 + xy + 2y^2$

In the last section, Lemma 2.3 was applied to derive iterated dissections for modular forms of fixed weight. However, interesting combinatorial information also results from a similar analysis of a single class of objects as the weight increases. In this section, we consider the number $Q_k(n)$ of representations by $x^2 + xy + 2y^2$ in exactly k ways. A family of new congruences will be developed for this well studied number theoretic function using the dissection method. The series

$$\sum_{n=0}^{\infty} Q_k(n) q^n = \left(\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2} \right)^k$$

is a modular form of weight k for $\Gamma_1(7)$, and the well known identity [18, Theorem 204]

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2} = 1 + 2 \sum_{j=1}^{\infty} \left(\frac{j}{7} \right) \frac{q^j}{1-q^j}, \tag{4.1}$$

indicates a simple divisor sum representation for the coefficients as well as a congruence modulo 2:

$$Q_1(n) = 2\sum_{d|n}\left(\frac{d}{7}\right).$$

Therefore, $Q_1(7n+r) = 0$ for r quadratic nonresidue modulo 7 since for $\left(\frac{n}{7}\right) = -1$,

$$\sum_{d|n}\left(\frac{d}{7}\right) = \sum_{d|n}\left(\frac{n/d}{7}\right) = -\sum_{d|n}\left(\frac{d}{7}\right).$$

This can also be seen by the relation

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2} = x - y + z, \tag{4.2}$$

where as before, $\langle x, y, z\rangle = \left\langle \frac{T_1^2}{T_3}, \frac{T_3^2}{T_5}, \frac{T_5^2}{T_1}\right\rangle$ form a basis for $\mathcal{M}_1(\Gamma_1(7))$ by Lemma 2.2. Application of Lemma 2.3 shows that the 7-dissection of $x - y + z$ does not involve any terms of the form

$$T_1^a T_3^b T_5^c, \qquad 6a + 5b + 3c \equiv 3, 5, 6 \pmod 7.$$

For the case $k = 2$, one has the following nice identity due to Ramanujan [1, p. 405]

$$\left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2}\right)^2 = 1 + 4\sum_{n=1}^{\infty}(\sigma(n) - 7\sigma(n/7))q^n. \tag{4.3}$$

Deriving additional congruences by employing Eisenstein series and divisor sum representations for larger values of k is problematic because of the appearance of rational coefficients. For example, if $k = 3$, one has that

$$\begin{aligned}
&\sum_{n=0}^{\infty} Q_3(n)q^n \\
&= 1 - \frac{7}{8}\sum_{j=1}^{\infty}\left(\frac{j}{7}\right)\frac{j^2q^j}{1-q^j} + \frac{49}{8}\sum_{j=1}^{\infty}\frac{j^2(q^j + q^{2j} - q^{3j} + q^{4j} - q^{5j} - q^{6j})}{1-q^{7j}} + \frac{3}{4}\eta(\tau)^3\eta(7\tau)^3.
\end{aligned}$$

In contrast, the dissection method introduced here applies to any k and uncovers a variety of new congruences. In the case $k = 2$, it is apparent from (4.3) that $Q_2(n) \equiv 0 \pmod 4$. What is less apparent is that $Q_2(7n+r) \equiv 0 \pmod 8$ for r a quadratic nonresidue modulo 7. This follows from the identity

$$\left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2}\right)^2 = x^2 + y^2 + z^2$$

resulting from Lemma 2.2, and the 7-dissections of each of x^2, y^2 and z^2 through Lemma 2.3. Explicitly, one has the following.

Theorem 4.1. *Let $Q_k(n)$ be the number of represenations of n by the binary quadratic form $m^2 + mn + 2n^2$ in exactly k ways. Then*

$$\sum_{n=0}^{\infty} Q_2(7n+1)q^{(7n+1)/7} = 4\left(3\frac{T_3^4}{T_1^2} + T_5T_3 + \frac{T_5^4}{T_1T_3} + 7\frac{T_1^2T_5^2}{T_3^2}\right), \tag{4.4}$$

$$\sum_{n=0}^{\infty} Q_2(7n+2)q^{(7n+2)/7} = -4\left(\frac{T_1^4}{T_3T_5} + 8\frac{T_3T_5^3}{T_1^2} - 11\frac{T_5^4}{T_3^2} + 7\frac{T_3^3}{T_1}\right), \tag{4.5}$$

$$\sum_{n=0}^{\infty} Q_2(7n+3)q^{(7n+3)/7} = 8\left(2\frac{T_5^3}{T_1} - 3\frac{T_1^2T_5}{T_3} + 6T_3^2\right), \tag{4.6}$$

$$\sum_{n=0}^{\infty} Q_2(7n+4)q^{(7n+4)/7} = 4\left(-\frac{T_3^4}{T_1T_5} + 11\frac{T_3^2T_5^2}{T_1^2} - 4\frac{T_5^3}{T_3} + 3\frac{T_1^3T_5}{T_3^2}\right), \tag{4.7}$$

$$\sum_{n=0}^{\infty} Q_2(7n+5)q^{(7n+5)/7} = 8\left(2\frac{T_3^3}{T_5} - 3\frac{T_5^2T_3}{T_1} + 6\frac{T_1T_5^3}{T_3^2}\right), \tag{4.8}$$

$$\sum_{n=0}^{\infty} Q_2(7n+6)q^{(7n+6)/7} = 8\left(-2\frac{T_1^3}{T_3} + 3T_5^2 + 3\frac{T_3^3T_5}{T_1^2}\right). \tag{4.9}$$

Similarly, by applying Lemma 2.3 to $(x-y+z)^k$ for any given positive integer k, one may find congruences for $Q_k(7n+r)$. For example, for $3 \le k \le 29$, we find the congruences in Example 4.2.

Example 4.2. *If ℓ is a quadratic residue modulo 7, then*

$$\begin{aligned}
&Q_3(7n+\ell) \equiv 0 \pmod{6}, \quad Q_4(7n+\ell) \equiv 0 \pmod{8}, \quad Q_5(7n+\ell) \equiv 0 \pmod{10},\\
&Q_6(7n+\ell) \equiv 0 \pmod{4}, \quad Q_7(7n+\ell) \equiv 0 \pmod{14}, \quad Q_8(7n+\ell) \equiv 0 \pmod{16},\\
&Q_9(7n+\ell) \equiv 0 \pmod{2}, \quad Q_{10}(7n+\ell) \equiv 0 \pmod{4}, \quad Q_{11}(7n+\ell) \equiv 0 \pmod{2},\\
&Q_{12}(7n+\ell) \equiv 0 \pmod{8}, \quad Q_{13}(7n+\ell) \equiv 0 \pmod{26}, \quad Q_{14}(7n+\ell) \equiv 0 \pmod{28},\\
&Q_{15}(7n+\ell) \equiv 0 \pmod{6}, \quad Q_{16}(7n+\ell) \equiv 0 \pmod{32}, \quad Q_{17}(7n+\ell) \equiv 0 \pmod{34},\\
&Q_{18}(7n+\ell) \equiv 0 \pmod{4}, \quad Q_{19}(7n+\ell) \equiv 0 \pmod{38}, \quad Q_{20}(7n+\ell) \equiv 0 \pmod{8},\\
&Q_{21}(7n+\ell) \equiv 0 \pmod{14}, \quad Q_{22}(7n+\ell) \equiv 0 \pmod{4}, \quad Q_{23}(7n+\ell) \equiv 0 \pmod{2},\\
&Q_{24}(7n+\ell) \equiv 0 \pmod{16}, \quad Q_{25}(7n+\ell) \equiv 0 \pmod{2}, \quad Q_{26}(7n+\ell) \equiv 0 \pmod{4},\\
&Q_{27}(7n+\ell) \equiv 0 \pmod{6}, \quad Q_{28}(7n+\ell) \equiv 0 \pmod{56}, \quad Q_{29}(7n+\ell) \equiv 0 \pmod{2}.
\end{aligned}$$

If r is a quadratic nonresidue modulo 7, then

$$\begin{aligned}
&Q_3(7n+r) \equiv 0 \pmod{56}, \quad Q_4(7n+r) \equiv 0 \pmod{16}, \quad Q_5(7n+\ell) \equiv 0 \pmod{48},\\
&Q_6(7n+r) \equiv 0 \pmod{8}, \quad Q_7(7n+r) \equiv 0 \pmod{56}, \quad Q_8(7n+r) \equiv 0 \pmod{224},\\
&Q_9(7n+r) \equiv 0 \pmod{96}, \quad Q_{10}(7n+r) \equiv 0 \pmod{56}, \quad Q_{11}(7n+r) \equiv 0 \pmod{88},\\
&Q_{12}(7n+r) \equiv 0 \pmod{16}, \quad Q_{13}(7n+r) \equiv 0 \pmod{16}, \quad Q_{14}(7n+r) \equiv 0 \pmod{56},\\
&Q_{15}(7n+r) \equiv 0 \pmod{56}, \quad Q_{16}(7n+r) \equiv 0 \pmod{64}, \quad Q_{17}(7n+r) \equiv 0 \pmod{448},\\
&Q_{18}(7n+r) \equiv 0 \pmod{8}, \quad Q_{19}(7n+r) \equiv 0 \pmod{8}, \quad Q_{20}(7n+r) \equiv 0 \pmod{16},\\
&Q_{21}(7n+r) \equiv 0 \pmod{112}, \quad Q_{22}(7n+r) \equiv 0 \pmod{56}, \quad Q_{23}(7n+r) \equiv 0 \pmod{184},\\
&Q_{24}(7n+r) \equiv 0 \pmod{224}, \quad Q_{25}(7n+r) \equiv 0 \pmod{160}, \quad Q_{26}(7n+r) \equiv 0 \pmod{8},\\
&Q_{27}(7n+r) \equiv 0 \pmod{8}, \quad Q_{28}(7n+r) \equiv 0 \pmod{112}, \quad Q_{29}(7n+r) \equiv 0 \pmod{3248}.
\end{aligned}$$

The modulus in each congruence of Example 4.2 is given by

$$d_{k,r} = \gcd_{\substack{n_0+n_1+n_2+n_4=k \\ n_1+2n_2+4n_4 \equiv r \pmod{7}}} \left[2^{n_1+n_2+n_4}\binom{k}{n_0,n_1,n_2,n_4}\right]. \tag{4.10}$$

This follows by appying the dissection identities in Lemma 2.3. The formula is derived in the proof of the next Theorem and implies that a number of simple congruences for $Q_k(7n+r)$ hold for k an odd prime.

Theorem 4.3. *Let $Q_k(n)$ denote the number of representations of a positive integer n by $x^2+xy+2y^2$ in exactly k ways. Then for k an odd prime, one has that*

(1) $Q_k(7n+r) \equiv 0 \pmod{2k}$ *if* $\left(\frac{r}{7}\right)=1$ *and* $\left(\frac{k}{7}\right)=-1$,

(2) $Q_k(7n+r) \equiv 0 \pmod{2}$ *if* $\left(\frac{r}{7}\right)=1$ *and* $\left(\frac{k}{7}\right)=1$,

(3) $Q_k(7n+r) \equiv 0 \pmod{8k}$ *if* $\left(\frac{r}{7}\right)=-1$ *and* $\left(\frac{k}{7}\right)=1$,

(4) $Q_k(7n+r) \equiv 0 \pmod{8}$ *if* $\left(\frac{r}{7}\right)=-1$ *and* $\left(\frac{k}{7}\right)=-1$.

Proof. We only give the proof of (3) since the proofs of other cases are similar. By (4.2) and Lemma 2.3, one has that

$$\sum_{n=0}^{\infty} Q_1(7n)q^n = \frac{T_1^2}{T_3} - \frac{T_3^2}{T_5} + \frac{T_5^2}{T_1} := x_0, \tag{4.11}$$

$$\sum_{n=0}^{\infty} Q_1(7n+1)q^{(7n+1)/7} = \frac{2T_5^2}{T_3} + \frac{4T_1T_3}{T_5} := x_1, \tag{4.12}$$

$$\sum_{n=0}^{\infty} Q_1(7n+2)q^{(7n+2)/7} = \frac{4T_3T_5}{T_1} - \frac{2T_1^2}{T_5} := x_2, \tag{4.13}$$

$$\sum_{n=0}^{\infty} Q_1(7n+4)q^{(7n+4)/7} = \frac{2T_3^2}{T_1} + \frac{4T_1T_5}{T_3} := x_4. \tag{4.14}$$

These imply

$$\sum_{n=0}^{\infty} Q_k(n)q^n = \left(\sum_{n=0}^{\infty} Q_1(n)q^n\right)^k = \sum_{n_0+n_1+n_2+n_4=k} \binom{k}{n_0, n_1, n_2, n_4} x_0^{n_0}x_1^{n_1}x_2^{n_2}x_4^{n_4}. \tag{4.15}$$

Thus, $d_{k,r} \mid Q_k(7n+r)$, where $d_{k,r}$ is defined by (4.10). Assuming $\left(\frac{r}{7}\right)=-1$, $\left(\frac{k}{7}\right)=1$, n_0, n_1, n_2 and n_4 are nonnegative integers satisfying $n_0+n_1+n_2+n_4=k$, and $n_1+2n_2+4n_4 \equiv r \pmod 7$. Since $\left(\frac{r}{7}\right)=-1$, then $(n_1,n_2,n_4) \neq (0,0,0)$, $(1,0,0)$, $(0,1,0)$ or $(0,0,1)$, and thus $n_1+n_2+n_4 \geq 2$. Therefore, one has that

$$\begin{aligned} d_{k,r} = 4 &\gcd_{\substack{n_0+n_1+n_2+n_4=k\\ n_1+2n_2+4n_4\equiv r \pmod 7\\ n_1+n_2+n_4=2}} \left[\binom{k}{n_0,n_1,n_2,n_4}\right] \\ &+ 8 \gcd_{\substack{n_0+n_1+n_2+n_4=k\\ n_1+2n_2+4n_4\equiv r \pmod 7\\ n_1+n_2+n_4\geq 3}} \left[2^{n_1+n_2+n_4-3}\binom{k}{n_0,n_1,n_2,n_4}\right]. \end{aligned}$$

Note that $\binom{k}{n_0,n_1,n_2,n_4}=1$ if and only if exactly one of n_0, n_1, n_2 and n_4 is k and the others are 0, which is impossible since $n_1+2n_2+4n_4 \equiv r \pmod 7$, $\left(\frac{r}{7}\right)=-1$ and $\left(\frac{k}{7}\right)=1$. Therefore, $\binom{k}{n_0,n_1,n_2,n_4} \neq 1$, and thus $k|\binom{k}{n_0,n_1,n_2,n_4}$ as k is an odd prime. Moreover, for $n_1+n_2+n_4=2$, one must have that $(n_1,n_2,n_4)=(1,1,0)$, $(1,0,1)$ or $(0,1,1)$, and each of these cases yields $n_0=k-2$. These imply that

$$\binom{k}{n_0,n_1,n_2,n_4} = k(k-1)$$

which is divisible by $2k$ since k is an odd prime. Hence, $8k|d_{k,r}$. □

References

[1] G. E. Andrews and B. C. Berndt, *Ramanujan's lost notebook Part I*, Springer, New York, 2005.

[2] A. Berkovich and H. Yesilyurt, *New identities for 7-cores with prescribed BG-rank*, Discrete Math. 308 (2008), no. 22, 5246–5259.

[3] M. Boylan, *Congruences for 2^t-core partition functions* J. Number Theory 92 (2002), no. 1, 131–138.

[4] S. Chen, *Arithmetical properties of the number of t-core partitions*, Ramanujan J. 18 (2009), no. 1, 103–112.

[5] S. Chen, *Congruences for t-core partition functions*, J. Number Theory 133 (2013), no. 12, 4036–4046.

[6] Du, Julia Q. D.; Liu, Edward Y. S.; Zhao, Jack C. D. Congruence properties of $p_k(n)$. Int. J. Number Theory 15 (2019), no. 6, 1267–1290.

[7] I. S. Eum, J. K. Koo, D. H. Shin *A modularity criterion for Klein forms, with an application to modular forms of level 13*, J. Math. Anal. Appl., **375** (2011), pp. 28-41.

[8] F. Garvan, D. Kim, D. Stanton *Cranks and t-cores*, Invent. Math., **101** (1990), No.1, pp. 1-17.

[9] T. Huber, M. Huerta and N. Mayes, *Arithmetic properties of septic partition functions*, International Journal of Number Theory, To Appear.

[10] T. Huber *A theory of theta functions to the quintic base*, J. Number Theory, **134** (2014), 49–92.

[11] T. Huber, D. Lara and E. Melendez *Balanced Modular Parameterizations*, In Frontiers in orthogonal polynomials and q-series, volume 1 of Contemp. Math. Appl. Monogr. Expo. Lect. Notes, pages 319–342. World Sci. Publ., Hackensack, NJ, 2018.

[12] T. Huber and D. Ye, *Ramanujan type congruences for modular forms of level p*, preprint.

[13] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Addison-Wesley, Reading, 1981.

[14] K. Khuri-Makdisi, *Moduli interpretation of Eisenstein series*, Int. J. Number Theory, **8** (2012), 715–748.

[15] F. Klein, *Ueber die Transformation siebenter Ordnung der elliptischen Functionen*, Math. Ann., **14** (1878), 428–471.

[16] F. Klein, *On the order-seven transformation of elliptic functions*, In The eightfold way, volume 35 of Math. Sci. Res. Inst. Publ., pages 287–331. Cambridge Univ. Press, Cambridge, 1999. Translated from the German and with an introduction by Silvio Levy.

[17] D Kubert and S. Lang, *Modular Units*, Springer-Verlag, 1981.

[18] E. Landau, Elementary number theory, Chelsea, New York, N.Y., 1958. Translated by J. E. Goodman.

[19] S. Ramanujan, *Congruence properties of partitions*, Proc. London Math. Soc. 19 (1919), 207–210.

School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, Edinburg, Texas 78539, USA

Email address: timothy.huber@utrgv.edu

School of Mathematics (Zhuhai), Sun Yat-sen University, Zhuhai 519082, Guangdong, People's Republic of China

Email address: lawrencefrommath@gmail.com