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ARITHMETIC PROPERTIES OF SEPTIC PARTITION FUNCTIONS

TIMOTHY HUBER, MAYRA HUERTA AND NATHANIEL MAYES

ABSTRACT. Congruences and related identities are derived for a set of colored and weighted partition functions whose generating functions generate the graded algebra of integer weight modular forms of level seven. The work determines a general strategy for identifying and proving identities and associated congruences for modular forms on the principal congruence subgroup of level 7. Ramanujan's partition congruence modulo 7 serves as a prototype for the process used to prove new congruences for modular forms of level 7.

1. INTRODUCTION

The integer partition function $p(n)$, defined for $|q| < 1$, by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}}, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_{\infty} = \lim_{n \rightarrow \infty} (a; q)_n, \quad (1.1)$$

satisfies congruences discovered by S. Ramanujan [14]

$$p(5n + 4) \equiv 0 \pmod{5}, \quad p(7n + 5) \equiv 0 \pmod{7}, \quad p(11n + 6) \equiv 0 \pmod{11}.$$

In this paper, arithmetic properties are developed for a set of infinite products spanning the vector space of modular forms of weight one on subgroups of $SL_2(\mathbb{Z})$ of level seven. Congruences are derived for the functions that count colored partitions according to weights determined by the parity of the number of parts in congruence classes modulo 7. The arithmetic functions are termed “septic partition functions” because of their association with modular forms of level 7. The functions are described in terms of 3 marked parameters.

Definition 1.1. Let $P_{a_1^*, a_2^*, a_3^*}(n)$ enumerate the number of colored partitions λ of n with the following conditions.

- Parts congruent to 0 modulo 7 appear with 2 colors, are distinct, and weighted by $(-1)^{\ell_0(\lambda)}$, where $\ell_0(\lambda)$ counts parts congruent to 0 (mod 7).
- Unmarked subscripts a_i indicate that parts congruent to $i, 7 - i \pmod{7}$ appear with a_i colors.
- Marked a_i^* indicate parts congruent to $i, 7 - i \pmod{7}$ appear with a_i -colors, are distinct, and contribute $(-1)^{\ell_i(\lambda)}$, where $\ell_i(\lambda)$ is the number of parts congruent to $i, 7 - i \pmod{7}$.

The definition is a natural arithmetic interpretation for Fourier coefficients of the corresponding infinite products. For example,

$$\sum_{n=0}^{\infty} P_{3^*, 1, 3}(n)q^n = \frac{(q^7; q^7)_{\infty}^2 (q; q^7)_{\infty}^3 (q^6; q^7)_{\infty}^3}{(q^2; q^7)_{\infty} (q^5; q^7)_{\infty} (q^3; q^7)_{\infty}^3 (q^4; q^7)_{\infty}^3}.$$

Radu [13] proved $p(An+B) \not\equiv 0 \pmod{2,3}$ for all positive integers A, B . In contrast, the septic partition functions satisfy many congruences modulo 2 and 3.

Theorem 1.2. *The following congruences are satisfied*

$$P_{0,1^*,2}(7n+1) \equiv P_{2,0,1^*}(7n+1) \equiv P_{1^*,2,0}(7n+2) \equiv 0 \pmod{2}, \quad (1.2)$$

$$P_{1^*,2,0}(7n+3) \equiv P_{0,1^*,2}(7n+4) \equiv P_{2,0,1^*}(7n+6) \equiv 0 \pmod{2}, \quad (1.3)$$

$$P_{3,3^*,1}(7n+3) \equiv P_{3^*,1,3}(7n+4) \equiv P_{1,3,3^*}(7n+5) \equiv 0 \pmod{2}, \quad (1.4)$$

$$P_{0,1^*,2}(7n+5) \equiv P_{1^*,2,0}(7n+4) \equiv P_{2,0,1^*}(7n+3) \equiv 0 \pmod{3}, \quad (1.5)$$

$$P_{3^*,1,3}(7n+1) \equiv P_{3,3^*,1}(7n+5) \equiv P_{1,3,3^*}(7n+6) \equiv 0 \pmod{3}, \quad (1.6)$$

$$P_{1,3,3^*}(7n+2) \equiv P_{3,3^*,1}(7n+4) \equiv P_{3^*,1,3}(7n+6) \equiv 0 \pmod{4}. \quad (1.7)$$

Ramanujan's analytic proof of the congruence for the classical partition function modulo 7 involves a dissections of the generating function for 7-core partitions [15]

$$\frac{(q^7; q^7)_\infty^7}{(q; q)_\infty} := \sum_{n=0}^{\infty} t(n)q^n, \quad t(7n+5) \equiv 0 \pmod{7}. \quad (1.8)$$

When an additional factor of q^2 is included, the generating function for $t(n)$ is a modular form of weight 3 and level 7 with Jacobi symbol nebentypus [11]. Theorem 2.1 shows that the septic partition functions appearing in Theorem 1.2 correspond to modular forms of weight one and level 7 that provide a factorization of (1.8)

$$\frac{(q^7; q^7)_\infty^7}{(q; q)_\infty} = \sum_{n=0}^{\infty} P_{a_1^*, a_2, a_3}(n)q^n \times \sum_{n=0}^{\infty} P_{a_3, a_1^*, a_2}(n)q^n \times \sum_{n=0}^{\infty} P_{a_2, a_3, a_1^*}(n)q^n. \quad (1.9)$$

This paper formulates decomposition formulas and congruences for a class of functions, including those listed in (1.8). The results link congruences modulo 2 and 3 for weight-one dissection components to congruences modulo 7 for counting functions associated to modular forms of higher weight such as $t(n)$ and $p(n)$. The dissections in Theorem 2.2 generate a multitude of identities featured in Theorem 3.1 between septic partition functions. Theorem 1.2 catalogues the congruences resulting from these identities. The decomposition formulas yield expansions for dissections of $p(n)$ and $t(n)$ in Corollary 3.2. The dissections also yield congruences for extensions of the septic partition functions. Since the associated generating functions generate the graded algebra of modular forms of level 7, the procedure outlined here determines a general strategy for identifying and proving congruences and related identities for modular forms of level 7. The paper concludes with identities in Corollary 3.6 for products formulated in terms of the septic partition functions. The congruences in Theorem 1.2 result from identities between spanning sets for the vector space of weight one modular forms of level seven. Relations with alternative spanning sets produce representations in terms of divisor sums $\sigma_\chi(n) = \sum_{d|n} \chi(d)$.

Theorem 1.3. *If χ is the Dirichlet character modulo 7 satisfying $\chi(3) = e^{\pi i/3}$,*

$$\begin{aligned} P_{1,0,0}(n) &= -P_{0,1^*,2}(7n+2) = \frac{1}{2}P_{1^*,2,0}(7n+2) = \frac{1}{3}P_{2,0,1^*}(7n+3) \\ &= \frac{3-i\sqrt{3}}{6}\sigma_\chi(7n+3), \\ P_{0,1,0}(n) &= P_{2,0,1^*}(7n+5) = \frac{1}{2}P_{0,1^*,2}(7n+4) = \frac{1}{3}P_{1^*,2,0}(7n+4) \\ &= \frac{3+i\sqrt{3}}{6}\sigma_\chi(7n+5), \\ P_{0,0,1}(n) &= -P_{1^*,2,0}(7n+5) = \frac{1}{2}P_{2,0,1^*}(7n+6) = -\frac{1}{3}P_{0,1^*,2}(7n+5) \\ &= -\frac{\sqrt{3}}{3}\sigma_\chi(7n+6). \end{aligned}$$

Theorem 1.4. *Let $\left(\frac{n}{7}\right)$ denote the Jacobi symbol modulo 7. Then*

$$P_{2,1^*,0}(n) \equiv \sum_{d|7n+1} \left(\frac{d}{7}\right) \pmod{2}, \quad (1.10)$$

$$P_{0,2,1^*}(n) \equiv \sum_{d|7n+4} \left(\frac{d}{7}\right) \pmod{2}, \quad (1.11)$$

$$P_{1^*,0,2}(n) \equiv \sum_{d|7n+9} \left(\frac{d}{7}\right) \pmod{2}. \quad (1.12)$$

Throughout the paper, we assume basic knowledge of the theory of modular forms available from [1, 2, 12, 16] and employ standard notation $\Gamma_0(7)$, $\Gamma_1(7)$, and $\Gamma(7)$ to denote relevant congruence subgroups of $SL_2(\mathbb{Z})$ consisting of the matrices $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, respectively.

2. A CHARACTERIZATION OF MODULAR FORMS OF LEVEL 7

A function $f(\tau)$ of the complex upper half plane $\mathbb{H}$ is a modular form of weight k for a finite index subgroup Γ of $SL_2(\mathbb{Z})$ if $f(\tau)$ is holomorphic in $\mathbb{H}$ and at the cusps of Γ and

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \text{ for all } M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

To verify the holomorphicity condition at the cusps, we need to show that $f(M\tau)/(c\tau+d)^k$ is holomorphic at $i\infty$ for each $M \in SL_2(\mathbb{Z})$. The vector space $\mathcal{M}_k(\Gamma)$ of weight k modular forms for a subgroup Γ of the full modular group is known to be finite dimensional. In what follows, we construct a basis of weight one forms in terms of the Dedekind eta function

$$\eta(\tau) = q^{1/24}(q; q)_\infty, \quad q = e^{2\pi i\tau} \quad (2.1)$$

and the theta constants given in product form by

$$\theta \begin{bmatrix} r/N \\ 1 \end{bmatrix} (N\tau) = \exp\left(\frac{\pi i r}{2N}\right) q^{r^2/(8N)} (q^{(N-r)/2}; q^N)_\infty (q^{(N+r)/2}; q^N)_\infty (q^N; q^N)_\infty. \quad (2.2)$$

Let $N = 7$ be fixed. Then for each odd positive integer $1 \leq r \leq 5$, let $T_r(\tau)$ denote

$$T_r(\tau) = \exp\left(\frac{\pi i r}{14}\right) \eta^3(7\tau) \cdot \theta^{-1} \begin{bmatrix} r/7 \\ 1 \end{bmatrix} (7\tau). \quad (2.3)$$

Theorem 2.1. *Define the following subscripted vectors:*

$$\vec{v}_0 = \left\langle \frac{T_1^2}{T_3}, \frac{T_3^2}{T_5}, \frac{T_5^2}{T_1} \right\rangle_0, \quad \vec{v}_1 = \left\langle \frac{T_5^2}{T_3}, \frac{T_3 T_1}{T_5} \right\rangle_1, \quad \vec{v}_2 = \left\langle \frac{T_1^2}{T_5}, \frac{T_3 T_5}{T_1} \right\rangle_2, \quad (2.4)$$

$$\vec{v}_3 = \langle T_5 \rangle_3, \quad \vec{v}_4 = \left\langle \frac{T_3^2}{T_1}, \frac{T_5 T_1}{T_3} \right\rangle_4, \quad \vec{v}_5 = \langle T_3 \rangle_5, \quad \vec{v}_6 = \langle T_1 \rangle_6. \quad (2.5)$$

- (1) *Components of (2.4)–(2.5) form a basis for $\mathcal{M}_1(\Gamma(7))$.*
- (2) *Components of $\vec{v}_0$ form a basis for $\mathcal{M}_1(\Gamma_1(7))$.*
- (3) *Components of $\vec{v}_j$ have the form*

$$q^{j/7} \sum_{n=0}^{\infty} a_n q^n, \quad a_n \in \mathbb{Z}.$$

- (4) *Components of $\vec{v}_0$ generate the graded ring of modular forms for $\Gamma_1(7)$.*
- (5) *Components of $\vec{v}_j$, $0 \leq j \leq 6$, generate the ring of modular forms for $\Gamma(7)$.*

Proof. Part (3) may be verified from the definition of T_i in (2.3). The relevant transformation formulas for parts (1) and (2) may be deduced from the transformation formulas for $\eta(\tau)$ and those for the theta functions given in [3, Lemma A.1], [9] under a suitable set of generators for $\Gamma(7)$ and $\Gamma_1(7)$. In particular, the transformation formula for $\eta(\tau)$ on $SL_2(\mathbb{Z})$ can be applied to derive transformations for $\eta(7\tau)$ under action by $\Gamma_0(7)$ since

$$7 \frac{a\tau + b}{c\tau + d} = \frac{a(7\tau) + 7b}{(c/7)(7\tau) + d} \quad \text{and} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} a & 7b \\ c/7 & d \end{pmatrix}.$$

From the product expansions of T_i given by (2.1)–(2.2), any quotient of powers of the T_i is holomorphic on $\mathbb{H}$. However, we must check the holomorphicity of each vector component at the cusps. The action of $SL_2(\mathbb{Z})$ may be characterized from its Bruhat decomposition as a disjoint union [10, p. 179]

$$SL_2(\mathbb{Z}) = \Gamma_0(7) \cup \Gamma_0(7) S \Gamma_0(7), \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2.6)$$

To trace the action on $T_i(\tau)$, we employ functions considered by Ibukiyama [5]. Define

$$\chi_i(\tau) = \exp\left(-\frac{\pi i r}{14}\right) \theta \begin{bmatrix} r/7 \\ 1 \end{bmatrix} (7\tau) \eta^{-3/7}(\tau). \quad (2.7)$$

The product representations for the T_i in terms of theta quotients may be used to show that, with an appropriately chosen branch of the square root,

$$T_1 = \sqrt{\chi_1 \chi_3^3 \chi_5^3}, \quad T_3 = \sqrt{\chi_1^3 \chi_3 \chi_5^3}, \quad T_5 = \sqrt{\chi_1^3 \chi_3^3 \chi_5}. \quad (2.8)$$

For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, define the action on $m \in \mathbb{Q}^2$ by $M \circ m = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} m + \begin{pmatrix} cd \\ ab \end{pmatrix}$. If $M = \begin{pmatrix} a & b \\ 7c & d \end{pmatrix} \in \Gamma_0(7)$, the transformation formula for the theta constants may be

written

$$\theta \begin{bmatrix} r/7 \\ 1 \end{bmatrix} \left(\frac{a(7\tau) + 7b}{c(7\tau) + d} \right) = \kappa \cdot (c(7\tau) + d)^{1/2} \theta \left[\begin{pmatrix} a & 7b \\ c & d \end{pmatrix}^{-1} \circ \begin{pmatrix} r/7 \\ 1 \end{pmatrix} \right] (7\tau), \quad (2.9)$$

for some κ depending only on M and r . Note that

$$\begin{pmatrix} a & 7b \\ c & d \end{pmatrix}^{-1} \circ \begin{pmatrix} r/7 \\ 1 \end{pmatrix} = \begin{pmatrix} (ar + 7c - 7ca)/7 \\ rb + d - 7bd \end{pmatrix}. \quad (2.10)$$

If ac is even, then $ar + 7c - 7ca$ is odd, since either a or c is odd. If ac is odd, then ar , $7c$, and $7ca$ are odd, so $ar + 7c - 7ca$ is odd. Similarly, we can see that $rb + d - 7bd$ is odd. Therefore, for some odd s , the vector in (2.10) is congruent to

$$\begin{pmatrix} s/7 \\ 1 \end{pmatrix} \pmod{1}. \quad (2.11)$$

This means that the action of $\Gamma_0(7)$ permutes the theta constants up to a constant, possibly different in each case. The action of $\Gamma_0(7)$ fixes $\eta(7\tau)$ up to a constant. Hence, the action of $\Gamma_0(7)$ permutes χ_i , $i = 1, 2, 3$ up to a constant. Therefore by (2.8), action by $\Gamma_0(7)$ permutes $T_i(\tau)$ up to a constant. Ibukiyama [5, Theorem 1.2] shows that the image of $\chi_i(\tau)$ under the action of S is a linear combination of χ_t , t odd $1 \leq t \leq 3$ with non-vanishing coefficients. Since the q -expansion of χ_1 starts from a constant, and χ_3, χ_5 start from some positive power of q , it follows that the image of χ_t under the action of S does not vanish at $i\infty$. Hence, the components of (2.4)–(2.5) are holomorphic at the cusps of $\Gamma_0(7)$. To complete the proof of (1) and (2), note the dimension formulas [2] $\dim M_1(\Gamma_1(7)) = 3$ and $\dim M_1(\Gamma(7)) = 12$ and use the q -expansions to prove linear independence. The claim for $\Gamma_1(7)$ is proven in [4]. The claim for $\Gamma(7)$ in part (4) follows from part (1) and [6]. $\square$

Theorem 2.2. *Let $(b_1, \dots, b_k)_j = \vec{v}_j(\tau) \cdot (b_1, \dots, b_k)$. Then*

$$\frac{T_1^2}{T_3} \left(\frac{\tau}{7} \right) = (1, 0, 0)_0 + (1, 1)_1 + (-2, 0)_2 + (-1)_3 + (-1, 3)_4 + (2)_5 + (-3)_6, \quad (2.12)$$

$$\frac{T_3^2}{T_5} \left(\frac{\tau}{7} \right) = (0, 1, 0)_0 + (1, -3)_1 + (1, -1)_2 + (2)_3 + (-2, 0)_4 + (3)_5 + (-1)_6, \quad (2.13)$$

$$\frac{T_5^2}{T_1} \left(\frac{\tau}{7} \right) = (0, 0, 1)_0 + (2, 0)_1 + (1, 3)_2 + (3)_3 + (1, 1)_4 + (1)_5 + (2)_6, \quad (2.14)$$

$$\frac{T_3 T_1^3}{T_5^3} \left(\frac{\tau}{7} \right) = (1, -1, 0)_0 + (0, 4)_1 + (-3, 1)_2 + (-3)_3 + (1, 3)_4 + (-1)_5 + (-2)_6, \quad (2.15)$$

$$\frac{T_1 T_5^3}{T_3^3} \left(\frac{\tau}{7} \right) = (1, 0, 1)_0 + (3, 1)_1 + (-1, 3)_2 + (2)_3 + (0, 4)_4 + (3)_5 + (-1)_6, \quad (2.16)$$

$$\frac{T_5 T_3^3}{T_1^3} \left(\frac{\tau}{7} \right) = (0, -1, 1)_0 + (1, 3)_1 + (0, 4)_2 + (1)_3 + (3, 1)_4 + (-2)_5 + (3)_6. \quad (2.17)$$

Proof. The left side of (2.12)–(2.14) are the vector components of $\vec{v}_0$ in Theorem 2.1 evaluated at $\tau/7$. It is straightforward to check that if $g(\tau)$ is a modular form for $\Gamma_0(7)$, then $g(\tau/7)$ is a modular form of the same weight for $\Gamma(7)$. Therefore, by Theorem 2.1, the left sides of (2.12)–(2.14) are elements of $\mathcal{M}_1(\Gamma(7))$. Hence, each expression on the left of (2.12)–(2.14) may be expressed as a linear combination of

the basis elements from (2.4)–(2.5). The decomposition formulas listed in (2.15)–(2.17) follow similarly from the fact that the quotients $T_5T_3^3/T_1^3$, $T_3T_1^3/T_5^3$, $T_1T_5^3/T_3^3$ constitute an alternative basis for $M_1(\Gamma_1(7))$. This can be verified using a procedure similar to that in the proof of Theorem 2.1. $\square$

3. ARITHMETIC CONSEQUENCES

The congruences in Theorem 1.2 result from a set of identities relating the septic partition functions. The identities follow by equating terms with indices in specific congruence classes modulo 7 in the $q^{1/7}$ expansions on both sides of (2.12)–(2.17). The congruences of Theorem 1.2 are consequences.

Corollary 3.1. *For each nonnegative integer n ,*

$$P_{0,1^*,2}(7n+8) = -2P_{1^*,0,2}(n) \quad (3.1)$$

$$P_{0,1^*,2}(7n+2) = -P_{1,0,0}(n), \quad (3.2)$$

$$P_{0,1^*,2}(7n+4) = 2P_{0,1,0}(n), \quad (3.3)$$

$$P_{0,1^*,2}(7n+5) = -3P_{0,0,1}(n), \quad (3.4)$$

$$P_{0,1^*,2}(7n+6) = P_{0,1^*,2}(n), \quad (3.5)$$

$$P_{1^*,2,0}(7n+2) = 2P_{1,0,0}(n), \quad (3.6)$$

$$P_{1^*,2,0}(7n+3) = -2P_{0,2,1^*}(n), \quad (3.7)$$

$$P_{1^*,2,0}(7n+4) = 3P_{0,1,0}(n), \quad (3.8)$$

$$P_{1^*,2,0}(7n+5) = -P_{0,0,1}(n), \quad (3.9)$$

$$P_{1^*,2,0}(7n+6) = P_{1^*,2,0}(n), \quad (3.10)$$

$$P_{2,0,1^*}(7n) = P_{2,0,1^*}(n), \quad (3.11)$$

$$P_{2,0,1^*}(7n+1) = 2P_{2,1^*,0}(n), \quad (3.12)$$

$$P_{2,0,1^*}(7n+3) = 3P_{1,0,0}(n), \quad (3.13)$$

$$P_{2,0,1^*}(7n+5) = P_{0,1,0}(n), \quad (3.14)$$

$$P_{2,0,1^*}(7n+6) = 2P_{0,0,1}(n), \quad (3.15)$$

$$P_{3^*,1,3}(7n+13) = 4P_{1^*,1,1}(n), \quad (3.16)$$

$$P_{3^*,1,3}(7n+1) = -3P_{1,0,0}(n), \quad (3.17)$$

$$P_{3^*,1,3}(7n+3) = -P_{0,1,0}(n), \quad (3.18)$$

$$P_{3^*,1,3}(7n+4) = -2P_{0,0,1}(n), \quad (3.19)$$

$$P_{3,3^*,1}(7n+3) = 2P_{1,0,0}(n), \quad (3.20)$$

$$P_{3,3^*,1}(7n+4) = 4P_{1,1^*,1}(n), \quad (3.21)$$

$$P_{3,3^*,1}(7n+5) = 3P_{0,1,0}(n), \quad (3.22)$$

$$P_{3,3^*,1}(7n+6) = -P_{0,0,1}(n), \quad (3.23)$$

$$P_{1,3,3^*}(7n+2) = 4P_{1,1,1^*}(n), \quad (3.24)$$

$$P_{1,3,3^*}(7n+3) = P_{1,0,0}(n), \quad (3.25)$$

$$P_{1,3,3^*}(7n+5) = -2P_{0,1,0}(n), \quad (3.26)$$

$$P_{1,3,3^*}(7n+6) = 3P_{0,0,1}(n). \quad (3.27)$$

Proof. For $a, b, c \in \mathbb{Z}$, Equation (2.3) implies

$$q^{-(6a+5b+3c)/7} T_1^a T_3^b T_5^c = (q, q^6; q^7)_\infty^{-c} (q^2, q^5; q^7)_\infty^{-b} (q^3, q^4; q^7)_\infty^{-a} (q^7; q^7)^{2(a+b+c)}. \quad (3.28)$$

If $a + b + c = 1$, then

$$T_1^a T_3^b T_5^c = q^{(6a+5b+3c)/7} \sum_{n=0}^{\infty} P_{\hat{c}, \hat{b}, \hat{a}}(n) q^n, \quad \hat{k} = \begin{cases} k^* & \text{if } k < 0, \\ k, & \text{otherwise.} \end{cases} \quad (3.29)$$

In particular,

$$\frac{T_5^2}{T_1}(\tau) = \sum_{n=0}^{\infty} P_{2,0,1^*}(n) q^n, \quad \frac{T_5^2}{T_3}(\tau) = q^{1/7} \sum_{n=0}^{\infty} P_{2,1^*,0}(n) q^n.$$

As a power series identity in $q^{1/7}$, Equation (2.14) takes the form

$$\sum_{n=0}^{\infty} P_{2,0,1^*}(n) q^{n/7} = \sum_{n=0}^{\infty} P_{2,0,1^*}(n) q^n + 2q^{1/7} \sum_{n=0}^{\infty} P_{2,1^*,0}(n) q^n + O(q^{2/7}). \quad (3.30)$$

Therefore, upon equating terms of index congruent to 1 modulo 7 in the $q^{1/7}$ expansion of both sides of (2.14), we obtain

$$\sum_{n=0}^{\infty} P_{2,0,1^*}(7n+1) q^{(7n+1)/7} = 2 \sum_{n=0}^{\infty} P_{2,1^*,0}(n) q^{(7n+1)/7}. \quad (3.31)$$

Hence, (3.31) implies

$$P_{2,0,1^*}(7n+1) = 2P_{2,1^*,0}(n).$$

The remaining identities are derived similarly by equating coefficients of terms corresponding to vectors on the right sides of (2.12)–(2.17) containing a single nonzero element. $\square$

Corollary 3.2. *Let $p(n)$ and $t(n)$ be as defined in the introduction. Then*

$$p(7n+5) \equiv t(7n+5) \equiv 0 \pmod{7}. \quad (3.32)$$

Proof. Equation (3.28) implies that

$$q^2 \frac{(q^7; q^7)_\infty^7}{(q; q)_\infty} = (T_1 T_2 T_3)(\tau). \quad (3.33)$$

Replace q by $q^{1/7}$ in (3.33) to derive

$$q^{2/7} \frac{(q; q)_\infty^7}{(q^{1/7}; q^{1/7})_\infty} = q^{2/7} (q; q)_\infty^7 \sum_{n=0}^{\infty} p(n) q^{n/7} = (T_1 T_3 T_5)(\tau/7). \quad (3.34)$$

Expand the right side of (3.34) using (2.12)–(2.14) and simplify using Klein's relation [7, 8]

$$\frac{T_1^2 T_3}{T_5} + \frac{T_3^2 T_5^2}{T_1} - \frac{T_1 T_5^2}{T_3} = 0. \quad (3.35)$$

Isolate terms of index congruent to 0 modulo seven in the $q^{1/7}$ -expansion on both sides of (3.34) to get

$$\begin{aligned} q^{2/7} \sum_{n=0}^{\infty} t(7n+5)q^{(7n+5)/7} &= q^{2/7}(q; q)_{\infty}^7 \sum_{n=0}^{\infty} p(7n+5)q^{(7n+5)/7} \\ &= 7 \left(\frac{2T_3^4}{T_1} + \frac{T_1^2 T_3^3}{T_5^2} - \frac{2T_1^4}{T_5} - 4T_1 T_3 T_5 + \frac{T_1^3 T_5^2}{T_3^2} - \frac{T_3^2 T_5^3}{T_1^2} + \frac{2T_5^4}{T_3} \right). \end{aligned} \quad (3.36)$$

The claimed congruences follow by noting that the expression on the right side of (3.36) is a power series in q . $\square$

Congruences modulo 3 result from relations between linear combinations of the partition functions.

Corollary 3.3. *For each nonnegative integer n ,*

$$P_{0,1^*,2}(7n+2) \equiv -P_{0,2,1^*}(n) \pmod{3}, \quad (3.37)$$

$$P_{0,1^*,2}(7n+2) \equiv P_{2,1^*,0}(n+1) \pmod{3}, \quad (3.38)$$

$$P_{2,0,1^*}(7n+9) \equiv P_{1^*,0,2}(n) \pmod{3}, \quad (3.39)$$

$$P_{3^*,1,3}(7n) \equiv P_{1,1,1^*}(n) \pmod{3}, \quad (3.40)$$

$$P_{3^*,1,3}(7n+2) \equiv P_{0,2,1^*}(n) \pmod{3}, \quad (3.41)$$

$$P_{1,3,3^*}(7n+1) \equiv P_{2,1^*,0}(n) \pmod{3}, \quad (3.42)$$

$$P_{1,3,3^*}(7n+4) \equiv P_{1,1^*,1}(n) \pmod{3}, \quad (3.43)$$

$$P_{3,3^*,1}(7n+8) \equiv P_{1^*,1,1}(n) \pmod{3}, \quad (3.44)$$

$$P_{3,3^*,1}(7n+9) \equiv -P_{1^*,0,2}(n) \pmod{3}. \quad (3.45)$$

Proof. Isolating powers of $q^{1/7}$ on both sides of (2.14) congruent to 2 modulo 7 yields

$$\sum_{n=0}^{\infty} P_{2,0,1^*}(n)q^{(7n+2n)/7} = \sum_{n=1}^{\infty} P_{1^*,0,2}(n-1)q^{(7n+2)/7} + 3 \sum_{n=0}^{\infty} P_{1,1,1^*}(n)q^{(7n+2)/7}.$$

Equating coefficients implies

$$P_{0,1^*,2}(7n+2) = 3P_{1,1^*,1}(n) - P_{0,2,1^*}(n), \quad n \geq 1.$$

Congruence (3.39) follows from this identity. The remaining congruences result from

$$P_{1^*,2,0}(7n+7) = P_{2,1^*,0}(n+1) - 3P_{1^*,1,1}(n), \quad (3.46)$$

$$P_{2,0,1^*}(7n+9) = P_{1^*,0,2}(n) + 3P_{1,1,1^*}(n+1), \quad (3.47)$$

$$P_{3^*,1,3}(7n+7) = P_{1,1,1^*}(n+1) - 3P_{1^*,0,2}(n), \quad (3.48)$$

$$P_{3^*,1,3}(7n+2) = P_{0,2,1^*}(n) + 3P_{1,1^*,1}(n), \quad (3.49)$$

$$P_{1,3,3^*}(7n+18) = P_{2,1^*,0}(n+1) + 3P_{1^*,1,1}(n), \quad (3.50)$$

$$P_{1,3,3^*}(7n+4) = 3P_{0,2,1^*}(n) + P_{1,1^*,1}(n), \quad (3.51)$$

$$P_{3,3^*,1}(7n+8) = 3P_{2,1^*,0}(n+1) + P_{1^*,1,1}(n), \quad (3.52)$$

$$P_{3,3^*,1}(7n+9) = 3P_{1,1,1^*}(n+1) - P_{1^*,0,2}(n). \quad (3.53)$$

Each relation is derived similarly by equating relevant terms in the $q^{1/7}$ expansions on both sides of the dissection formulas from Theorem 2.2. $\square$

Recall that the vector components of $\vec{v}_0$ in Theorem 2.1 form a basis for $\mathcal{M}_1(\Gamma_1(7))$. An alternative basis is given in terms of Eisenstein series defined for the 3 Dirichlet characters χ modulo 7

$$E_{\chi,k}(\tau) = 1 + \frac{2}{L(1-k, \chi)} \sum_{n=1}^{\infty} \chi(n) \frac{n^{k-1} q^n}{1-q^n}, \quad (3.54)$$

where $L(1-k, \chi)$ is the analytic continuation of the associated Dirichlet L -series. The congruences in Theorem 1.3 follow from a dissection of $E_{\chi,(\frac{\cdot}{7})}(\tau)$ in terms of the basis for $\mathcal{M}_1(\Gamma_1(7))$ from Theorem 2.1. These dissections, in turn, yield the identities of Corollary 3.4.

Corollary 3.4.

$$\sum_{d|7n+1} \left(\frac{d}{7} \right) = P_{2,1^*,0}(n) + 2P_{1^*,1,1}(n-1), \quad (3.55)$$

$$\sum_{d|7n+2} \left(\frac{d}{7} \right) = 2P_{1,1,1^*}(n) - P_{1^*,0,2}(n-1), \quad (3.56)$$

$$\sum_{d|7n+4} \left(\frac{d}{7} \right) = P_{0,2,1^*}(n) + 2P_{1,1^*,1}(n), \quad (3.57)$$

$$\sum_{d|7n+j} \left(\frac{d}{7} \right) = 0, \quad j = 3, 5, 6. \quad (3.58)$$

Proof. The identities follow from representing $E_{\chi,(\frac{\cdot}{7})}(\tau/7) \in \mathcal{M}_1(\Gamma(7))$ in terms of the basis given in Theorem 2.1 and equating dissection components. $\square$

Several additional congruences result from relations with Eisenstein series.

Corollary 3.5. *For each nonnegative integer n ,*

$$P_{0,1^*,2}(n-1) - P_{1^*,2,0}(n-1) + P_{2,0,1^*}(n) \equiv 0 \pmod{2}, \quad (3.59)$$

$$P_{3^*,1,3}(n-2) + P_{3,3^*,1}(n) + P_{1,3,3^*}(n) \equiv 0 \pmod{4}. \quad (3.60)$$

Proof. Equation (3.59) follows from the relation

$$1 + 2 \sum_{n=1}^{\infty} \left(\frac{n}{7} \right) \frac{q^n}{1-q^n} = \frac{T_1^2}{T_3} - \frac{T_3^2}{T_5} + \frac{T_5^2}{T_3}. \quad (3.61)$$

This implies

$$2 \sum_{d|n} \left(\frac{d}{7} \right) = P_{0,1^*,2}(n-1) - P_{1^*,2,0}(n-1) + P_{2,0,1^*}(n). \quad (3.62)$$

Equation (3.60) is similarly derived from

$$4 \sum_{d|n} \left(\frac{d}{7} \right) = P_{3^*,1,3}(n-2) + P_{3,3^*,1}(n) + P_{1,3,3^*}(n). \quad (3.63)$$

Equation (3.63) results from the expansion of $E_{\chi,(\frac{\cdot}{7})}(\tau)$ in terms of the alternative basis for $\mathcal{M}_1(\Gamma_1(7))$ given by $T_5 T_3^3 / T_1^3$, $T_3 T_1^3 / T_5^3$, $T_1 T_5^3 / T_3^3$. $\square$

We conclude the paper with congruences resulting from dissections of products and powers of the generating functions from Theorem 2.1. In order to simplify subsequent references, we extend the notation given in Definition 1.1 for the septic theta functions to 4 marked parameters,

$$P_{a_0, a_1, a_2, a_3}(n), \quad (3.64)$$

where the additional initial subscript a_0 refers to parts in the partition λ congruent to 0 modulo 7; unmarked subscripts a_i , for $0 \leq i \leq 3$, indicate that parts congruent to $i, 7-i \pmod{7}$ appear with a_i colors; and marked a_i^* indicate parts congruent to $i, 7-i \pmod{7}$ are a_i -colored, distinct, and contribute $(-1)^{\ell_i(\lambda)}$ to the enumeration, where $\ell_i(\lambda)$ is the number of parts congruent to $i, 7-i \pmod{7}$. In particular, using the notation of (3.29),

$$(q^7; q^7)_\infty^{-a_0} (q, q^6; q^7)_\infty^{-a_1} (q^2, q^5; q^7)_\infty^{-a_2} (q^3, q^4; q^7)_\infty^{-a_3} = \sum_{n=0}^{\infty} P_{\hat{a}_0, \hat{a}_1, \hat{a}_2, \hat{a}_3}(n) q^n. \quad (3.65)$$

Corollary 3.6. *For each nonnegative integer n ,*

$$P_{4^*, 0, 2^*, 4}(7n+1) \equiv 0 \pmod{14}, \quad (3.66)$$

$$P_{4^*, 2^*, 4, 0}(7n+4) \equiv 0 \pmod{14}, \quad (3.67)$$

$$P_{4^*, 4, 0, 2^*}(7n+5) \equiv 0 \pmod{14}, \quad (3.68)$$

$$P_{6^*, 0, 3^*, 6}(7n+5) \equiv 0 \pmod{7}, \quad (3.69)$$

$$P_{6^*, 0, 3^*, 6}(7n+1) \equiv 0 \pmod{21}, \quad (3.70)$$

$$P_{6^*, 3^*, 6, 0}(7n+6) \equiv 0 \pmod{7}, \quad (3.71)$$

$$P_{6^*, 3^*, 6, 0}(7n+5) \equiv 0 \pmod{21}, \quad (3.72)$$

$$P_{6^*, 6, 0, 3^*}(7n+4) \equiv 0 \pmod{7}, \quad (3.73)$$

$$P_{6^*, 6, 0, 3^*}(7n+2) \equiv 0 \pmod{21}, \quad (3.74)$$

$$P_{6^*, 1^*, 0, 4}(7n+2) \equiv 0 \pmod{7}, \quad (3.75)$$

$$P_{6^*, 1^*, 0, 4}(7n+4) \equiv 0 \pmod{7^2}, \quad (3.76)$$

$$P_{6^*, 2^*, 3, 2}(7n+4) \equiv 0 \pmod{7}, \quad (3.77)$$

$$P_{6^*, 2^*, 3, 2}(7n+6) \equiv 0 \pmod{7}, \quad (3.78)$$

$$P_{6^*, 2, 2^*, 3}(7n+5) \equiv 0 \pmod{7}, \quad (3.79)$$

$$P_{6^*, 2, 2^*, 3}(7+6n) \equiv 0 \pmod{7}, \quad (3.80)$$

$$P_{6^*, 4, 1^*, 0}(7n+5) \equiv 0 \pmod{7}, \quad (3.81)$$

$$P_{6^*, 4, 1^*, 0}(7n+6) \equiv 0 \pmod{7^2}, \quad (3.82)$$

$$P_{6^*, 0, 4, 1^*}(7n+1) \equiv 0 \pmod{7}, \quad (3.83)$$

$$P_{6^*, 0, 4, 1^*}(7n+5) \equiv 0 \pmod{7^2}, \quad (3.84)$$

$$P_{6^*, 3, 0, 0}(7n+3) \equiv 0 \pmod{7}, \quad (3.85)$$

$$P_{6^*, 3, 0, 0}(7n+6) \equiv 0 \pmod{7}, \quad (3.86)$$

and

$$P_{4^*, 2, 1^*, 1}(n) = P_{4^*, 1^*, 1, 2}(n) + P_{4^*, 1, 2, 1^*}(n). \quad (3.87)$$

Proof. Apply the dissection formulas in Theorem 2.2 to the square of the first component of $\vec{v}_0$ in Theorem 2.1, namely

$$q^2 \sum_{n=0}^{\infty} P_{4^*,0,2^*,4}(n)q^n = \left(q \sum_{n=1}^{\infty} P_{0,1^*,2}(n)q^n \right)^2. \quad (3.88)$$

In particular, replace q by $q^{1/7}$ in (3.88) and apply (2.12) and Klein's relation (3.35) to derive

$$q^{2/7} \sum_{n=0}^{\infty} P_{4^*,0,2^*,4}(7n+1)q^{(7n+1)/7} = 14 \left(T_3^2 - 2 \frac{T_1^2 T_5}{T_3} \right). \quad (3.89)$$

The congruence claimed by (3.66) follows by equating coefficients in the power series expansion in $q^{1/7}$ on both sides of (3.89). The remaining congruences are derived similarly. Equation (3.87) is the combinatorial interpretation of Klein's relation (3.35). $\square$

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