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Identification of parameters and the distribution of the maximum and the minimum

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IDENTIFICATION OF PARAMETERS AND THE DISTRIBUTIONS OF THE MAXIMUM AND THE MINIMUM

A Thesis

by

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ABSTRACT

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In problems of competing risks, where an individual may be subject to m causes of death and X(i) is the lifetime of an individual exposed to the ith cause, the X(i)s are not observable but only is their minimum, and inference is needed on the X(i) based on their minimum.

The same is the case with a m-component system, where the components are connected in series, and we are interested in inference on the lifetimes of the individual components. These examples motivate problems on parameter identification by the distribution of the minimum.

Similarly, the example of a m-component system where the components are connected in parallel motivates the corresponding maximum problem. In this thesis, these problems will be discussed in the context of Gamma, Weibull, Cauchy, Normal, and Poisson distributions.

DEDICATION

The completion of my master studies would not have been possible without the love and support of parents. My mother, Ruiqin Wang and my father Yuantong Bi, wholeheartedly inspired, motivated and support me by all means to accomplish this degree. Thank you for your love and patience.

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CHAPTER I

In this thesis, we consider the problem of identification of parameters of a random vector
$(X_1, X_2, \ldots, X_n)$, whose distribution is given in the sense of whether it is continuous or discrete, and if continuous, whether it is normal, Gamma, Weibull, Cauchy, etc, and similarly, if discrete, but the parameters of whose distributions are not known. The problem is to determine if there is a unique set of parameters of its distribution when the distribution of the maximum or the minimum of $\{X_1, X_2, \ldots, X_n\}$ is known or given. It only makes sense estimating these unknown parameters using standard methods of estimation such as the method of moments or the method of maximum likelihood, once the above uniqueness property has been established. For example, if X and Y are independent exponential random variables with parameters $a > 0,\ b > 0$, then

$$\begin{aligned} P(min\{X,Y\} > x) &= P(X > x)P(Y > x) \\ &= e^{-(a+b)x}, \quad x \geq 0, \end{aligned}$$

and consequently, if Z and W are another two independent exponential random variables with parameters $c > 0,\ d > 0$ such that $\{a, b\} \neq \{c, d\}$, but $a + b = c + d$, then it is clear that $min\{X, Y\}$ and $min\{Z, W\}$ have exactly the same distribution, but the parameters in the distribution of (X, Y) can be totally different from those for (Z, W).

Identification problems were considered in the literature for the normal distributions in the univariate and bivariate cases many years ago by Anderson and Ghurye (see [1] and [2]). They mentioned in their papers that their problems came up naturally to deal with certain supply and demand problems in econometrics. Later on, many authors contributed in this area including Basu and Ghosh, Gilliland and Hannan, Mukherjea, Nakassis and Miyashita, Dai and Mukherjea, and many others. One can look into the references included in the references at the end of this thesis.

In Chapter 2 of this thesis, we consider an identification problem considered by Davis and Mukherjea in [6]. They considered there a tri-variate normal random vector $\{X_1, X_2, X_3\}$ with zero means and a non-singular covariance matrix where all the correlations were assumed negative or zero. It was shown in [7] that one could then determine the three correlations and three variances uniquely based only on the knowledge of the density of the minimum of $\{X_1, X_2, X_3\}$. In Chapter 2 here, we take this problem again, but with the assumption that the means are not necessarily zeros, and we solve the problem of identifying the means.

In Chapter 3, we have the identification problems in the context of other continuous distributions such as Gamma, Weibull and Cauchy. We also considered here a discrete distribution, namely, Possion. This distribution turned out to be a very interesting one in the sense that the identification problem seemed to be difficult specially when we consider independent Possion random variables $\{X_1, X_2, \ldots, X_n\}$ with parameters $\lambda_1 > 0, \lambda_2 > 0, \ldots, \lambda_n > 0$ for $n \geq 5$, in the case when we assume knowledge of the distribution of the $max\{X_1, X_2, \ldots, X_n\}$, and even for $n \geq 2$, in the case when we assume only the knowledge of the distribution of the $min\{X_1, X_2, \ldots, X_n\}$. These problems lead to some very interesting open problems involving partial sums of the series for $exp(x)$.

In the context of Gamma and Weibull, we have been very much helped by the papers of Basu and Ghosh in [3], and in the case of Cauchy by the paper [8].

CHAPTER II

The univariate normal density is

$$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}, -\infty < x < \infty \tag{2.1}$$

where μ is the mean and σ^2 is the variance. Rewrite (2.1) as

$$(constant)e^{-\frac{1}{2}(x-\mu)(\sigma^2)^{-1}(x-\mu)}, -\infty < x < \infty$$

For the n-dimensional analogue, we replace x by $(x_1, x_2, \ldots, x_n)$, still call it x, replace μ by $(\mu_1, \mu_2, \ldots, \mu_n)$, call it m, and σ^2 by a symmetric positive definite $n \times n$ matrix. The constant is chose so that we have a n-dimensional density function (that is , the total integral must be 1). Thus, the n-dimensional normal density is

$$(2\pi)^{\frac{n}{2}} \left(\sqrt{det\Sigma}\right)^{-1} e^{-\frac{1}{2}(x-m)(\Sigma)^{-1}(x-m)^T},$$
$$x = (x_1, x_2, \ldots, x_n), -\infty < x_i < \infty$$

for all i. It turns out that Σ is the covariance matrix. It is well-known that if $X = (X_1, X_2, \ldots, X_n)$ is a n-dimensional normal random vector with density given as above, its characteristic function

$$\varphi_X(t) = e^{imt^T - \frac{1}{2}t\Sigma t^T}, \ -\infty < t < \infty.$$

Also, it is well-known that if $X = (X_1, X_2, \ldots, X_n)$ has a multivariate normal density, then any linear combination $\sum_{i=1}^{n} a_i X_i$ is normally distributed for $(a_1, a_2, \ldots, a_n)$ in R^n, and conversely, if every such linear combination is normally distributed, then X has a multivariate normal density.

The minimum problem mentioned in chapter 1 was considered by Davis and Mukherjea in [6] for a tri-variate non-singular normal vector all of whose correlations are negative and which has zero means. An example of such a normal vector, though not given in [6], can be constructed easily. For example, let $(Z_1, Z_2, \ldots, Z_n)$ be i.i.d. normal random variables with zero means and variances 1. Consider the matrix A given by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

and

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = A \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}$$

so that

$$\begin{cases} X_1 = a_{11}Z_1 + a_{12}Z_2 + a_{13}Z_3 \\ X_2 = a_{22}Z_2 + a_{23}Z_3 \\ X_3 = a_{33}Z_3 \end{cases}$$

Notice that

$$\begin{aligned} E(X_1X_3) =& a_{13}a_{33} \\ E(X_2X_3) =& a_{23}a_{33} \\ E(X_1X_2) =& a_{12}a_{22} + a_{13}a_{23} \end{aligned}$$

so that if $a_{12} > 0, a_{22} < 0, a_{23} < 0, a_{33} > 0, a_{13} < 0$,
$a_{12}a_{22} + a_{13}a_{23} < 0$, X_1, X_2, X_3 are a tri-variate normal with all its correlation negative.
Now let us consider a non-singular tri-variate normal X_1, X_2, X_3 with means μ_1, μ_2, μ_3, variance $\sigma_1^2, \sigma_2^2, \sigma_3^2$.
Let we write

$$X = min\{X_1, X_2, X_3\}.$$

Then the distribution of X is given by

$$\begin{aligned} P(X \leq t) =& P(X_1 \leq t, X_1 \leq X_2, X_3 \leq X_3) \\ &+ P(X_2 \leq t, X_2 \leq X_1, X_2 \leq X_3) \\ &+ P(X_3 \leq t, X_3 \leq X_1, X_3 \leq X_2) \end{aligned}$$

$$= \int_{-\infty}^{t} P(X_2 \le x_1, X_3 \le x_1 | X_1 = t) f_{X_1}(x_1) dx_1$$
$$+ \int_{-\infty}^{t} P(X_1 \le x_2, X_3 \le x_2 | X_2 = t) f_{X_2}(x_2) dx_2$$
$$+ \int_{-\infty}^{t} P(X_1 \le x_3, X_2 \le x_3 | X_3 = t) f_{X_3}(x_3) dx_3$$

Differentiating with respect to t, we have

$$\begin{aligned} f(t) = & \tfrac{1}{\sigma_1}\varphi(\tfrac{t-\mu_1}{\sigma_1}) P(X_2 \ge t, X_3 \ge t | X_1 = t) \\ + & \tfrac{1}{\sigma_2}\varphi(\tfrac{t-\mu_2}{\sigma_2}) P(X_1 \ge t, X_3 \ge t | X_2 = t) \\ + & \tfrac{1}{\sigma_3}\varphi(\tfrac{t-\mu_3}{\sigma_3}) P(X_1 \ge t, X_2 \ge t | X_3 = t) \end{aligned}$$

Notice that

$$\begin{aligned} \mu_{21} =& E(X_2|X_1 = t) = \mu_2 + \rho_{12}\frac{\sigma_2}{\sigma_1}(t - \mu_1) \\ \mu_{31} =& E(X_3|X_1 = t) = \mu_3 + \rho_{13}\frac{\sigma_3}{\sigma_1}(t - \mu_1) \\ \mu_{12} =& E(X_1|X_2 = t) = \mu_1 + \rho_{12}\frac{\sigma_1}{\sigma_2}(t - \mu_2) \\ \mu_{32} =& E(X_3|X_2 = t) = \mu_3 + \rho_{13}\frac{\sigma_3}{\sigma_2}(t - \mu_2) \\ \mu_{13} =& E(X_1|X_3 = t) = \mu_1 + \rho_{13}\frac{\sigma_1}{\sigma_3}(t - \mu_3) \\ \mu_{23} =& E(X_2|X_3 = t) = \mu_2 + \rho_{23}\frac{\sigma_2}{\sigma_3}(t - \mu_3) \end{aligned}$$

$$\begin{aligned} {\sigma_{21}}^2 =& V(X_2|X_1 = t) = {\sigma_2}^2(1 - {\rho_{12}}^2) \\ {\sigma_{31}}^2 =& V(X_3|X_1 = t) = {\sigma_3}^2(1 - {\rho_{13}}^2) \\ {\sigma_{12}}^2 =& V(X_1|X_2 = t) = {\sigma_1}^2(1 - {\rho_{12}}^2) \\ {\sigma_{32}}^2 =& V(X_3|X_2 = t) = {\sigma_3}^2(1 - {\rho_{23}}^2) \\ {\sigma_{13}}^2 =& V(X_1|X_3 = t) = {\sigma_1}^2(1 - {\rho_{13}}^2) \\ {\sigma_{23}}^2 =& V(X_2|X_3 = t) = {\sigma_2}^2(1 - {\rho_{23}}^2) \end{aligned}$$

Then, we have

$$\begin{aligned} f(t) =&\frac{1}{\sigma_1}\varphi(\frac{t-\mu_1}{\sigma_1})P(W_{21} \geq \frac{t-\mu_2-\rho_{12}(\frac{\sigma_2}{\sigma_1})(t-\mu_1)}{\sigma_2\sqrt{1-\rho_{12}{}^2}}, \\ &W_{31} \geq \frac{t-\mu_3-\rho_{13}(\frac{\sigma_3}{\sigma_1})(t-\mu_1)}{\sigma_3\sqrt{1-\rho_{13}{}^2}}) \\ &+ \textit{two other similar expressions} \end{aligned}$$

where (W_{21}, W_{31}) is a bivariate normal with zero means, variances each one, and correlations, $\rho_{23.1} = \frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}{}^2}\sqrt{1-\rho_{13}{}^2}}$ and (W_{12}, W_{32}) and (W_{13}, W_{23}) are both bivariate normals with zero means and variances all ones, and correlations given,respectively, by $\rho_{13.2} = \frac{\rho_{13}-\rho_{12}\rho_{23}}{\sqrt{1-\rho_{12}{}^2}\sqrt{1-\rho_{23}{}^2}}$ and $\rho_{12.3} = \frac{\rho_{12}-\rho_{13}\rho_{23}}{\sqrt{1-\rho_{13}{}^2}\sqrt{1-\rho_{23}{}^2}}$. In the paper of Davis and Mukherjea, under the assumption that means are all zeros, the variances all distinct, and the correlations all negative, it was shown that when $f(t)$ is given, then the three variances and the three correlations can be uniquely determined or identified, based on the knowledge of $f(t)$.

Let's assume that the means are not necessarily zeros, $\sigma_1^2 > \sigma_2^2 > \sigma_3^2$, and the correlations all negative. Now, in what follows, we assume that the density function $f(t)$ of $min\{X_1, X_2, X_3\}$, where X_1, X_2, X_3 is a tri-variate normal with means μ_1, μ_2, μ_3, variances $\sigma_1{}^2, \sigma_2{}^2, \sigma_3{}^2$ and correlations $\rho_{12}, \rho_{13}, \rho_{23}$, is given. We assume that the variances are distinct (though this assumption can be easily dropped with some extra work) and correlations all negative. The assumption on the correlations is important in our solution of the problem to identify the parameters when $f(t)$ is given. In [**?**], this problem was solves when the means are all zeros. We show in the rest of this chapter how to identify all the parameters (nine in all, the three means, the three variances, and the three correlations) when $f(t)$ is given (or known). We use the method used by Davis and Mukherjea. We do not assume the means to be zeros here. First, we need two lemmas.

Lemma 2.1 Let (X_1, X_2) be a non-singular bivariate normal with zero means, variances each 1 and negative correlation ρ. Let $\alpha_1,\ \alpha_2, \alpha_3,\ \beta_1,\ \beta_2,\ \beta_3$ be all constants, and

$\alpha_1 > 0,\ \alpha_2 > 0$. Then as $t \to \infty$,

$$P(X_1 \geq \alpha_1 t + \beta_1,\ X_2 \geq \alpha_2 t + \beta_2)$$

has the same order as

$$\frac{(1-\rho^2)^{\frac{3}{2}}}{2\pi t^2} \frac{1}{(\alpha_1 - \rho\alpha_2)(\alpha_2 - \rho\alpha_1)} e^{-\frac{1}{2}u(t)\Sigma^{-1}u(t)^T},$$

where Σ is the covariance matrix of (X_1, X_2) and the vector
$u(t) = (\alpha_1 t + \beta_1,\ \alpha_2 t + \beta_2)$

Proof: Write:

$$I(t) = P(X_1 \geq \alpha_1 t + \beta_1,\ X_2 \geq \alpha_2 t + \beta_2)$$

Then we have

$$\begin{aligned} I(t) =& \frac{1}{2\pi\sqrt{|det\Sigma|}} \int_{\alpha_1 t+\beta_1}^{\infty} \int_{\alpha_2 t+\beta_2}^{\infty} e^{-\frac{1}{2}x\Sigma^{-1}x^T} dx \\ =& \frac{1}{2\pi\sqrt{|det\Sigma|}} \int_0^{\infty} \int_0^{\infty} e^{-\frac{1}{2}(x+u(t))\Sigma^{-1}(x+u(t))^T} dx \\ =& \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2}u(t)\Sigma^{-1}u(t)^T} \int_0^{\infty} \int_0^{\infty} e^{-\frac{1}{2}x\Sigma^{-1}x^T} e^{-x\Sigma^{-1}u(t)^T} dx \end{aligned}$$

Then after substituting $x = \frac{y}{t}$ [that is, $(x_1, x_2)t = (y_1, y_2)$], taking $t \to \infty$, we obtain:

$$\begin{aligned} & I(t)\ e^{\frac{1}{2}u(t)\Sigma^{-1}u(t)^T}\ t^2 \\ =& \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^{\infty} \int_0^{\infty} e^{-\frac{1}{2}\frac{1}{t^2}y\Sigma^{-1}(y)^T - y\Sigma^{-1}(\alpha_1,\alpha_2)^T - \frac{y}{t}\Sigma^{-1}(\beta_1,\beta_2)^T} dy \\ \to& \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^{\infty} \int_0^{\infty} e^{-y\Sigma^{-1}(\alpha_1,\alpha_2)^T} dy \end{aligned}$$

From here, the result of this lemma follows after integration and taking the limits. Q.E.D

Note that in Lemma 2.1, after working out $u(t)\Sigma^{-1}u(t)^T$, it follows that as $t \to \infty$,

$$I(t) \approx \frac{(1-\rho^2)^{\frac{3}{2}}}{2\pi t^2(\alpha_1 - \rho\alpha_2)(\alpha_2 - \rho\alpha_1)} e^{-\frac{1}{2}\ \frac{Ct^2+2Dt+E}{1-\rho^2}},$$

where

$$\begin{aligned} C &= \alpha_1^2 - 2\rho\alpha_1\alpha_2 + \alpha_2^2 \\ D &= \alpha_1\beta_1 + \alpha_2\beta_2 - \rho(\alpha_1\beta_2 + \alpha_2\beta_1) \\ E &= \beta_1^2 - 2\rho\beta_1\beta_2 + \beta_2^2 \end{aligned}$$

Corollary 2.2 Write (W_{21}, W_{31}) for (X_1, X_2), $\rho = \rho_{23.1}$ (the correlation of $W_{21},\ W_{31}$)

$$\begin{aligned} \alpha_1 &= \frac{\sigma_1 - \rho_{12}\sigma_2}{\sigma_1\sigma_2\sqrt{1-\rho_{12}{}^2}}, \quad \alpha_2 = \frac{\sigma_1 - \rho_{13}\sigma_3}{\sigma_1\sigma_3\sqrt{1-\rho_{13}{}^2}} \\ \beta_1 &= \frac{\mu_2\sigma_1 - \rho_{12}\sigma_2\mu_1}{\sigma_1\sigma_2\sqrt{1-\rho_{12}{}^2}}, \quad \beta_2 = \frac{\mu_3\sigma_1 - \rho_{13}\sigma_3\mu_1}{\sigma_1\sigma_3\sqrt{1-\rho_{13}{}^2}} \end{aligned}$$

in Lemma 2.1. Then the following result holds:

$$\begin{aligned} &P(Z \geq \alpha_1 t + \beta_1) - P(W_{21} \geq \alpha_1 t + \beta_1, W_{31} \geq \alpha_2 t + \beta_2) \\ \approx&\frac{1}{\sqrt{2\pi}|\alpha_2 t + \beta_2|}\ e^{-\frac{1}{2}(\alpha_2 t + \beta_2)^2},\ as\ t \to\ -\infty \end{aligned}$$

Proof: Write:

$$\begin{aligned} &P(Z \geq \alpha_1 t + \beta_1) - P(W_{21} \geq \alpha_1 t + \beta_1, W_{31} \geq \alpha_2 t + \beta_2) \\ =&P(W_{21} \geq \alpha_1 t + \beta_1, W_{31} \leq \alpha_2 t + \beta_2) \\ =&P(W_{31} \leq \alpha_2 t + \beta_2) - P(W_{21} \leq \alpha_1 t + \beta_1, W_{31} \leq \alpha_2 t + \beta_2) \\ =&P(-W_{31} \geq \alpha_2(-t) - \beta_2) \\ -&P(-W_{21} \geq \alpha_1(-t) - \beta_1,\ -W_{31} \leq \alpha_2(-t) - \beta_2) \end{aligned}$$

We use Lemma 2.1 and notice that

$$\frac{\alpha_1^2 - 2\rho\alpha_1\alpha_2 + \alpha_2^2}{1-\rho^2} > \alpha_2{}^2\ as\ \rho < 0$$

Then the result follows. Q.E.D.

Notice that in the corollary above,

$$\begin{aligned} \alpha_2{}^2 &= \left(\frac{\sigma_1 - \rho_{13}\sigma_3}{\sigma_1\sigma_3\sqrt{1-\rho_{13}{}^2}}\right)^2 \\ &> \frac{1}{\sigma_3^2} - \frac{1}{\sigma_2^2} \end{aligned}$$

and this means that as $t \to -\infty$, for any real x, if α_1 and α_2 are as in Corollary 2.1, then

$$\begin{aligned}&P(Z \geq \alpha_1 t + \beta_1) - P(W_{21} \geq \alpha_1 t + \beta_1, W_{31} \geq \alpha_2 t + \beta_2)\\ =&o\left(e^{-\frac{1}{2}t^2(\frac{1}{\sigma_3^2} - \frac{1}{\sigma_1^2}) + tx}\right),\end{aligned}$$

and similarly, as $t \to -\infty$,

$$\begin{aligned}&P(Z \geq \alpha_2 t + \beta_2) - P(W_{21} \geq \alpha_1 t + \beta_1, W_{31} \geq \alpha_2 t + \beta_2)\\ =&o\left(e^{-\frac{1}{2}t^2(\frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}) + tx}\right).\end{aligned}$$

Lemma 2.3 Let (X_1, X_2) be as in Lemma 2.1. Then as $t \to -\infty$

$$1 - P(X_1 \geq \alpha_1 t + \beta_1,\ X_2 \geq \alpha_2 t + \beta_2)$$

has the same order as either

$$\frac{1}{\sqrt{2\pi|\alpha_1 t + \beta_1|}} e^{-\frac{1}{2}(\alpha_1 t + \beta_1)^2}$$

or

$$\frac{1}{\sqrt{2\pi|\alpha_2 t + \beta_2|}} e^{-\frac{1}{2}(\alpha_2 t + \beta_2)^2},$$

depending on which one of these two are dominant. For instance, if $\alpha_1^2 > \alpha_2^2$, then the second one (the one containing α_2) is dominant, and in this case,

$$\begin{aligned}&1 - P(X_1 \geq \alpha_1 t + \beta_1,\ X_2 \geq \alpha_2 t + \beta_2)\\ \approx&\frac{1}{\sqrt{2\pi|\alpha_2 t + \beta_2|}} e^{-\frac{1}{2}(\alpha_2 t + \beta_2)^2} \ \textit{as}\ t \to -\infty.\end{aligned}$$

In case $\alpha_1 = \alpha_2,\ \beta_1 = \beta_2$, then the terms above have the same order, and in this case,

$$\begin{aligned}&1 - P(X_1 \geq \alpha_1 t + \beta_1,\ X_2 \geq \alpha_2 t + \beta_2)\\ \approx&\frac{2}{\sqrt{2\pi|\alpha_1 t + \beta_1|}} e^{-\frac{1}{2}(\alpha_1 t + \beta_1)^2} \ \textit{as}\ t \to -\infty.\end{aligned}$$

Proof: Using Lemma 2.1, we observe that

$$\begin{aligned} T_1 &\equiv P(X_1 \leq \alpha_1 t + \beta_1,\ X_2 \leq \alpha_2 t + \beta_2) \\ &= P(-X_1 \geq \alpha_1(-t) - \beta_1,\ -X_2 \geq \alpha_2(-t) - \beta_2) \\ &\approx (constant) e^{-\frac{1}{2}\frac{Ct^2+2Dt+E}{1-\rho^2}}\ as\ t \to\ -\infty. \end{aligned}$$

where the constant C is given by

$$C = \alpha_1^2 - 2\rho\alpha_1\alpha_2 + \alpha_2^2$$

and this means that

$$\begin{aligned} \frac{C}{1-\rho^2} - {\alpha_1}^2 &= \frac{(\alpha_2 - \rho\alpha_1)^2}{1-\rho^2} \\ &> 0, \end{aligned}$$

and also,

$$\begin{aligned} \frac{C}{1-\rho^2} - {\alpha_2}^2 &= \frac{(\alpha_1 - \rho\alpha_2)^2}{1-\rho^2} \\ &> 0 \end{aligned}$$

Let us also observe that

$$\begin{aligned} T_2 &\equiv P(X_1 \leq \alpha_1 t + \beta_1,\ X_2 \geq \alpha_2 t + \beta_2) \\ &= P(X_1 \leq \alpha_1 t + \beta_1) - P(X_1 \leq \alpha_1 t + \beta_1,\ X_2 \leq \alpha_2 t + \beta_2) \\ &\approx \frac{1}{\sqrt{2\pi|\alpha_1 t + \beta_1|}} e^{-\frac{1}{2}(\alpha_1 t + \beta_1)^2}\ as\ t \to\ -\infty. \end{aligned}$$

Similarly,

$$\begin{aligned} T_3 &\equiv P(X_1 \geq \alpha_1 t + \beta_1,\ X_2 \leq \alpha_2 t + \beta_2) \\ &\approx \frac{1}{\sqrt{2\pi|\alpha_2 t + \beta_2|}} e^{-\frac{1}{2}(\alpha_2 t + \beta_2)^2}\ as\ t \to\ -\infty. \end{aligned}$$

The result now follows observing that

$$1 - P(X_1 \geq \alpha_1 t + \beta_1,\ X_2 \geq \alpha_2 t + \beta_2) = T_1 + T_2 + T_3$$

and the orders of $T_1,\ T_2,\ and\ T_3$ obtained above.Q.E.D

Now in what follows, we describe our method of identifying the parameters of a non-singular tri-variate normal (X_1, X_2, X_3) with means μ_1, μ_2, μ_3, variances ${\sigma_1}^2, {\sigma_2}^2, {\sigma_3}^2$ and correlations $\rho_{12}, \rho_{13}, \rho_{23}$ all three negative, where the density function $f(t)$ of the $min\{X_1, X_2, X_3\}$ is given. Thus, we have

$$\begin{aligned} f(t) = & \ \tfrac{1}{\sigma_1}\varphi(\tfrac{t-\mu_1}{\sigma_1})P(W_{21} \geq \tfrac{1-\rho_{12}\frac{\sigma_2}{\sigma_1}}{\sigma_2\sqrt{1-{\rho_{12}}^2}}t + \tfrac{\mu_1\rho_{12}\frac{\sigma_2}{\sigma_1}-\mu_2}{\sigma_2\sqrt{1-{\rho_{12}}^2}}, \\ & \qquad W_{31} \geq \tfrac{1-\rho_{13}\frac{\sigma_3}{\sigma_1}}{\sigma_3\sqrt{1-{\rho_{13}}^2}}t + \tfrac{\mu_1\rho_{13}\frac{\sigma_3}{\sigma_1}-\mu_3}{\sigma_3\sqrt{1-{\rho_{13}}^2}}) \\ + & \qquad \textit{two other similar expressions}, \end{aligned}$$

where $(W_{21},\ W_{31}),\ (W_{12},\ W_{32}),\ (W_{13},\ W_{23})$ are as described in the beginning of this chapter. We follow the method in [?]; however, some substantial modifications in arguments are needed to deal with the present situation of non-zero means. With no loss of generality, we assume :

$${\sigma_1}^2 > {\sigma_2}^2 > {\sigma_3}^2.$$

(We could have assumed only ${\sigma_1}^2 \geq {\sigma_2}^2 \geq {\sigma_3}^2$; but this case is no more difficult than our "greater than" assumption, and can be easily treated following the method here.

First, we notice that ${\sigma_1}^2$ is uniquely identified by $f(t)$, as in [?], by the fact that

$$\frac{1}{{\sigma_1}^2} = sup\{m^2| \lim_{t \to -\infty} \frac{f(t)}{\varphi(mt)} = 0\} \tag{2.2}$$

Now in terms of $f(t)$ and ${\sigma_1}^2$, we can determine the mean μ_1 as follows. There exists $x < \mu_1$, such that

$$\lim_{t \to -\infty} \sigma_1 f(t)\ e^{\frac{t^2}{2\sigma_1^2} - \frac{tx}{\sigma_1^2}} = 0 \tag{2.3}$$

and this means that

$$sup\{x \in R| \lim_{t \to -\infty} \sigma_1 f(t)\ e^{\frac{1}{\sigma_1^2}(\frac{t^2}{2} - tx)} = 0\} \tag{2.4}$$

must equal μ_1; thus, this "sup" identifies μ_1 as the only possible mean value of X_1 for the given f(t).

Next, with ${\sigma_1}^2$ and μ_1 identified, we wish to identify ${\sigma_2}^2$ by considering the function

$$f(t) - \frac{1}{\sigma_1}\varphi(\frac{t-\mu_1}{\sigma_1}),$$

which is equal to

$$-\frac{1}{\sigma_1}\varphi(\frac{t-\mu_1}{\sigma_1})[1-P(W_{21}\geq\frac{\sigma_1-\rho_{12}\sigma_2}{\sigma_1\sigma_2\sqrt{1-\rho_{12}{}^2}}t-\frac{\sigma_1\mu_2-\rho_{12}\sigma_2\mu_1}{\sigma_1\sigma_2\sqrt{1-\rho_{12}{}^2}},$$
$$W_{31}\geq\frac{\sigma_1-\rho_{13}\sigma_3}{\sigma_1\sigma_3\sqrt{1-\rho_{13}{}^2}}t-\frac{\sigma_1\mu_3-\rho_{13}\sigma_3\mu_1}{\sigma_1\sigma_3\sqrt{1-\rho_{13}{}^2}})]$$

Note that we can now use Lemma 2.3, and since

$$\left(\frac{\sigma_1-\rho_{12}\sigma_2}{\sigma_1\sigma_2\sqrt{1-\rho_{12}{}^2}}\right)^2+\frac{1}{\sigma_1^2}>\frac{1}{\sigma_2^2} \tag{2.5}$$

and also,

$$\left(\frac{\sigma_1-\rho_1\sigma_3}{\sigma_1\sigma_3\sqrt{1-\rho_{13}{}^2}}\right)^2+\frac{1}{\sigma_1^2}>\frac{1}{\sigma_3^2}>\frac{1}{\sigma_2^2} \tag{2.6}$$

by direct calculations, it is clear that

$$\frac{1}{\sigma_2^2}=sup\{m^2|\lim_{t\to-\infty}e^{\frac{1}{2}m^2t^2}\left[f(t)-\frac{1}{\sigma_1}\varphi(\frac{t-\mu_1}{\sigma_1})\right]=0\} \tag{2.7}$$

This identifies σ_2. With $\sigma_1,\ \mu_1,\ and\ \sigma_2$ identified uniquely, we will now identify μ_2. To do this, we consider the function $g(x,t)$ given by

$$g(x,t)=\left[\sigma_1 f(t)\ e^{\frac{1}{2}(\frac{t-\mu_1}{\sigma_1})^2}-1\right]e^{\frac{1}{2}t^2(\frac{1}{\sigma_2^2}-\frac{1}{\sigma_1^2})-tx} \tag{2.8}$$

where x is a real number.

Notice that $g(x,t)$ has three terms. The first term is

$$P\left(W_{21}\geq\frac{t-\mu_{21}}{\sigma_{21}},\ W_{31}\geq\frac{t-\mu_{31}}{\sigma_{31}}\right)e^{\frac{1}{2}t^2(\frac{1}{\sigma_2^2}-\frac{1}{\sigma_1^2})-tx},$$

and this term goes to zero as $t\to-\infty$, by our earlier remarks. The third term of $g(x,t)$ is

$$\frac{\sigma_1}{\sigma_2}e^{-\frac{1}{2}(\frac{t-\mu_3}{\sigma_3})^2}e^{\frac{1}{2}t^2(\frac{1}{\sigma_2^2}-\frac{1}{\sigma_1^2})-tx}e^{\frac{1}{2}(\frac{t-\mu_1}{\sigma_1})^2}$$

let

$$B(t,x)=P\left(W_{21}\geq\frac{t-\mu_{21}}{\sigma_{21}},W_{31}\geq\frac{t-\mu_{31}}{\sigma_{31}}\right).$$

Note that as $t\to-\infty$, for $x<\frac{\mu_2}{\sigma_2^2}-\frac{\mu_1}{\sigma_1^2}$, $B(t,x)\to 0$. Thus, it is clear that

$$\frac{\mu_2}{\sigma_2^2}-\frac{\mu_1}{\sigma_1^2}=sup\{x\in R|\lim_{t\to-\infty}B(t,x)=0\}$$

In other words, $\frac{\mu_2}{\sigma_2^2} - \frac{\mu_1}{\sigma_1^2}$ is uniquely determined and therefore, μ_2 is also.

With $\sigma_1,\ \mu_1,\ \sigma_2,\ \mu_2$ identified, we now consider the function

$$f(t) - \frac{1}{\sigma_1}\varphi(\frac{t-\mu_1}{\sigma_1}) - \frac{1}{\sigma_2}\varphi(\frac{t-\mu_2}{\sigma_2})$$

Using Lemma 2.3, we notice the following. If

$$\begin{aligned} A_3 &= \frac{1}{\sigma_1^2} + \frac{(\sigma_1 - \rho_{12}\sigma_2)^2}{\sigma_1^2\sigma_2^2(1-{\rho_{12}}^2)} \\ &= \frac{1}{\sigma_2^2} + \frac{(\sigma_2 - \rho_{12}\sigma_1)^2}{\sigma_1^2\sigma_2^2(1-{\rho_{12}}^2)} \\ &= \frac{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}{\sigma_1^2\sigma_2^2(1-{\rho_{12}}^2)}, \end{aligned}$$

then defining s by

$$s = sup\{v| \lim_{t\to\infty} e^{vt^2}[f(t) - \frac{1}{\sigma_1}\varphi\left(\frac{t-\mu_1}{\sigma_1}\right) - \frac{1}{\sigma_2}\varphi\left(\frac{t-\mu_2}{\sigma_2}\right)] = 0\}, \qquad (2.9)$$

we see that this s is well-defined as whenever

$$v \le \frac{1}{2}A_3 \ and\ v < \frac{1}{2\sigma_3^2},$$

then

$$\lim_{t\to-\infty} e^{vt^2}[f(t) - \frac{1}{\sigma_1}\varphi\left(\frac{t-\mu_1}{\sigma_1}\right) - \frac{1}{\sigma_2}\varphi\left(\frac{t-\mu_2}{\sigma_2}\right)] = 0$$

In case

$$\frac{1}{\sigma_3^2} > A_3,\ then\ s = \frac{1}{2}A_3,$$

and in this case, using Lemma 2.3, we see that

$$\lim_{t\to-\infty} e^{st^2}\left[f(t) - \frac{1}{\sigma_1}\varphi\left(\frac{t-\mu_1}{\sigma_1}\right) - \frac{1}{\sigma_2}\varphi\left(\frac{t-\mu_2}{\sigma_2}\right)\right] = 0 \qquad (2.10)$$

because of a term containing t in the denominator in the "order" term of Lemma 2.3. However, if

$$A_3 \ge \frac{1}{\sigma_3^2},$$

then

$$s = \frac{1}{2\sigma_3^2},$$

and in this case, we must have, unlike in (2.10),

$$\lim_{t \to -\infty} e^{st^2}[f(t) - \frac{1}{\sigma_1}\varphi\left(\frac{t-\mu_1}{\sigma_1}\right) - \frac{1}{\sigma_2}\varphi\left(\frac{t-\mu_2}{\sigma_2}\right)] \neq 0 \tag{2.11}$$

Thus, in case of (2.11), $s = \frac{1}{2\sigma_3^2}$ and s identifies σ_3^2, and in case of (2.10), $2s$ identifies A_3 and then $s = \frac{1}{2}A_3$. In this case,

$$\frac{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}{\sigma_1^2\sigma_2^2(1-{\rho_{12}}^2)} = 2s \tag{2.12}$$

and it can be easily verified by solving the equation (2.12) that in this case

$$\rho_{12} = \frac{1}{2s\sigma_1\sigma_2}\left[1 - \sqrt{(2s\sigma_1^2-1)(2s\sigma_2^2-1)}\right] \tag{2.13}$$

identifying ρ_{12} uniquely by (2.13). Thus, after we have identified $\sigma_1,\ \mu_1,\ \sigma_2,\ \mu_2$, we can either identify σ_3 or ρ_{12}. In what follows, we show that in either case whether $\sigma_1,\ \mu_1,\ \sigma_2,\ \sigma_3$ have been identified or $\sigma_1,\ \mu_1,\ \sigma_2,\ \rho_{12}$ have been identified, we can proceed to identify μ_3 as follows.

In case $\sigma_1,\ \mu_1,\ \sigma_2,\ \mu_2$ *and* σ_3 have been identified, we know that $A_3 \geq \frac{1}{\sigma_3^2}$. In this case, we consider the function

$$g(t) = f(t) - \frac{1}{\sigma_1}\varphi\left(\frac{t-\mu_1}{\sigma_1}\right) - \frac{1}{\sigma_2}\varphi\left(\frac{t-\mu_2}{\sigma_2}\right) \tag{2.14}$$

as before. We observe that in this case, since $A_3 \geq \frac{1}{\sigma_3^2}$ for any real $x < \mu_3$,

$$\lim_{t \to -\infty} g(t)e^{\frac{1}{2}\frac{t^2}{\sigma_3^2} - \frac{tx}{\sigma_3^2}} = 0 \tag{2.15}$$

This identifies μ_3, since then

$$\mu_3 = sup\{x|\lim_{t \to -\infty} g(t)e^{\frac{1}{2}\frac{t^2}{\sigma_3^2} - \frac{tx}{\sigma_3^2}} = 0\} \tag{2.16}$$

In case $\sigma_1,\ \mu_1,\ \sigma_2,\ \mu_2$ and ρ_{12} (instead of σ_3) were identified first, then to identify μ_3, we proceed as follows. We know that in this case $A_3 < \frac{1}{\sigma_3^2}$, and also, then the constants $\alpha_1,\ \beta_1\ \ \alpha_2,\ \beta_2$ given by

$$\alpha_1 = \frac{\sigma_1 - \rho_{12}\sigma_2}{\sigma_1\sigma_2\sqrt{1-{\rho_{12}}^2}},\quad \beta_1 = \frac{\sigma_1\mu_2 - \rho_{12}\sigma_2\mu_1}{\sigma_1\sigma_2\sqrt{1-\rho_{12}}}$$

$$\alpha_2 = \frac{\sigma_2 - \rho_{12}\sigma_1}{\sigma_1\sigma_2\sqrt{1-\rho_{12}}^2}, \quad \beta_2 = \frac{\sigma_2\mu_1 - \rho_{12}\sigma_1\mu_2}{\sigma_1\sigma_2\sqrt{1-\rho_{12}}}$$

are all known (or already identified), and thus, we can then consider the function $h(v,t)$ given by

$$\begin{aligned} h(v,t) =& [f(t) - \frac{1}{\sigma_1}\varphi(\frac{t-\mu_1}{\sigma_1})P(Z \geq \alpha_1 t + \beta_1) \\ & - \frac{1}{\sigma_2}\varphi(\frac{t-\mu_2}{\sigma_2})P(Z \geq \alpha_2 t + \beta_2)]\ e^{vt^2} \end{aligned}$$

Note that $h(v,t)$ has three terms. The first term of $h(v,t)$ is given by

$$\begin{aligned} & -\frac{e^{\frac{1}{2}vt^2}}{\sigma_1}\varphi(\frac{t-\mu_1}{\sigma_1})[P(Z \geq \alpha_1 t + \beta_1) \\ & - P(W_{21} \geq \alpha_1 t + \beta_1, W_{31} \geq \alpha_1^{'} t + \beta_1^{'})] \end{aligned}$$

where

$$\alpha_1^{'} = \frac{\sigma_1 - \rho_{13}\sigma_3}{\sigma_1\sigma_3\sqrt{1-\rho_{13}}^2}, \quad \beta_1^{'} = \frac{\sigma_1\mu_3 - \rho_{13}\sigma_3\mu_1}{\sigma_1\sigma_3\sqrt{1-\rho_{13}}^2}.$$

By Corollary 2.2, this term, as $t \to -\infty$, has the same order as

$$(constant)\ \frac{1}{|\alpha_1^{'}t + \beta_1^{'}|}\ e^{\frac{1}{2}vt^2 - \frac{1}{2}(\alpha_1^{'}t+\beta_1^{'})^2 - \frac{1}{2}\frac{t^2}{\sigma_1^2}} \tag{2.17}$$

Note that if $v < \frac{1}{\sigma_3^2}$, then since

$$\begin{aligned} \alpha_1^{'\,2} + \sigma_1^2 &= (\frac{\sigma_1 - \rho_{13}\sigma_3}{\sigma_1\sigma_3\sqrt{1-\rho_{13}}^2})^2 + \frac{1}{\sigma_1^2} \\ &> \frac{1}{\sigma_3^2} \\ &> v \end{aligned}$$

it follows that the expression in (2.18) goes to zero, as $t \to -\infty$. Similarly, the second term of $h(v,t)$, given by

$$\begin{aligned} & -\frac{e^{\frac{1}{2}vt^2}}{\sigma_2}\varphi(\frac{t-\mu_2}{\sigma_2})[P(Z \geq \alpha_2 t + \beta_2)P(W_{12} \geq \alpha_2 t + \beta_2, \\ & W_{32} \geq \alpha_2^{'} t + \beta_2^{'})] \end{aligned}$$

where

$$\alpha_2^{'} = \frac{\sigma_2 - \rho_{23}\sigma_3}{\sigma_2\sigma_3\sqrt{1-\rho_{23}}^2}, \quad beta_2^{'} = \frac{\sigma_1\mu_3 - \rho_{23}\sigma_3\mu_2}{\sigma_2\sigma_3\sqrt{1-\rho_{23}}^2}.$$

Thus, it follows that for h given by (2.17),

$$\frac{1}{\sigma_3^2} = sup\{v | \lim_{t \to -\infty} h(v,t) = 0\} \tag{2.18}$$

Thus, (2.18) now identified σ_3, so that we have now identified $\sigma_1,\ \mu_1,\ \sigma_2,\ \mu_2$ and at least σ_3, if not both ρ_{12} and σ_3. Thus, we can now also identify μ_3 as in (2.15). To identify the remaining parameters, we can now follow the method of Davis and Mukherjea by replacing there $\varphi(\frac{x}{\sigma_1}),\ \varphi(\frac{x}{\sigma_2}),\ \varphi(\frac{x}{\sigma_3})$ by $\varphi(\frac{x-\mu_1}{\sigma_1}),\ \varphi(\frac{x-\mu_2}{\sigma_2}),\ \varphi(\frac{x-\mu_3}{\sigma_3})$.

CHAPTER III

In this chapter, we consider the Exponential, Gamma and Weibull distribution in the context of the Maximum and Minimum problem. For Exponential distribution, if X_1 and X_2 are independent and exponential with parameters $\lambda_1 > 0$ and $\lambda_2 > 0$, then

$$\begin{aligned} &P\,(min\{X_1,\ X_2\}) \\ =&P(X_1 > x)P(X_2 > x) \\ =&e^{-\lambda_1 x}e^{-\lambda_2 x} \\ =&e^{-(\lambda_1+\lambda_2)x} \end{aligned}$$

And this means that we cannot identify λ_1 or λ_2 , knowing the distribution of the minimum.

Theorem 3.1 Let the pdf of X_i be given by

$$f(x;\lambda_i) = \begin{cases} f(x) & x \leq 0 \\ 0 & x < 0 \quad where\ \lambda_i > 0 \end{cases}$$

Let $X_1,\ X_2,\ \ldots,\ X_p$ be independent random variables. Similarly, let $X_{p+1},\ X_{p+2},\ \ldots,\ X_{p+q}$ be independent random variables. If the distribution of $max\{X_1,\ X_2,\ \ldots,\ X_p\}$ is identical with that of $max\{X_{p+1},\ X_{p+2},\ \ldots,\ X_{p+q}\}$ then we have $(\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_p)$ is a permutation of $(\lambda_{p+1},\ \lambda_{p+2},\ \ldots,\ \lambda_{p+q})$.

Proof: Let F_i be the cdf of X_i. Since

$$\begin{aligned} &P(max\{X_1,\ X_2,\ \ldots,\ X_p\} \leq x) \\ =&P(max\{X_{p+1},\ X_{p+2},\ \ldots,\ X_{p+q}\} \leq x),\ where\ x > 0 \end{aligned}$$

which is,

$$\begin{aligned} &P(X_1 \leq x,\ \ X_2 \leq x,\ \ldots,\ X_p \leq x) \\ =&P(X_{p+1} \leq x,\ X_{p+2} \leq x,\ \ldots,\ X_{p+q} \leq x). \end{aligned}$$

Since $X_1,\ X_2,\ \ldots,\ X_p$ are independent and $X_{p+1},\ X_{p+2},\ \ldots,\ X_{p+q}$ are independent. We have

$$\begin{aligned}&P(X_1 \leq x)P(X_2 \leq x)\ldots P(X_p \leq x)\\ =&P(X_{p+1} \leq x)P(X_{p+2} \leq x)\ldots P(X_{p+q} \leq x)\end{aligned}$$

that is,

$$F_1(x)F_2(x)\ldots F_p(x) = F_{p+1}(x)F_{p+2}(x)\ldots F_{p+q}(x).$$

Differentiating the logarithm of both sides with respect to x . Then we have

$$\frac{f_1(x)}{F_1(x)} + \frac{f_2(x)}{F_2(x)} + \ldots + \frac{f_p(x)}{F_p(x)} = \frac{f_{p+1}(x)}{F_{p+1}(x)} + \frac{f_{p+2}(x)}{F_{p+2}(x)} + \ldots + \frac{f_{p+q}(x)}{F_{p+q}(x)} \tag{3.1}$$

hence,

$$\sum_{i=1}^{p+q} a_i \frac{f_i(x)}{F_i(x)} = 0, \tag{3.2}$$

where $a_i = 1$ for $i = 1, 2, \ldots, p$, and $a_i = -1$ for $i = p+1, p+2, \ldots, p+q$.

$$\begin{aligned}&\lim_{x\to\infty} \frac{f_i(x)}{f_j(x)}\\ =&\lim_{x\to\infty} \frac{\lambda_i e^{-\lambda_i x}}{\lambda_j e^{-\lambda_j x}}\\ =&\lim_{x\to\infty} \frac{\lambda_i}{\lambda_j} e^{(\lambda_j - \lambda_i)x}\end{aligned}$$

Therefore,

$$\lim_{x\to\infty} \frac{f_i(x)}{f_j(x)} = \begin{cases} 1 & if \lambda_i = \lambda_j \\ \infty & if \lambda_i > \lambda_j \\ 0 & if \lambda_i < \lambda_j \end{cases}$$

Consider the case $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p$, dividing (3.2) throughout by $f_1(x)$ and making $x \to \infty$. Then $\lim_{x\to\infty} = 0\ or\ 1$. Let $I = \{i :\ \lambda_i = \lambda_1\}$. Hence $\sum' a_i = 0$ where $\sum'$ denotes the sum over all $i \in I$. Therefore I contains an even number of elements, half of which are from $i = 1, 2, \ldots, p$, and the other half are from $i = p+1, p+2, \ldots, p+q$.Thus a certain number of f_i, where $i = 1, 2, \ldots, p$ in (3.1), are identical to one another and to the same number of f_i, where $i = p+1, p+2, \ldots, p+q$. Subtracting these identical terms from both sides of (3.1), we have a new equation of the same form but with fewer terms. The process can be repeated until each term on one side of (3.1) is matched with one term on the other. If $p = q$, the proposition is proved. On the other hand, if $p \neq q$, say $p < q$, we have $p - q$ of the $f_i(x) = 0$ where $i = p+1, p+2, \ldots, p+q$, contrary to the definition of pdf. Hence, $p = q$. Q.E.D.

Next, we consider the Gamma distribution. These distributions were first considered in our context by Basu and Ghosh in [3]. They considered our problem by considering only the minimum. We consider the corresponding maximum problem here.

Lemma 3.1

$$J = \int_x^\infty e^{-y} y^{\alpha-1} dy \approx (e^{-x})(x^{\alpha-1}) \ as \ x \to \infty, \ where \ \alpha > 0,$$

In other words,

$$\lim_{x\to\infty} \frac{(e^{-1})(x^{\alpha-1})}{J} = 1$$

Proof: Suppose first when $0 < \alpha \leq 1$, then integrating by parts, we have

$$J = -\int_x^\infty y^{\alpha-1} d(e^{-y}) = (e^{-x})(x^{\alpha-1})$$
$$+(\alpha-1)\int_x^\infty (e^{-y})(y^{\alpha-2}) dy \leq (e^{-x})(x^{\alpha-1})$$

Similarly,

$$\begin{aligned} J &= (e^{-x})(x^{\alpha-1}) + (\alpha-1)\int_x^\infty (e^{-y})(y^{\alpha-2}) dy \\ &= (e^{-x})(x^{\alpha-1}) + (\alpha-1)(\alpha-2)\int_x^\infty (e^{-y})(y^{\alpha-3}) dy \\ &= (e^{-x})(x^{\alpha-1})(1 + \frac{\alpha-1}{x}) \\ &\geq (e^{-x}(x^{\alpha-1}) \end{aligned}$$

Hence , it follows that

$$J = \int_x^\infty e^{-y} y^{\alpha-1} dy \approx (e^{-x}) x^{\alpha-1} \ as \ x \to \infty \ for \ 0 < \alpha \leq 1$$

Now let $m < \alpha \leq m+1$ where m is a positive integer. Then by integrating by parts

$$J = (e^{-x})(x^{\alpha-1}) + (\alpha-1)(e^{-x})(x^{\alpha-2}) + \ldots$$
$$+(\alpha-1)\ldots(\alpha-m)\int_x^\infty e^{-y}(y^{\alpha-m-1}) dy \geq (e^{-x})(x^{\alpha-1})$$

Since the other terms on the right hand side above are all nonnegative. Also,

$$
\begin{aligned}
J =&(e^{-x})(x^{\alpha-1}) + \ldots + (\alpha-1)\ldots(\alpha-m)\int_x^\infty e^{-y}(y^{\alpha-m-1})dy \\
=&(e^{-x})(x^{\alpha-1}) + \ldots + (\alpha-1)\ldots(\alpha-m)(e^{-x})(x^{\alpha-m})[(e^{-x})(x^{\alpha-m-1}) \\
+&(\alpha-m-1)\int_x^\infty (e^{-y})(y^{\alpha-m-2})dy]
\end{aligned}
$$

since all the terms in the previous step are positive except for the last term. Thus,

$$J \leq (e^{-x})(x^{\alpha-1})[1 + \frac{\alpha-1}{x} + \ldots + \frac{(\alpha-1)\ldots(\alpha-m)}{x^{m+1}}]$$

This proves that

$$J = \int_x^{-y} y^{\alpha-1}dy \approx (e^{-x})(x^{\alpha-1}) \ as \ x \to \infty, \ where \ \alpha > 0$$

We remark that this result was used by Basu and Ghosh to identify the parameters for the minimum problem when two independent Gamma distributions are given. We do not use this lemma in our results on Gamma distributions (see Theorems 3.2 and 3.3 below). But will use it in the proof of Theorem 3.5.

Theorem 3.2 Let the pdf of X_i be given by

$$
\begin{aligned}
f_i(x) =&f(x;\alpha_i;\beta_i) \\
=&\frac{(e^{-\frac{x}{\beta_i}})(x^{\alpha_i-1})}{(\beta_i^{\alpha_i})[\gamma(\alpha_i)]} \ where \ \alpha_i > 0 \ and \ \beta_i > 0.
\end{aligned}
$$

Let the pdf of Y_i be given by

$$
\begin{aligned}
g_i(x) =&g(x;\alpha_i^{'};\beta_i^{'}) \\
=&\frac{(e^{-\frac{x}{\beta_i{'}}})(x^{\alpha_i{'}-1})}{(\beta_i{'}^{\alpha_i{'}})[\gamma(\alpha_i{'})]} \ where \ \alpha_i^{'} > 0 \ and \ \beta_i^{'} > 0.
\end{aligned}
$$

Let $F_i(x)$ be the cdf ofX_i and $G_i(x)$ be the cdf of Y_i. We also assume that $X_1, \ X_2, \ \ldots, X_p$ are independent and $Y_1, \ Y_2, \ \ldots, \ Y_q$ are independent. And we also assume that the distribution of

$max\{X_1, \ X_2, \ \ldots, \ X_p\}$ is identical with that of $max\{Y_1, \ Y_2, \ \ldots, \ Y_q\}$ and the conditions

$\beta_1 \geq \beta_2 \geq \ldots \geq \beta_p$ and

$\beta_1{}' \geq \beta_2{}' \geq \ldots \geq \beta_q{}'$ hold. Then we have $p = q$ and (α_i, β_i) is a permutation of $(\alpha_i{}', \beta_i{}')$, where $i = 1, 2, \ldots, p$.

Proof: Since

$$\lim_{x\to\infty} \frac{f_i(x)}{f_j(x)} = \begin{cases} 0 & if\ \beta_j > \beta_i \\ \infty & if\ \beta_j < \beta_i \\ \begin{cases} 1 \quad if\ \alpha_j = \alpha_i \\ 0 \quad if\ \alpha_j > \alpha_i \\ \infty \quad if\ \alpha_j = \alpha_i \end{cases} & if\ \beta_j = \beta_i \end{cases}$$

Since

$$\begin{aligned} &P(max\{X_1,\ X_2,\ \ldots,\ X_p\} \leq x \\ =&P(max\{Y_1,\ Y_2,\ \ldots,\ Y_q\}) \leq x,\ where\ x > 0 \end{aligned}$$

which is,

$$\begin{aligned} &P(X_1 \leq x, X_2 \leq x, \ldots, X_p \leq x) \\ =&P(Y_1 \leq x, Y_2 \leq x, \ldots, Y_p \leq x) \end{aligned}$$

Since $X_1,\ X_2,\ \ldots,\ X_p$ are independent and $Y_1,\ Y_2,\ \ldots,\ Y_q$ are independent. We have

$$\begin{aligned} &P(X_1 \leq x)P(X_2 \leq x)\ldots P(X_p \leq x) \\ =&P(Y_1 \leq x)P(Y_2 \leq x)\ldots P(Y_p \leq x) \end{aligned}$$

that is,

$$F_1(x)F_2(x)\ldots F_p(x) = G_1(x)G_2(x)\ldots G_q(x)$$

Differentiating the logarithm of both sides with respect to x. Then we have

$$\frac{f_1(x)}{F_1(x)} + \frac{f_2(x)}{F_2(x)} + \ldots + \frac{f_p(x)}{F_p(x)} = \frac{g_1(x)}{G_1(x)} + \frac{g_2(x)}{G_2(x)} + \ldots + \frac{g_q(x)}{G_q(x)} \tag{3.3}$$

Let $A_1 = i : \beta_i = \beta_1,\ 1 \leq i \leq p$ and let $\alpha_{i_o} = max\{\alpha_i :\ i \in A_1,\ 1 \leq i \leq p\}$. We divide both sides of(3.3) by $f_{i_o}(x)$ and then let $x \to \infty$. Then on the left side of (3.3), we get α, the cardinality of the set

$$B_1 = \{i :\ \beta_i = \beta_1,\ \alpha_i = \alpha_{i_o},\ 1 \leq i \leq p\}.$$

Therefore, we must also get on the right hand side of (3.3), the same quantity, namely, as $x \to \infty$. In other words, there will be exactly a many f on each side of (3.3), which are all equal to each other. This means we can match a positive number of f_is on the left hand side of with an equal number of f_is on the right hand side of (3.3). Now we can cancel these terms from both sides of (3.3), and get again an equation similar to (3.3), but this time with fewer number of terms on each side. We repeat the procedure. After repeating a finite number of procedure, we are done. Q.E.D.

We also include below a result in the minimum case, similar to that given in Basu and Ghosh in [3], but in the general case.

Theorem 3.3 Let the pdf of X_i be given by

$$f_i(x) = f(x;\alpha_i;\beta_i)$$
$$=\frac{(e^{-\frac{x}{\beta_i}})(x^{\alpha_i-1})}{(\beta_i{}^{\alpha_i})[\gamma(\alpha_i)]} \quad where\ \alpha_i > 0\ and\ \beta_i > 0.$$

Let the pdf of Y_i be given by

$$g_i(x) = g(x;\alpha_i{}';\beta_i{}')$$
$$=\frac{(e^{-\frac{x}{\beta_i{}'}})(x^{\alpha_i{}'-1})}{(\beta_i{}'^{\alpha_i{}'})[\gamma(\alpha_i{}')]} \quad where\ \alpha_i{}' > 0\ and\ \beta_i{}' > 0.$$

Let $F_i(x)$ be the cdf of X_i and $G_i(x)$ be the cdf of Y_i. $X_1,\ X_2,\ \ldots,\ X_p$ are independent and $Y_1,\ Y_2,\ \ldots,\ Y_q$ are independent. And If the distribution of $minX_1,\ X_2,\ \ldots,\ X_p$ is identical with that of $minY_1,\ Y_2,\ \ldots,\ Y_q$ under the conditions $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_p$. Then we have $p = q$ and (α_i, β_i) is a permutation of $(\alpha_i{}', \beta_i{}')$, where $i = 1, 2, \ldots, p$.

Proof: Since

$$\lim_{x\to\infty}\frac{f_i(x)}{f_j(x)} = \begin{cases} 0 & if \quad \alpha_j < \alpha_i \\ \infty & if \quad \alpha_j > \alpha_i \\ c & if \quad \alpha_j = \alpha_i \quad where\ c = (\frac{\beta_j}{\beta_i})^{\alpha_i} \end{cases}$$

Since

$$P(min\{X_1,\ X_2,\ \ldots,\ X_p\} \leq x$$
$$=P(min\{Y_1,\ Y_2,\ \ldots,\ Y_q\}) \leq x,\ where\ x > 0$$

which is,

$$\begin{aligned} &P(X_1 \geq x, X_2 \geq x, \ldots, X_p \geq x) \\ =&P(Y_1 \geq x, Y_2 \geq x, \ldots, Y_p \geq x) \end{aligned}$$

Since $X_1,\ X_2,\ \ldots,\ X_p$ are independent and $Y_1,\ Y_2,\ \ldots,\ Y_q$ are independent. We have

$$\begin{aligned} &P(X_1 \geq x)P(X_2 \geq x)\ldots P(X_p \geq x) \\ =&P(Y_1 \geq x)P(Y_2 \geq x)\ldots P(Y_p \geq x) \end{aligned}$$

that is,

$$\begin{aligned} &[1-F_1(x)][1-F_2(x)]\ldots[1-F_p(x)] \\ =&[1-G_1(x)][1-G_2(x)]\ldots[1-G_q(x)]. \end{aligned}$$

Differentiating the logarithm of both sides with respect to x. Then we have

$$\frac{f_1(x)}{1-F_1(x)} + \frac{f_2(x)}{1-F_2(x)} + \ldots + \frac{f_p(x)}{1-F_p(x)} = \frac{g_1(x)}{1-G_1(x)} + \frac{g_2(x)}{1-G_2(x)} + \ldots + \frac{g_q(x)}{1-G_q(x)} \tag{3.4}$$

Dividing (3.4) by $f_1(x)$ and making $x \to 0$, on the left hand side of (3.4) we get 1. Hence on the right hand side of (3.4) it must equal 1. Thus,

$$(\alpha_1, \beta_1) = ({\alpha_1}', {\beta_1}'),$$

where ${\alpha_1}' = max\{{\alpha_1}', {\alpha_2}', \ldots, {\alpha_q}'\}$. Otherwise the equation will not hold. Subtracting these identical terms from both sides of (4), we have a new equation of the same form but with fewer terms. Repeating the same procedure we for finite steps, we will get $p = q$ and (α_i, β_i) is a permutation of $({\alpha_i}', {\beta_i}')$, where $i = 1, 2, \ldots, p$.

Note that it follows from lemma 3.1 that as $x \to \infty$,
$\left[\frac{f_i(x)}{1-F_i(x)}\right] = \left[\frac{1-F_j(x)}{f_j(x)}\right]$ has the same order that of

$$\left[\frac{(e^{-\frac{x}{\beta_i}})(x^{\alpha_i-1})}{(\beta_i)(e^{-\frac{x}{\beta_i}})(x^{\alpha_i-1})}\right]\left[\frac{(\beta_j)(e^{-\frac{x}{\beta_j}})(x^{\alpha_j-1})}{(e^{-\frac{x}{\beta_j}})(x^{\alpha_j-1})}\right] = \frac{\beta_j}{\beta_i}$$

Since

$$
\begin{aligned}
1 - F_i(x) =& \frac{1}{\beta_i^{\alpha_i}} \int_x^\infty e^{-\frac{y}{\beta_i}} (y^{\alpha_i - 1}) dy \\
=& \frac{1}{\gamma(\alpha_i)} \int_{\frac{x}{\beta_i}}^\infty (e^{-y})(y^{\alpha_i - 1}) dy \\
\approx& \frac{1}{(\beta_i^{\alpha_i})[\gamma(\alpha_i)]} (e^{-\frac{x}{\beta_i}})(x^{\alpha_i - 1}), \ as\ x \to \infty
\end{aligned}
$$

by lemma 3.1. Now we consider Weibull distributions. The distributions were also considered by Basu and Ghosh in [3]. We present the result for the maximum problem.

Theorem 3.4 Let the pdf of X_i be given by

$$f_i(x) = \frac{p_i}{\theta_i} x^{p_i - 1} e^{-\frac{x^{p_i}}{\theta_i}}, \ where\ x > 0,\ \theta_i > 0,\ p_i > 0$$

Let the pdf of Y_i be given by

$$g_i(x) = \frac{p_i'}{\theta_i'} x^{p_i' - 1} e^{-\frac{x^{p_i'}}{\theta_i'}}, \ where\ x > 0,\ \theta_i' > 0,\ p_i' > 0$$

Let F_i be the cdf of X_i and G_i be the cdf of Y_i. Let $X_1,\ X_2,\ \ldots,\ X_m$ be independent and $Y_1,\ Y_2,\ \ldots,\ Y_n$ be also independent. If the distribution of $max X_1,\ X_2,\ \ldots,\ X_m$ is identical with that of $Y_1,\ Y_2,\ \ldots,\ Y_n$ under the conditions $p_1 < p_2 < \ldots < p_m$ and $p_1' < p_2' < \ldots < p_m'$. Then we have $m = n$, $p_i = p_i'$ and $\theta_i = \theta_i'$, where $i = 1, 2, \ldots, m$.

Proof: Since

$$\lim_{x \to \infty} \frac{f_i(x)}{f_j(x)} = \begin{cases} 0 & if\ p_i < p_j \\ \infty & if\ p_i > p_j \end{cases}$$

Since

$$
\begin{aligned}
& P(max\{X_1,\ X_2,\ \ldots,\ X_m\} \leq x) \\
=& P(max\{Y_1,\ Y_2,\ \ldots,\ Y_n\} \leq x)
\end{aligned}
$$

$$
\begin{aligned}
& P(X_1 \leq x, X_2 \leq x, \ldots, X_m \leq x) \\
=& P(Y_1 \leq x, Y_2 \leq x, \ldots, Y_n \leq x)
\end{aligned}
$$

Since $X_1,\ X_2,\ \ldots,\ X_m$ are independent and $Y_1,\ Y_2,\ \ldots,\ Y_n$ are independent. We have

$$\begin{aligned} &P(X_1 \leq x)P(X_2 \leq x)\ldots P(X_m \leq x)\\ =&P(Y_1 \leq x)P(Y_2 \leq x)\ldots P(Y_n \leq x)\end{aligned}$$

that is,

$$F_1(x)F_2(x)\ldots F_m(x) = G_1(x)G_2(x)\ldots G_n(x)$$

Differentiating the logarithm of both sides with respect to x. Then we have

$$\frac{f_1(x)}{F_1(x)} + \frac{f_2(x)}{F_2(x)} + \ldots + \frac{f_m(x)}{F_m(x)} = \frac{g_1(x)}{G_1(x)} + \frac{g_2(x)}{G_2(x)} + \ldots + \frac{g_n(x)}{G_n(x)} \tag{3.5}$$

Dividing (3.5) by $f_1(x)$ and making $x \to \infty$, on the left hand side of (3.5) we get 1. Hence on the right hand side of (3.5) it must equal 1. Thus,$p_1 = p_1^{'}$, otherwise the equation will not hold. Subtracting these identical terms from both sides of (3.5), we have a new equation of the same form but with fewer terms, which is

$$\frac{f_2(x)}{F_2(x)} + \frac{f_3(x)}{F_3(x)} + \ldots + \frac{f_m(x)}{F_m(x)} = \frac{g_2(x)}{G_2(x)} + \frac{g_3(x)}{G_3(x)} + \ldots + \frac{g_n(x)}{G_n(x)} \tag{3.6}$$

Dividing (3.6) by $f_2(x)$ and making $x \to \infty$, we get $p_2 = p_2^{'}$. Repeating the same procedure we for finite steps, we will get $m = n$ and $p_i = p_i^{'}$, where $i = 1, 2, \ldots, m$. Notice that if

$$f_i(x) = \frac{p_i}{\theta_i}\ x^{p_i-1}\ e^{-\frac{x^{p_i}}{\theta_i}},\ where\ x > 0$$

and

$$g_i(x) = \frac{p_i}{\theta_i^{'}}\ x^{p_i-1}\ e^{-\frac{x^{p_i}}{\theta_i}^{'}},\ where\ x > 0$$

Then if $\theta_i < \theta_i^{'}$, then

$$\lim_{x\to\infty} \frac{f_i(x)}{g_i(x)} = \frac{\theta_i}{\theta_i^{'}} \lim_{x\to\infty} e^{x^{p_i\left(\frac{1}{\theta_i} - \frac{1}{\theta_i^{'}}\right)}} = 0 \tag{3.7}$$

This observation immediately shows us that once the parameters p_i and $p_i^{'}$ are equal, the other two associated parameters θ_i and $\theta_i^{'}$ also must be so, for each i, $1 \leq i \leq n$. Q.E.D.

Theorem 3.5 Let the pdf of X_i be given by

$$f_i(x) = \frac{p_i}{\theta_i}\ x^{p_i-1}\ e^{-\frac{x^{p_i}}{\theta_i}},\ where\ x > 0,\ \theta_i > 0,\ p_i > 0$$

Let the pdf of Y_i be given by

$$g_i(x) = \frac{p_i{'}}{\theta_i{'}}\ x^{p_i{'}-1}\ e^{-\frac{x^{p_i{'}}}{\theta_i{'}}},\ where\ x > 0,\ \theta_i^{'} > 0,\ p_i^{'} > 0$$

Let $F_i(x)$ be the cdf of X_i and $G_i(x)$ be the cdf of Y_i. Let $X_1,\ X_2,\ \ldots,\ X_m$ be independent and $Y_1,\ Y_2,\ \ldots,\ Y_n$ be also independent. Assume that the distribution of $min\{X_1,\ X_2,\ \ldots,\ X_m\}$ is identical with that of $min\{Y_1,\ Y_2,\ \ldots,\ Y_n\}$ under the conditions $p_1 < p_2 < \ldots < p_m$. Then we have $m = n$, $p_i = p_i^{'}$ and $\theta_i = \theta_i^{'}$ where $i = 1, 2, \ldots, m$.

Since

$$\lim_{x\to 0} \frac{f_i(x)}{f_j(x)} = \begin{cases} 0 & if\ p_j < p_i \\ \infty & if\ p_j > p_i \end{cases}$$

$$\begin{aligned} &P(min\{X_1,\ X_2,\ \ldots,\ X_m\} \geq x) \\ =&P(min\{Y_1,\ Y_2,\ \ldots,\ Y_n\} \geq x) \end{aligned}$$

Since $X_1,\ X_2,\ \ldots,\ X_m$ are independent and $Y_1,\ Y_2,\ \ldots,\ Y_n$ are independent. We have

$$P(X_1 \geq x)P(X_2 \geq x)\ldots P(X_m \geq x) = P(Y_1 \geq x)P(Y_2 \geq x)\ldots P(Y_n \geq x) \quad (3.8)$$

that is,

$$[1 - F_1(x)][1 - F_2(x)]\ldots[1 - F_m(x)] = [1 - G_1(x)][1 - G_2(x)]\ldots[1 - G_n(x)] \quad (3.9)$$

Differentiating the logarithm of both sides with respect to x. Then we have

$$\frac{f_2(x)}{1 - F_2(x)} + \frac{f_3(x)}{1 - F_3(x)} + \ldots + \frac{f_m(x)}{1 - F_m(x)} = \frac{g_2(x)}{1 - G_2(x)} + \frac{g_3(x)}{1 - G_3(x)} + \ldots + \frac{g_n(x)}{1 - G_n(x)} \quad (3.10)$$

Dividing (3.10) by f_1 and making $x \to 0$, on the left hand side of (3.10) we get 1. Hence on the right hand side of (3.10) it must equal 1. Thus, $p_1 = p_1^{'}$, otherwise the equation will not hold. Subtracting these identical terms from both sides of (3.10), we have a new equation of the same form but with fewer terms, which is

$$\frac{f_2(x)}{1-F_2(x)}+\frac{f_3(x)}{1-F_3(x)}+\ldots+\frac{f_m(x)}{1-F_m(x)}=\frac{g_2(x)}{1-G_2(x)}+\frac{g_3(x)}{1-G_3(x)}+\ldots+\frac{g_n(x)}{1-G_n(x)} \tag{3.11}$$

Dividing (3.11) by f_2 and making $x \to 0$, we get $p_2 = p_2^{'}$. Repeating the same procedure we for finite steps, we will get $m = n$ and $p_i = p_i^{'}$, where $i = 1, 2, \ldots, m$. Now we will use lemma 3.1 . By the lemma,

$$\begin{aligned} 1 - F_i(x) &= \int_x^\infty \frac{p_i}{\theta_i} y^{p_i - 1} e^{\frac{y^{p_i}}{\theta_i}} dy \\ &= \int_{\frac{x^{p_i}}{\theta_i}}^\infty e^{-y} dy \\ &\approx e^{-\frac{x^{p_i}}{\theta_i}} \end{aligned}$$

It follows from (3.11) above that

$$e^{-(\frac{x^{p_1}}{\theta_1}+\frac{x^{p_2}}{\theta_2}+\ldots+\frac{x^{p_n}}{\theta_n})} = e^{-(\frac{x^{p_1}}{\theta_1^{'}}+\frac{x^{p_2}}{\theta_2^{'}}+\ldots+\frac{x^{p_n}}{\theta_n^{'}})}$$

It is not difficult to see from this that we must now have $\theta_i = \theta_i^{'}$ Q.E.D.

The "maximum" problem was considered earlier for Cauchy distributions in [MNM]. I present here the minimum problem.

Theorem 3.6 Let the pdf of X_i be given by

$$f_i(x) = \frac{1}{\pi} \frac{a_i}{1 + a_i^2 x^2}, \ \textit{where } i = 1, 2, \ldots, p.$$

Let the pdf of Y_i be given by

$$g_i(x) = \frac{1}{\pi} \frac{b_i}{1 + b_i^2 x^2}, \ \textit{where } i = 1, 2, \ldots, q.$$

Let $F_i(x)$ be the cdf of X_i and $G_i(x)$ be the cdf of Y_i.Let $X_1, \ X_2, \ \ldots, \ X_p$ be independent and $X_{p+1}, \ X_{p+2}, \ \ldots, \ X_{p+q}$ be also independent. Let

$$f(x) = \frac{1}{\pi} \frac{1}{1 + x^2}$$

and $F_i(X)$ be the cdf of x, where

$$F(x) = \frac{1}{2} + \frac{1}{\pi} tan^{-1}(x).$$

If the distribution of $min\{X_1,\ X_2,\ \ldots,\ X_p\}$ is same as that of $min\{X_{p+1},\ X_{p+2},\ \ldots,\ X_{p+q}\}$. Then we have $m = n$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ is a permutation of $\{\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{p+q}\}$, where $i = 1, 2, \ldots, p$.

Proof: Since $\frac{f(x)}{1-F(X)}$ is infinitely differentiable, then we have

$$\begin{aligned} h(x) &= \frac{f(x)}{1 - F(x)} \\ &= \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

where

$$c_n = \frac{h^{(n)}(0)}{n!}.$$

Since

$$\begin{aligned} h_i(x) &= \frac{f_i(x)}{1 - F_i(x)} \\ &= \frac{a_i f(a_i x)}{1 - F_i(a_i x)} \\ &= a_i \sum_{n=0}^{\infty} c_n (a_i x)^n \\ &= \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} a_i^{n+1} x^n, \ where\ i = 1, 2, \ldots, p+q \end{aligned}$$

If

$$\sum_{i=1}^{p} h_i(x) = \sum_{i=p+1}^{p+q} h_j(x) \tag{3.12}$$

which is,

$$\begin{aligned} &\sum_{i=1}^{p} \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} a_i^{n+1} x^n \\ &= \sum_{i=p+1}^{p+q} \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} a_i^{n+1} x^n. \end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty}\left[\sum_{i=1}^{p} a_i^{n+1}\right]\frac{h^{(n)}(0)}{n!}\ x^n$$
$$=\sum_{n=0}^{\infty}\left[\sum_{i=p+1}^{p+q} a_i^{n+1}\right]\frac{h^{(n)}(0)}{n!}\ x^n.$$

Then we have

$$\sum_{i=1}^{p} a_i^{n+1}=\sum_{i=p+1}^{p+q} a_i^{n+1}$$

for infinite many n. Since

$$\left[\sum_{i=1}^{p}(a_i^{n+1})\right]^{\frac{1}{n+1}} \to max\{a_i : 1\leq i\leq p\},\ as\ n+1\to\infty$$

Therefore,

$$max\{a_i : 1\leq i\leq p\}=max\{a_i : p+1\leq i\leq p+q\}$$

Canceling these terms from both sides of (3.12) and repeating same procedure, after a finite number of steps we obtain $p=q$ and $\{a_1,a_2,\ldots,a_p\}$ is a permutation of $\{a_{p+1},a_{p+2},\ldots,a_{p+q}\}$ where $i=1,2,\ldots,p$.

In what follows, we will consider Poisson distributions.

Theorem 3.7 Let the probability mass function (pmf) of X_i be given by

$$f_i(k)=\frac{e^{-\lambda_i}\ {\lambda_i}^k}{k!},$$
$$where\ k=0,1,2,\ldots\ i=1,2,\ldots,m\ and\ each\ \lambda_i>0.$$

Let the pmf of Y_i be given by

$$g_j(k)=f_{j+m}(k),\ where\ k=0,1,2,\ldots\ and\ j=1,2,\ldots,n$$

Let $F_i(x)$ be the cdf of X_i and $G_i(x)$ be the cdf of Y_i.Let $X_1,X_2,\ldots,X_m$ be independent and $X_{m+1},X_{m+2},\ldots,X_{m+n}$ be also independent. If the distribution of $max\{X_1,X_2,\ldots,X_m\}$ is the same as that of $max\{X_{m+1},X_{m+2},\ldots,X_{m+n}\}$, then we have $m=n$ and $\{\lambda_1,\lambda_2,\ldots,\lambda_m\}$ is a permutation of $\{\lambda_{m+1},\lambda_{m+2},\ldots,\lambda_{m+n}\}$ where $i=1,2,\ldots,m$.

We believe that this theorem is true for every $n \geq 1$. However, we can only prove it for $n \leq 3$. Below we give the proofs for $n = 2$ and $n = 3$.

Proof: First when

$$P(max\{X_1, X_2\} \leq)0$$
$$=P(max\{X_3, X_4\}) \leq 0)$$

then

$$P(X_1 = 0, X_2 = 0)$$
$$=P(X_3 = 0, X_4 = 0)$$

Since X_1, X_2 are independent and X_3, X_4 are independent. We have

$$P(X_1 = 0)(X_2 = 0)$$
$$=P(X_3 = 0)(X_4 = 0)$$

that is,

$$e^{-\lambda_1}e^{-\lambda_2} = e^{-\lambda_3}\ e^{-\lambda_4}$$

Which is,

$$e^{-(\lambda_1+\lambda_2)} = e^{-(\lambda_3+\lambda_4)}$$

We get

$$\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4 \tag{3.13}$$

When

$$P(max\{X_1, X_2\} \leq 1)$$
$$=P(max\{X_3, X_4\} \leq 1)$$

Which is

$$[P(X_1 = 0) + P(X_1 = 0)][P(X_2 = 0) + P(X_2 = 1)]$$
$$=[P(X_3 = 0) + P(X_3 = 1)][P(X_4 = 0) + P(X_4 = 1)]$$

Then we have,

$$(1+\lambda_1)(1+\lambda_2) = (1+\lambda_3)(1+\lambda_4) \tag{3.14}$$

We can obtain,

$$\lambda_1\lambda_2 = \lambda_3\lambda_4 \tag{3.15}$$

By (3.13) and (3.15), we have $\{\lambda_1, \lambda_2\}$ is a permutation of $\{\lambda_3, \lambda_4\}$. When

$$P(max\{X_1, X_2, X_3\} \leq 0)$$
$$=P(max\{X_4, X_5, X_6\} \leq 0)$$

then

$$P(X_1 = 0, X_2 = 0, X_3 = 0)$$
$$=P(X_4 = 0, X_5 = 0, X_6 = 0)$$

Since X_1, X_2, X_3 are independent and X_4, X_5, X_6 are independent. We have

$$P(X_1 = 0)P(X_2 = 0)P(X_3 = 0)$$
$$=P(X_4 = 0)P(X_5 = 0)P(X_6 = 0)$$

that is,

$$e^{-\lambda_1}\ e^{-\lambda_2}\ e^{-\lambda_3}$$
$$=e^{-\lambda_4}\ e^{-\lambda_5}\ e^{-\lambda_6}$$

Which is,

$$e^{-(\lambda_1+\lambda_2+\lambda_3)} = e^{-(\lambda_4+\lambda_5+\lambda_6)}$$

We get

$$\lambda_1 + \lambda_2 + \lambda_3 = \lambda_4 + \lambda_5 + \lambda_6 \tag{3.16}$$

When

$$P(maxX_1, X_2, X_3) \leq 1$$
$$=P(maxX_4, X_5, X_6 \leq 1)$$

then

$$P(X_1 \leq 1, X_2 \leq 1, X_3 \leq 1)$$
$$= P(X_4 \leq 1, X_5 \leq 1, X_6 \leq 1)$$

Which is

$$\left[e^{-\lambda_1}\ (1+\lambda_1)\right]\left[e^{-\lambda_2}\ (1+\lambda_2)\right]\left[e^{-\lambda_3}\ (1+\lambda_3)\right]$$
$$= \left[e^{-\lambda_4}\ (1+\lambda_4)\right]\left[e^{-\lambda_5}\ (1+\lambda_5)\right]\left[e^{-\lambda_6}\ (1+\lambda_6)\right]$$

Then we have,

$$(1+\lambda_1)(1+\lambda_2)(1+\lambda_3) = (1+\lambda_4)(1+\lambda_5)(1+\lambda_6) \tag{3.17}$$

When

$$P(max\{X_1, X_2, X_3\} \leq 2) = P(max\{X_4, X_5, X_6\} \leq 2) \tag{3.18}$$

Assume

$$1+\lambda_i = \lambda_i^*,\ where\ i = 1, 2, \ldots, 6$$

By (3.16), (3.17) and (3.18), we can have

$$\sum_{i=1}^{3} \lambda_i^* = \sum_{i=4}^{6} \lambda_i^* \tag{3.19}$$

$$\prod_{i=1}^{3} \lambda_i^* = \prod_{i=4}^{6} \lambda_i^* \tag{3.20}$$

$$\prod_{i=1}^{3} 1+\lambda_i^{*2} = \prod_{i=4}^{6} 1+\lambda_i^{*2} \tag{3.21}$$

Since

$$\begin{aligned}
&\prod_{i=1}^{3} 1+\lambda_i^{*2} \\
=&1+\sum_{i=1}^{3}\lambda_i^{*2}+\sum_{i<j}\lambda_i^{*2}\lambda_j^{*2}+\prod_{i=1}^{3}\lambda_i{}^{*2} \\
=&1+[\left(\sum_{i=1}^{3}\lambda_i{}^{*}\right)^2-2\sum_{i<j}\lambda_i{}^{*}\lambda_j{}^{*}] \\
+&[\left(\sum_{i<j}\lambda_i^{*}\lambda_j^{*}\right)^2-2\lambda_1^{*}\lambda_2^{*}\lambda_3^{*}\sum_{i=1}^{3}\lambda_i{}^{*}]+(\lambda_1{}^{*}\lambda_2{}^{*}\lambda_3{}^{*})^2 \\
=&(1-\sum_{i<j}\lambda_i{}^{*}\lambda_j{}^{*})^2
\end{aligned}$$

By (3.21), we can have

$$\begin{aligned}
&(1-\sum_{i<j;\ i=1,2,3;\ j=1,2,3}\lambda_i{}^{*}\lambda_j{}^{*})^2 \\
=&(1-\sum_{i<j;\ i=4,5,6;\ j=4,5,6}\lambda_i{}^{*}\lambda_j{}^{*})^2
\end{aligned}$$

Or

$$\sum_{i<j;\ i=1,2,3;\ j=1,2,3}\lambda_i{}^{*}\lambda_j=\sum_{i<j;\ i=4,5,6;\ j=4,5,6}\lambda_i{}^{*}\lambda_j{}^{*} \tag{3.22}$$

If this does not happen, then

$$\sum_{i<j;\ i=1,2,3;\ j=1,2,3}\lambda_i{}^{*}\lambda_j+\sum_{i<j;\ i=4,5,6;\ j=4,5,6}\lambda_i{}^{*}\lambda_j{}^{*}=2$$

which is not possible, since each $\lambda_i{}^{*}>1$, where $i=1,2,\ldots,6$. Because of (3.19), (3.20) and (3.22), it follows that $\{\lambda_1^{*},\lambda_2^{*},\lambda_3^{*}\}$ is a permutation of $\{\lambda_4^{*},\lambda_5^{*},\lambda_6^{*}\}$. The corresponding minimum problem for Poisson distributions are in the following.

In the simplest case, when $n=2$, l et X_1 and X_2 be independent Poisson with parameters λ_1 and λ_2 where $\lambda_1>0$ and $\lambda_2>0$, and let X_3 and X_4 be also independent. Suppose that for each integer $k\geq 0$,

$$\begin{aligned}
&p(min\{X_1,X_2\}\geq k) \\
=&P(min\{X_3,X_4\})\geq k.
\end{aligned}$$

Then taking $k = 0$ and 1, we have

$$\begin{aligned}&(1 - e^{-\lambda_1})(1 - e^{-\lambda_2})\\=&(1 - e^{-\lambda_3})(1 - e^{-\lambda_4})\end{aligned}$$

$$\begin{aligned}&[1 - e^{-\lambda_1}(1 + \lambda_1)][1 - e^{-\lambda_2}(1 + \lambda_2)]\\=&[1 - e^{-\lambda_3}(1 + \lambda_3)][1 - e^{-\lambda_4}(1 + \lambda_4)]\end{aligned}$$

and many other equations corresponding to values of $k \geq 2$. But even though we feel confident that is true. However, we are unable to solve these equations.

REFERENCES

[1] T.W. Anderson and S.G. Ghurye, Identification of parameters by the distribution of maximum random variable, J. of Royal Stat. Society 39B (1977), 337–342.

[2] T.W. Adnerson and S.G. Ghurye, Unique factorization of products of bivariate normal cumulative distribution functions, Annals of the Institute of Statistical Math.30 (1978), 63–69.

[3] A.P. Basu, Identifiability problems in the theory of competing and complementary risks: a survey in "Statistical Distributions in Scientific Work", vol.5, (C.Taillie, et. al, eds.), D.Reidel Publishing Co.,Dordrecht, Holland, 1981, pp. 335-347.

[4] A.P. Basu and J.K. Ghosh, Identifiability of the multinormal and other distributions under competing risks model, J. Multivariate Analysis 8 (1978), 413-429.

[5] A.P. Basu and J.K. Ghosh, Identifiablility of distributions under competing risks and complementary risks model, Commun. Statistics A9 (14) (1980), no.14, 1515-1525.

[6] J.Davis and A.Mukherjea, Identification of parameters by the distribution of the minimum, J. Multivariate Analysis 9, 1141-1159, 2007.

[7] M.Dai and A.Mukherjea, Identification of the parameters of a multivariate normal vector by the distribution of the maximum, J. Theoret, Probability 14, 267-298, 2001.

[8] A.Mukherjea, A.Nakassis, and J.Miyashita, Identification of parameters by the distribution of the maximum random variable: The Anderson-Ghuiye theorem, J. Multivariate Analysis 18, 178-186, 1986.

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