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Multi-Soliton solutions to a model equation for shallow water waves

Zhijiang Qiao
University of Texas-Pan American

MULTI-SOLITON SOLUTIONS TO A MODEL EQUATION FOR SHALLOW WATER WAVES

A Thesis

by

ZHIJIANG QIAO

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FOR SHALLOW WATER WAVES

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ZHIJIANG QIAO

COMMITTEE MEMBERS

Dr. Baofeng Feng
Chair of Committee

Dr. Paul Bracken
Committee Member

Dr. Kenichi Maruno
Committee Member

Dr. Andras Balogh
Committee Member

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ABSTRACT

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In Soliton theory, Hirota direct method is most efficient tool for seeking one soliton solutions or multi-soliton solutions of integrable nonlinear partial differential equations. The key step of the Hirota direct method is to transform the given equation into its Hirota bilinear form. Once the bilinear form of the given equation is found, we can construct the soliton and multi-soliton solutions of that model. Many interesting characteristics of Pfaffians were discovered through studies of soliton equations. In this thesis, a shallow water wave model and its bilinear equation are investigated. Using Hirota direct method, we obtain the multi-soliton solutions and Pfaffian solutions for a shallow water wave model.

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Finally, I dedicate this thesis to my family, for their continued love and support throughout my graduate school career.

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CHAPTER I

INTRODUCTION

A soliton is a solitary wave which preserves its well-defined shape after it collides with another wave of the same kind. In the last fifty years there have been important developments in the soliton theory. Solitons have been studied by mathematicians, physicists and engineers for their applicability in physical applications. The initial observation of a solitary wave in shallow water was made by John Scott Russell. Russell was a Scottish engineer and naval architect who was conducting experiments for the Union Canal Company to design a more efficient canal boat. In Russell's (1844) own words: "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped–not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of

Translation.” Russell built a water tank to replicate the phenomenon and research the properties of the solitary wave he had observed. In 1895, the Dutch professor Diederik Korteweg and his doctoral student Gustav de Vries (1895) derived a partial differential equation (PDE) which models the solitary wave that Russell had observed. Parenthetically, the equation which now bears their name had already appeared in seminal work on water waves published by Boussinesq (1872, 1877) and Rayleigh (1976). The solitary wave was considered a relatively unimportant curiosity in the field of nonlinear waves. That all changed in 1965, when Zabusky and Kruskal realized that the KdV equation arises as the continuum limit of a one dimensional anharmonic lattice used by Fermi, Pasta, and Ulam (1955) to investigate “thermalization” – or how energy is distributed among the many possible oscillations in the lattice. Zabusky and Kruskal (1965) simulated the collision of solitary waves in a nonlinear crystal lattice and observed that they retain their shapes and speed after collision. Interacting solitary waves merely experience a phase shift, advancing the faster and retarding the slower. In analogy with colliding particles, they coined the word “solitons” to describe these elastically colliding waves. A narrative of the discovery of solitons can be found in Zabusky (2005).

Since the 1970s, the KdV equation and other equations that admit solitary wave and soliton solutions have been the subject of intense study. Indeed, scientists remain intrigued by the physical properties and elegant mathematical theory of the shallow water wave models. It is well known that the multi-soliton solutions of certain nonlinear evolution equations can be obtained by the three different methods, the inverse scattering method (ISM), the Bäcklund transformation (BT) and the dependent variable transformation (DVT) developed by Hirota. The three methods are closely related to each other.

Shallow water wave models are applicable where the water depth is much less than the horizontal scale of motion. Shallow water models are also called long wave models as the wavelength is long compared with the depth. They are used to represent the flow of water waves in coastal seas and estuaries. They can also be used to model lake flows. Such models are used to predict the water velocity and water height at various points within a region of flow at different times. Shallow water wave models are important because they describe real life situations that are critical to large numbers of people.

In this thesis, a shallow water wave model and its bilinear equation will be investigated; Hirota's direct method and Pfaffian technique will be applied to solve the nonlinear partial differential equations; finally, we obtain the multi-soliton solutions for the shallow water wave model. The outline of this thesis is as follows. In Chapter I, we just give a quick introduction on the history of solitary wave in shallow water. In Chapter II, a shallow water wave model and its bilinear equation will be investigated in detail. Horita Direct Method is introduced and applied to obtain the bilinear form for the model. In Chapter III, we focus on the BKP equation and its pfaffian solutions. Pfaffian technique is discussed and applied to solve the BKP equation. In Chapter IV, based on the previous study we get multi-soliton solution for the shallow water wave model. We present the one and two- soliton solutions of the model. The conclusion of this thesis and future work are given in Chapter V. Some formulas for D-operators and Pfaffians are given in appendix.

CHAPTER II

A SHALLOW WATER WAVE MODEL AND ITS BILINEAR EQUATION

1. Hirota Direct Method

In 1971, Hirota, who is a Japanese mathematician, developed an ingenious method for obtaining the exact multi-soliton solutions of the KdV equation and derived an explicit expression for its N-soliton solution. An elegant formulation of this method requires the use of bilinear operators; therefore it is called Hirota's bilinear method [1]. Over the last several decades this method has been shown to be applicable to a large class of nonlinear evolution equations, including difference-differential and integral-differential equations. Hirota's bilinear method, which is usually applied to completely integrable equations, is well suited for partially integrable equations as well. Hirota's method has been one of the most successful direct techniques for constructing exact solutions to various nonlinear PDEs from mathematical physics and soliton theory. This technique applies to any equation that can be written in bilinear form, either as a single bilinear equation or as a system of coupled bilinear equations. Once the bilinear form is obtained the method becomes algorithmic.

A. The Bilinear Operator

Hirota introduced the differential operator D_x, defined on ordered pairs of functions $f(x)$ and $g(x)$, as follows

$$D_x(f\cdot g)=(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'})f(x)g(x')\Big|_{x'=x} \tag{2.1}$$

More general, Hirota defined

$$D_x^m D_t^n(f\cdot g)=(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'})^m(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'})^n f(x,t)g(x',t')\Big|_{x'=x,t'=t} \tag{2.2}$$

for non-negative integers m and n. This type of differential operator is called a bilinear operator, due to the obvious linearity in both its arguments. The following properties for the bilinear operators in (2.1) and (2.2) are easily verified:

$$D_x^m(f\cdot 1)=\frac{\partial^m f}{\partial x^m} \tag{2.3}$$

$$D_x^m(f\cdot g)=(-1)^m D_x^m(g\cdot f) \tag{2.4}$$

$$D_x^m(f\cdot f)=0\text{, for odd m} \tag{2.5}$$

B. The Hirota Bilinear Forms

We take the KdV equation as an example to explain how the Hirota's direct method works and how the KdV equation is transferred to the bilinear form by using D-operator.

The KdV equation is given by

$$u_t+6uu_x+u_{xxx}=0 \tag{2.6}$$

The solution of the above KdV equation is

$$u=\frac{p^2}{2}\sec h^2\frac{\eta}{2}=\frac{2p^2\exp\eta}{(1+\exp\eta)^2} \tag{2.7}$$

Where $\eta=Px-\Omega t$, $\Omega=P^3$, P is arbitrary parameter. We use dependent variable transformation by letting $u=G/F$, to derive differential equations for F and G and then to find F and G as solutions to this differential equation.

It follows

$$u_t = \frac{G_t F - G F_t}{F^2}$$

$$u_x = \frac{G_x F - G F_x}{F^2}$$

$$u_{xxx} = \frac{G_{xxx}}{F} - \frac{3G_{xx}F_x + 3G_x F_{xx} + GF_{xxx}}{F^2} + 6\frac{G_x F_x^2 + GF_{xx}F_x}{F^3} - \frac{GF_x^3}{F^4}$$

Substitutions of above relations into the KdV equation (2.6), and reorganize the terms yield the surprisingly complicated equation

$$\frac{G_t F - GF_t + G_{xxx}F - 3G_{xx}F_x + 3G_x F_{xx} - GF_{xxx}}{F^2} + 6(G_x F - GF_x)\frac{GF - (FF_{xx} - F_x^2)}{F^4} = 0$$

and we choose to adopt, as decoupled equations,

$$G_t F - GF_t + G_{xxx}F - 3G_{xx}F_x + 3G_x F_{xx} - GF_{xxx} = 0 \tag{2.8}$$

$$FF_{xx} - F_x^2 - GF = 0 \tag{2.9}$$

By using D-operator and its notation, we have

$$D_t G \cdot F = G_t F - GF_t$$

$$D_x G \cdot F = G_x F - GF_x$$

$$D_x^3 G \cdot F = G_{xxx}F - 3G_{xx}F_x + 3G_x F_{xx} - GF_{xxx}$$

$$D_x^2 F \cdot F = 2(F_{xx}F - F_x^2)$$

So, the decoupled KdV equations (2.8) and (2.9) can be concisely rewritten as

$$(D_t + D_x^3)G \cdot F = 0 \tag{2.10}$$

$$D_x^2 F \cdot F - 2GF = 0 \tag{2.11}$$

Alternatively, the KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0$$

can be transformed, through the dependent variable transformation,

$$u = 2(\log f)_{xx} \tag{2.12}$$

into

$$\frac{\partial}{\partial x}\left[\frac{f_{xt}f - f_x f_t + f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2}{f^2}\right] = 0 \tag{2.13}$$

from which we obtain the bilinear equation

$$f_{xt}f - f_x f_t + f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2 = cf^2 \tag{2.14}$$

where c is a constant of integration [2]. Equation (2.14), with c = 0, may also written concisely in terms of D-operators as

$$(D_x D_t + D_x^4) f \cdot f = 0 \tag{2.15}$$

The constant c in (2.14) can be chosen to be zero when seeking a solitary wave (or soliton) solution. Equations (2.10), (2.11) and (2.15) are called Hirota bilinear forms. We have employed two dependent variable transformations for u, which are

$$u = \frac{G}{F}$$

$$u = 2(\log f)_{xx}$$

From this, we might assume that the following relations hold.

$$F = f^2 \tag{2.16}$$

$$G = 2(f_{xx}f - f_x^2) \tag{2.17}$$

2. A shallow water wave model

The shallow water wave equation, which is derived using the Boussinesq approximation, includes two special cases; one equation is discussed by Ablowitz, et al [3] and the other is established by Hirota and Satsuma [4]. The shallow water wave equation introduced by Hirota and Satsuma is

$$u_t - u_{xxt} - 3uu_t + 3u_x \int_x^{\infty} u_t dx \ + u_x = 0 \tag{2.18}$$

which reduces to the KdV equation.

Now, we use dependent variable transformation and D-operator to obtain the bilinear form of the above shallow water wave equation, as follows,

First let $u = w_x$, we get

$$w_{xt} - w_{xxxt} - 3w_x w_{xt} - 3w_{xx} w_t + w_{xx} = 0 \tag{2.19}$$

or,

$$w_{xt} - w_{xxxt} - 3(w_x w_t)_x + w_{xx} = 0 \tag{2.20}$$

Integrating once on both sides of (2.20) gives us

$$w_t - w_{xxt} - 3w_x w_t + w_x = 0 \tag{2.21}$$

Introducing dependent variable transformation, $u = 2(\log f)_{xx}$, or $w = 2(\log f)_x$ we have

$$\frac{(D_x^2 f \cdot f)}{f^2} = u \tag{2.22}$$

$$\frac{(D_x^4 f \cdot f)}{f^2} = u_{xx} + 3u^2 \tag{2.23}$$

$$\frac{(D_x^6 f \cdot f)}{f^2} = u_{xxxx} + 15uu_{xx} + 15u^3 \tag{2.24}$$

which is from (A.29)—(A.32).

Since $u = w_x$, we have following identities,

$$w_x = \frac{D_x^2 f \cdot f}{f^2} \tag{2.25}$$

$$w_t = \frac{D_x D_t f \cdot f}{f^2} \tag{2.26}$$

$$w_x w_t = \frac{D_x^2 f \cdot f}{f^2} \cdot \frac{D_x D_t f \cdot f}{f^2} \tag{2.27}$$

look back to the equation (2.21), we still need to find w_{xxt}, and we already let $w = 2(\log f)_x$, so,

$$w_{xxt} = 2(\log f)_{xxxt} \tag{2.28}$$

By definition of D-operator D_z and differential operator ∂_z, which are $D_z = D_t + \varepsilon D_x$,

$\partial_z = \partial_t + \varepsilon \partial_x$, and identity $\frac{D_x^4 f \cdot f}{f^2} = u_{xx} + 3u^2$, we have following relations

$$u_{xx} = \frac{D_x^4 f \cdot f}{f^2} - 3\left(\frac{D_x^2 f \cdot f}{f^2}\right)^2 \tag{2.29}$$

since $u = 2(\log f)_{xx}$, it gives

$$2\frac{\partial^4}{\partial z^4} \log f(z) = \frac{D_z^4 f \cdot f}{f^2} - 3\left(\frac{D_z^2 f \cdot f}{f^2}\right)^2 \tag{2.30}$$

$$2\left(\frac{\partial}{\partial z^4} + \varepsilon \frac{\partial}{\partial t}\right)^4 \log f = \frac{\left(D_x + \varepsilon D_t\right)^4 f \cdot f}{f^2} - 3\left(\frac{\left(D_x + \varepsilon D_t\right)^2 f \cdot f}{f^2}\right)^2 \tag{2.31}$$

We only consider terms containing ε with $D_x^3 D_t f \cdot f$, because we just need to find w_{xxt},

which is $2(\log f)_{xxxt}$. Expanding on both sides, disregarding all other terms, we have

$$2 \cdot 4 \cdot \varepsilon (\log f)_{xxxt} = 4 \cdot \varepsilon \cdot \frac{D_x^3 D_t f \cdot f}{f^2} - 3 \cdot 4 \cdot \varepsilon \frac{(D_x^2 f \cdot f)(D_x D_t f \cdot f)}{f^4} \tag{2.32}$$

So,

$$2(\log f)_{xxxt} = \frac{D_x^3 D_t f \cdot f}{f^2} - 3\frac{(D_x^2 f \cdot f)(D_x D_t f \cdot f)}{f^4} = w_{xxt} \tag{2.33}$$

Now, substitute into equation (2.21), gives

$$\frac{D_x^2 f\cdot f}{f^2}+\frac{D_xD_t f\cdot f}{f^2}-3\frac{D_x^2 f\cdot f}{f^2}\cdot\frac{D_xD_t f\cdot f}{f^2}=\frac{D_x^3D_t f\cdot f}{f^2}-3\frac{(D_x^2 f\cdot f)(D_xD_t f\cdot f)}{f^4} \tag{2.34}$$

Obviously, it is true that

$$\frac{D_x^2 f\cdot f}{f^2}+\frac{D_xD_t f\cdot f}{f^2}-\frac{D_x^3D_t f\cdot f}{f^2}=0 \tag{2.35}$$

So, we get the following bilinear form for the shallow water wave model

$$D_x^2 f\cdot f+D_xD_t f\cdot f-D_x^3D_t f\cdot f=0 \tag{2.36}$$

which is equivalent to

$$D_x(D_t-D_tD_x^2+D_x)f\cdot f=0 \tag{2.37}$$

CHAPTER III

THE BKP EQUATION AND ITS PFAFFIAN SOLUTIONS

The BKP hierarchy was introduced by Date, Jimbo, Kashiwara and Miwa [5], which is a particular reduction of the KP hierarchy of integrable equations [6, 7]. Like the well-known KP hierarchy, the BKP hierarchy possesses multi-soliton solutions. It is well known that both polynomial and soliton-type solutions to equations associated with the bilinear KP hierarchy [8] can be conveniently expressed in terms of Wronskian determinants [9, 10, 11]. (Freeman and Nimmo 1983, Ohta et a1 1988, Nimmo 1989). For the BKP hierarchy, its solutions are expressed as Pfaffians rather than determinants. In the early 1990s Ohta and Hirota [12] introduced a procedure for generalizing equations from the KP hierarchy to produce coupled systems of equations with solutions in the form of Pfaffians [13]. Like the KP equations that produced them, these ‘Pfaffianized’ coupled equations are integrable and have soliton solutions. The majority of integrable nonlinear equations can be written in bilinear form and the solutions of these bilinear equations can usually be expressed as determinants. Pfaffians have a richer structure than determinants and hence the process of Pfaffianization can provide new systems of integrable equations with a broader class of solutions than those having determinantal solutions. The shallow water wave model belongs to the class of BKP-type equations, and the most suitable expressions for its soliton solutions are pfaffians [14].

1. The Pfaffian Technique

1.1 Definition of pfaffians

Let $A = \det\left|a_{jk}\right|_{1\le j,k\le n}$ be an antisymmetric determinant, in which $a_{j,k} = -a_{k,j}$ for $j,k = 1,2,\cdots,n.$ It is known that A of odd order vanishes but A of even order $2m$ is the square of a "pfaffians". We present this pfaffian P by

$$P = (1,2,\cdots,2m) \tag{3.1}$$

For example, we have $m=2$

$$\det\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{vmatrix} = \left[a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}\right]^2 \tag{3.2}$$

and

$$(1,2,3,4) = (1,2)(3,4) - (1,3)(2,4) + (1,4)(2,3) \tag{3.3}$$

Where $(j,k) = -(k,j)$ denotes the elements $a_{j,k}$ with $j<k$.

The expansion rule for pfaffians is given by

$$(1,2,\cdots,2m) = \sum_{j=2}^{2m}(-1)^j (1,j)(2,3,\cdots,\widehat{j},\cdots,2m), \tag{3.4}$$

$$(1,2,\cdots,2m) = \Sigma'(-1)^p (i_1,i_2)(i_3,i_4)\cdots(i_{2m-1},i_{2m}), \tag{3.5}$$

Where $\hat{}$ indicates deletion of the letter under it and Σ' denotes summation over all permutations $i_1,i_2,\cdots,i_{2m}$ of $(1,2,\cdots,2m)$ which satisfy the inequalities

$$i_1<i_2, i_3<i_4,\cdots,i_{2m-1}<i_{2m},$$
$$i_1<i_3<\cdots<i_{2m-1},$$

Where $(-1)^p$ is $+1$ or -1 according as the sequence $(i_1, i_2, \cdots, i_{2m})$ is an even or odd permutation of $(1, 2, \cdots, 2m)$. Conversely, the pfaffians can be defined by this expansion rule. Note that the pfaffian $(i_1, i_2, \cdots, i_{2n})$ vanishes if $i_l = i_m$ foe any pair of m and l chosen from $1, 2, \cdots, 2n$.

We have defined the pfaffians through the determinant of the antisymmetric matrix A. On the other hand, the determinant of an $n \times n$ matrix B is expressed by the pfaffian. Let $(j, k^*) = b_{j,k}, (j,k) = 0, (j^*, k^*) = 0,$ for $j, k = 1, 2, \cdots, n.$ then we have

$$\det \left| b_{j,k} \right|_{1 \le j,k \le n} = (1, 2, \cdots, n, n^*, \cdots, 2^*, 1^*) \tag{3.6}$$

For example, taking $n = 2$, we obtain,

$$\begin{aligned} \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} &= (1, 2, 2^*, 1^*) = -(1, 2^*)(2, 1^*) + (1, 1^*)(2, 2^*) \\ &= -b_{12} b_{21} + b_{11} b_{22} \end{aligned} \tag{3.7}$$

We introduce pfaffians which represent derivatives of functions $f_j(x)$ for $j = 1, 2, \cdots,$ let

$$(d_n, j) = \frac{\partial^n}{\partial x^n} f_j(x) \quad \text{for} \quad n = 0, 1, \cdots, \infty \tag{3.8}$$

$$(d_n, d_m) = 0 \quad \text{for} \quad n, m = 0, 1, \cdots, \infty \tag{3.9}$$

Then a Wronskian of order n defined by

$$Wr(f_1, f_2, \cdots, f_n) = \det \left| \frac{\partial^{k-1}}{\partial x^{k-1}} f_j \right|_{1 \le j,k \le n} \tag{3.10}$$

is expressed by the pfaffian:

$$Wr(f_1, f_2, \cdots, f_n) = (d_0, d_1, \cdots, d_{n-1}, n, \cdots, 2, 1) \tag{3.11}$$

For example, taking $n = 2$, we have

$$\begin{vmatrix} f_1 & \frac{\partial f_1}{\partial x} \\ f_2 & \frac{\partial f_2}{\partial x} \end{vmatrix} = (d_0, d_1, 2, 1) = -(d_0, 2)(d_1, 1) + (d_0, 1)(d_1, 2)$$

$$= -f_2 \frac{\partial f_1}{\partial x} + f_1 \frac{\partial f_2}{\partial x} \tag{3.12}$$

1.2 Expansion Theorems for Pfaffians

$$\left(a_1, a_2, 1, 2, \cdots, 2n\right) = \sum_{j=2}^{2n} (-1)^j \begin{bmatrix} \left(a_1, a_2, 1, j\right)\left(2, 3, \cdots, \hat{j}, \cdots, 2n\right) \\ +\left(1, j\right)\left(a_1, a_2, 2, 3, \cdots, \hat{j}, \cdots, 2n\right) \end{bmatrix} - \left(a_1, a_2\right)\left(1, 2, \cdots, 2n\right) \tag{3.13}$$

Let $A\left(j,k,\cdots,r\right)$ be the antisymmetric determinant obtained from A by striking out the $j-th, k-th, \cdots r-th$ rows and columns with $j < k < \cdots < r$, and let $M\left(j,k,\cdots,r\right)$ be the corresponding pfaffian. $M\left(j,k,\cdots,r\right)$ is called the minor of the $\left(j,k,\cdots,r\right)$ element of the pfaffian $\left(1,2,\cdots,2m\right)$, in analogy with the theory of determinants. We have following expansion theorem:

Let $\left(b_j, b_k\right) = 0$ for $j,k = 1,2,\cdots,l$, $M\left(j_1, j_2, \cdots j_l\right)$ be the minor of the pfaffian $\left(1,2,\cdots,2m\right)$, and m,l be positive integers.

$$\begin{aligned} \left(b_1, b_2, \cdots, b_l, 1, 2, \cdots, 2m\right) = \sum \mathop{\cdots}_{1 \le j_1 < j_2 \cdots < j_l \le 2m} \sum (-1)^{P_l} \left(b_1, b_2, \cdots, b_l, j_1, j_2, \cdots, j_l\right) \\ \times M\left(j_1, j_2, \cdots, j_l\right) \end{aligned} \tag{3.14}$$

where

$$P_l = j_1 + j_2 + \cdots + j_l - \frac{l\left(l+1\right)}{2} \tag{3.15}$$

We have for $l = 2$

$$\left(b_1,b_2,1,2,\cdots,2m\right)=\sum_{j=1}^{2m}\sum_{k=j+1}^{2m}(-1)^{j+k-1}\left(b_1,b_2,j,k\right)\left(1,2,\cdots,\hat{j},\cdots\hat{k},\cdots,2m\right) \tag{3.16}$$

which gives us the following expansion of the pfaffian:

$$\begin{aligned}\left(b_1,b_2,c_1,c_2,1,2,\cdots,2n\right)=&\sum_{j=1}^{2n}\sum_{k=j+1}^{2n}(-1)^{j+k-1}\left(b_1,b_2,j,k\right)\\&\times\left(c_1,c_2,1,2,\cdots,\hat{j},\cdots,\hat{k},\cdots,2n\right)\end{aligned} \tag{3.17}$$

provided that $\left(b_j,c_k\right)=0$ for $j,k=1,2$.

1.3 Identities of pfaffians

We have following identities of pfaffians which correspond to the Jacobi identity of determinants. All proofs are given in Appendix. Let m and n be positive integers.

$$\begin{aligned}&\left(a_1,a_2,\cdots,a_{2m},1,2,\cdots,2n\right)\left(1,2,\cdots,2n\right)\\&=\sum_{s=2}^{2m}(-1)^{s}\left(a_1,a_s,1,2,\cdots 2n\right)\left(a_2,a_3,\cdots,\hat{a}_s,\cdots,a_{2m},1,2,\cdots 2n\right)\end{aligned} \tag{3.18}$$

$$\begin{aligned}&\left(a_1,a_2,\cdots,a_{2m-1},1,2,\cdots,2n-1\right)\left(1,2,\cdots,2n\right)\\&=\sum_{s=1}^{2m-1}(-1)^{s-1}\left(a_s,1,2,\cdots 2n-1\right)\left(a_1,a_2,\cdots,\hat{a}_s,\cdots,a_{2m-1},1,2,\cdots 2n\right)\end{aligned} \tag{3.19}$$

For $m=2$, we have

$$\begin{aligned}&\left(a_1,a_2,a_3,a_4,1,2,\cdots,2n\right)\left(1,2,\cdots,2n\right)=\\&\left(a_1,a_2,1,2,\cdots,2n\right)\left(a_3,a_4,1,2,\cdots,2n\right)-\\&\left(a_1,a_3,1,2,\cdots,2n\right)\left(a_2,a_4,1,2,\cdots,2n\right)+\\&\left(a_1,a_4,1,2,\cdots,2n\right)\left(a_2,a_3 1,2,\cdots,2n\right)\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}&\left(a_1,a_2,a_3,1,2,\cdots,2n-1\right)\left(1,2,\cdots,2n\right)=\\&\left(a_1,1,2,\cdots,2n-1\right)\left(a_2,a_3,1,2,\cdots,2n\right)-\\&\left(a_2,1,2,\cdots,2n-1\right)\left(a_1,a_3,1,2,\cdots,2n\right)+\\&\left(a_3,1,2,\cdots,2n-1\right)\left(a_1,a_2 1,2,\cdots,2n\right)\end{aligned} \tag{3.21}$$

2. Pfaffian solutions to the BKP Equations

The BKP hierarchy was introduced by Date, Jimbo, Kashiwara and Miwa [5], and is expressed by the bilinear forms

$$\left[\left(D_3 - D_1^3\right)D_{-1} + 3D_1^2\right]\tau\cdot\tau = 0 \tag{3.22}$$

$$\left[D_1^6 - 5D_1^3 D_3 - 5D_3^2 + 9D_1 D_5\right]\tau\cdot\tau = 0 \tag{3.23}$$

We will express the soliton solutions of the BKP equation in terms of a pfaffian, and we will show that the bilinear BKP equation is equivalent to a pfaffian identity.

The N-soliton solution τ_N can also be expressed as the Nth-order pfaffian,

$$\tau_N = \left(1,2,3,\cdots,2N\right) \tag{3.24}$$

where $(i,j) = c_{ij} + \int^x D_x f_i(x)\cdot f_j(x)\,dx \quad (x = x_1)$ and $f_i(x)$ for $i = 1,2,3,\cdots$, $c_{ij} = -c_{ji}$, satisfy the linear differential equations

$$\frac{\partial}{\partial x_n} f_i(x) = \frac{\partial^n}{\partial x^n} f_i(x) \quad \left(n = -1,1,3,5,\cdots\right) \tag{3.25}$$

For $n = -1$, we have following definition

$$\frac{\partial}{\partial x_{-1}} f_i(x) = \int^x f_i(x)\,dx \tag{3.26}$$

So, we confirm that the pfaffian expression for τ satisfies the BKP-type equation (3.22), which is expressed in terms of normal derivatives as

$$\begin{aligned}&\left[\frac{\partial}{\partial x_{-1}}\left(\frac{\partial}{\partial x_3} - \frac{\partial^3}{\partial x^3}\right)\tau + 3\frac{\partial^2}{\partial x^2}\tau\right]\tau + 3\left[\left(\frac{\partial^3}{\partial x_{-1}\partial x^2} - \frac{\partial}{\partial x}\right)\tau\right]\frac{\partial \tau}{\partial x}\\ &-3\left[\frac{\partial^2}{\partial x_{-1}\partial x}\tau\right]\frac{\partial^2 \tau}{\partial x^2} - \left[\left(\frac{\partial}{\partial x_3} - \frac{\partial^3}{\partial x^3}\right)\tau\right]\frac{\partial \tau}{\partial x_{-1}} = 0\end{aligned} \tag{3.27}$$

From the formula

$$\left(i,j\right)=c_{ij}+\int^{x}D_{x}f_{i}\left(x\right)\cdot f_{j}\left(x\right)dx \tag{3.28}$$

We obtain

$$\begin{aligned}\frac{\partial}{\partial x}\left(i,j\right)&=\left[\frac{\partial}{\partial x}f_{i}\right]f_{j}-f_{i}\left[\frac{\partial}{\partial x}f_{j}\right]\\&=\left(d_{0},d_{1},i,j\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x_{-1}}\left(i,j\right)&=\int^{x}\left[\frac{\partial^{2}f_{i}}{\partial x_{-1}\partial x}+\frac{\partial f_{i}}{\partial x}\frac{\partial f_{j}}{\partial x_{-1}}-\frac{\partial f_{i}}{\partial x_{-1}}\frac{\partial f_{j}}{\partial x}-f_{i}\frac{\partial^{2}f_{j}}{\partial x_{-1}\partial x}\right]dx\\&=f_{i}\left[\frac{\partial}{\partial x_{-1}}f_{j}\right]-\left[\frac{\partial}{\partial x_{-1}}f_{i}\right]f_{j}\\&=\left(d_{-1},d_{0},i,j\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x_{3}}\left(i,j\right)&=\int^{x}\left[\frac{\partial^{2}f_{i}}{\partial x_{3}\partial x}f_{j}+\frac{\partial f_{i}}{\partial x}\frac{\partial f_{j}}{\partial x_{3}}-\frac{\partial f_{i}}{\partial x_{3}}\frac{\partial f_{j}}{\partial x}-f_{i}\frac{\partial^{2}f_{j}}{\partial x_{3}\partial x}\right]dx\\&=\frac{\partial^{3}f_{i}}{\partial x^{3}}f_{j}-f_{i}\frac{\partial^{3}f_{j}}{\partial x^{3}}-2\left[\frac{\partial^{2}f_{i}}{\partial x^{2}}\frac{\partial f_{j}}{\partial x}-\frac{\partial f_{i}}{\partial x}\frac{\partial^{2}f_{j}}{\partial x^{2}}\right]\\&=\left(d_{0},d_{3},i,j\right)-2\left(d_{1},d_{2},i,j\right)\end{aligned}$$

we define

$$\left(d_{n},i\right)=\frac{\partial^{n}}{\partial x^{n}}f_{i}\left(x\right)\qquad\left(n=-1,0,1,2,\cdots\right) \tag{3.29}$$

Consequently, the derivatives of τ_{N} ,

$$\tau_{N}=\left(1,2,3,\cdots,2N\right)=\left(\bullet\right) \tag{3.30}$$

are given by

$$\frac{\partial}{\partial x}\tau_{N}=\left(d_{0},d_{1},\bullet\right) \tag{3.31}$$

$$\frac{\partial^{2}}{\partial x^{2}}\tau_{N}=\left(d_{0},d_{2},\bullet\right) \tag{3.32}$$

$$\frac{\partial^3}{\partial x^3}\tau_N = \left(d_1, d_2, \bullet\right) + \left(d_0, d_3, \bullet\right) \tag{3.33}$$

$$\frac{\partial^3}{\partial x_{-1}\partial x^2}\tau_N = \left(d_{-1}, d_2, \bullet\right) + \left(d_0, d_1, \bullet\right) \tag{3.34}$$

$$\frac{\partial}{\partial x_{-1}}\tau_N = \left(d_{-1}, d_0, \bullet\right) \tag{3.35}$$

$$\frac{\partial^2}{\partial x_{-1}\partial x}\tau_N = \left(d_{-1}, d_1, \bullet\right) \tag{3.36}$$

$$\frac{\partial}{\partial x_3}\tau_N = \left(d_0, d_3, \bullet\right) - 2\left(d_1, d_2, \bullet\right) \tag{3.37}$$

$$\frac{\partial}{\partial x_{-1}}\left(\frac{\partial}{\partial x_3} - \frac{\partial^3}{\partial x^3}\right)\tau_N = -3\left[\left(d_0, d_2, \bullet\right) + \left(d_{-1}, d_0, d_1, d_2, \bullet\right)\right] \tag{3.38}$$

Substitute the above results into (3.27), we obtain

$$\begin{array}{l}
\quad\overset{d_{-1}}{\boxed{\circ}}\overset{d_0}{\boxed{\circ}}\overset{d_1}{\boxed{\circ}}\overset{d_2}{\boxed{\circ}} \times \overset{d_{-1}}{\boxed{}}\overset{d_0}{\boxed{}}\overset{d_1}{\boxed{}}\overset{d_2}{\boxed{}} \\
-\boxed{\circ}\boxed{\circ}\boxed{}\boxed{} \times \boxed{}\boxed{}\boxed{\circ}\boxed{\circ} \\
+\boxed{\circ}\boxed{}\boxed{\circ}\boxed{} \times \boxed{}\boxed{\circ}\boxed{}\boxed{\circ} \\
-\boxed{\circ}\boxed{}\boxed{}\boxed{\circ} \times \boxed{}\boxed{\circ}\boxed{\circ}\boxed{} = 0
\end{array} \tag{3.39}$$

which is nothing but a pfaffian identity as (3.20). Therefore, τ_N solves the BKP-type equation (3.22). Through similar calculations, we can confirm that τ_N also satisfies(3.23).

CHAPTER IV

MULTI-SOLITON SOLUTION FOR THE SHALLOW WATER WAVE MODEL

1. A reduction from the pfaffian solutions of the BKP equations

In chapter II, we have obtained the bilinear form for the shallow water wave equation, which is

$$D_x(D_t - D_t D_x^2 + D_x) f \cdot f = 0 \tag{4.1}$$

It is known that the reduction of the BKP hierarchy expressed by

$$\left[\left(D_3 - D_1^3\right) D_{-1} + 3D_1^2\right] \tau \cdot \tau = 0 \tag{4.2}$$

$$\left[D_1^6 - 5D_1^3 D_3 - 5D_3^2 + 9D_1 D_5\right] \tau \cdot \tau = 0 \tag{4.3}$$

reproduces various soliton equations.

In equation (4.2), if we let $D_3 = D_1 = D_x$, and $D_{-1} = D_t$, then we get

$$\left[\left(D_x - D_x^3\right) D_t + 3D_x^2\right] \tau \cdot \tau = 0 \tag{4.4}$$

which is equivalent to

$$D_x(D_t - D_t D_x^2 + 3D_x) \tau \cdot \tau = 0 \tag{4.5}$$

Compare equation (4.1) with equation (4.5), the only differences are in the coefficients of their nonlinear terms for D_x^2, they have the same nonlinearity and dispersion. Therefore, the shallow water wave model equation is a reduction from the BKP equations we discussed in chapter III and the pfaffian solutions of the BKP equations $\tau_N = (1,2,3,\cdots,2N)$ satisfies the equation (4.1) as well.

2. One and Two- soliton solutions for SWW

Let's review the N-soliton solution τ_N expressed by the Nth-order pfaffian

$$\tau_N = (1,2,3,\cdots,2N) \tag{4.6}$$

where each pfaffian entry (i,j) is

$$(i,j) = c_{ij} + \int^x D_x f_i(x) \cdot f_j(x)\,dx \tag{4.7}$$

and $f_i(x)$ for $i = 1,2,3,\cdots$, $c_{ij} = -c_{ji}$, satisfy the linear differential equations

$$\frac{\partial}{\partial x_n} f_i(x) = \frac{\partial^n}{\partial x^n} f_i(x) \quad (n = -1,1,3,5,\cdots) \tag{4.8}$$

for $n = -1$, we have following definition

$$\frac{\partial}{\partial x_{-1}} f_i(x) = \int^x f_i(x)\,dx \tag{4.9}$$

In particular, f_i takes the form

$$f_i(x) = p_i^{-1} x_{-1} + p_i x_1 + p_i^3 x_3 + p_i^5 x_5 + \cdots \tag{4.10}$$

Since we have restriction $D_3 = D_1 = D_x$ from the BKP equation, which is equivalent to

$$p_i + q_i = p_i^3 + q_i^3 \tag{4.11}$$

For one-solition solution τ_1,

$$\begin{aligned} \tau_1 &= (1,2) \\ &= C_{12} + \frac{p_1 - q_1}{p_1 + q_1} \exp\left(\xi_1 + \hat{\xi}_1\right) \end{aligned} \tag{4.12}$$

Let $C_{12}=1$, $p_1+q_1=k_1$, we have $p_1 q_1=\frac{1}{3}\left(k_1^2-1\right)$ since $p_1+q_1=p_1^3+q_1^3$, after scaling, we can rewrite τ_1 as,

$$\begin{aligned}\tau_1&=1+\exp\left(\widehat{\eta}_1\right)\\&=1+\exp\left[\left(p_1+q_1\right)x+\left(\frac{1}{p_1}+\frac{1}{q_1}\right)t\right]\\&=1+\exp\left[k_1\left(x-v_1t\right)\right]\end{aligned}\tag{4.13}$$

where $v_1=-\frac{1}{p_1q_1}=\frac{3}{1-k_1^2}$, $\widehat{\eta}_1=k_1\left(x-v_1t\right)$. Since $u=2\left(\log\tau_1\right)_{xx}$, we substitute τ_1 and calculate partial derivatives, then we get,

$$\begin{aligned}u&=\frac{2k_1^2}{4}\frac{1}{\left(\dfrac{\exp\left(\frac{\eta_1}{2}\right)+\exp\left(-\frac{\eta_1}{2}\right)}{2}\right)^2}\\&=\frac{2k_1^2}{4}\sec h^2\left(\frac{\eta_1}{2}\right)\\&=A\sec h^2\left[\frac{k_1}{2}\left(x-v_1t\right)\right]\end{aligned}\tag{4.14}$$

where $A=\frac{k_1^2}{2}$, and $v_1=\frac{3}{1-k_1^2}$, the physical meaning of A is amplitude of one-solition, while v_1 is the velocity of this wave; if $k_1>1$, then $v_1<0$, the wave moves to left; if $k_1<1$, then $v_1>0$, the wave moves to right; if $k_1\to 1$, then $v_1\to\infty$, there is no soliton.

By choosing $c_{12}=c_{34}=1$, $c_{13}=c_{14}=c_{23}=c_{24}=0$, $f_1=\exp\xi_1$, $f_2=\exp\widehat{\xi}_1$, $f_3=\exp\xi_2$, and $f_4=\exp\widehat{\xi}_2$, the two-soliton solution is

$$\tau_2 = (1,2)(3,4) - (1,3)(2,4) + (1,4)(2,3)$$
$$= \left[1 + \frac{p_1 - q_1}{p_1 + q_1} \exp\left(\xi_1 + \widehat{\xi}_1\right)\right]\left[1 + \frac{p_2 - q_2}{p_2 + q_2} \exp\left(\xi_2 + \widehat{\xi}_2\right)\right]$$
$$- \frac{p_1 - p_2}{p_1 + p_2} \exp\left(\xi_1 + \xi_2\right) \times \frac{q_1 - q_2}{q_1 + q_2} \exp\left(\widehat{\xi}_1 + \widehat{\xi}_2\right) \tag{4.15}$$
$$+ \frac{p_1 - q_2}{p_1 + q_2} \exp\left(\xi_1 + \widehat{\xi}_2\right) \times \frac{q_1 - p_2}{q_1 + p_2} \exp\left(\widehat{\xi}_1 + \xi_2\right)$$

Putting

$$\widehat{\eta}_i = \xi_i + \widehat{\xi}_i + \delta_i \qquad \textit{where} \quad \exp \delta_i = \frac{p_i - q_i}{p_i + q_i} \tag{4.16}$$

We may rewrite τ_2 as

$$\tau_2 = 1 + \exp\left(\widehat{\eta}_1\right) + \exp\left(\widehat{\eta}_2\right) + b_{12} \exp\left(\widehat{\eta}_1 + \widehat{\eta}_2\right) \tag{4.17}$$

which coincides with the two-soliton solution found by the perturbation method. Where p_i and q_i are parameters, b_{12} is the phase shift term given by

$$b_{12} = \frac{\left(p_1 - p_2\right)\left(p_1 - q_2\right)\left(q_1 - p_2\right)\left(q_1 - q_2\right)}{\left(p_1 + p_2\right)\left(p_1 + q_2\right)\left(q_1 + p_2\right)\left(q_1 + q_2\right)} \tag{4.18}$$

which shows to be of another form,

$$b_{12} = \frac{(k_1 - k_2)^2 (k_1^{\ 2} - k_1 k_2 + k_2^{\ 2} - 3)}{(k_1 + k_2)^2 (k_1^{\ 2} + k_1 k_2 + k_2^{\ 2} - 3)} \tag{4.19}$$

if we let $p_1 + q_1 = k_1$, and $p_2 + q_2 = k_2$.

We use software Maltlab simulate the τ_1 and τ_2 functions and get the following figures which demonstrate the physical behavior of one-soliton and two-solitons for the shallow water wave model.

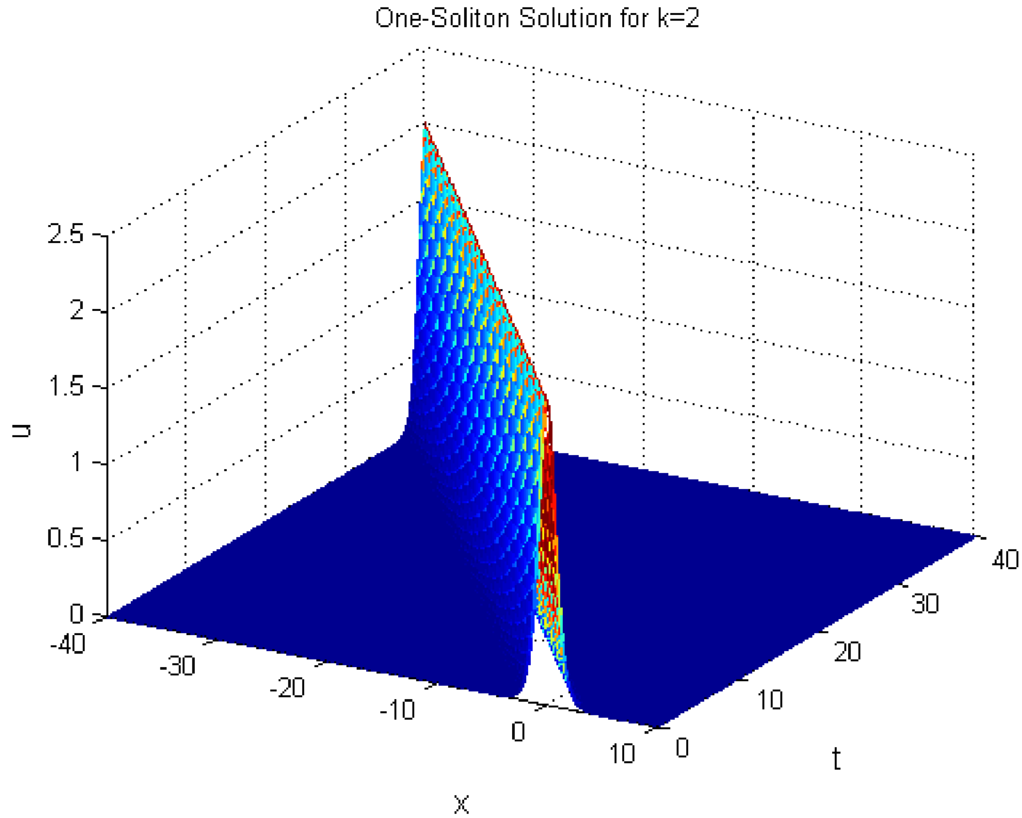

Figure 1. one-soliton,   k=2

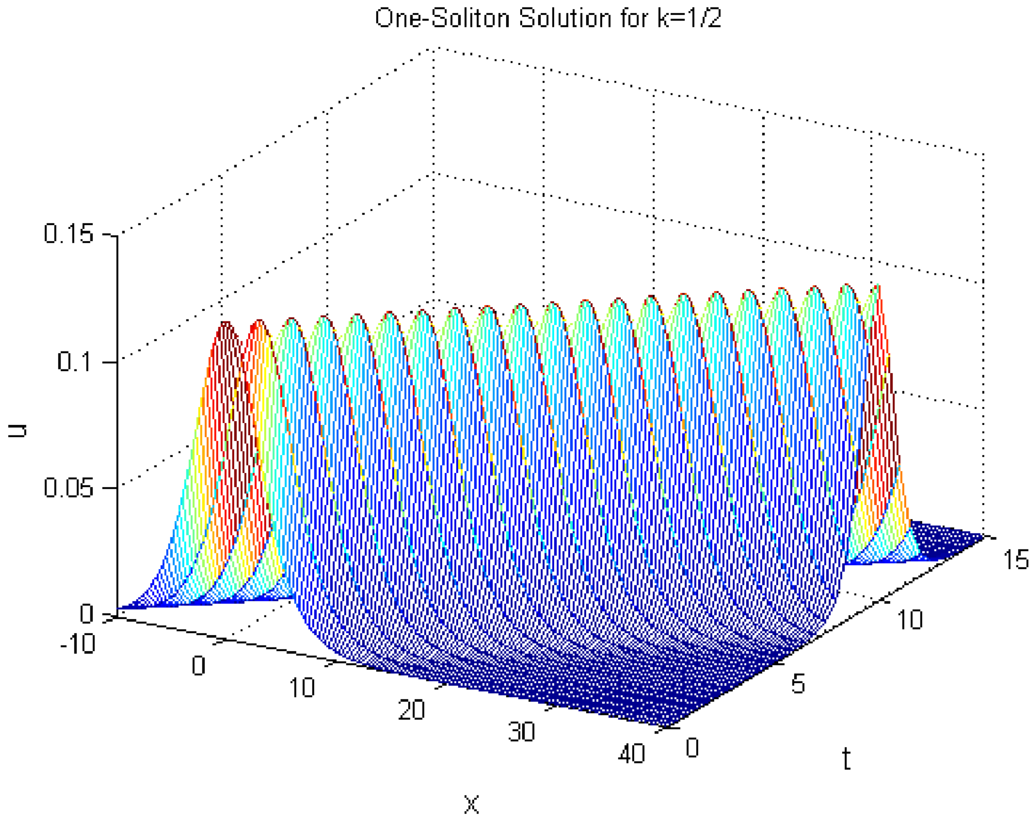

Figure 2. one-soliton, k=1/2

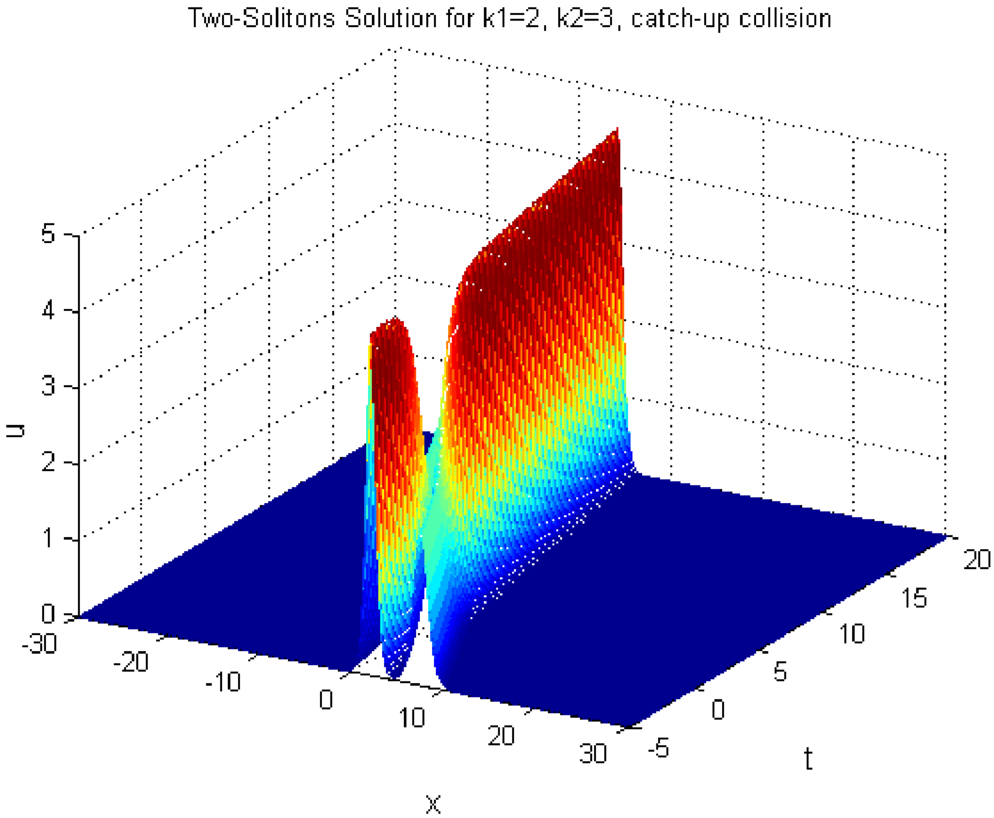

Figure 3. two-solitons, k1=2, k2=3

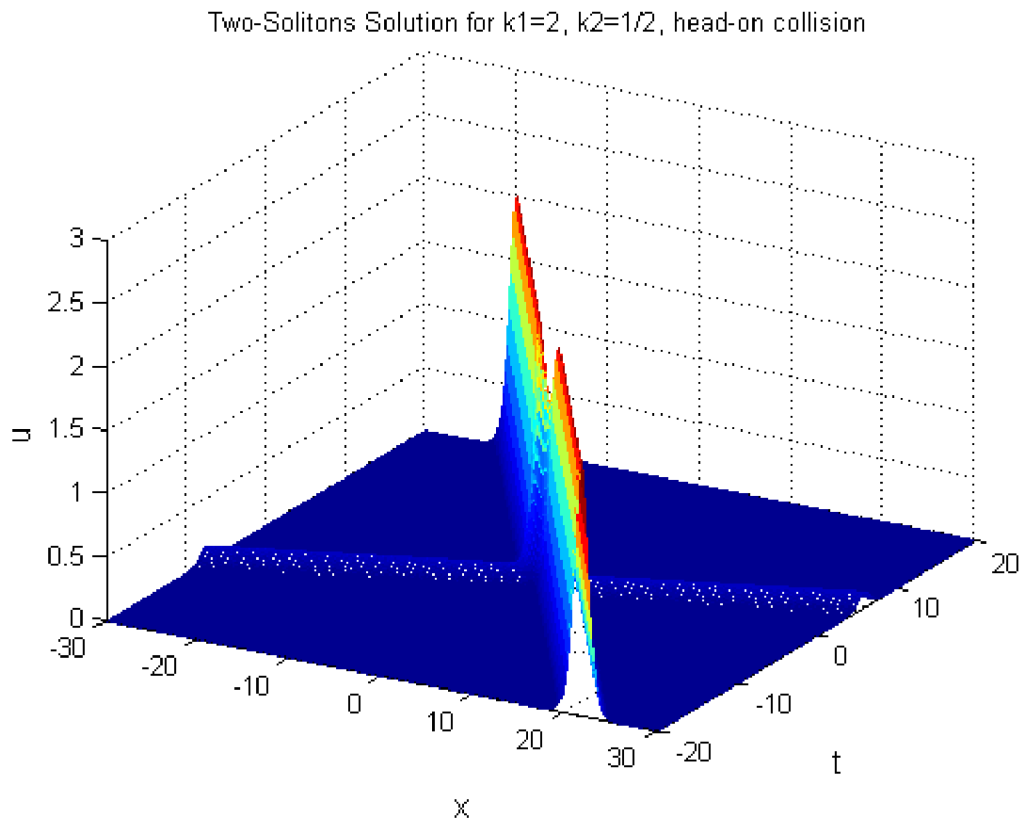

Figure 4. two-solitons, k1=2, k2=1/2

From the above figures, we note that one soliton is the simplest case, which has only one solitary wave moves toward right if the velocity is positive or left directions if the velocity is negative, and the amplitudes and velocities are controlled by parameter k; while the two solitons have two solitary waves interact each other and move toward right or left directions with more complicated behaviors which are controlled by two parameters k_1 and k_2. Since the solutions depend on arbitrary functions, we choose different parameters as input to our simulations.

CHAPTER V

CONCLUSION AND FUTURE TOPICS

In this thesis, we have investigated a shallow water wave model equation and obtained its bilinear form by using Hirota direct method. Then we discussed the BKP equations and obtained its pfaffians solutions through pfaffian technique. From the reduction of the BKP equations, we got the shallow water wave model equation, which shows the shallow water wave model is in BKP hierarchy. Also, by reduction from the pfaffian solutions of the BKP equations, the pfaffian solution of the shallow water wave is obtained, which is nothing but pfaffian identities. The one-soliton and two-solitons solutions to the shallow water wave model are given and the physical behaviors are illustrated.

In future work, we will investigate more integrable nonlinear partial differential equations and try to get the pfaffians solutions of them.

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APPENDIX A

APPENDIX A

SOME FORMULAS FOR D-OPERATOR

Hirota's D-operator is defined by

$$D_t^m D_x^n a(t,x)\cdot b(t,x) = \frac{\partial^m}{\partial s^m}\frac{\partial^n}{\partial y^n} a(t+s,x+y)\,b(t-s,x-y)\Big|_{s=0,y=0} \qquad m,n=0,1,2,3,\cdots \tag{A.1}$$

For the sake of comparison, the Leibniz rule for differentiation of a product of functions may be written in a similar way as

$$\frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial x^n} a(t,x)\cdot b(t,x) = \frac{\partial^m}{\partial s^m}\frac{\partial^n}{\partial y^n} a(t+s,x+y)\,b(t+s,x+y)\Big|_{s=0,y=0} \qquad m,n=0,1,2,3,\cdots \tag{A.2}$$

The following is some simple examples:

$$\begin{aligned} D_x a\cdot b &= a_x b - ab_x \\ D_x^2 a\cdot b &= a_{xx} b - 2a_x bx + ab_{xx} \\ D_x^3 a\cdot b &= a_{xxx} b - 3a_{xx} b_x + 3a_x b_{xx} - ab_{xxx} \end{aligned} \tag{A.3}$$

For comparison, the corresponding normal derivatives of a product of functions are

$$\begin{aligned} \partial_x a\cdot b &= a_x b + ab_x \\ \partial_x^2 a\cdot b &= a_{xx} b + 2a_x bx + ab_{xx} \\ \partial_x^3 a\cdot b &= a_{xxx} b + 3a_{xx} b_x + 3a_x b_{xx} + ab_{xxx} \end{aligned} \tag{A.4}$$

From the definition, we have

$$D_t^m D_x^n a \cdot b = D_x^n D_t^m a \cdot b = D_x^{n-1} D_t^m D_x a \cdot b$$
$$D_t^m D_x^n a \cdot 1 = \partial_t^m \partial_x^n a \tag{A.5}$$

Also, Hirota defined the D-operator D_z and the differential operator ∂_z by

$$D_z = D_t + \varepsilon D_x$$
$$\partial_z = \partial_t + \varepsilon \partial_x \tag{A.6}$$

The following properties are easily seen from the definition of D-operator

$$D_x^m (f \cdot 1) = \left(\frac{\partial}{\partial x} \right)^m f \tag{A.7}$$

$$D_x^m (f \cdot g) = (-1)^m D_x^m (g \cdot f) \tag{A.8}$$

$$D_x^m (f \cdot f) = 0 \qquad \textit{for odd } m \tag{A.9}$$

$$D_x^m (f \cdot g) = D_x^{m-1} (f_x \cdot g - f \cdot g_x) \tag{A.10}$$

$$D_x^m (f \cdot f) = 2 D_x^{m-1} (f_x \cdot f) \qquad \textit{for even } m \tag{A.11}$$

$$D_x D_t (f \cdot f) = 2 D_x (f_t \cdot f) = 2 D_t (f_x \cdot f) \tag{A.12}$$

$$D_x^m \exp(p_1 x) \cdot \exp(p_2 x) = (p_1 - p_2)^m \exp\left[(p_1 + p_2) x \right] \tag{A.13}$$

$$\exp(\varepsilon D_x)\big(f(x) \cdot g(x)\big) = f(x+\varepsilon) g(x-\varepsilon) \tag{A.14}$$

$$\exp(\varepsilon D_x)(fg \cdot hl) = \left[\exp(\varepsilon D_x)(f \cdot h) \right]\left[\exp(\varepsilon D_x)(g \cdot l) \right] \tag{A.15}$$

$$D_x (fg \cdot h) = \frac{\partial f}{\partial x} gh + f \big(D_x (g \cdot h) \big) \tag{A.16}$$

$$D_x^2 (fg \cdot h) = \frac{\partial^2 f}{\partial x^2} gh + 2 \frac{\partial f}{\partial x} D_x (g \cdot h) + f D_x^2 (g \cdot h) \tag{A.17}$$

$$D_x^m \exp(px) f \cdot \exp(px) g = \exp(2px) D_x^m f \cdot g \tag{A.18}$$

The following equalities are useful for transforming nonlinear differential equations into the bilinear forms.

$$\exp\left(\varepsilon\frac{\partial}{\partial x}\right)\left(\frac{f}{g}\right)=\frac{\exp\left(\varepsilon D_x\right)f\cdot g}{\cosh\left(\varepsilon D_x\right)g\cdot g} \tag{A.19}$$

$$\frac{\partial}{\partial x}\left(\frac{f}{g}\right)=\frac{D_x\left(f\cdot g\right)}{g^2} \tag{A.20}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{f}{g}\right)=\frac{D_x^2\left(f\cdot g\right)}{g^2}-\left(\frac{f}{g}\right)\frac{D_x^2\left(f\cdot g\right)}{g^2} \tag{A.21}$$

$$\frac{\partial^3}{\partial x^3}\left(\frac{f}{g}\right)=\frac{D_x^3\left(f\cdot g\right)}{g^2}-3\left[\frac{D_x^2\left(f\cdot g\right)}{g^2}\frac{D_x^2\left(f\cdot f\right)}{g^2}\right] \tag{A.22}$$

$$2\cosh\left(\varepsilon\frac{\partial}{\partial x}\right)\log f=\log\left[\cosh\left(\varepsilon D_x\right)f\cdot f\right] \tag{A.23}$$

$$\frac{\partial^2}{\partial x^2}\left(\log f\right)=\frac{D_x^2\left(f\cdot f\right)}{2f^2} \tag{A.24}$$

$$\frac{\partial^4}{\partial x^4}\left(\log f\right)=\frac{D_x^4\left(f\cdot f\right)}{2f^2}-6\left[\frac{D_x^2\left(f\cdot f\right)}{2f^2}\right]^2 \tag{A.25}$$

If we let $u\left(x,t\right)=2\left(\log f\right)_{xx}$, where $f\left(x,t\right)$ is given by the perturbation expansion,

$$f\left(x,t\right)=1+\sum_{n=1}^{\infty}\varepsilon^n f_n\left(x,t\right) \tag{A.26}$$

Some of the properties of the D-operators are as follows:

$$\frac{D_t^2 f\cdot f}{f^2}=\iint u_{tt}\,dx\,dx \tag{A.27}$$

$$\frac{D_t D_x^3 f\cdot f}{f^2}=u_{xt}+3u\int xu_t\,dx' \tag{A.28}$$

$$\frac{D_x^2 f \cdot f}{f^2} = u \tag{A.29}$$

$$\frac{D_x^4 f \cdot f}{f^2} = u_{xx} + 3u^2 \tag{A.30}$$

$$\frac{D_t D_x f \cdot f}{f^2} = \log\left(f^2\right)_{xt} \tag{A.31}$$

$$\frac{D_x^6 f \cdot f}{f^2} = u_{xxxx} + 15uu_{xx} + 15u^3 \tag{A.32}$$

APPENDIX B

APPENDIX B

SOME FORMULAS FOR PFAFFIANS

First we prove the identity of the pfaffian (3.19). Let m be an odd integer. We have

$$\begin{aligned}&(a_1,a_2,\cdots,a_m,1,2,\cdots,2n-1)(1,2,\cdots,2n)\\&=\sum_{s=1}^{m}(-1)^{s-1}(a_s,1,2,\cdots 2n-1)(a_1,a_2,\cdots,\hat{a}_s,\cdots,a_m,1,2,\cdots 2n)\end{aligned} \tag{B.1}$$

We note the following relation. Let j belong to a set $(1,2,\cdots,2n)$. Then we have for odd m

$$\begin{aligned}&\sum_{k=1}^{2n}(-1)^k(j,k)(a_1,a_2,\cdots,a_m,1,2,\cdots,\hat{k},\cdots,2n)\\&=\sum_{s=1}^{m}(-1)^s(j,a_s)(a_1,a_2,\cdots,\hat{a}_s,\cdots,a_m,1,2,\cdots,2n)\end{aligned} \tag{B.2}$$

Equation (B.2) is obtained by expanding the pfaffian of value 0:

$$\begin{aligned}0&=(j,a_1,a_2,\cdots,a_m,1,2,\cdots,2n)\\&=\sum_{k=1}^{2n}(-1)^k(j,k)(a_1,a_2,\cdots,a_m,1,2,\cdots,\hat{k},\cdots,2n)\\&+\sum_{s=1}^{m}(-1)^{s-1}(j,a_s)(a_1,a_2,\cdots,\hat{a}_s,\cdots,a_m,1,2,\cdots,2n)\end{aligned} \tag{B.3}$$

Consider the *r.h.s* of equation (B.1):

$$\begin{aligned}&\sum_{s=1}^{m}(-1)^{s-1}(a_s,1,2,\cdots 2n-1)(a_1,a_2,\cdots,\hat{a}_s,\cdots,a_m,1,2,\cdots 2n)\\&=\sum_{s=1}^{m}(-1)^{s-1}\left\{\sum_{j=1}^{2n-1}(-1)^{j-1}(a_s,j)M(j)\right\}(a_1,a_2,\cdots,\hat{a}_s,\cdots,a_m,1,2,\cdots 2n)\end{aligned} \tag{B.4}$$

where $M(j) = \left(1,2,\cdots,\hat{j},\cdots,2n-1\right)$,

$$= \sum_{j=1}^{2n-1}(-1)^{j-1} M(j) \sum_{s=1}^{m}(-1)^{s-1}(a_s, j)(a_1, a_2, \cdots, \hat{a}_s, \cdots, a_m, 1, 2, \cdots 2n) \tag{B.5}$$

$$= \sum_{j=1}^{2n-1}(-1)^{j-1} M(j) \sum_{k=1}^{2n}(-1)^{k}(j,k)\left(a_1, a_2, \cdots, a_m, 1, 2, \cdots, \hat{k}, \cdots, 2n\right) \tag{B.6}$$

where we have used equation (B.2)

$$= -\sum_{k=1}^{2n}(-1)^{k}(k,1,2,\cdots,2n-1)\left(a_1, a_2, \cdots, a_m, 1, 2, \cdots, \hat{k}, \cdots, 2n\right) \tag{B.7}$$

where we have used the expansion rule

$$\begin{aligned} &\sum_{j=1}^{2n-1}(-1)^{j-1}(k,j)M(j) = (k,1,2,\cdots,2n-1) \\ &= -(2n,1,2,\cdots,2n-1)(a_1, a_2, \cdots, a_m, 1, 2, \cdots, 2n-1) \end{aligned} \tag{B.8}$$

$$= (a_1, a_2, \cdots, a_m, 1, 2, \cdots, 2n-1)(1,2,\cdots,2n) \tag{B.9}$$

which is the *l.h.s* of equation (B.1). Hence the identity of the pfaffian has been proved.

The identity (3.18), which is

$$\begin{aligned} &(a_1, a_2, \cdots, a_{2l}, 1, 2, \cdots, 2n)(1,2,\cdots,2n) \\ &= \sum_{s=2}^{2l}(-1)^{s}(a_1, a_s, 1, 2, \cdots 2n)(a_2, a_3, \cdots, \hat{a}_s, \cdots, a_{2l}, 1, 2, \cdots 2n) \end{aligned} \tag{B.10}$$

is proved using the identity (3.19). We rewrite the identity (3.19) for later use as

$$\begin{aligned} &\left(a_2, a_3, \cdots, a_{2l}, 1, 2, \cdots \hat{j}, \cdots, 2n\right)(1,2,\cdots,2n) \\ &= \sum_{s=2}^{l}(-1)^{s}\left(a_s, 1, 2, \cdots \hat{j}, \cdots, 2n\right)\left(a_2, a_3, \cdots, \hat{a}_s, \cdots, a_{2l}, 1, 2, \cdots, 2n\right) \end{aligned} \tag{B.11}$$

Consider the *r.h.s* of equation (B.10)

$$\sum_{s=2}^{2l}(-1)^s\left(a_1,a_s,1,2,\cdots 2n\right)\left(a_2,a_3,\cdots,\hat{a}_s,\cdots,a_{2l},1,2,\cdots 2n\right)$$

$$=\sum_{s=2}^{2l}(-1)^s\left[\sum_{j=1}^{2n}(-1)^j\left(a_1,j\right)\left(a_s,1,2,\cdots,\hat{j},\cdots,2n\right)\right] \tag{B.12}$$

$$\times\left(a_2,a_3,\cdots,\hat{a}_s,\cdots,a_{2l},1,2,\cdots 2n\right)$$

where we have expended $\left(a_1,a_s,1,2,\cdots,2n\right)$

$$=\sum_{j=1}^{2n}(-1)^j\left(a_1,j\right)\left(a_2,a_3,\cdots,a_{2l},1,2,\cdots,\hat{j},\cdots,2n\right)\left(1,2,\cdots,2n\right) \tag{B.13}$$

$$=\left(a_1,a_2,\cdots,a_{2l},1,2,\cdots,2n\right)\left(1,2,\cdots,2n\right) \tag{B.14}$$

which is the *l.h.s* of equation (B.10). Hence we have proved the identity (3.18).

BIOGRAPHICAL SKETCH

Zhijiang Qiao, who lives in 508 E Redbud Ave., McAllen, TX 78504, is graduate student in the Department of Mathematics at The University of Texas-Pan American. Zhijiang received his B.S. from the Liaoning University in 2003, M.S. in Computer Science from The University of Texas-Pan American in 2007, and M.S. in Mathematics from The University of Texas-Pan American in 2010. From 2002 to 2005, he was on the faculty at the Shenyang Institute of Engineering, where he taught undergraduate courses in computer science and telecommunication. From 1994 to 2002, he was on the research position at the Shenyang Institute of Engineering. Zhijiang joined the University of Texas-Pan American in 2006, pursued master degree in computer science and mathematics. He served as research assistant and teaching assistant for department of computer science and department of mathematics from 2006 to 2010. His research covers applied mathematics, computational mathematics, and digital image and video processing.