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Social Security as Markov Equilibrium in OLG Models: Clarifications and New Insights

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Social Security as Markov Equilibrium in OLG Models: Clarifications and Some New Insights*

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Abstract

This paper studies the politico-economic sustainability of pay-as-you-go social security in OLG models under Markovian strategies as first studied by Forni (2005). Under logarithmic utility, the paper shows that equilibria with social security can only exist if the underlying economy is dynamically inefficient. The paper also derives the exact parametric conditions that allow for the existence of equilibria and shows that among all the admissible (arbitrary) constants that produce a Markov perfect equilibrium, the maximum constant in such set yields the only equilibrium that solves dynamic inefficiency.

Keywords: *social security, political economy model, overlapping generations, Markov perfect, golden rule, dynamic inefficiency*

JEL: *C72, H55, E24*

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1 Introduction

This paper characterizes the Markov Perfect Equilibria (MPE) of social security in 2-period OLG economies as first studied by Forni (2005). The economy is the canonical 2-period OLG model with capital accumulation and 100% depreciation rate, Cobb-Douglas production and logarithmic utility, augmented by a politico-economic decision on payroll taxes to finance a pay-as-you-go (PAYGO) transfer to the old.

Forni discovered that the model has a closed-form solution, which displays a multiplicity of equilibria indexed by an arbitrary constant of integration. Gonzalez-Eiras (2011) refined Forni's results by identifying additional conditions that would allow for such equilibria to exist. This paper analyzes the same model and provides several new findings. In particular, the paper shows that (1) for a PAYGO to exist in a stable steady state, the economy ought to be dynamically inefficient, (2) the paper completely characterizes the parametric conditions that result in equilibria under differentiability of the policy function, (3) shows that among all possible constants that produce a MPE, the equilibrium induced by the maximum constant in this set is the only one that solves dynamic inefficiency; moreover (4) such equilibrium produces a PAYGO system with a tax rate that induces the golden rule level of capital accumulation; (5) there exists a closed-form solution for this maximum constant; and finally (6) it is shown that any other admissible constant yields equilibria with 2 steady states, a dynamically-stable but dynamically-inefficient steady state, and a dynamically-efficient but dynamically-unstable steady state.[1]

2 The Model Economy

Population evolves according to $N_{t+1} = \eta N_t$, where N_t is the size of the young generation alive at time t, η is the gross growth rate of the population (i.e. $\eta \equiv 1 + n$ is the number of children that agents have and thus n is also the population growth rate). Following Forni, it is assumed that the young is always the majority, so that $\eta \geq 1$.[2]

Wages are taxed at the rate τ_t, with transfers b_{t+1} to the old alive in period $[t+1]$ defined by $b_{t+1} = \tau_{t+1} w_{t+1} \eta$. Production is Cobb-Douglas $F(K_t, L_t) = K^{\alpha} L^{1-\alpha}$, which implies factor prices given by $w_t = (1-\alpha) k_t^{\alpha}$ and $R_t = \alpha k_t^{\alpha - 1}$. Capital depreciates over a generation.

[1] Related literature on the political sustainability of social security in OLG settings include the seminal papers of Cooley and Soares (1999) and Boldrin and Rustichini (2000) which use subgame perfection, of which MPE is a refinement. Sand and Razin (2007) generalize Forni's model for the joint decision on immigration and social security. Gonzalez-Eiras and Niepelt (2008) study a variation of the model under stochastic growth of the population. Song (2011) considers inequality and social security under a more general specification of the model which requires computational solutions. Lopez-Velasco (2016) considers guest-worker programs in an otherwise similar economy to Forni's and finds that the Markovian solution also displays multiple equilibria that depends on an arbitrary constant. Dynamic inefficiency is also a necessary condition for the MPE.

[2] Workers are assumed the majority and hence their vote on taxes (and implied pensions) determines policy. For simplicity, if $\eta = 1$, which implies that workers and retirees cohorts are of the same size, it is assumed that the vote of the young still determines policy.

The equilibrium concept is Markov-Perfect and this in turn restricts the strategies (voting over the tax rate that defines transfers) to be functions of the state variable -the capital stock. Forni finds that the following two equations determine the evolution of the system in a MPE

$$\tau^{*}\left(k_{t}\right)=\left(\frac{\alpha}{1-\alpha}\right)\left(Ck_{t}^{-(1+\alpha\beta)/(1+\beta)}-1\right) \tag{1}$$

$$\beta k_{t+1}+Ck_{t+1}^{\frac{\beta(1-\alpha)}{(1+\beta)}}=\frac{\beta}{\eta}\left(k_{t}^{\alpha}-\alpha Ck_{t}^{-\frac{(1-\alpha)}{(1+\beta)}}\right) \tag{2}$$

With numerical analysis, Forni typically finds 2 steady states (given that a parameterization admits a solution), one with a high level of capital and dynamically-stable and another one with low-capital but dynamically-unstable. Gonzalez-Eiras elaborates on this model and finds that (i) if an equilibrium exists, then there must exist at most 2 steady states (but doesn't elaborate on circumstances that produce a single steady state); (ii) suggests that equilibria can exist whenever the arbitrary constant is in $\left[\underline{C},\overline{C}\right]$, with $\underline{C}$ given by

$$\underline{C}=\left(\frac{(1-\alpha)\beta}{\eta(1+\beta)}\right)^{\frac{1+\alpha\beta}{(1-\alpha)(1+\beta)}}, \tag{3}$$

and which is the level of the arbitrary constant C that yields a tax rate of 0 in equation (1) when evaluated at a dynamically-stable steady state. The existence of the specific upper bound $\overline{C}$ (the highest admissible constant) is identified by Gonzalez-Eiras, who mentions that there is no closed-form expression for $\overline{C}$. The current paper shows among other findings that a closed-form expression for $\overline{C}$ does indeed exist.

Gonzalez-Eiras also establishes that differentiability of the policy function yields a system that is stable for any capital in $k\in\left[k_{u}^{ss}(C),\overline{k}(C)\right]$, where $k_{u}^{ss}(C)$ refers to the steady state with low-capital but which is dynamically-unstable and where $\overline{k}(C)$ refers to the capital level that yields a zero tax rate in the equilibrium policy function (1) and which is higher than the high-capital steady state level $k_{s}^{ss}(C)$, with $0<k_{u}^{ss}(C)<k_{s}^{ss}(C)<\overline{k}(C)$.

I start by characterizing steady states in a more detailed way than previously found, which leads to clarifications and new insights. For several expressions, it will be convenient to work in terms of the steady-state gross interest rate R (a strictly decreasing function of capital), as opposed to working in terms of capital, and study how the possible solutions depend on the arbitrary constant C. To do this, rewrite the induced capital evolution equation (2) at steady state in terms of R. The appendix shows that the equation defining steady state interest rates (conditional on C) can be written as

$$C(\eta+\beta R)\left(\frac{\alpha}{R}\right)^{\frac{\beta}{1+\beta}}=\beta\left(\frac{\alpha}{R}\right)^{\frac{1}{1-\alpha}}\left(\frac{R}{\alpha}-\eta\right). \tag{4}$$

Solutions to the above equation represent steady states. Define the left and right hand sides as a pair of functions H_A and H_B given by

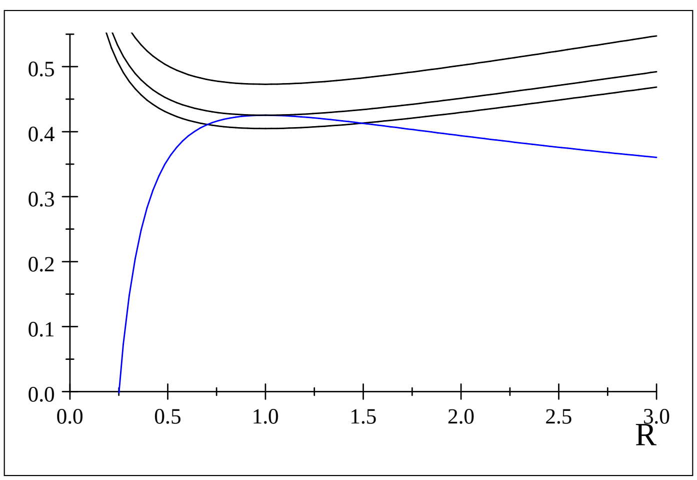
$$H_A(R;C) \equiv C(\eta + \beta R)\left(\frac{\alpha}{R}\right)^{\frac{\beta}{1+\beta}},$$
$$H_B(R) \equiv \beta\left(\frac{\alpha}{R}\right)^{\frac{1}{1-\alpha}}\left(\frac{R}{\alpha} - \eta\right).$$

Hence, steady states are found whenever some $R > 0$ satisfies $H_A(R, C) = H_B(R)$.

The appendix shows several properties of these functions. Of particular importance are that (i) given $C > 0$ and $R > 0$, H_A is always positive, (ii) H_A has an asymmetric "U" shape in the space of H_A vs R, (iii) H_A has a minimum at $R = \eta$, irrespective of C, and (iv) a higher constant C shifts the curve H_A up. The curve H_B in turn (i) doesn't depend on C, (ii) is positive only if $R > \alpha\eta$, (iii) has an asymmetric inverted "U" shape, (iv) has a maximum when $R = \eta$ that is independent of C, and (v) H_B asymptotically reaches 0 as $R \longrightarrow \infty$. The shape of these curves imply that if the parameterization allows for multiple solutions indexed by C, then there exists a maximum constant $\overline{C}$ which yield steady states. More specifically, when $C = \overline{C}$, H_A and H_B are tangent at the point $R = \eta$ (producing a single steady state as opposed to two). Then for any $\underline{C} < C < \overline{C}$ there always exist 2 intersections, one to the left of η and another to the right of η. These intersections get closer to each other (and to η) the higher the constant C is, where $\overline{C}$ yields the case with a single steady state. Finally, for any $C > \overline{C}$ there are no intersections.

For illustrative purposes, figure 1 presents these curves for Forni's parameterization, with $\alpha = 0.25$, $\beta = 0.9$, $\eta = 1$, for 3 different constants ($\underline{C} = 0.4108$ which yields 2 intersections, $\overline{C} = .43157$ which yields the tangency $R = \eta = 1$ and for $C = 0.48 > \overline{C}$ which yields no intersections).

Figure 1. Steady states as intersections of H_A and H_B



The constant $\overline{C}$ induces a single steady state which implies $R = \eta$ and thus the implied steady-state capital is consistent with the golden-rule level of capital accumulation, $k^*(\overline{C}) = \left[\frac{\alpha}{\eta}\right]^{\frac{1}{1-\alpha}}$. Also because $R = \eta$, one can solve for $\overline{C}$ in (4). This yields

$$\overline{C} = \left(\frac{\beta}{1+\beta}\right)\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha}{\eta}\right)^{\frac{(1+\alpha\beta)}{(1-\alpha)(1+\beta)}} > 0. \tag{5}$$

The steady state induced by $\overline{C}$ produces an analytic solution for its associated tax rate, which is found if one replaces (5) into (1) and also using $k^*\left(\overline{C}\right) = \left[\frac{\alpha}{\eta}\right]^{\frac{1}{1-\alpha}}$, which yields

$$\tau_{ss}^*\left(\overline{C}\right) = \frac{\beta}{1+\beta} - \frac{\alpha}{1-\alpha}. \tag{6}$$

Notice that in order for (6) to be positive, the parameters ought to satisfy $\frac{\beta}{1+\beta} > \frac{\alpha}{1-\alpha}$ (alternatively, $\alpha < \frac{\beta}{1+2\beta}$), which is precisely the condition that defines a dynamically inefficient economy in absence of social security.[3] The appendix shows that dynamic inefficiency is a necessary condition for the existence of PAYGO in any dynamically stable steady state (for any admissible C).

For a full characterization of the case with 2 steady states it is required for C to satisfy $\underline{C} < C < \overline{C}$ (Gonzalez-Eiras specifies $\underline{C} \leq \overline{C}$ but the inequality ought to be strict since $\overline{C}$ yields a single steady state, while $\underline{C}$ yields a tax rate of zero in the dynamically stable steady state).[4]

The equilibrium evolution of capital in (2) under the optimal policy function is such that k_{t+1} is strictly increasing in k_t, with a derivative $\frac{d\ln k_{t+1}}{d\ln k_t}$ (see appendix) given by

$$\frac{d\ln k_{t+1}}{d\ln k_t} = \left(\frac{R_t}{\eta}\right)\frac{J\left(R_t; C\right)}{J\left(R_{t+1;}C\right)} > 0$$

for a strictly positive function $J(\cdot) > 0$ defined as

$$J\left(R_t; C\right) \equiv \beta\left(\frac{\alpha}{R_t}\right)^{\frac{1}{1-\alpha}} + C\left(\frac{\beta\left(1-\alpha\right)}{1+\beta}\right)\left(\frac{\alpha}{R_t}\right)^{\frac{\beta(1-\alpha)-\alpha(1+\beta)}{(1-\alpha)(1+\beta)}} > 0.$$

The above expression holds for any steady state induced by $C \in (\underline{C}, \overline{C}]$. This implies that in any steady state, the slope of the capital-evolution function is given by $\frac{d\ln k_{t+1}}{d\ln k_t}|_{k^{ss}} = \frac{R}{\eta}$. Any dynamically-stable steady state satisfies $\frac{d\ln k_{t+1}}{d\ln k_t}|_{k^{ss}} = \frac{R}{\eta} < 1$ and it is therefore dynamically-inefficient, while the dynamically-unstable steady state ($\frac{d\ln k_{t+1}}{d\ln k_t}|_{k^{ss}} = \frac{R}{\eta} > 1$) is dynamically-efficient.

For the case with $C = \overline{C}$, the capital evolution equation is tangent to the 45 degree line in the space (k_{t+1}, k_t) and this happens when capital is at the golden-rule level. This curve is necessarily below the 45 degree line at any other points and therefore convergence to the steady state can only happen from above (i.e., $k > \left(\frac{\alpha}{\eta}\right)^{\frac{1}{1-\alpha}}$) and hence it is semi-stable. Figure 2 shows the capital evolution function (upward sloping curve), the equilibrium policy function (downward sloping)

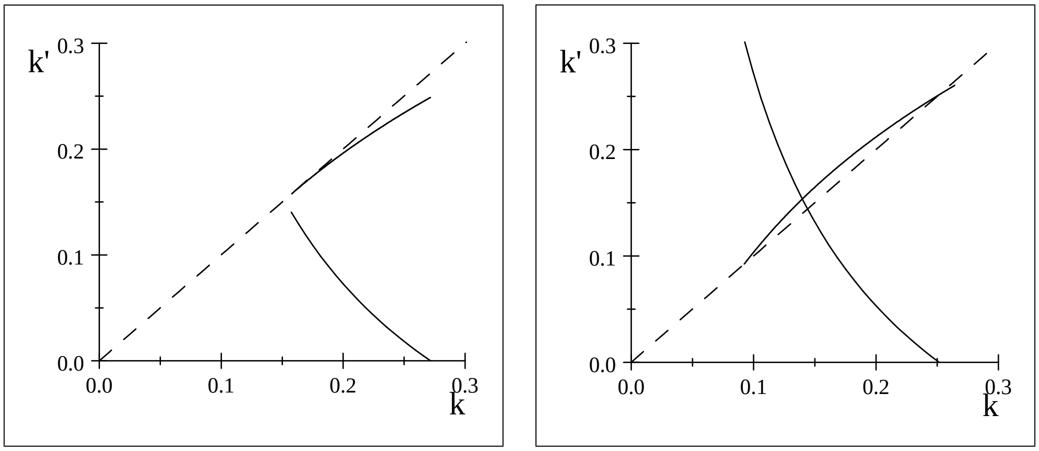
[3] The economy without social security has capital evolution given by $k_{t+1} = \frac{\beta(1-\alpha)}{(1+\beta)\eta}k_t^\alpha$ and steady state $k = \left[\frac{\beta(1-\alpha)}{(1+\beta)\eta}\right]^{\frac{1}{1-\alpha}}$. At steady state $R = \alpha k^{\alpha-1} = \frac{\alpha(1+\beta)\eta}{\beta(1-\alpha)}$. Therefore, $R < \eta$ if and only if $\frac{\beta}{(1+\beta)} > \frac{\alpha}{(1-\alpha)}$.

[4] It can also be easily proven that the condition $\underline{C} < \overline{C}$ implies that the economy is dynamically inefficient (just by using equations (3) and (5)).

and the 45 degree line (dashed line) under $\overline{C}$ and $\underline{C}$ for the parameterization considered above, taking into consideration the domain restriction for the existence of an equilibrium. [5]

Figure 2. Dynamic System under $\overline{C}$ & $\underline{C}$ $(\alpha = 0.25, \beta = 0.9, \eta = 1)$

Case with $C = \overline{C}$ Case with $C = \underline{C}$



Finally, the appendix proves that among all stable steady states, the one with the largest steady state tax rate is the one that is consistent with the golden rule of capital accumulation, while other dynamically-stable steady states do not solve the problem of dynamic inefficiency (since dynamic inefficiency is required for an MPE to exist). That the tax rate from (1) always satisfies $0 < \tau_t (k_t) < 1$ for the relevant domain of capital is also relegated to the appendix.

In summary, for all parameterizations such that $\eta \geq 1$, $\beta \in (0, 1)$ and $\alpha \in \left(0, \frac{\beta}{1+2\beta}\right)$, there exists a multiplicity of MPE indexed by the constant $C \in (\underline{C}, \overline{C}]$. For the case with $C = \overline{C}$ given by (5), the economy has a single steady state consistent with the golden-rule level of capital ($k^* \left(\overline{C}\right) = (\alpha/\eta)^{\frac{1}{1-\alpha}}$), a steady state tax rate given by (6) and which is dynamically stable for all $k \geq \left(\frac{\alpha}{\eta}\right)^{\frac{1}{1-\alpha}}$ (semi-stable). The economy converges to the steady state monotonically from above in $[(\alpha/\eta)^{\frac{1}{1-\alpha}}, \overline{k}(C)]$ and taxes during the transition necessarily satisfy $0 < \tau_t < 1$. For the cases with $C \in (\underline{C}, \overline{C})$, there exist 2 steady states denoted by $k_u^{ss}(C)$ and $k_s^{ss}(C)$, where the low-capital steady state $k_u^{ss}(C)$ is dynamically-efficient but dynamically-unstable and the high-capital steady state $k_s^{ss}(C)$ is dynamically-inefficient but dynamically-stable, where $0 < k_u^{ss}(C) < (\alpha/\eta)^{\frac{1}{1-\alpha}} < k_s^{ss}(C) < \overline{k}(C)$. For all initial capital in $(k_u^{ss}(C), \overline{k}(C)]$, capital converges monotonically to the stable steady state and the sequence of equilibrium taxes $\{\tau_t\}$ satisfy $0 \leq \tau_t < 1$. Only if one were to assume that the initial condition is given by $k_u^{ss}(C)$ then the economy would stay there (absence any perturbation), with a payroll tax satisfying $0 \leq \tau_t < 1$.

[5] For an equilibrium to exist, the domain of the curves is to satisfy $k \in \left[k_u^{ss}(C), \overline{k}(C)\right]$, where the lower limit $k_u^{ss}(C)$ refers to the unstable steady state (semi-stable for $C = \overline{C}$, which yields the case with a single steady state), and the upper limit $\overline{k}(C)$ refers to the equilibrium policy function that yields a zero tax rate. Gonzales-Eiras shows how to extend the dominion of admissible capital stocks, by means of assuming continuity in the policy function (as opposed to the more stringent requirement of differentiability). His conclusions about the shape of the capital evolution and for the equilibrium policy for $k < k_u^{ss}(C)$ and for $k > \overline{k}(C)$ still apply to this paper (under continuity of the policy function).

3 Concluding Remarks

This paper shows that Forni's (2005) political-economic equilibrium of PAYGO social security requires the underlying economy to be dynamically inefficient. Whether this is a satisfactory result for the existence and sustainability of social security is an empirical question. Abel et.al (1989) find that the US and six other developed economies are dynamically efficient, but these conclusions have recently been challenged by Geerolf (2018) who finds in a larger sample that these and other economies might not be. Since dynamic inefficiency might be a restrictive assumption, the literature has considered alternatives that do not require this assumption, including the use of an alternative equilibrium concept like subgame perfection (see Cooley and Soares (1999) and Boldrin and Rustichini (2000)) which allow for trigger strategies and which is a concept that permits for social security to depend on other variables, or a different way of aggregating preferences (e.g. probabilistic voting as opposed to a median voter framework) as in Gonzalez-Eiras and Niepelt (2008).

Finally, the analysis presented in this note holds for the case of logarithmic utility. Forni also considers the case with isoelastic preferences (CRRA), which is analyzed numerically. Whether the conclusions of this note hold in the isoelastic case is left for future research.

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4 Appendix

4.1 Proof on Steady States of the Model

The evolution of capital in (2) can be written for simplicity as

$\beta k_{t+1} + C k_{t+1}^{1-\upsilon} = \frac{\beta}{\eta}\left(k_t^{\alpha} - \alpha C k_t^{\alpha-\upsilon}\right)$, where $\upsilon \equiv \frac{1+\alpha\beta}{1+\beta}$.

Rewrite the expression at steady state, and group the terms with C on one side to obtain

$C\left(\eta k^{1-\alpha} + \alpha\beta\right) = \beta k^{\upsilon}\left(1 - \eta k^{1-\alpha}\right)$.

Since $R = \alpha k^{\alpha-1}$, one can substitute α/R instead of $k^{1-\alpha}$. Then multiply both sides of the above equation by $\frac{R}{\alpha}$ to obtain

$C\left(\eta + \beta R\right) = \frac{\beta}{\alpha} k^{\upsilon}\left(R - \eta\alpha\right)$

Since $k = \left[\frac{\alpha}{R}\right]^{\frac{1}{1-\alpha}}$, then $k^{\upsilon} = \left[\frac{\alpha}{R}\right]^{\frac{\upsilon}{1-\alpha}}$. Given that $\upsilon \equiv \frac{1+\alpha\beta}{1+\beta}$, then k^{υ} is given by $\left[\alpha^{\frac{1}{1-\alpha}-\frac{\beta}{1+\beta}}\right] R^{\frac{\beta}{1+\beta}-\frac{1}{1-\alpha}}$. Replace and rearrange to obtain

$$C\left(\eta + \beta R\right)\left(\frac{\alpha}{R}\right)^{\frac{\beta}{1+\beta}} = \beta\left(\frac{\alpha}{R}\right)^{\frac{1}{1-\alpha}}\left(\frac{R}{\alpha} - \eta\right).$$

Define the functions H_A and H_B as

$$H_A\left(R;C\right) \equiv C\left(\eta + \beta R\right)\left(\frac{\alpha}{R}\right)^{\frac{\beta}{1+\beta}}$$
$$H_B\left(R\right) \equiv \beta\left(\frac{\alpha}{R}\right)^{\frac{1}{1-\alpha}}\left(\frac{R}{\alpha} - \eta\right)$$

The strategy is to characterize the shapes of these functions since their intersections represent steady states.

The shape of H_A.

1. Given the space for parameters, it trivially follows that $H_A > 0$ for all $R > 0$.
2. The derivative $\frac{dH_A}{dC}$ is always positive, given by

$\frac{dH_A}{dC} = \alpha\left(\eta + \beta R\right)\left(\frac{\alpha}{R}\right)^{\frac{\beta}{1+\beta}} > 0$,

so that H_A is increasing in C.

3. From the definition of H_A, it trivially follows that $H_A > 0$ if and only if $C > 0$.
4. The derivative $\frac{dH_A}{dR}$ is given by

$\frac{dH_A}{dR} = \alpha^{\frac{\beta}{1+\beta}} C\left[\beta\left(R\right)^{\frac{-\beta}{1+\beta}} + \left(\eta + \beta R\right)\left(\frac{-\beta}{1+\beta}\right)\frac{\left(R\right)^{\frac{-\beta}{1+\beta}}}{R}\right]$

which simplifies to

$$\frac{dH_A}{dR} = \frac{\alpha^{\frac{\beta}{1+\beta}} C\beta\left(R\right)^{\frac{-\beta}{1+\beta}}}{1+\beta}\left[\frac{R-\eta}{R}\right]$$

The sign of $\frac{dH_A}{dR}$ is determined by the difference $R - \eta$, with $\frac{dH_A}{dR} < 0$ if and only if $R < \eta$ and

$\frac{dH_A}{dR} > 0$ if and only if $R > \eta$, with $\frac{dH_A}{dR} = 0$ when $R = \eta$. It follows that in the space of H_A *vs* R, H_A has a "U" shape with a minimum at $R = \eta$. In addition $sign\left\{\frac{d^2H_A}{dRdC}\right\} = sign\left\{\frac{dH_A}{dR}\right\}$ and hence a larger constant C becomes "steeper" away from the point where $R = \eta$.

The shape of H_B

5. Since $H_B = \beta\left(\frac{\alpha}{R}\right)^{\frac{1}{1-\alpha}}\left[\frac{R}{\alpha} - \eta\right]$, then $H_B > 0$ for all $R > \alpha\eta$, $H_B = 0$ when $R = \alpha\eta$ and $H_B < 0$ for all $0 < R < \alpha\eta$.

6. Taking limits when $R \to \infty$, yields $\lim_{R\to\infty} H_B(R) = 0$.

Proof. First write $H_B = \beta\alpha^{\frac{1}{1-\alpha}}\frac{1}{R^{\frac{1}{1-\alpha}}}\left[\frac{R-\alpha\eta}{\alpha}\right] = \beta\frac{\alpha^{\frac{1}{1-\alpha}}}{\alpha}\frac{[R-\alpha\eta]}{R^{1/(1-\alpha)}}$

$\lim_{R\to\infty} H_B(R) = \beta\alpha^{\frac{\alpha}{1-\alpha}}\lim_{R\to\infty}\frac{[R-\alpha\eta]}{R^{1/(1-\alpha)}} = \frac{\infty}{\infty}$. Thus one can use L'Hospital rule for the term $\lim_{R\to\infty}\frac{[R-\alpha\eta]}{R^{1/(1-\alpha)}}$, then taking the derivatives on numerator and denominator and then taking limits yields $\lim_{R\to\infty}\frac{[R-\alpha\eta]}{R^{1/(1-\alpha)}} = \frac{1}{\lim_{R\to\infty}\left[\frac{1}{(1-\alpha)}R^{\frac{\alpha}{(1-\alpha)}}\right]} = \frac{(1-\alpha)}{\infty} = 0$.

7. The slope of the H_B function is obtained by simple differentiation as

$\frac{dH_B}{dR} = \beta\left[\left(\frac{\alpha}{R}\right)^{\frac{1}{1-\alpha}}\frac{1}{\alpha} + \left[\frac{R}{\alpha} - \eta\right](\alpha)^{\frac{1}{1-\alpha}}\left(-\frac{1}{1-\alpha}\right)\frac{R^{-\frac{1}{1-\alpha}}}{R}\right]$

Which simplifies to

$$\frac{dH_B}{dR} = \left(\frac{\beta\alpha^{\frac{1}{1-\alpha}}}{1-\alpha}\right)\left[\frac{\eta - R}{R^{\frac{2-\alpha}{1-\alpha}}}\right]$$

Therefore $\frac{dH_B}{dR} > 0 \longleftrightarrow R < \eta$ and $\frac{dH_B}{dR} < 0 \longleftrightarrow R > \eta$ while $\frac{dH_B}{dR} = 0 \longleftrightarrow R = \eta$.

Hence H_B has an inverted "U" shape, with H_B being negative for all $R < \alpha\eta$, crossing the H_B plane (in the space of H_B vs R) when $R = \alpha\eta$; reaches a maximum when $R = \eta$, and then decreases asymptotically to 0.

Intersections of the curves (the steady states).

Given the shapes of the H_A and H_B functions, it follows from (1) - (7) above that

(i) There exists a unique value $C^* > 0$ such that H_A and H_B intersect at a unique point, which is when $R = \eta$ (the minimum of H_A and the maximum of H_B). Since this is the only value for C that yields the unique intersection (steady state), this value is by definition $C^* = \overline{C}$ and is given by equation (5). Hence $H_A\left(\eta, \overline{C}\right) = H_B(\eta)$.

(ii) If $C > \overline{C}$, then $H_A > H_B$ for all $R > 0$. There is no R^* that solves $H_A = H_B$. There doesn't exist a steady state if $C > \overline{C}$.

(iii) If $C = \overline{C}$ then curves H_A and H_B are tangent and intersect at a single point, with $R = \eta$. That is, $H_A\left(\eta; \overline{C}\right) = H_B(\eta)$. This yields a single steady state which yields the golden rule level of capital accumulation.

(iv) If $0 < C < \overline{C}$ then there exist 2 intersections (steady states), one for some R to the left of η (define it as $R_s^{ss}(C)$) and another to the right of η (define it by $R_u^{ss}(C)$). Since $H_B > 0$ only for $R > \alpha\eta$ and $H_A > 0$ for all $R > 0$, it follows that the 2 steady states satisfy $0 < \alpha\eta <$

$R_s^{ss}(C) < \eta < R_u^{ss}(C)$ where $R_s^{ss}(C)$ and $R_u^{ss}(C)$ represent each of the 2 steady states. Since $H_A - H_B > 0$ for all $0 < R \le \alpha\eta$, and since a steady state is found whenever $H_A - H_B = 0$, then there doesn't exist any steady state with an interest rate lower than $\alpha\eta$. Finally, since $R = \alpha k^{\alpha-1}$, the inequalities on the induced steady state interest rates imply that in capital levels we have that $0 < k_u^{ss}(C) < \left[\frac{\alpha}{\eta}\right]^{\frac{1}{1-\alpha}} < k_s^{ss}(C) < \left[\frac{1}{\eta}\right]^{\frac{1}{1-\alpha}}$.

Q.E.D.

4.2 The derivation of $\overline{C}$

Consider equation (2), which defines the evolution of capital. The equation at steady state is given by

$$\beta k + Ck^{1-\upsilon} = \frac{\beta}{\eta}\left(k^{\alpha} - \alpha C k^{\alpha-\upsilon}\right).$$

Divide by k to get

$$\beta + \frac{C}{k^{\upsilon}} = \frac{\beta}{\eta}k^{\alpha-1} - \alpha\frac{\beta}{\eta}C\frac{k^{\alpha-1}}{k^{\upsilon}}.$$

Since $k^{\alpha-1} = \frac{R}{\alpha}$, one can write

$$\beta + \frac{C}{k^{\upsilon}} = \frac{\beta}{\eta}\frac{R}{\alpha} - \frac{\beta}{\eta}C\frac{R}{k^{\upsilon}}.$$

Since at the steady state induced by $\overline{C}$ it is the case that $R = \eta$, one can rewrite the above equation as

$$\beta + \frac{\overline{C}}{k^{\upsilon}} = \frac{\beta}{\alpha} - \frac{\beta\overline{C}}{k^{\upsilon}},$$

then solving for $\overline{C}$ yields

$$\overline{C} = \left(\frac{\beta}{1+\beta}\right)\left(\frac{1-\alpha}{\alpha}\right)k^{\upsilon}.$$

Finally, since (i) $k = \left(\frac{\alpha}{R}\right)^{\frac{1}{1-\alpha}}$, (ii) $R = \eta$ when using $\overline{C}$, (iii) $k = \left(\frac{\alpha}{\eta}\right)^{\frac{1}{1-\alpha}}$ when using $\overline{C}$ and from the definition of $\upsilon \equiv \left(\frac{1+\alpha\beta}{1+\beta}\right)$, one can finally write

$$\overline{C} = \left(\frac{\beta}{1+\beta}\right)\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha}{\eta}\right)^{\frac{(1+\alpha\beta)}{(1-\alpha)(1+\beta)}},$$

which is the constant that yields the golden rule level of capital.

Q.E.D.

4.3 The slope of the capital evolution function

This section obtains $\frac{d\ln k_{t+1}}{d\ln k_t}$. Start from equation (2), given by

$$\beta k_{t+1} + Ck_{t+1}^{1-\upsilon} = \frac{\beta}{\eta}\left(k_t^{\alpha} - \alpha C k_t^{\alpha-\upsilon}\right).$$

Take a derivative with respect to k_t on both sides, obtain

$$\beta\frac{dk_{t+1}}{dk_t} + C(1-\upsilon)k_{t+1}^{-\upsilon}\frac{dk_{t+1}}{dk_t} = \frac{\beta}{\eta}\left(\alpha k_t^{\alpha-1} - \alpha(\alpha-\upsilon)Ck_t^{\alpha-\upsilon-1}\right).$$

Solving for $\frac{dk_{t+1}}{dk_t}$, this yields

$$\frac{dk_{t+1}}{dk_t} = \frac{\beta\alpha k_t^{\alpha-1}}{\eta}\frac{1-(\alpha-\upsilon)Ck_t^{-\upsilon}}{\beta+C(1-\upsilon)k_{t+1}^{-\upsilon}} = \frac{R_t}{\eta}\frac{\beta-\beta(\alpha-\upsilon)Ck_t^{-\upsilon}}{\beta+C(1-\upsilon)k_{t+1}^{-\upsilon}},$$

where the last equality is obtained when one uses $R_t = \alpha k_t^{\alpha-1}$.

Given that $\upsilon \equiv \frac{(1+\beta\alpha)}{(1+\beta)}$, then $(\alpha-\upsilon)$ and $(1-\upsilon)$ are given by $(\alpha-\upsilon) = -\left(\frac{1-\alpha}{1+\beta}\right)$ and $(1-\upsilon) = \frac{\beta(1-\alpha)}{1+\beta}$. Replacing them in the above equation yields

$$\frac{dk_{t+1}}{dk_t} = \frac{R_t}{\eta}\frac{\beta-\beta\left(-\left(\frac{1-\alpha}{1+\beta}\right)\right)Ck_t^{-\upsilon}}{\beta+C\left(\frac{\beta(1-\alpha)}{1+\beta}\right)k_{t+1}^{-\upsilon}} = \frac{R_t}{\eta}\frac{\beta+C\left(\frac{\beta(1-\alpha)}{1+\beta}\right)k_t^{-\upsilon}}{\beta+C\left(\frac{\beta(1-\alpha)}{1+\beta}\right)k_{t+1}^{-\upsilon}}.$$

Given that $R_t = \alpha k_t^{\alpha-1}$ and $\upsilon \equiv \frac{(1+\beta\alpha)}{(1+\beta)}$, then $k_t^{-\upsilon} = \left(\frac{\alpha}{R_t}\right)^{\frac{-(1+\beta\alpha)}{(1-\alpha)(1+\beta)}}$. Hence one can write

$$\frac{dk_{t+1}}{dk_t} = \frac{R_t}{\eta}\frac{\beta+C\left(\frac{\beta(1-\alpha)}{1+\beta}\right)\left(\frac{\alpha}{R_t}\right)^{\frac{-(1+\beta\alpha)}{(1-\alpha)(1+\beta)}}}{\beta+C\left(\frac{\beta(1-\alpha)}{1+\beta}\right)\left(\frac{\alpha}{R_{t+1}}\right)^{\frac{-(1+\beta\alpha)}{(1-\alpha)(1+\beta)}}} > 0,$$

which is clearly positive as all elements in numerator and denominator are positive. In terms of an elasticity, multiply both sides by k_t/k_{t+1} in order to obtain $\frac{d\ln k_{t+1}}{d\ln k_t}$. This yields the expressions as claimed in the text

$$\frac{d\ln k_{t+1}}{d\ln k_t} = \frac{R_t}{\eta}\frac{J(R_t)}{J(R_{t+1})}$$
$$J(R) = \beta\left(\frac{\alpha}{R}\right)^{\frac{1}{1-\alpha}} + C\left(\frac{\beta(1-\alpha)}{1+\beta}\right)\left(\frac{\alpha}{R}\right)^{\frac{\beta(1-\alpha)-\alpha(1+\beta)}{(1-\alpha)(1+\beta)}} > 0$$

The above expression, when evaluated at steady state yields

$$\frac{d\ln k_{t+1}}{d\ln k_t}|_k = \frac{R}{\eta} > 0$$

The proof on steady states above shows that there can be 1 or 2 steady states for this model. Consider the case with 2 steady states (i.e. when $C < \overline{C}$). Let $R_s^{ss}(C)$ (with associated steady state capital stock k_s^{ss}) and $R_u^{ss}(C)$ (with associated steady state capital stock k_u^{ss}) represent the steady state gross interest rates where (as shown in the proof in the previous section) it is the case that $0 < \alpha\eta < R_s^{ss}(C) < \eta < R_u^{ss}(C)$, and where $R_s^{ss}(C)$ is the interest rate associated to the dynamically inefficient equilibrium while $R_u^{ss}(C)$ is associated to the dynamically efficient equilibrium. Then

(i) $\alpha < \frac{d\ln k_{t+1}}{d\ln k_t}|_{k_s^{ss}} = \frac{R_s^{ss}(C)}{\eta} < 1$: *the dynamically inefficient equilibrium ($\alpha\eta < R_s^{ss}(C) < \eta$) is dynamically stable.*

(ii) $\frac{d\ln k_{t+1}}{d\ln k_t}|_{k_u^{ss}} = \frac{R_u^{ss}(C)}{\eta} > 1$: *the dynamically efficient equilibrium ($R_u^{ss}(C) > \eta$) is dynamically unstable.*

For the case with a single steady state, which is the case where $C = \overline{C}$, yields $R = \eta$. Hence in this case

(iii) $\frac{dk_{t+1}}{dk_t}|_{k(\overline{C})} = 1$

Hence whenever 2 steady states exist (that is, whenever $C \in (0, C\max)$, only the dynamically inefficient steady state ($R_s^{ss}(C) < \eta$) is dynamically stable since $\frac{dk_{t+1}}{dk_t}|_{k^*} = \frac{R_s^{ss}(C)}{\eta} < 1$. In addition, for the case where $C = \overline{C}$, the steady state is dynamically efficient (Since $R = \eta$) and consistent with the golden-rule level of capital accumulation, and it is dynamically stable for all k above the steady state level (thus it is semi-stable).

Q.E.D.

4.4 Stability when $C = \overline{C}$

1. From the proofs on steady states, we have that $R\left(\overline{C}\right) = \eta$.

2. The second derivative$\frac{d^2 \ln k_{t+1}}{d \ln k_t^2}|_k$ is negative when evaluated at the steady state induced by $\overline{C}$, the golden rule level of capital.

Proof:

Start from the fact that

$\frac{d \ln k_{t+1}}{d \ln k_t} = \frac{d \ln R_{t+1}}{d \ln R_t}$ and $\frac{d \ln R_{t+1}}{d \ln R_t} = \frac{R_t}{\eta} \frac{J(R_t)}{J(R_{t+1})}$

Then taking a derivative with respect to $\ln k_t$ on both sides of the first equality yields

$\frac{d^2 \ln k_{t+1}}{d \ln k_t^2} = \frac{d^2 \ln R_{t+1}}{d \ln R_t^2} \frac{d \ln R_t}{d \ln k_t}$,

since $\frac{d \ln R_t}{d \ln k_t} = (\alpha - 1)$ (from the definition of $R_t = \alpha k_t^{\alpha-1}$), then $\frac{d^2 \ln R_{t+1}}{d \ln R_t^2} > 0 \iff \frac{d^2 \ln k_{t+1}}{d \ln k_t^2} < 0$.

For simplicity, in what follows, write $J(R_t)$ as J_t and $J(R_{t+1})$ as J_{t+1}.

Compute $\frac{d^2 \ln R_{t+1}}{d \ln R_t^2}$ from $\frac{d \ln R_{t+1}}{d \ln R_t} = \frac{R_t}{\eta} \frac{J(R_t)}{J(R_{t+1})}$, obtain

$$\frac{d^2 \ln R_{t+1}}{d \ln R_t^2} = \frac{1}{\eta} \left[\frac{J_{t+1} \frac{d(R_t J_t)}{d \ln R_t} - R_t J_t \frac{d J_{t+1}}{d \ln R_{t+1}} \frac{d \ln R_{t+1}}{d \ln R_t}}{J_{t+1}^2} \right]$$

Then use $\frac{d(R_t J_t)}{d \ln R_t} = \frac{d(R_t J_t)}{d R_t} R_t$; $\frac{d J_{t+1}}{d \ln R_{t+1}} = J'_{t+1} R_{t+1}$ to get

$$\frac{d^2 \ln R_{t+1}}{d \ln R_t^2} = \frac{1}{\eta} \frac{R_t}{J_{t+1}} \left[\frac{d(R_t J_t)}{d R_t} - \frac{J_t J'_{t+1} R_{t+1}}{J_{t+1}} \frac{d \ln R_{t+1}}{d \ln R_t} \right]$$

Replace $\frac{d \ln R_{t+1}}{d \ln R_t}$ by $\frac{R_t}{\eta} \frac{J_t}{J_{t+1}}$ and replace $\frac{d(R_t J_t)}{d R_t} = R_t J'_t + J_t$. This yields

$$\frac{d^2 \ln R_{t+1}}{d \ln R_t^2} = \frac{1}{\eta} \frac{R_t}{J_{t+1}} \left[R_t J'_t + J_t - \left(\frac{J_t}{J_{t+1}} \right)^2 \left(J'_{t+1} R_{t+1} \right) \frac{R_t}{\eta} \right]$$

At any steady state, the above simplifies to

$$\frac{d^2 \ln R_{t+1}}{d \ln R_t^2} = \frac{1}{\eta} \frac{R}{J} \left[J + R J' \left(1 - \frac{R}{\eta} \right) \right]$$

Therefore at the golden-rule (for $R\left(\overline{C}\right) = \eta$, which implies the capital level of $k\left(\overline{C}\right) = \left[\frac{\alpha}{\eta}\right]^{\frac{1}{1-\alpha}}$), it is the case that

$\frac{d^2 \ln R_{t+1}}{d \ln R_t^2}|_{k\left(\overline{C}\right)} = \frac{1}{\eta} \frac{R}{J} J = \frac{R}{\eta} = 1$, which implies that $\frac{d^2 \ln k_{t+1}}{d \ln k_t^2}|_{k\left(\overline{C}\right)} = (\alpha - 1) < 0$.

3. The slope $\frac{d \ln k_{t+1}}{d \ln k_t}$ is always positive, and from item (2) above, $\frac{d^2 \ln k_{t+1}}{d \ln k_t^2}|_k < 0$ at the steady state. This implies that local stability at the point that yields $R = \eta$ is guaranteed.

4. To establish stability for any initial level of capital $k_0 > \left[\frac{\alpha}{\eta}\right]^{\frac{1}{1-\alpha}}$, notice that $\frac{d \ln k_{t+1}}{d \ln k_t}$ is always positive but step (1) above implies that there exists a unique steady state. This implies that the capital evolution equation is always below the 45 degree line in the phase diagram depicting the transition of capital (k_{t+1} vs k_t) except at the steady state, which is when $k = \left[\frac{\alpha}{\eta}\right]^{\frac{1}{1-\alpha}}$. This shape of the phase diagram implies that capital diverges to a negative number whenever the initial condition is $k_0 < \left[\frac{\alpha}{\eta}\right]^{\frac{1}{1-\alpha}}$. However, for $k_0 > \left[\frac{\alpha}{\eta}\right]^{\frac{1}{1-\alpha}}$ the shape of the phase diagram (positive

sloped but below the 45 degree line for all $k \neq \left[\frac{\alpha}{\eta}\right]^{\frac{1}{1-\alpha}}$) implies that capital monotonically converges to the steady state ($k_t \downarrow \left[\frac{\alpha}{\eta}\right]^{\frac{1}{1-\alpha}}$).

Q.E.D.

4.5 Proof that $\overline{C}$ yields the largest tax rate at stable steady states

1. From expression (1), C affects the steady state tax rate τ_{ss} directly, and it also affects the steady state level of capital. The total effect can be written as

$$\frac{d\tau_{ss}(C)}{dC} = \frac{\partial \tau_{ss}(C)}{\partial C} + \frac{\partial \tau_{ss}(C)}{\partial \ln k}\frac{d\ln k}{dC}.$$

The next lines show that $\frac{\partial \tau_{ss}(C)}{\partial C} > 0$, $\frac{\partial \tau_{ss}(C)}{\partial \ln k} < 0$ & $\frac{d\ln k}{dC} < 0$ when $k = k_s^{ss}$, which in turn imply that $\frac{d\tau_{ss}(C)}{dC} > 0$.

2. The term $\frac{\partial \tau_{ss}(C)}{\partial C}$ is easily obtained from (1) as $\frac{\partial \tau_{ss}(C)}{\partial C}|_k = \left(\frac{\alpha}{1-\alpha}\right) k_t^{-(1+\alpha\beta)/(1+\beta)} > 0$, therefore it is always positive.

3. The term $\frac{\partial \tau_{ss}(C)}{\partial \ln k}$ is given by $\frac{\partial \tau_{ss}(C)}{\partial \ln k} = -\left(\frac{1+\alpha\beta}{1+\beta}\right)\left(\frac{\alpha}{1-\alpha}\right)\frac{k_t^{-(1+\alpha\beta)/(1+\beta)}}{k_t} < 0$, always negative.

4. The term $\frac{d\ln k}{dC}$ is to be evaluated at the dynamically stable steady state ($\frac{d\ln k}{dC}|_{k_s^{ss}}$) and in turn can be written as $\frac{d\ln k_s^{ss}}{dC}$ and can be computed as the following product $\frac{d\ln k_s^{ss}}{dR_s^{ss}}\frac{dR_s^{ss}}{dC}$. Since from the proof above on steady states and their relationship to C, the dynamically stable steady state interest rate is increasing in C for all $C \in \left(\underline{C}, \overline{C}\right)$, hence $\frac{d\ln \underline{R}(C)}{dC} > 0$. Also since $R = \alpha k^{\alpha-1}$, then it trivially follows that $\frac{d\ln k_s^{ss}}{dR_s^{ss}} < 0$. Hence, $\frac{d\ln k_s^{ss}}{dC} = \frac{d\ln k_s^{ss}}{dR_s^{ss}}\frac{dR_s^{ss}}{dC} < 0$, thus always negative.

5. Using the above lines, then $\frac{d\tau_{ss}(C)}{dC} > 0$. It follows that since $\overline{C}$ is the maximum constant that can yield a dynamically stable steady state, then $\overline{C}$ dictates the highest possible tax rate in a dynamically stable steady state.

Q.E.D.

4.6 Proof that the tax rate is lower than 100% in any steady state

First compute the tax rate as a function of the current gross interest rate. Rewrite (1) in terms of the interest rate using the fact that $R = \alpha k^{\alpha-1}$, this yields

$$\tau^*(R_t) = \left(\frac{\alpha}{1-\alpha}\right)\left(C\left[\frac{\alpha}{R_t}\right]^{\frac{\beta}{1+\beta}-\frac{1}{1-\alpha}} - 1\right)$$

Now from the equation determining the steady states in the interest rates, we have that

$$C\left(\eta + \beta R_{ss}\right)\left(\frac{\alpha}{R_{ss}}\right)^{\frac{\beta}{1+\beta}} = \beta\left(\frac{\alpha}{R_{ss}}\right)^{\frac{1}{1-\alpha}}\left(\frac{R_{ss}}{\alpha} - \eta\right)$$

One can solve for $C\left[\frac{\alpha}{R_{ss}}\right]^{\frac{\beta}{1+\beta}-\frac{1}{1-\alpha}}$ in the above equation as

$C\left(\frac{\alpha}{R_{ss}}\right)^{\frac{\beta}{1+\beta}-\frac{1}{1-\alpha}} = \frac{\beta\left(\frac{R_{ss}}{\alpha}-\eta\right)}{(\eta+\beta R_{ss})}$

Therefore the equation $\tau^{*}(R_t)$, when evaluated at R_{ss} (some steady state interest rate) can be written as

$\tau^{*}(R_{ss}) = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{\beta\left(\frac{R_{ss}}{\alpha}-\eta\right)}{(\eta+\beta R_{ss})} - 1\right)$

which can be rewritten as

$$\tau_{ss}\left(R_{ss}\right) = \frac{1}{1-\alpha}\left[\left(\frac{R_{ss}-\alpha\eta}{R_{ss}}\right)\left(\frac{\beta R_{ss}}{\eta+\beta R_{ss}}\right) - \alpha\right] \tag{A.1}$$

This function shows the tax rate as a function of any interest rates R that can be generated as steady state.The above equation is useful in that one can study the bounds on the tax rates given our knowledge on the possible steady states. From the proof on steady states above, necessarily it is the case that $R_{ss} - \alpha\eta > 0$.

Notice that in any steady state, we have that $\tau_{ss}\left(R_{ss}\right) < 1$ since $0 < \left(\frac{R_{ss}-\alpha\eta}{R_{ss}}\right) < 1$ and $0 < \left(\frac{\beta R_{ss}}{\eta+\beta R_{ss}}\right) < 1$, which in turn implies that the term in brackets in the above expression is lower than $1-\alpha$, hence in any steady state associated to any $C \in (\underline{C}, \overline{C}]$, it is the case that $\tau_{ss}\left(R_{ss}\right) < 1$.

Q.E.D.