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Soliton solutions to integrable equations

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SOLITON SOLUTIONS TO INTEGRABLE EQUATIONS

A Thesis

by

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Abstract

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In recent years, integrable systems and soliton theory play an important role in the study of nonlinear water wave equations. In this thesis, we will focus on the procedure of how to get soliton solutions for integrable equations. The fundamental idea is to use the traveling wave setting to convert a partial differential equation to an ordinary differential equation and to solve ordinary differential equations yields soliton solutions for the integrable equations under certain boundary conditions at both negative and positive infinities. In our work, we will consider five integrable equations and present their solitons solutions, one of which will be solved using the so-called bilinear approach. All the solutions will be given in either an explicit or an implicit form, and we will show how their graphs look like. In our future work, we will work on multi-solitons and Lax pair scheme.

Dedication

The completion of my graduate studies would not have been possible without the love and support of my family. My parents, and Liang Ding, wholeheartedly inspired, motivated and supported me by all means to accomplish this degree.

Thank you for your love and patience.

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I will always be grateful to Dr. Zhijun Qiao, chair of my master thesis committee and thesis advisor, for all his mentoring and advice. Many thanks go to my master thesis committee members. Their advice, input, and comments on my thesis helped to ensure the quality of my intellectual work.

I would also like to thank the integrable system research group who helped me generate the initial ideal for my research. Also, I would like to acknowledge my teachers in mathematics department who participated in my focus research.

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Chapter I

Introduction

Integrable systems and soliton theory have played an important role in the study of nonlinear wave equations. In this thesis, we will focus on using the traveling wave setting and bilinear approach to solve the following physical partial differential equations, and to find their soliton solutions. The traveling-wave solution of the wave equation was first published by d'Alembert in 1747. In recent years, the traveling wave setting has been widely used to find the soliton solutions. Bilinear approach was proposed by Ryogo Hirota in the book "The Direct Method in Soliton Theory"[**8**].

Here are the first two equations we will solve:

$$u_t = (\frac{1}{\sqrt{u}})_{xxx} - (\frac{1}{\sqrt{u}})_x, \tag{I.1}$$

$$u_t = (\frac{1}{\sqrt{u}})_{xxx} + (\frac{1}{\sqrt{u}})_x. \tag{I.2}$$

The two equations are both Harry-Dym type equations, equation (1.1) is the second number in the positive Camassa-Holm hierarchy [**6**][**5**].

Harry Dym is a professor at the Weizmann Institute of Science, Israel, whose research interests include operator theory, interpolation theory, and inverse problems. He is famous for the discovery of the Dym equation. In mathematics, and particular in the theory of solitons, the Dym equation (HD) is the third-order partial differential equation:

$$u_t = u^3 u_{xxx}.$$

If we let $u = \frac{1}{\sqrt{v}}$, the previous equation is often written as the following equivalent form:

$$v_t = (v^{-\frac{1}{2}})_{xxx}.$$

The Dym equation first appeared in Kruskal paper [**14**], and it is attributed to an unpublished paper by Harry Dym.

Here is the third equation:

$$u_{txx} + 2k^2 u_x + 2u_x u_{xx} + u u_{xxx} = 0. \tag{I.3}$$

This is the short wave model for Camassa-Holm. In fluid dynamics, the Camassa-Holm equation is the integrable, dimensionless and non-linear partial differential equation:

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + u u_{xxx}.$$

The equation was introduced by Camassa and Holm as a bi-Hamiltonian model for waves in shallow water, and in this context the parameter is positive and the solitary wave solutions are smooth solitons. In the special case when it is equal to zero, the Camassa-Holm equation has peakon solutions.

Here is the forth equation:

$$v_{xt} = 2vv_{xx} + v_x^2. \tag{I.4}$$

The name of this equation is Hunter-Zheng equation, this equation is proposed by John K. Hunter and Yuxi Zheng [**3**], and it arises in two different physical contexts in two nonequivalent variational forms. It describes the propagation of the weekly nonlinear orientation waves in a massive nematic liquid crystal director field, and it is the high-frequency limit of the Camassa-Holm equation, which is an integrable model equation for shallow water waves.

The following equation is the last one:

$$u_{xt} = u - \frac{1}{6}u^3. \tag{I.5}$$

This is an exercise of the book "The Soliton Theory", written by Dengyuan Chen[**7**].

Chapter II

Harry-Dym Type Equation, Subtraction

We give the first integrable equation

$$u_t = (\frac{1}{\sqrt{u}})_{xxx} - (\frac{1}{\sqrt{u}})_x. \tag{II.1}$$

This above equation is called the Harry-Dym type Equation. It is the first member in the positive Camassa-Holm hierarchy[**6**][**5**].

Here u is a function of two variables: x and t. If we let $v = (\frac{1}{\sqrt{u}}) = u^{-1/2}$, then $u = v^{-2}$, $u_t = -2v^{-3}v_t$, equation (II.1) will be changed to

$$-2v^{-3}v_t = v_{xxx} - v_x. \tag{II.2}$$

Let $v(x,t) = V(x-ct) = V(\xi)$, we call this transformation the traveling wave setting, here c is a constant, which is the traveling wave speed.

By using the traveling wave setting, equation (II.2) will be changed to

$$2cV^{-3}V' = V''' - V'. \tag{II.3}$$

After integrating, we get

$$-cV^{-2} = V'' - V + A, \tag{II.4}$$

here A is the integral constant, and $A \neq 0$. Because if $A = 0$, the result we find does not satisfy the boundary values.

Assume $V'(\xi) = W(V), V = V(\xi)$, so $V'' = W'(V) \cdot V'(\xi) = W \cdot W'(V) = W\frac{dW}{dV}$, substituting these into equation (II.5) will have the following

$$-cV^{-2} = W\frac{dW}{dV} - V + A \Longrightarrow 2WdW = (2V - 2cV^{-2} + A)dV. \tag{II.5}$$

After integrating, we have

$$W^2 = V^2 + 2cV^{-1} + AV + B, \tag{II.6}$$

here B is the integral constant.

So

$$W = \frac{dV}{d\xi} = \pm\sqrt{\frac{V^3+AV^2+BV+2c}{V}}$$
$$\Longrightarrow \sqrt{\frac{V}{V^3+AV^2+BV+2c}}dV = \pm d\xi$$

$$\Longrightarrow \int \sqrt{\frac{V}{V^3+AV^2+BV+2c}}dv = \pm\xi + D. \tag{II.7}$$

D is a constant.

If we let $A = 3(2c)^{\frac{1}{3}}, B = 3(2c)^{\frac{2}{3}}$, then

$$\begin{aligned} L.H.S &= \int \sqrt{\frac{V}{[V+(2c)^{\frac{1}{3}}]^3}}dV \\ &= \int \frac{\sqrt{V}}{[V+(2c)^{\frac{1}{3}}]^{\frac{3}{2}}}dV \\ &= -2\int \sqrt{V}d(\frac{1}{\sqrt{V+(2c)^{\frac{1}{3}}}}). \end{aligned}$$

Integral by parts, we have

$$= -2\frac{\sqrt{V}}{\sqrt{V+(2c)^{\frac{1}{3}}}} + 2\int \frac{1}{\sqrt{V+(2c)^{\frac{1}{3}}}}d\sqrt{V}$$
$$= -2\frac{\sqrt{V}}{\sqrt{V+(2c)^{\frac{1}{3}}}} + 2\ln|\sqrt{V} + \sqrt{V+(2c)^{\frac{1}{3}}}|.$$

So we say that:

$$-2\frac{\sqrt{V}}{\sqrt{V+(2c)^{\frac{1}{3}}}}+2\ln|\sqrt{V}+\sqrt{V+(2c)^{\frac{1}{3}}}|=\pm\xi+D. \tag{II.8}$$

Replacing $V(\xi)=v(x,t)=u^{-\frac{1}{2}}$ in equation (II.8)gives the final answer:

$$-2\frac{u^{-\frac{1}{4}}}{\sqrt{u^{-\frac{1}{2}}+(2c)^{\frac{1}{3}}}}+2\ln|u^{-\frac{1}{4}}+\sqrt{u^{-\frac{1}{2}}+(2c)^{\frac{1}{3}}}|$$
$$=\pm(x-ct)+D.$$

After graphing, the graph looks like the cuspon soliton solution.

Chapter III

Harry-Dym Type Equation, Addition

Let us consider the second equation:

$$u_t = (\frac{1}{\sqrt{u}})_{xxx} + (\frac{1}{\sqrt{u}})_x. \tag{III.1}$$

This equation is still the Harry-Dym type Equation.

Let $v = (\frac{1}{\sqrt{u}}) = u^{-1/2}$, then $u = v^{-2}$, $u_t = -2v^{-3}v_t$. So equation (III.1) is changed to

$$-2v^{-3}v_t = v_{xxx} + v_x. \tag{III.2}$$

Let $v(x,t) = V(x-ct) = V(\xi)$, equation(III.2) is changed to

$$2cV^{-3}V' = V''' + V'. \tag{III.3}$$

After integrating, we get

$$-cV^{-2} = V'' + V + B, \tag{III.4}$$

here B is an arbitrary constant.

If we assume $\lim_{\xi\to\pm\infty} V = A$, A is an arbitrary constant, we can separate the equation into two cases: $A = 0$ and $A \neq 0$.

Case 1: If $A = 0$, then there is no soliton solution.

Case 2: If $A \neq 0$, using the boundary value to equation(III.4),we find that $B = -cA^{-2} - A$. Multiply V' for both sides, integrate,combine the similar terms and use boundary values so that we get

$$(V')^2 = -V^2 + 2cV^{-1} + 2AV + 2cA^{-2}V - A^2 - 4cA^{-1}$$
$$= \tfrac{-(V-A)^2(V-2cA^{-2})}{V}.$$

So

(III.5) $$V' = \frac{dV}{d\xi} = \pm\sqrt{\frac{-(V-A)^2(V-2cA^{-2})}{V}}.$$

Rewrite equation (III.5) into:

(III.6) $$\frac{1}{V-A}\sqrt{\frac{V}{2cA^{-2}-V}}dV = \pm d\xi.$$

Here we let $\sqrt{\frac{V}{2cA^{-2}-V}} = W$, then

$$V = \tfrac{2cA^{-2}W^2}{1+W^2} \Longrightarrow dV = \tfrac{4cA^{-2}W^3}{(1+W^2)^2}dW.$$

Using this transformation to the equation(III.6), by integrating, the left side is changed into

$$\textstyle\int \frac{1}{V-A}\sqrt{\frac{V}{2cA^{-2}-v}}dV$$

(III.7) $$= \quad 2\int \frac{1}{(2cA^{-3}-1)W^2-1}dW + 2\arctan W.$$

We sperate this case into three subcases:

Subcase 1: If $2cA^{-3} - 1 > 0$, then previous equation (III.7) is changed to the following:

$$2\int \frac{1}{(2cA^{-3}-1)W^2-1}dW + 2\arctan W$$

$$\text{(III.8)} \qquad = \frac{1}{\sqrt{2cA^{-3}-1}}\ln|\frac{\sqrt{2cA^{-3}-1}W-1}{\sqrt{2cA^{-3}+1}W+1}| + 2\arctan t = \pm\xi + F,$$

here F is the integral constant.

In terms of u, we have

$$\frac{1}{\sqrt{2cA^{-3}-1}}\ln|\frac{\sqrt{2cA^{-3}\sqrt{u}-1}-\sqrt{2cA^{-2}-\sqrt{u}}}{\sqrt{2cA^{-3}\sqrt{u}-1}+\sqrt{2cA^{-2}-\sqrt{u}}}| + 2\arctan\sqrt{\frac{1}{2cA^{-2}\sqrt{u}-1}}$$

$$\text{(III.9)} \qquad = \pm(x-ct)+F.$$

Subcase 2: If $2cA^{-3}-1<0$, then previous equation (III.7)is changed to:

$$-2\int \frac{1}{(1-2cA^{-3})W^2+1}dW + 2\arctan W$$

$$\text{(III.10)} \qquad = -\frac{2}{\sqrt{1-2cA^{-3}}}\arctan(\sqrt{1-2cA^{-3}}W) + 2\arctan W = \pm\xi + F,$$

here F is the integral constant.

In terms of u, we have

$$-\frac{2}{\sqrt{1-2cA^{-3}}}\arctan\sqrt{\frac{1-2cA^{-3}}{2cA^{-2}\sqrt{u}-1}} + 2\arctan\sqrt{\frac{1}{2cA^{-2}\sqrt{u}-1}}$$

$$\text{(III.11)} \qquad = \pm(x-ct)+F.$$

Subcase 3: If $2cA^{-3} - 1 = 0$, then previous equation (III.7) is changed to:

$$
\begin{aligned}
&-2\int \tfrac{1}{(1-2cA^{-3})W^2+1} dW + 2\arctan W \\
&= -2\int dW + 2\ln W = -2W + 2\ln W = \pm\xi + F,
\end{aligned} \tag{III.12}
$$

here F is the integral constant.

In terms of u, we have

$$
-2\sqrt{\frac{1}{2cA^{-2}\sqrt{u}-1}} + 2\arctan\sqrt{\frac{1}{2cA^{-2}\sqrt{u}-1}} = \pm(x-ct) + F. \tag{III.13}
$$

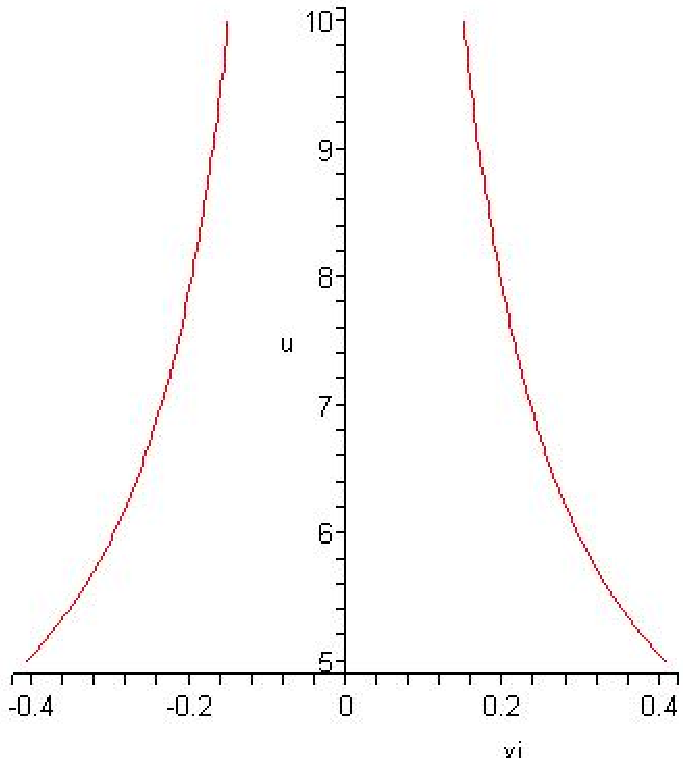

Figure III.1. 2-D graph for the subcase 2,$c = 0.25, A = 1, F = 0, \xi = -10..10, u = 5..10$

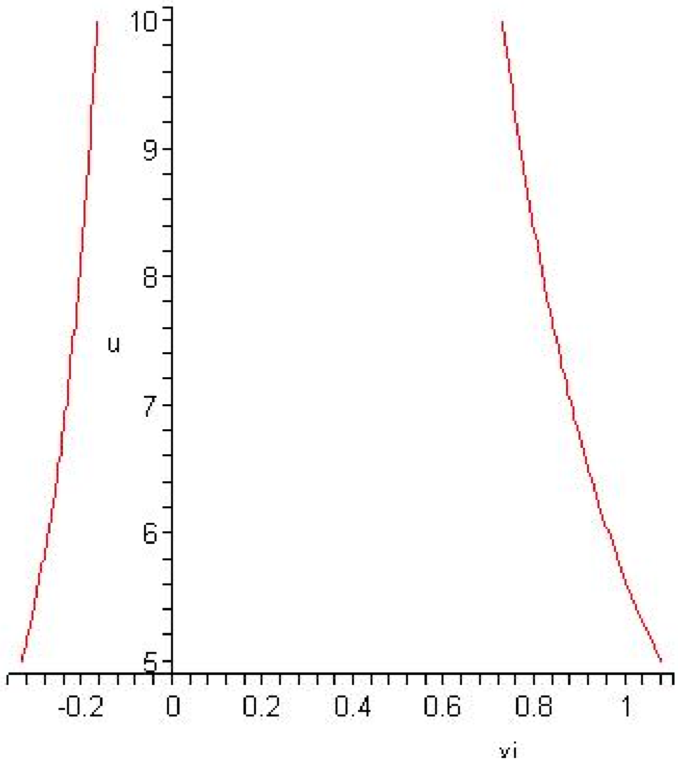

Figure III.2. 2-D graph for the subcase 3,$c = 0.5, A = 1, F = 0, \xi = -5..5, u = 5..10$

Chapter IV

Short wave model for Camassa-Holm

The next equation we will solve is the short wave model for Camassa-Holm

$$u_{txx} + 2k^2 u_x + 2u_x u_{xx} + u u_{xxx} = 0. \tag{IV.1}$$

This equation was proposed by Yoshimasa Matsuno in 2006[**2**].

If we let $u(x,t) = U(x - ct) = U(\xi)$, then equation (IV.1) becomes

$$-cU''' + 2k^2 U' + 2U'U'' + UU''' = 0. \tag{IV.2}$$

After integrating, equation(IV.2) is changed to

$$-cU'' + 2k^2 U + (U')^2 + UU'' - \frac{1}{2}(U')^2 = A. \tag{IV.3}$$

If we let $A = 0$, then

$$-cU'' + 2k^2 U + \frac{1}{2}(U')^2 + UU'' = 0. \tag{IV.4}$$

We multiply 2 to both sides to equation(IV.4)

$$-2cU'' + 2k^2 U + (U')^2 + 2UU'' = 0. \tag{IV.5}$$

Assume $U'(\xi) = V(U)$, $(U = U(\xi))$. Let $U''(\xi) = V'(U)U'(\xi) = V'V$ and substitute into equation (IV.5), so that

$$-2cV'V + 4k^2U + V^2 + 2UVV' = 0. \tag{IV.6}$$

By rewriting equation(IV.6)

$$-c(V^2)' + (UV^2)' + 4k^2V = 0. \tag{IV.7}$$

After integrating, we get

$$cV^2 - UV^2 = 2k^2U^2. \tag{IV.8}$$

In terms of V^2, then

$$V^2 = \frac{2k^2U^2}{c-U}. \tag{IV.9}$$

Going back to V, we have

$$V = \frac{dU}{d\xi} \pm \sqrt{2}|k|\frac{U}{\sqrt{c-U}}. \tag{IV.10}$$

Separating variables,

$$\frac{\sqrt{c-U}}{U}dU = \pm\sqrt{2}|k|d\xi. \tag{IV.11}$$

After integrating, we go back to u, so that

$$\pm\sqrt{2}|k|(x-ct) + c_0 = 2\sqrt{c-u} - \frac{\sqrt{2}}{2}ln\frac{\sqrt{c}+\sqrt{c-u}}{\sqrt{c}-\sqrt{c-u}}. \tag{IV.12}$$

Chapter V

Hunter-Zheng Equation

The following equation is Hunter-Zheng equation:

$$v_{xt} = 2vv_{xx} + v_x^2. \tag{V.1}$$

This equation was proposed by John K. Hunter and Yuxi Zheng in 1994[**3**].

Using the traveling wave setting $v(x,t) = V(x-ct) = V(\xi)$, we can change equation into:

$$-cV'' = 2VV'' + (V')^2, \tag{V.2}$$

after integrating, we get

$$\ln|V'|^{-1} = \ln|2V+c|^{1/2} + A, \tag{V.3}$$

where A is the integral constant.

Solving for V' gives,

$$V' = \frac{dV}{d\xi} = \pm\frac{1}{A\sqrt{|2V+c|}}. \tag{V.4}$$

If $2V + c > 0$, we get:

(V.5) $$(2V+c)^{\frac{3}{2}} = A|\xi| + B.$$

Here B is the integral constant.

In terms of v, we have

(V.6) $$v = \frac{(A|x-ct|+B)^{\frac{2}{3}} - c}{2}.$$

Now we will prove that when $B = 0$, this solution is a weak solution.

We know that $(2V+c)^{\frac{3}{2}} = A|\xi| + B$, differentiating both sides, we get

(V.7) $$3(2V+c)^{\frac{1}{2}}V' = Asgn\xi.$$

After another differentiating, we have

(V.8) $$3(2V+c)^{-\frac{1}{2}}(V')^2 + 3(2V+c)^{\frac{1}{2}}V'' = 2A\delta(\xi).$$

We divide this equation by 3 and multiply $(2v+c)^{\frac{1}{2}}$ for both sides:

$$\begin{aligned}
(V')^2 + (2V+c)V'' &= \frac{2}{3}A(2V+c)^{\frac{1}{2}}\delta(\xi) && \text{(V.9)}\\
&= \frac{2}{3}A(A|\xi|+B)^{\frac{1}{3}}\delta(\xi) && \text{(V.10)}\\
&= \frac{2}{3}AB^{\frac{1}{3}}\delta(\xi). && \text{(V.11)}
\end{aligned}$$

So it is clear that when $B = 0$, the equation is equal to 0. We can determine that: the solution (IV.1) is a weak solution.

Then we explain the transition from (V.10) to (V.11):

$$\frac{2}{3}A(A|\xi| + B)^{\frac{1}{3}}\delta(\xi) = \frac{2}{3}AB^{\frac{1}{3}}\delta(\xi).$$

For $\forall\varphi(\xi)\epsilon C_0^\infty(R)$

$$\begin{aligned}
&\langle \tfrac{2}{3}A(A|\xi| + B)^{\frac{1}{3}}\delta(\xi), \varphi(\xi)\rangle \\
&= \tfrac{2}{3}A \int \delta(\xi)(A|\xi| + B)^{\frac{1}{3}}\varphi(\xi)d\xi \\
&= \tfrac{2}{3}A(A|0| + B)^{\frac{1}{3}}\varphi(0) \\
&= \tfrac{2}{3}AB^{\frac{1}{3}}\varphi(0) \\
&= \langle \tfrac{2}{3}AB^{\frac{1}{3}}\delta(\xi), \varphi(\xi)\rangle.
\end{aligned}$$

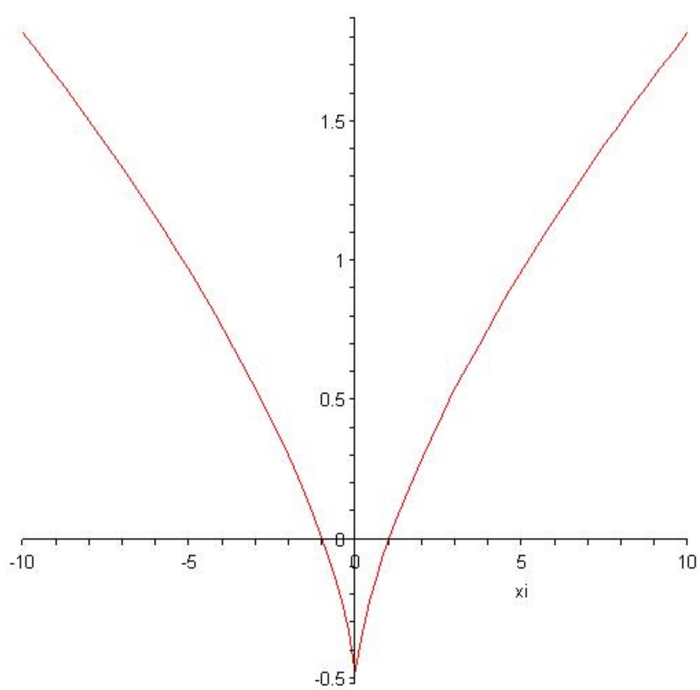

Figure V.1. 2-D graph for the solution, $c = 1, A = 1, B = 0$

Chapter VI

Application of Bilinear Approach

In this case we consider the of use the bilinear approach to analyze the equation:

$$u_{xt} = u - \frac{1}{6}u^3. \tag{VI.1}$$

First we give the definition of D-operator:

$$D_t^m D_x^n f \cdot g = (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n f(t,x) g(t',x')|_{t'=t, x'=x}. \tag{VI.2}$$

Assume $u = g/f$ and substitute it to the original equation so that:

$$(\frac{g}{f})_{xt} = \frac{1}{f^3}[g_{xt}f^2 + 2f_t f_x - g f_{xt} f - g_t f_x f - g_x f_t f] = \frac{g}{f} - \frac{g^3}{6f^3}. \tag{VI.3}$$

By Subtracting $\frac{1}{f^3} g f_{xt} f$ from both sides, multiplying by f^3, and combining the similar terms, and we have

$$g[2f_x f_t - 2f f_{xt} + \frac{1}{6}g^2] = f[gf - g f_{xt} - g_{xt} f_t g_t f_x + g_x f_t]. \tag{VI.4}$$

Let $12(f f_{xt} - f_x f_t) = g^2$, $f g_{xt} - f_x g_t - f_t g_x + f_{xt} g = fg$, we can get a system of quadratic equations from equation(VI.4):

$$6D_x D_t f \cdot f = g^2,$$

$$D_t D_x f \cdot g = fg.$$

Assume f and g can expand in the small parameter ε:

$$f = 1 + \varepsilon^2 f^{(2)} + \cdots\cdots + \varepsilon^{2j} f^{(2j)} + \cdots, \tag{VI.5}$$

$$g = \varepsilon g^{(1)} + \varepsilon^3 g^{(3)} + \cdots\cdots + \varepsilon^{2j+1} g^{(2j+1)} + \cdots. \tag{VI.6}$$

Substitute them into the system so that

$$\begin{aligned} & 12(1 + \varepsilon^2 f^{(2)} + \cdots + \varepsilon^{2j} f^{(2j)} + \cdots)(f_{xt}^{(2)}\varepsilon^2 + \cdots + f_{xt}^{(2j)}\varepsilon^{2j} + \cdots) \\ & +2(f_x^{(2)}\varepsilon^2 + \cdots + f_x^{(2j)}\varepsilon^{2j} + \cdots)(f_t^{(2)}\varepsilon^2 + \cdots + f_t^{(2j)}\varepsilon^{2j} + \cdots) \\ = \; & (\varepsilon g^{(1)} + \varepsilon^3 g^{(3)} + \cdots + \varepsilon^{2j+1} g^{(2j+1)} + \cdots)^2, \end{aligned} \tag{VI.7}$$

and

$$\begin{aligned} & (1 + \varepsilon^2 f^{(2)} + \cdots\cdots + \varepsilon^{2j} f^{(2j)} + \cdots)(g_{xt})^{(1)}\varepsilon + g_{xt})^{(3)}\varepsilon^3 + \cdots) \\ & -(f_x^{(2)}\varepsilon^2 + \cdots + f_x^{(2j)}\varepsilon^{2j} + \cdots)(g_t^{(1)}\varepsilon + g_t^{(3)}\varepsilon^3 + \cdots) \\ & -(f_t^{(2)}\varepsilon^2 + \cdots + f_t^{(2j)}\varepsilon^{2j} + \cdots)(g_x^{(1)}\varepsilon + g_x^{(3)}\varepsilon^3 + \cdots) \\ & +(f_{xt}^{(2)}\varepsilon^2 + \cdots + f_{xt}^{(2j)}\varepsilon^{2j} + \cdots)(g_x^{(1)}\varepsilon + g_x^{(3)}\varepsilon^3 + \cdots) \\ = \; & (1 + \varepsilon^2 f^{(2)} + \cdots\cdots + \varepsilon^{2j} f^{(2j)} + \cdots) \\ & (\varepsilon g^{(1)} + \varepsilon^3 g^{(3)} + \cdots\cdots + \varepsilon^{2j+1} g^{(2j+1)} + \cdots). \end{aligned} \tag{VI.8}$$

We compare the similar terms of the power series:

$$\varepsilon^2 : 12 f^{(2)}_{xt} = (g^{(1)})^2, \tag{VI.9}$$

$$\varepsilon^4 : 12 f^{(4)}_{xt} + 12 f^{(2)} f^{(2)}_{xt} - 12 f^{(2)}_x f^{(2)}_t = 2 g^{(1)} g^{(3)}, \tag{VI.10}$$

$$\varepsilon^6 : 12 f^{(6)}_{xt} + 12 f^{(2)} f^{(4)}_{xt} + 12 f^{(4)} f^{(2)}_{xt} - 12 f^{(4)}_x f^{(2)}_t - 12 f^{(2)}_x f^{(4)}_t$$

$$= 2 g^{(1)} g^{(5)} + (g^{(3)})^2, \tag{VI.11}$$

$$\cdots\cdots\cdots\cdots$$

$$\varepsilon^1 : g^{(1)}_{xt} = g^{(1)}, \tag{VI.12}$$

$$\varepsilon^3 : g^{(3)}_{xt} + f^{(2)} g^{(1)}_{xt} + g^{(1)} f^{(2)}_{xt} - f^{(2)}_t g^{(1)}_x - f^{(2)}_x g^{(1)}_t = g^{(3)} + f^{(2)} g^{(1)}, \tag{VI.13}$$

$$\varepsilon^5 : g^{(5)}_{xt} + f^{(2)} g^{(3)}_{xt} + f^{(4)} g^{(1)}_{xt} - f^{(2)}_x g^{(3)}_t - f^{(4)}_x g^{(1)}_t - f^{(2)}_t g^{(3)}_x - f^{(4)}_t g^{(1)}_x$$

$$+ g^{(3)} f^{(2)}_{xt} + g^{(1)} f^{(4)}_{xt} = g^{(5)} + f^{(2)} g^{(3)} + f^{(4)} g^{(1)}, \tag{VI.14}$$

$$\cdots\cdots\cdots\cdots$$

from (VI.12), let $g^{(1)} = \gamma e^{\frac{\theta}{2}}$, $\theta = 2\alpha x + \beta t + \delta$, here $\alpha, \beta, \gamma, \delta$ are arbitrarily constants. Replace them in (VI.12) to get:

$$\gamma e^{\frac{\theta}{2}} \cdot \tfrac{1}{4}\alpha\beta = \gamma e^{\frac{\theta}{2}}, \Longrightarrow \alpha\beta = 4,$$

replace $g^{(1)} = \gamma e^{\frac{\theta}{2}}$ in (21) so that:

$$f^{(2)}_{xt} = \tfrac{1}{12}(\gamma^2 e^{\theta}).$$

After integrating, we take the integral constant to equal θ so that

$$f^{(2)} = \tfrac{\gamma^2}{48} e^{\theta}.$$

Let $\gamma = \sqrt{48}$, then $f^{(2)} = e^{\theta}$.

Substitute $f^{(2)} = e^{\theta}$, $g^{(1)} = 4\sqrt{3}e^{\frac{\theta}{2}}$ into (VI.13) so we have

$$g_{xt}^{(3)} + e^{\theta} g_{xt}^{(1)} + 4\sqrt{3}e^{\frac{\theta}{2}} \cdot f_{xt}^{(2)} - f_x^{(2)} g_t^{(1)} - f_t^{(2)} g_x^{(1)} = g^{(3)} + f^{(2)} g^{(1)}.$$

Also since $f^{(2)} g_{xt}^{(1)} + g^{(1)} f_{xt}^{(2)} - f_x^{(2)} g_t^{(1)} - f_t^{(2)} g_x^{(1)} = f^{(2)} g^{(1)}$,

we have

$$g_{xt}^{(3)} = g^{(3)},$$

it goes back to (VI.12).

Let $g^{(3)} = 0$, then substitute $f^{(2)} = e^{\theta}$, $g^{(1)} = 4\sqrt{3}e^{\frac{\theta}{2}}$ and $g^{(3)} = 0$ into (VI.10) so that

$$12 f_{xt}^{(4)} = 0 \text{ (since } f^{(2)} f_{xt}^{(2)} - f_x^{(2)} f_t^{(2)} = 0).$$

Therefore, we take $f^{(4)} = 0$. Continue this way, the series can be cut off as

$$f = 1 + f^{(2)}\varepsilon,\ g = g^{(1)}\varepsilon.$$

If we let $\varepsilon = 1$, we have:

$$f = 1 + e^{\theta},\ g = 4\sqrt{3}e^{\frac{\theta}{2}}.$$

Since we already have

$$\theta = 2\alpha x + \beta t + \delta,\ \alpha\beta = 4.$$

Thus

$$\begin{aligned} u &= g/f \\ &= 4\sqrt{3} \cdot \tfrac{e^{\frac{\theta}{2}}}{1+e^{\theta}} \\ &= 4\sqrt{3} \cdot \tfrac{1}{e^{-\frac{\theta}{2}}+e^{\frac{\theta}{2}}} \\ &= 2\sqrt{3} \cdot \tfrac{1}{ch\frac{\theta}{2}} \\ &= 2\sqrt{3} \cdot sech\frac{\theta}{2}, \end{aligned}$$

so a soliton solution of nonlinear equation $u_{xx} = u - \frac{1}{6}u^3$ is obtained.

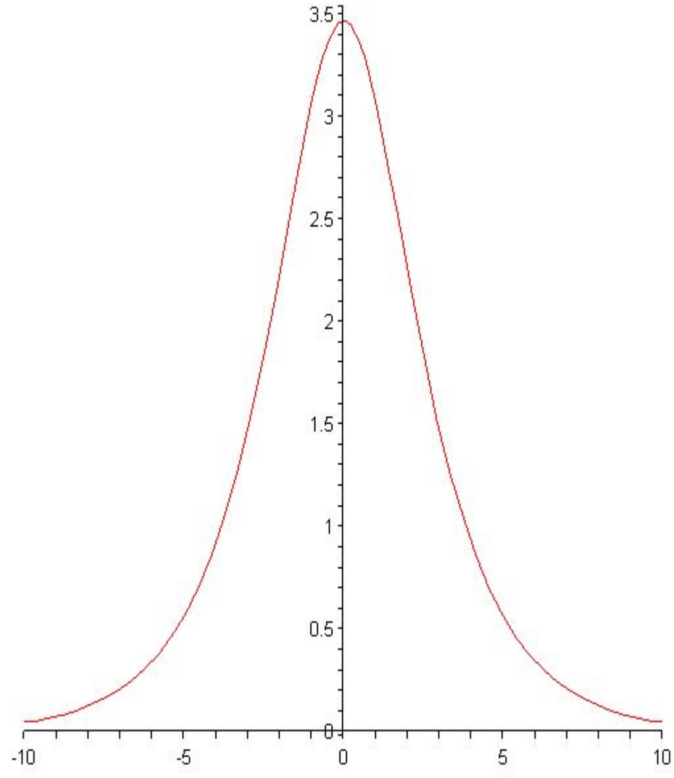

Figure VI.1. 2-D Graph for the soliton solution

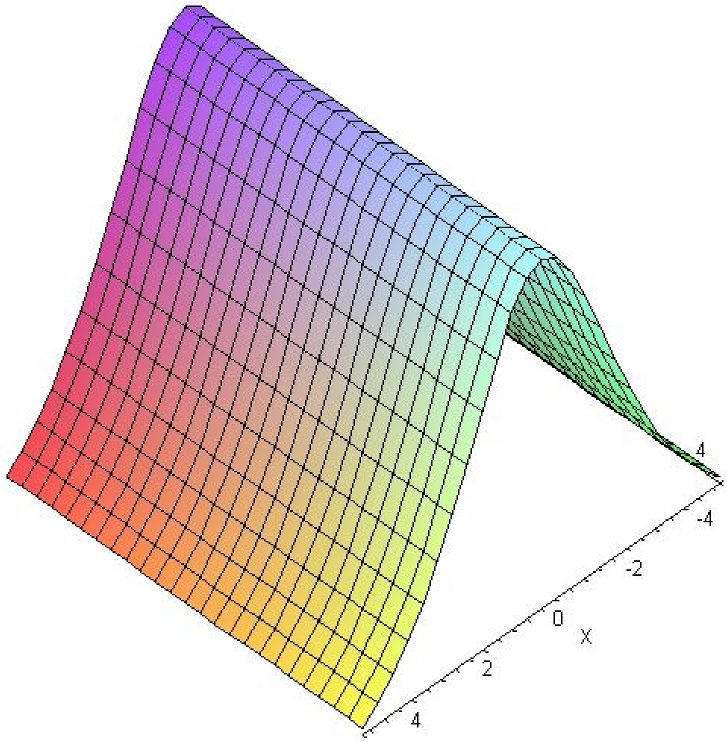

Figure VI.2. 3-D Graph for the soliton solution

Chapter VII

Conclusion

In this thesis, we use the traveling wave setting and bilinear approach to analyze some physical partial differential equation, and to find soliton solutions. For the Harry-Dym type equation, after traveling wave setting we find the traveling wave solutions. For the short wave model for Camassa-Holm equation, we find the special case when the integration constant is zero. For the Hunter-Zheng equation, we find the weak solution. And at last, by using the bilinear approach, we can not only find the soliton solution, but also show the graphs. In the further, we will study some problems, and find the multiple solitons.

References

[1] Jibin Li and Zhijun Qiao, Bifurcations of traveling wave solutions for an integrable equation, journal of mathematical physics 51. 042703(2010).

[2] Yoshimasa Matsuno, Cusp and loap soliton solutions of short-wave models for the Camassa-Holm and Degasperis-Procesi equation,Physics Letter A, volume 359, Issue 5, 4 December 2006, pages 451-457..

[3] John K. Hunter and Yuxi Zheng, On a Completely Integrable Nonlinear Hyperbolic Variational Equation,Physica D., 79(1994), pages 361-386.

[4] Xianqi Li,Zhijun Qiao, A New Peakon Equation, Master Thesis,UTPA, 2009.

[5] M.S.Alber,R.Camassa,Y.N.Fedorov,D.D.Holm,J.E.Marsden,Commun, Math.Phys.221,197(2001).

[6] Zhijun Qiao, The Camassa-Holm hierarchy, N-dimensional integrable systems, and algebro-geometric solution on a symplectic submanifold, Communications in Mathematical Physics 239, 309-341, 2003.

[7] Dengyuan Chen, The soliton Theory,2006.

[8] Ryogo Hirota, The Direct Method in Soliton Theory, 2004.

[9] Cercignani, Carlo; David H. Sattinger (1998). Scaling limits and models in physical processes. Basel: Birkh?user Verlag. ISBN 0817659854.

[10] Kichenassamy, Satyanad (1996). Nonlinear wave equations. Marcel Dekker. ISBN 0824793285.

[11] Gesztesy, Fritz; Holden, Helge (2003). Soliton equations and their algebro-geometric solutions. Cambridge University Press. ISBN 0521753074.

[12] Olver, Peter J. (1993). Applications of Lie groups to differential equations, 2nd ed. Springer-Verlag. ISBN 0387940073.

[13] Vassiliou, P.J. (2001), "Harry Dym equation", in Hazewinkel, Michiel, Encyclopaedia of Mathematics, Springer, ISBN 978-1556080104

[14] Kruskal, M. Nonlinear Wave Equations. In J. Moser, editor, Dynamical Systems, Theory and Applications, volume 38 of Lecture Notes in Physics, pages 310-354. Heidelberg. Springer. 1975.

Biographical Sketch

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