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Approximate analysis solution of Burgers-KdV equation

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APPROXIMATE ANALYSIS SOLUTION OF BURGERS-KDV EQUATION

A Thesis

by

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ABSTRACT

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In this thesis, we study the Two-Dimensional Burgers-Kortewege-de Vries(2D-BKdV) equation by analyzing the equivalent Abel equation, which indicates that under some particular conditions, the 2D-BKdV equation has a unique bounded traveling wave solution. By using the theorem of contractive mapping, a traveling wave solution to the 2D-BKdV equation is expressed explicitly.

DEDICATION

The completion of my masters studies would not have been possible without the love and support of my my family and professors, including my mother, my father, my brothers, my sister, my uncle Kamal, my wife, and my advisor Dr. Zhaosheng Feng.

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CHAPTER I

INTRODUCTION

Consider the Burgers-Korteweg-de Vries equation (Burgers-KdV)

$$U_t + \alpha U U_x + \beta U_{xx} + s U_{xxx} = 0, \tag{1}$$

Where α,β,s are constants, and $\alpha\beta s \neq 0$. Eq.(1) arises from many different physical phenomena as a nonlinear model equation incorporating the effects of dispersion, dissipation and nonlinearity. Johnson [1] derived (1) as the governing equation for waves propagating in a liquid-filled elastic tube in which the weak effects of dispersion, dissipation and nonlinearity are present. vanWijngaarden [2] and Gao and co-workers [3] [4] used it as a nonlinear model in the flow of liquids containing gas bubbles and turbulence, respectively. Grua and Hu [5] used a steady-state version of (1) to describe a weak shock profile in plasmas. In the limits $\beta \to 0$ or $S \to 0$, i.e. when the weak effect of dissipation or dispersion is comparatively insignificant and so can be neglected, the Burgers-KdV equation can approximated by the KdV equation:

$$U_t + \alpha U U_x + s U_{xxx} = 0, \tag{2}$$

or the Burgers equation:

$$U_t + \alpha U U_x + \beta U_{xx} = 0, \tag{3}$$

Consider the 2D-BKdV equation

$$(U_t + \alpha U U_x + \beta U_{xx} + s U_{xxx})_x + \gamma U_{yy} = 0, \tag{4}$$

Where α, β, γ are constants, and $\alpha\beta\gamma \neq 0$.

Equation (4) is a two-dimentional generalization of the Burgers-Korteweg-de Vries equation

$$U_t + \alpha U U_x + \beta U_{xx} + s U_{xxx} = 0. \tag{5}$$

During the past two decades,much attention has been received to the 2D-BKdV equation. Barrera and Brugarino applied Lie group analysis to study the similarity solutions of (4) and examined some features of these invariant solutions, but explicit exact traveling wave solution to (4) was not shown [6]. Li and Wang used the Holf-Cole transformation and a computer algebra system to study (4) and obtained an exact traveling wave solution to (4) [7]. In the mean time, Ma proposed a bounded traveling wave solution to (4) by applying a special solution of square Holf-Cole type to an ordinary differential equation [8]. These two methods were compared with each other, and the solutions are proven to be equivalent by Parkes [9].

In papers[10] [11][12], Feng studied equation (4) by utilizing the first integral method and Painleve analysis, respectively, and obtained a more general travelling wave solution in terms of elliptic functions, and in paper[13] Feng studied equation (4) by analyzing an equivalent two-dimensional autonomous system and travelling solitary wave solution to the 2D-BKdV equation in expressed explicity.

In the present thesis, our purpose is to apply the contractive mapping theory to the studies of traveling wave solution and proper solution of the 2D-BKdV equation. A traveling wave solution is obtained more efficiently by a direct method and the asymptotic behavior of proper solution is presented.

The rest of the thesis is organized as follows. In Chapter 2, we give a short introduction of the contractive mapping theorem, Abel equation. In Chapter 3, we apply the contractive mapping theorem to study the solution of the Abel equation. Chapter 4 is the process to find the traveling wave solution of 2D-KdV equation. Chapter 5 is the conclusion we made.

CHAPTER II

PRELIMINARIES

2.1 Contraction Mapping and the Banach Fixed Point Theorem

We will give a brief introduction of the contraction Mapping and the Banach Fixed Point Theorem.

The name, fixed point theorem, is usually given to a result which says that, if a mapping f satisfies certain conditions, then there is a point z such that $f(z) = z$. Such a point z is called a fixed point of f.

Definition 1. (Norm) A function $x \to ||x||$ from a vector space E into R is called a norm if it satisfies the following conditions:

(a) $||x|| = 0$ implies $x = 0$;

(b) $||\lambda x||= ||\lambda|| ||x||$ for every $x \in E$ and $\lambda \in F$;

(c) $||x+y|| \leq ||x|| + ||y||$ for every $x, y \in E$.

Definition 2. (Normed Space) A vector space with a norm is called a normed space.

Definition 3. (Convergence in a normed Space) Let $(E, ||.||)$ be a normed space. We say that a sequence (x_n) of elements of E converges to some $x \in E$, if for every $\varepsilon > 0$ there exists a number M such that for every $n \geq M$ we have $||x_n - x|| < \varepsilon$. In such a case we write or simply $x_n \to x$.

Definition 4. (Cauchy Sequence) A sequence of vectors x_n in a normed space is called a Cauchy sequence if for any $\varepsilon > 0$ there exists a number M such that $||x_n - x|| < \varepsilon$ for all $m, n > M$

Definition 5. (Complete normed space)A normed space E is called complete if every Cauchy sequence in E converges to an element of E. Acomplete normed space is called a Banach space.

Definition 6. (Contraction mapping) A mapping f from a subset A of a normed space E into E is called contraction mapping (or simply a contraction) if there exists a positive number $\alpha < 1$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|.$$

Theorem 1. *Let F be a closed subset of a Banach space E and let f be a contraction mapping from F to F. Then there exists a unique fix point $z \in F$ of f [14].*

Proof: Let $0 < \alpha < 1$ be such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|$$

for all $x, y \in F$. Let x_0 be an arbitrary point in F and let $x_n = f(x_{n-1})$ for n=1,2,....
We will show that (x_n) is a Cauchy sequence. We First observe that, for any $n, m \in N$,

$$\begin{aligned}
\|x_n - x_m\| \leq& \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + ... + \|x_{m+1} - x_m\| \\
\leq& (\alpha^{n-1} + \alpha^{n-2} + ... + \alpha^m)\|x_1 - x_0\| \\
\leq& \frac{\|x_1 - x_0\|}{1 - \alpha}\alpha^m \to 0 \quad as \quad m \to \infty
\end{aligned}$$

Thus,(x_n) is a Cauchy sequence. Since F is closed subset of a complete space, there exists a $z \in F$ such that $x_n \to z$ as $n \to \infty$. are going to show that z is the unique point such that $f(z) = z$. Indeed, since

$$\begin{aligned}
\|f(z) - z\| \leq& \|f(z) - x_n\| + \|x_n - z\| \\
=& \|f(z) - f(x_n)\| + \|x_n - z\| \\
\leq& \alpha \|z - x_{n-1}\| + \|x_n - z\| \to 0 \quad as \quad n \to \infty
\end{aligned}$$

We have $\|f(z)-z\|$ =0, and thus $f(z)=z$. Suppose now $f(w)=w$ for some $w \in F$. Then

$$\|z-w\| = \|f(z)-f(w)\| \leq \|z-w\|.$$

Since $0 < \alpha < 1$, we must have $\|z-w\| = 0$, which implies $z = w$.

2.2 Abel Equation

2.2.1 Abel Equation of the First Kind

Based on reference [15], we will give a brief introduction of the Abel Equation of the First Kind

An Abel equation of the first kind is an equation of the form,

$$y' = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x)$$

where $f_3(x) \neq 0$. If $f_3(x) = 0$ and $f_0(x) = 0$, or $f_2(x) = 0$ and $f_0(x) = 0$, the equation reduces to a Bernoulli equation, while if $f_3(x) = 0$ the equation reduces to a Riccati equation.

(a). If f_1 is continuous, f_2 and f_2 are differentiable, $f_3 \neq 0$, then the substitution

$$y = w(x)\eta(\zeta) - \frac{f_2}{3f_3}$$

where

$$\zeta = \int f_3 w^2 dx,$$

$$w(x) = exp \int (f_1 - \frac{f_2^2}{3f_3})dx$$

brings the Abel equation of the first kind to the canonical form

$$\eta' = \eta^3 + I(x),$$

where

$$f_3 w^3 I = f_0 + \frac{d}{dx}\frac{f_2}{3f_3} - \frac{f_1 f_2}{3f_3} + \frac{2f_2^3}{27f_3^2}$$

Panayotounakos and Zarmpoutis discovered an analytic method to solve the above equation generally.

(b). The substitution $y = \frac{1}{u}$ brings the Abel equation of the first kind to the Abel equation of the second kind of the form

$$uu' = -f_0(x)u^3 - f_1(x)u^2 - f_2(x)u - f_3(x)$$

2.2.2 Abel Equation of the Second Kind

We will give a brief introduction of the Abel equation of the second kind is an equation of the form,

$$yy' = f(x)y^2 + g(x)y + h(x)$$

where f, g, h are continuous functions. (a). The substitution $y = E(x)w$,
where $E(x) = exp(\int f(x)dx)$, brings this equation to the simper form,

$$ww' = F_1(x) + F_0(x), \tag{6}$$

where

$$F_1(x) = g(x)/E(x),$$

$$F_0(x) = h(x)/E^2(x).$$

(b). By introducing the new independent Variable

$$z = \int F_1(x)dx,$$

the Abel equation of the second kind can be reduced to the canonical form,

$$ww' - w = \Phi(z). \tag{7}$$

Here the function $\Phi(z)$ is defined parametrically (x is the parameter) by the relations

$$\Phi = \frac{F_0(x)}{F_1(x)},$$

$$z = \int F_1(x)dx.$$

The books by Zaitsev & Polyanin (1994) and Polyanin & Zaitsev (2003) present a large number of solutions to the Abel equation of the form (6) and (7).

CHAPTER III

PROPER SOLUTION TO ABEL EQUATION

3.1 Integral Form of Abel Equation

In this section, we present a simple integral form of Abel equations which plays a fundamental role in this section. Let us consider the Abel equation

$$r' = a(t)r^2 + b(t)r^n, \quad t \in [t_0, t_1], \quad n \geq 3. \tag{8}$$

Dividing equation (8) by r^2 where $r \neq 0$ on both sides, then

$$\frac{r'}{r^2} = a(t) + b(t)r^{n-2} \tag{9}$$

Integrating equation (9) from t_0 to t, where $t \in [t_0, t_1]$, we obtain

$$\int_{t_0}^{t} \frac{r'}{r^2} d\tau = \int_{t_0}^{t} \Big(a(\tau) + b(\tau)r^{n-2} \Big) d\tau + c_1,$$

where c_1 is an integral constant.

so we have,

$$\int_{r(t_0)}^{r(t)} \frac{1}{r^2} dr = \int_{t_0}^{t} a(\tau) d\tau + \int_{t_0}^{t} b(\tau) r^{n-2} d\tau + c_1$$

$$-\frac{1}{r(t)}-(-\frac{1}{r(t_0)})=A(t)+\int_{t_0}^{t}b(\tau)r^{n-2})d\tau+c_1$$

where $A(t)=\int_{t_0}^{t}a(\tau)d\tau$

$$\Rightarrow -\frac{1}{r(t)}=A(t)+\int_{t_0}^{t}b(\tau)r^{n-2}d\tau+c_2$$

where $c_2=c_1-\frac{1}{r(t_0)}$

$$r(t)=\frac{1}{-A(t)-\int_{t_0}^{t}b(\tau)r^{n-2}d\tau-c_2}$$
$$r(t)=\frac{-\frac{1}{c_2}}{\frac{1}{c_2}A(t)+\frac{1}{c_2}\int_{t_0}^{t}b(\tau)r^{n-2}d\tau+1}$$
$$r(t)=\frac{c}{1-cA(t)-c\int_{t_0}^{t}b(\tau)r^{n-2}d\tau}$$

where $c=-\frac{1}{c_2}$.

So,

$$r(t)\left(1-cA(t)-c\int_{t_0}^{t}b(\tau)r^{n-2}d\tau\right)=c$$

or equivalently,

$$r(t)=c\left(1+r(t)A(t)+r(t)\int_{t_0}^{t}b(\tau)r^{n-2}d\tau\right) \tag{10}$$

Proposition 1. *A continuous function $r(.)$ on the closed interval $[t_0,t_1]$ satisfies the integral equation (10) if and only if it is continuously differentible on the open interval (t_0,t_1) and satisfies the Abel equation (8) with the initial condition $r(t_0)=c$.*

Proof: The conclusion is obvious.

3.2 Solutions to Abel Equation

In this section we first define a nonlinear operator T_c for a given continuous functions a and b and a constant c. Then we prove that T_c is contractive and that an iterated sequence $\{T_c^n(f)\}$ with a suitable function f converges to the solution of the Abel equation (8).

For convenience we take $[t_0, t_1]$ to be [0,1]. Let $C[0,1]$ denote the Banach space of all continious functions on the interval [0,1] with the norm $\|f\| = max_{0 \leq t \leq 1}|f(t)|$. With the equation (10) in mind we define a nonlinear operator,

$$T_c : C[0,1] \rightarrow C[0,1]$$

$$T_c(f)(t) = c\left(1 + f(t)A(t) + f(t)\int_{t_0}^{t} b(\tau)r^{n-2}d\tau\right)$$

for given a,b $\in [0,1]$, $c \in R$, and $A(t) = \int_0^t a(\tau)d\tau$.

Lemma 1. *If* $\|f\| \leq 1$*,* $\|f(t)A(t)\| \leq 1$*, and* $\|c\|(2 + \|b\|) \leq 1$*, then* $\|T_c f\| \leq 1$

Proof:

$$\begin{aligned}
\|T_c(f)(t)\| =& \|c(1 + f(t)A(t) + f(t)\int_{t_0}^{t} b(\tau)f^{n-2}d\tau)\| \\
\leq& \|c\|(\|1\| + \|f(t)A(t)\| + \|f(t)\int_{t_0}^{t} b(\tau)f^{n-2}d\tau)\|) \\
\leq& \|c\|[1 + 1 + \|b\|] \\
=& \|c\|[2 + \|b\|] \leq 1
\end{aligned}$$

Lemma 2. *If* $\|c\|\left(\|A\| + \|b\|(n-1)\right) \leq 1$*, then* T_c *is a contractive mapping on the closed unit ball* $\beta_1 = \{f \in C[0,1], \|f\| \leq 1\} of C[0,1]$.

Proof: From lemma (1), we have for any $f, g \in \beta_1, T_c(f), T_c(g) \in \beta_1$.

Moreover,

$$\begin{aligned}
\|T_c(f)(t) - T_c(g)(t)\| =& \|c(1 + f(t)A(t) + f(t)\int_0^t b(\tau)f^{n-2}d\tau) - c(1 + g(t)A(t) + g(t)\int_0^t b(\tau)g^{n-2}d\tau)\| \\
\leq& \|c\| \|f(t)A(t) + f(t)\int_0^t b(\tau)f^{n-2}d\tau - g(t)A(t) - g(t)\int_0^t b(\tau)g^{n-2}d\tau\| \\
\leq& \|c\|(\|A(t)(f-g)\| + \|(f-g)\int_0^t b(\tau)f^{n-2}d\tau + g(t)\int_0^t b(\tau)(f^{n-2} - g^{n-2}d\tau)\| \\
\leq& \|c\|\Big(\|A(t)(f-g)\| + \|(f-g)\|\|b\| + \|b\|\|f^{n-2} - g^{n-2}\|\Big)
\end{aligned}$$

but,

$$\begin{aligned}
\|f^{n-2} - g^{n-2}\| =& ||(f-g)(f^{n-3} + f^{n-4}g + \cdots + fg^{n-4} + g^{n-3})\| \\
\leq& \|f-g\|(n-2)
\end{aligned}$$

So,

$$\begin{aligned}
\|T_c(f) - T_c(g)\| \leq& \|c\|\Big(\|A(t)(f-g)\| + \|(f-g)\| \cdot \|b\|(1+n-2)\Big) \\
\leq& \|c\|\Big(\|A(t)(f-g)\| + \|b\|(n-1)\|(f-g)\|\Big) \\
\leq& \|c\|\Big(\|A(t)||(f-g)\| + \|b\|(n-1)\|(f-g)\|) \\
\leq& \|c\|\Big(\|(f-g)(\|A(t)\| + \|b\|(n-1)\Big) \\
\leq& \|f-g\|
\end{aligned}$$

Therefore, T_c is contractive on β_1

According to the well known Banach contraction principle, an iterated sequence $\{T_c^n(f)\}$ with $f \in \beta_1$ converges uniformly for $t \in [0,1]$ to the unique fixed point of T_c in β_1. Proposition (1) shows that the fixed point is no other than the solution $r(t)$ of the equation (8).

Theorem 2. *For given a, $b \in [0,1]$, and for any $t \in [0,1]$ with $\|f(t)A(t)\| \leq 1$, and $\|C\|(2+\|b\|) \leq 1$, and $\|A\| + \|B(n-1)\|)\|c\| \leq 1$, the solution $r(t)$ of equation (4) with $r(0) = c$ can be uniformly approximated by an iterated sequence $\{T_c^n(f)(t)\}$, i.e.,*

$$r(t) = \lim_{x \to \infty} T_c^n(f)(t), 0 \leq t \leq 1 \tag{11}$$

for arbitrary $f \in [0,1]$ with $\|f\| \leq 1$.

Proof: As we mentioned above, the conclusion follows from the Banach contraction principle.

Now, $r(t) = c\left(1 + r(t)A(t) + r(t)\int_{t_0}^{t} b(\tau)r^{n-2}d\tau\right)$

If we set n=3, and for $r_i(t) \in \beta_1 (i = 1,2,3,....)$, let $h_i(t) = \int_0^t b(\tau)r_i(\tau)d\tau$ with $b \in C[0,1]$, and $\|b\| < 1$, then it gives

$$\begin{aligned}
r_2(t) =& c\left(1 + r_1(t)A(t) + r_1(t)h_1(t)\right) \\
=& c\left[1 + r_1(t)\left(A(t) + h_1(t)\right)\right] \\
\Rightarrow& r_2(t) = c + cr_1(t)\left(A(t) + h_1(t)\right)
\end{aligned}$$

$$\begin{aligned}
r_3(t) =& c\left(1 + r_2(t)A(t) + r_2(t)h_2(t)\right) \\
=& c + cr_2(t)\left(A(t) + h_2(t)\right) \\
=& c + c\left(c + cr_1(t)\left(A(t) + h_1(t)\right)\right)\left(A(t) + h_2(t)\right) \\
\Rightarrow& r_3(t) = c + c^2\left(A(t) + h_2(t)\right) + c^2 r_1\left(A(t) + h_1(t)\right)\left(A + h_2(t)\right)
\end{aligned}$$

$$
\begin{aligned}
r_4(t) =& c\Big(1+r_3(t)A(t)+r_3(t)h_3(t)\Big) \\
=& c+cr_3(t)\Big(A(t)+h_3(t)\Big) \\
=& c+c\Big(A(t)+h_3(t)\Big)\Big[c+c^2\Big(A(t)+h_2(t)\Big)+c^3r_1\Big(A(t)+h_1(t)\Big)\Big(A(t)+h_2(t)\Big)\Big] \\
\Rightarrow & r_4(t) = c+c^2\Big(A(t)+h_3(t)\Big)+c^3\Big(A(t)+h_3(t)\Big)\Big(A(t)+h_2(t)\Big) \\
+& c^3r_1\Big(A(t)+h_3(t)\Big)\Big(A(t)+h_2(t)\Big)\Big(A(t)+h_1(t)\Big)
\end{aligned}
$$

$$
\begin{aligned}
r_5(t) =& c+cr_4(t)\Big(A(t)+h_4(t)\Big) \\
=& c+c\Big(A(t)+h_4(t)\Big)\Big[c+c^2\Big(A(t)+h_3(t)\Big)+c^3\Big(A(t)+h_3(t)\Big)\Big(A(t)+h_2(t)\Big) \\
+& c^3r_1\Big(A(t)+h_1(t)\Big)\Big(A(t)+h_2(t)\Big)\Big] \\
\Rightarrow & r_5(t) = c+c^2\Big(A(t)+h_4(t)\Big)+c^3\Big(A(t)+h_4(t)\Big)\Big(A(t)+h_3(t)\Big) \\
+& c^4\Big(A(t)+h_4(t)\Big)\Big(A(t)+h_3(t)\Big)\Big(A(t)+h_2(t)\Big) \\
+& c^4\Big(A(t)+h_4(t)\Big)\Big(A(t)+h_3(t)\Big)\Big(A(t)+h_2(t)\Big)\Big(A(t)+h_1(t)\Big) \\
& \vdots \\
& \vdots
\end{aligned}
$$

So,

$$
r_n(t) = c+c^2\Big(A(t)+h_{n-1}(t)\Big)+c^3\Big(A(t)+h_{n-1}(t)\Big)\Big(A(t)+h_{n-2}(t)\Big)+\cdots
$$

Since $\|h_i(t)\| = \| \int_0^t b(\tau)r_i(\tau)d\tau\| \leq \|b\| < 1 \Rightarrow h_1, h_2, ..., h_n$ are bounded.

If c is sufficiently small, then

$$
r_n(t) \approx c+c^2\Big(A(t)+h_{n-1}(t)\Big), n \geq 3
$$

Take *lim* as $n \to \infty$

$\Rightarrow r_n(t) \approx c + c^2\Big(A(t) + \int_0^t b(\tau)r(\tau)d\tau\Big)$, where $t \in [0,1]$.

Furthermore, $r(t)$ is between $c + c^2\Big(A(t)+1\Big)$ and $c + c^2\Big(A(t)-1\Big)$,

That is, $c + c^2\Big(A(t)-1\Big) < r(t) < c + c^2\Big(A(t)+1\Big)$.

CHAPTER IV

TRAVELING WAVE SOLUTIONS TO 2D-KDV EQUATION

4.1 From PDE to ODE

In this section, we will transform the 2D-BKdV Equation to a second order nonlinear ODE. Consider the 2D-BKdV equation

$$(U_t + \alpha U U_x + \beta U_{xx} + s U_{xxx})_x + \gamma U_{yy} = 0$$

Where α,β,γ are constants, and $\alpha\beta\gamma \neq 0$,

Assume that equation (4) has an exact solution in the form

$$U(x,y,t) \equiv U(\xi), \xi = hx + ly - wt \tag{12}$$

where h,l,w are real constants to be determined. Substitution of (12) into equation (4) yields

$$-whU_{\xi\xi} + \alpha h^2 (UU_\xi)_\xi + \beta h^3 U_{\xi\xi\xi} + sh^4 U_{\xi\xi\xi\xi} + \gamma l^2 U_{\xi\xi} = 0$$

Integrating the above equation twice with respect to ξ, we have

$$sh^4 U_{\xi\xi} + \beta h^3 U_\xi + \frac{\alpha}{2} h^2 U^2 + \gamma l^2 U - whU = C$$

where we set the first integration constant to zero and set the second one as C. Rewrite this second order ordinary differential equation as

$$U''(\xi)+\lambda U'(\xi)+aU^2(\xi)+bU(\xi)+d=0 \tag{13}$$

where

$$\lambda=\frac{\beta}{sh}, a=\frac{\alpha}{2sh^2}, b=\frac{\gamma l^2-wh}{sh^4}, \text{and } d=-\frac{C}{sh^4}$$

4.2 From Second Order ODE to Abel Equation

In this section, we will transform the second order nonlinear ODE to an Abel Equation. From equation (13), let $v=U(\xi)$ and $y=U'(\xi)$, then,

$$U''(\xi)=\frac{dU'(\xi)}{d\xi}=\frac{dy}{d\xi}=\frac{dy}{dv}\frac{dv}{d\xi}=\frac{dy}{dv}y$$

So we can rewrite the equation (13) into

$$\frac{dy}{dv}y+\lambda y+av^2+bv+d=0 \tag{14}$$

Solving for $\frac{dy}{dv}$ we get,

$$\frac{dy}{dv}=-\lambda-(av^2+bv+d)y^{-1} \tag{15}$$

Let $z=\frac{1}{y}$, then $y=\frac{1}{z}$, $\frac{dy}{dv}=\frac{dy}{dz}\frac{dz}{dv}=-\frac{1}{z^2}\frac{dz}{dv}$ so,

$$-\frac{1}{z^2}\frac{dz}{dv}=-\lambda-(av^2+bv+d)z$$

By multiplying $-z^2$ both sides, we have

$$\frac{dz}{dv} = \lambda z^2 + (av^2 + bv + d)z^3 \tag{16}$$

Let $f(v) = \lambda, g(v) = av^2 + bv + d$, then

$$z' = f(v)z^2 + g(v)z^3$$

where $v = U(\xi) \in [v_0, v_1]$. Also $U(\xi_0) = v_0, U'(\xi_0) = \frac{1}{c}$, $z(v_0) = \frac{1}{U'(\xi_0)} = c$, and c is a real constant.

Let $\eta = \frac{v - v_0}{v_1 - v_0}$, then $\eta \in [0, 1]$ and $v = v_0 + (v_1 - v_0)\eta$, so let

$$\begin{aligned} r(\eta) &= z(v), \\ h(\eta) &= (v_1 - v_0)f(v) = (v_1 - v_0)\lambda, \\ k(\eta) &= (v_1 - v_0)g(v), \end{aligned}$$

then we have

$$r' = h(\eta)r^2 + k(\eta)r^3, \quad with \quad initial \quad condition \quad r(0) = c \tag{17}$$

where $h(\eta), k(\eta) \in C[0, 1]$.

By Theorem 1, if $\|f(t)h(\eta)\| \leq 1$, and $\|c\|(2 + \|k\|) \leq 1$, and $\|c\|\Big(\|H(\eta)\| + \|k\|(n-1)\Big) \leq 1$, the solution to the equation (17) is

$$r(\eta) = lim_{n \to +\infty} T_C^n(w)(\eta) \tag{18}$$

where $0 \leq \eta \leq 1$ for any $w \in [0, 1]$ with $|w| \leq 1$ and

$$T_C(w)(\eta) = c\Big(1 + w(\eta)H(\eta) + w(\eta)\int_0^{\eta} k(x)wdx\Big)$$

with

$$H(\eta) = \int_0^{\eta} h(x)dx = \int_0^{\eta} (v_1 - v_0)\lambda dx = (v_1 - v_0)\lambda\eta,$$
$$k(x) = (v_1 - v_0)\Bigg(a\Big(v_0 + (v_1 - v_0)x\Big)^2 + b\Big(v_0 + (v_1 - v_0)x\Big) + d\Bigg)$$

Let $v_2 = v_1 - v_0$, then

$$k(x) = v_2\Bigg(a\big(v_0 + v_2 x\big)^2 + b\big(v_0 + v_2 x\big) + d\Bigg)$$

$$= v_2\Big(av_2^2 x^2 + (2av_2 v_0 + bv_2)x + av_0^2 + bv_0 + d\Big)$$

Let $\bar{\alpha} = av_2^3$,$\bar{\beta} = v_2(2av_2v_0 + bv_2)$, $\bar{\mu} = v_2(av_0^2 + bv_0 + d)$, then

$$k(x) = \bar{\alpha}x^2 + \bar{\beta}x + \bar{\mu}.$$

4.3 Application of Contraction Mapping

Since $c + c^2\Big(A(t) - 1\Big) < r(t) < c + c^2\Big(A(t) + 1\Big)$

Then,

$$\frac{1}{c + c^2(A(t) + 1)} < \frac{1}{r(t)} < \frac{1}{c + c^2(A(t) - 1)}$$

$\Rightarrow U'(\xi) = \frac{1}{r(\eta)}$ is between $F(\xi)$ and $G(\xi)$

then $F(\xi) = \frac{1}{c+c^2(A(t)+1)}$, $G(\xi) = \frac{1}{c+c^2(A(t)-1)}$

but $H(\eta) = \lambda(U(\xi) - v_0)$, then

$$F(\xi) = \frac{1}{c+c^2[\lambda(U(\xi)-v_0)+1]}, G(\xi) = \frac{1}{c+c^2[\lambda(U(\xi)-v_0)-1]}$$

Now, $\frac{1}{c+c^2[\lambda(U(\xi)-v_0)\pm 1]} = \frac{du}{d\xi} = U'(\xi)$

$$\Rightarrow \frac{d(u-v_0)}{d\xi} = \frac{1}{c+c^2[\lambda(U(\xi)-v_0)\pm 1]}$$

Let $v = U - v_0$, then $\frac{dv}{d\xi} = \frac{1}{c+c^2[\lambda v \pm 1]}$

$$\Rightarrow c + c^2[\lambda v \pm 1]dv = d\xi$$
$$\Rightarrow c + c^2[\frac{\lambda v^2}{2} \pm v] = \xi + c^*$$
$$\Rightarrow c + c^2[\frac{\lambda v^2 \pm 2v}{2}] = \xi + c^*$$
$$\Rightarrow \frac{2}{c^2}[\xi + c^* - c] = \lambda v^2 \pm 2v$$

Case 1: $\frac{2}{c^2}[\xi + c^* - c] = \lambda v^2 + 2v$

$$\Rightarrow v(\lambda v + 2) = \frac{2}{c^2}[\xi + c^* - c]$$

but $U - v_0 = v$, then

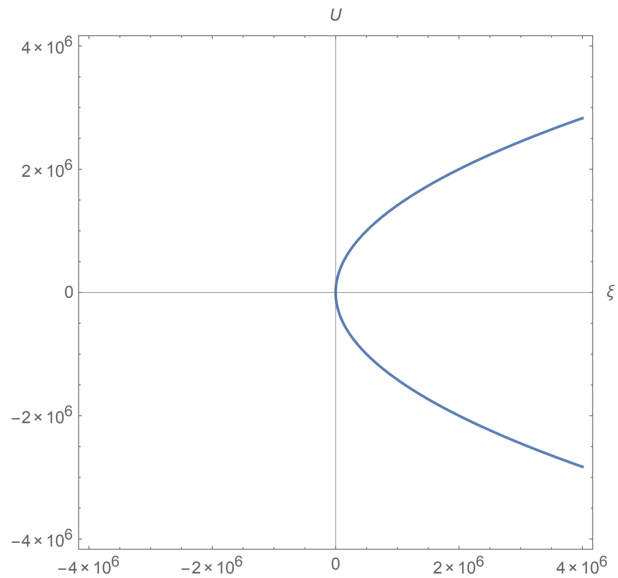

Figure 4.1: $U^2 + U - 2000000\lambda = 20000.75$

$$(U - v_0)[\lambda U + \lambda v_0 + 2) = \frac{2}{c^2}[\xi + c^* - c]$$

$$\Rightarrow \lambda U^2 - \lambda U v_0 + 2U - v_0 \lambda U - \lambda v_0^2 - 2v_0 = \frac{2}{c^2}[\xi + c^* - c]$$
$$\Rightarrow \lambda U^2 + (2 - 2\lambda v_0)U - \frac{2}{c^2}[\xi + c^* - c] = -\lambda v_0^2 + 2v_0$$
$$\Rightarrow \lambda U^2 + (2 - 2\lambda v_0)U - \frac{2}{c^2}\xi = -\lambda v_0^2 + 2v_0 + \frac{2}{c^2}(c^* - c)$$

To graph the equation,let for example $\lambda = 1$, $v_0 = \frac{1}{2}$,$c = .001$,$c^* = 0$,

then

$$U^2 + U - 2000000\xi = 20000.75$$

Case 2: $\frac{2}{c^2}[\xi+c^*-c]=\lambda v^2-2v$

$$\Rightarrow v(\lambda v-2)=\frac{2}{c^2}[\xi+c^*-c]$$

but $U-v_0=v$, then

$$(U-v_0)[\lambda U+\lambda v_0-2)=\frac{2}{c^2}[\xi+c^*-c]$$

$$\Rightarrow \lambda U^2-\lambda U v_0-2U-v_0\lambda U-\lambda v_0^2-2v_0=\frac{2}{c^2}[\xi+c^*-c]$$
$$\Rightarrow \lambda U^2-(2+2\lambda v_0)U-\frac{2}{c^2}[\xi+c^*-c]=-\lambda v_0^2+2v_0$$
$$\Rightarrow \lambda U^2+(2-2\lambda v_0)U-\frac{2}{c^2}\xi=-\lambda v_0^2+2v_0+\frac{2}{c^2}(c^*-c)$$

let $\lambda=1$, $v_0=\frac{1}{2}$,$c=.001$,$c^*=0$,

then

$$U^2-3U-2000000\lambda=-20000$$

4.4 The Study of Equilibrium Points

Consider the KdV Equation:

$$U''(\xi)+\lambda U'(\xi)+aU^2(\xi)+bU(\xi)+d=0$$

Let $u'=v$, then $u''=v'$

Then,$u''=v'=-lu'-au^2-bu-d$

$\Rightarrow v'=-lv-au^2-bu-d$

The above equations is equivalent to

$$u' = v = P(u,v)$$
$$v' = -lv - au^2 - bu - d = Q(u,v)$$

Let $P(u,v) = 0$, $Q(u,v) = 0$, then

$$\begin{cases} v = 0 \\ au^2 + bu + d = 0 \end{cases}$$

then, $u = \frac{-b \pm \sqrt{b^2 - 4ad}}{2a}$

Now we have 2 cases:

Case1:$b^2 - 4ad = 0$, then $u = \frac{-b}{2a}$

so the equilibrium point is $(\frac{-b}{2a}, 0)$

The Jacobian Matrix of the system is

$$\begin{pmatrix} P_u & P_v \\ Q_u & Q_u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2au - b & -l \end{pmatrix}$$

For the point $(\frac{-b}{2a}, 0)$,

$J(\frac{-b}{2a}, 0)$= $\begin{pmatrix} 0 & 1 \\ 0 & -l \end{pmatrix} \to |\lambda I - J| = \begin{vmatrix} \lambda & -1 \\ 0 & \lambda + l \end{vmatrix}$ $=\lambda^2 + \lambda l = o$

then $\lambda = 0$, or $\lambda = -l$

$\Rightarrow$ A$(\frac{-b}{2a}, 0)$ is unstable

Case2:$b^2 - 4ad > 0$. Let $D^2 = b^2 - 4ad$, then $u = \frac{-b \pm D}{2a}$

$u = \frac{-b \pm \sqrt{b^2-4ad}}{2a}$

$\Rightarrow$ we have 2 cases: Case 2.A: $u = \frac{-b+\sqrt{b^2-4ad}}{2a}$

Case 2.B: $u = \frac{-b-\sqrt{b^2-4ad}}{2a}$

Case 2.A: $u = \frac{-b+\sqrt{b^2-4ad}}{2a}$

$$J(\frac{-b+\sqrt{b^2-4ad}}{2a}, 0) = \begin{pmatrix} 0 & 1 \\ -\sqrt{b^2-4ad} & -l \end{pmatrix}$$

$$\Rightarrow |\lambda I - J| = \begin{vmatrix} \lambda & -1 \\ \sqrt{b^2-4ad} & \lambda + l \end{vmatrix} = \lambda^2 + \lambda l + \sqrt{b^2-4ad} = 0$$

Let $D^2 = b^2 - 4ad$, then$\lambda^2 + \lambda l + D = 0$

$\Rightarrow \lambda_1 + \lambda_2 = -l$, $\lambda_1 \cdot \lambda_2 = D$, $\Rightarrow \lambda = \frac{-l \pm \sqrt{l^2-4D}}{2}$, $D > 0$

- Case 2.A.1: $l^2 - 4D = 0 (\lambda_1 = \lambda_2), D > 0 (\lambda_1 \cdot \lambda_2 > 0), l < 0$, then$\lambda_1, \lambda_2 > 0$,

  so $A(\frac{-b+D}{2a}, 0)$ is unstable node or spiral point.

- Case 2.A.2:$l^2 - 4D = 0 (\lambda_1 = \lambda_2), D > 0 (\lambda_1 \cdot \lambda_2 > 0), l > 0$, then$\lambda_1, \lambda_2 < 0$,

  so $A(\frac{-b+D}{2a}, 0)$ is stable node or spiral point.

- Case 2.A.3: $l^2 - 4D = 0 (\lambda_1 \neq \lambda_2), D > 0 (\lambda_1 \cdot \lambda_2 > 0), l > 2\sqrt{D}$, then$\lambda_1, \lambda_2 < 0$,

  so $A(\frac{-b+D}{2a}, 0)$ is stable node.

- Case 2.A.4: $l^2 - 4D = 0 (\lambda_1 \neq \lambda_2), D > 0 (\lambda_1 \cdot \lambda_2 > 0), l < -2\sqrt{D}$, then$\lambda_1, \lambda_2 > 0$,

  so $A(\frac{-b+D}{2a}, 0)$ is unstable node.

- Case 2.A.5: $l^2 < 4D$, and $l > 0$, then $\lambda = \frac{-l \pm i\sqrt{4D-l^2}}{2}$,

  so $A(\frac{-b+D}{2a}, 0)$ is stable spiral point.

- Case 2.A.6: $l^2 < 4D$, and $l < 0$, then $\lambda = \frac{-l \pm i\sqrt{4D-l^2}}{2}$,

  so $A(\frac{-b+D}{2a}, 0)$ is unstable spiral point.

Case 2.B: $u = \frac{-b-\sqrt{b^2-4ad}}{2a}$

$$J\left(\frac{-b-\sqrt{b^2-4ad}}{2a},0\right)=\begin{pmatrix} 0 & 1 \\ \sqrt{b^2-4ad} & -l \end{pmatrix}$$

$$\Rightarrow |\lambda I-J|=\begin{vmatrix} \lambda & -1 \\ -\sqrt{b^2-4ad} & \lambda+l \end{vmatrix}=\lambda^2+\lambda l+\sqrt{b^2-4ad}=0$$

Let $D^2=b^2-4ad$, then$\lambda^2+\lambda l+D=0$

$\Rightarrow \lambda_1+\lambda_2=-l$, $\lambda_1\cdot\lambda_2=-D$, $\Rightarrow \lambda=\frac{-l\pm\sqrt{l^2+4D}}{2}$, $D>0$

then $\lambda=\frac{-l\pm\sqrt{4D+l^2}}{2}$, so $\lambda_1\cdot\lambda_2>0$,so $A(\frac{-b-D}{2a},0)$ is unstable saddle point.

CHAPTER V

CONCLUSION

In this thesis, we study the Two-Dimensional Burgers-Kortewege-de Vries(2D-BKdV) equation by analyzing the equivalent Abel equation, wich indicates that under some particular conditions, the 2D-BKdV equation has a unique bounded travelling wave solution. By using the theorem of contractive mapping, a traveling wave solution to the 2D-BKdV equation is expressed explicitly.

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BIOGRAPHICAL SKETCH

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