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An Introduction to Generalized Entropy and Some Quantum Applications

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Abstract

The concept of generalized entropy is introduced and some of its properties are studied. Irreversible time evolution can be generated by a non-Hermitian superoperator on the states of the system. The case when irreversibility comes about from embedding the system in a thermal reservoir is looked at. The time evolution is found compatible both with equilibrium thermodynamics and entropy production near the final state. Some examples are presented as well as a longer introduction as to how this might play a role in the black hole information loss paradox.

Keywords: thermodynamics, entropy, density matrix, state, equilibrium, quantum, eigenvalues

1 Introduction

Thermodynamic entropy is a classical concept and can be defined at thermal equilibrium or perhaps in an immediate neighborhood of it [?, ?]. At present, the concept of entropy is experiencing many new advances. There are many reasons for this. One reason is to try to extend and understand the concept at the quantum level. There has also been development in the area of nonequilibrium thermodynamics as well as applications of the concept to other emerging areas of physics such as information theory, chaos [?] and blackhole physics [?, ?, ?, ?]. Open systems are distinguished by their ability to recognize a direction of time, and it is manifested by means of a collective entity having a monotonic evolution toward the future. It might be asked whether time can be characterized by the evolution of a monotonically changing function of the quantum state which would play the role of generalized entropy. The main difference in considering such a problem is that the concept of entropy can be formulated in alternate ways depending on the situation. In order to gain perspective with regard to some of the main concepts, an overview of the subject is presented. The first approach to entropy is the thermophysical entropy which has appeared in classical thermodynamics and has temperature as its conjugate variable. Boltzmann [?] made a critical contribution by introducing his variational H-function which illustrates the flow or approach to equilibrium. Shannon more recently introduced a kind of entropy into information theory at the classical level. It was subsequently generalized and applied to quantum systems by von Neumann and is referred to as von Neumann entropy [?, ?].

Boltzmann's integro-differential equation for the time evolution of the velocity distribution in a gas is not symmetrical under time reversal and Boltzmann showed in the H theorem that the entropy like quantity H increases with time or stays constant and is defined for all states [?]. It provides a connection to statistical physics. Entropy has the interpretation here of a measure of disorder or uncertainty in the knowledge one has of the system. It has the disadvantage that it depends on an irreversible dynamical function with its form subject to some approximation. Boltzmann initiated the study of irreversible dynamics by originating the Boltzmann equation, the first entropy expression which is defined arbitrarily far from equilibrium. In the limit of a final thermal equilibrium state, the entity introduced becomes the thermodynamic entropy. At present, it seems to be known that there can be no unique extension of the thermal equilibrium [?]. Boltzmann was able to extract a quantity which explicitly displays the difference between forward and backward time evolution, and only in one direction does it increase. All irreversible evolution models should show some imprint of the direction of time. In dynamical systems, such an entity exists, and has for a long time been referred to as a Lyapunov function which uniquely defines the direction of time and physically may be thought of as a generalized entropy.

The Shannon entropy was originally motivated by the conduction of message transmission. Its subsequent generalization to quantum mechanics has proved particularly efficient in the study of the information content in quantum systems and quantum processes [?, ?]. It is interesting to note that many of the classical properties have quantum analogs and classical concepts related to communication and computation can generalize to describe the states of quantum systems. There has also been interest in extending these ideas so as to apply them to gravity and cosmology [?]. It is also remarkable that the entropies mentioned so far do not have any necessary logical connections and should be investigated in turn. These ideas have initiated a research area which is called open systems dynamics. It is the intention here to study this emerging concept known as generalized entropy. This more general concept has developed from the need to extend and investigate entropy at the quantum level [?]. Several examples are also given. As well a discussion as to how these ideas can be incorporated into the study of the information paradox which is central to black hole physics [?, ?, ?]. It is the information theoretic approach that has introduced a collection of generalized entropies with different properties and applications [?]. The associated state functional displays the direction of time by changing monotonically and may be thought of as a generalized entropy, or equivalently, a Lyapunov function. There has also been the realization that the quantum vacuum plays a very significant role in the study of gravity than previously thought [?, ?, ?, ?].

2 Density Matrices and States

2.1 Liouville Representation

It is possible to represent quantum states of a system by means of a density matrix which has the following properties [?, ?]. Each element of this matrix represents the mean value of an observable physical quantity. The entire square array of elements ρ_{mn} may be regarded as the components of a vector in an appropriate space and replaced by a set of real components. The Schrödinger-von Neumann equation which governs the time dependence of ρ represents a rotation of this vector. Physical questions related to a large physical system are represented by the time variation of this projection. This leads to a vector space formulation which is called the Liouville representation. This representation has useful applications to systems which has a rather small number of degrees of freedom but interacts with a much larger system whose role is to act as a thermal reservoir.

The state of a physical system is the information about the system required in addition to knowledge of its fixed properties and physical laws to predict an outcome for any experiment performed on it. In quantum mechanics, such predictions take the form of ensemble mean values of Hermitian operators, denoted as $\langle \mathbf{A} \rangle$. The mean value of an operator $\mathbf{A}$ with matrix representation A_{mn} can then be put in the form,

$$\langle \mathbf{A} \rangle = \sum_{m,n} A_{mn} \, \langle U^{(mn)} \rangle. \tag{2.1}$$

In (**??**), $U^{(mn)}$ is the operator whose matrix has element (m, n) equal to one and all others equal to zero. This operator can be expressed in terms of pure state representatives ψ_m belonging to a complete orthonormalized set of states and their conjugates $\psi_m^{\dagger}$ so we have

$$U^{(mn)} = \psi_m \psi_n^{\dagger}. \tag{2.2}$$

By (**??**) $U^{(nm)} = U^{(mn)\dagger}$ the mean values $\langle U^{(mn)} \rangle = \langle U^{(nm)} \rangle^*$ generate a Hermitian matrix when placed in a square array. The density matrix is defined as

$$\rho_{mn} = \langle U^{(mn)} \rangle = \langle U^{(nm)} \rangle^* \tag{2.3}$$

and it is a Hermitian matrix given as a square array. Thus (**??**) can be written as

$$\langle \mathbf{A} \rangle = \sum_{m,n} A_{mn} \rho_{mn} = \mathrm{Tr}(\mathbf{A}\,\rho). \tag{2.4}$$

Knowledge of $\langle U^{(mn)} \rangle$ for all ordered pairs (m, n) describes the state of the system since (**??**) applies to any operator $\mathbf{A}$.

Any set of independent parameters that can be related to the $\langle U^{(mn)} \rangle = \rho_{nm}$ by a transformation that has an inverse can serve to identify the state of the system and is evaluated as the mean value of all its operators as in (**??**) and (**??**).

The Liouville representation allows us to consider in general a set of operators $U^{(j)}$ and to identify the state of a system by means of the set of mean values

$$\rho_j = \langle U^{(j)\,\dagger} \rangle = \langle U^{(j)} \rangle^* = \sum_{m,n} U^{(j)*}_{mn}\, \rho_{mn}. \tag{2.5}$$

Suppose for convenience the set of operators is orthonormalized in the sense that

$$\mathrm{Tr}\big(U^{(j)}\, U^{(k)\,\dagger}\big) = \delta_{jk}. \tag{2.6}$$

This ensures the set of all matrix elements $U^{(j)}_{mn}$ arranged in a rectangular array with the row index j and the column index (m, n) constitutes a unitary square matrix. A very familiar example of this procedure consists of the replacement of a 2×2 spin orientation density matrix by the mean values of the 4 operators $I/\sqrt{2}$, $\sigma_x/\sqrt{2}$, $\sigma_y/\sqrt{2}$ and $\sigma_z/\sqrt{2}$ where I and σ_k are the identity and Pauli matrices respectively.

The operator $U^{(j)}$ can then be regarded as unit coordinate vectors in a vector space such that all operators are represented generally

$$\mathbf{A} = \sum_j A^*_j\, U^{(j)} = \sum_j A_j\, U^{(j)\dagger}. \tag{2.7}$$

The state vector can be represented as

$$\rho = \sum_j \rho^*_j\, U^{(j)} = \sum_j \rho_j U^{(j)\,\dagger},$$

and (**??**) generalizes to

$$\langle \mathbf{A} \rangle = \mathrm{Tr}(\mathbf{A}\,\rho) = \sum_j A_j \rho_j^* = \mathbf{A} \cdot \rho. \tag{2.8}$$

The density matrix ρ_{mn} of any state can be reduced to diagonal form $\rho_{nm} = \rho_n \, \delta_{nm}$ which is indicated in the Liouville representation by

$$\rho = \sum_a p_a \, \rho^{(a)}, \tag{2.9}$$

In (**??**), $\rho^{(a)}$ are the eigenvectors of ρ, which constitute an orthonormal system in the sense that

$$\rho^{(a)} \cdot \rho^{(b)} = \delta_{ab} \tag{2.10}$$

The eigenvalues p_a of ρ_{mn} must be nonnegative in order that any operator with nonnegative eigenvalues has a nonnegative mean value. In the notation of (**??**), $\rho^{(a)}$ are designated $U^{(aa)}$. It follows that, from the condition $p_a \geq 0$,

$$\sum_a p_a = 1, \qquad \sum_a p_a^r = \mathrm{Tr}(\rho^r) \leq 1, \quad r > 1. \tag{2.11}$$

The case $r = 2$ in (**??**) will be important in what follows. The constraint limits the length of vector ρ

$$\sum_a p_a^2 = \mathrm{Tr}(\rho^2) = \rho \cdot \rho \leq 1. \tag{2.12}$$

An entropy based on this has been proposed by Fano [**?**] and is defined as

$$S_2 = -\log\big(\mathrm{Tr}(\rho^2)\big) = -\log\big(\langle \rho \rangle\big). \tag{2.13}$$

In the Liouville representation, the mean values of the operators of interest characterize the projection of ρ on a very small subspace of the entire state space of the state vector. A geometric interpretation is helpful in describing the nature of irreversibility.

The Schrödinger-von Neumann equation of motion for density matrices is

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}(H\rho - \rho H). \tag{2.14}$$

This equation conserves the value of $\mathrm{Tr}(\rho^r)$, hence conserves, in particular, $\mathrm{Tr}(\rho^2)$, the length of ρ. It can be represented in the alternative way

$$\frac{\partial \rho}{\partial t} = \mathbf{R}\rho = -i\mathbf{L}\rho. \tag{2.15}$$

where $\mathbf{R}$ and $\mathbf{L}$ are operators on the vector space with $\mathbf{L}$ and $\mathbf{R}$ anti-Hermitian. The eigenvalues of $\mathbf{L}$ are either of equal modulus and opposite sign in pairs or else equal to zero. In terms of real cartesian coordinates of the vector space , $\mathbf{R}$ is represented by a real skew-symmetric matrix. The eigenvalues of $\mathbf{L}$ are the non-Hermitian $U^{(r,s)}$, where r, s indicate eigenstates of the Hamiltonian and the corresponding eigenvalues are the proper frequencies

$$\omega_{rs} = \frac{1}{\hbar}(E_r - E_s). \tag{2.16}$$

In the Limiting case of a classical system with canonical coordinates q_i and p_i in which

$$\rho \to f(q_i, p_i), \qquad \mathbf{R} \to \sum_i [\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}],$$

(**??**) reduces to the Liouville equation for the distribution function $f(q_i, p_i)$, hence the name of this vector space representation.

2.2 Irreverssible Evolution in Time

Since this is a vector space based representation, and normalized superpositions of allowed states are also admissible, the set of physical states can be extended to a more general vector space on which the time evolution takes place. In general, this is not a Hilbert space but a generalized state space. In order to emphasize this, state ρ is written as $|\rho\rangle\rangle$. It is the case that the object ρ which represents the state of a physical system obeys a general time evolution equation of a similar form to (**??**), which we write as

$$\frac{\partial \rho}{\partial t} = \mathcal{L}\rho. \tag{2.17}$$

The operator $\mathcal{L}$ acts on all physically admissible states which can be taken to belong to the linear manifold composed of elements $\{|\rho\rangle\rangle\}$. There is an inner product on this space defined as,

$$\langle\langle \rho_1 | \rho_2 \rangle\rangle = \mathrm{Tr}(\rho_1^\dagger \rho_2). \tag{2.18}$$

Lindblad [**?**] gave the most general Markovian generator of the time evolution in the quantum case. The operator is non-Hermitian with respect to inner product (**??**). In general, these will be right and left eigenvectors which are introduced as

$$\mathcal{L}\,|x_\nu\rangle\rangle = \lambda_\nu\,|x_\nu\rangle\rangle, \qquad \mathcal{L}^\dagger\,|y_\nu\rangle\rangle = \lambda_\nu^* |y_\nu\rangle\rangle. \tag{2.19}$$

These relations imply that

$$\langle\langle y_\mu|\mathcal{L}|x_\nu\rangle\rangle = \lambda_\nu\langle\langle y_\mu|x_\nu\rangle\rangle, \quad \langle\langle\mathcal{L}^\dagger y_\mu|x_\nu\rangle\rangle = \lambda_\mu\langle\langle y_\mu|x_\nu\rangle\rangle. \tag{2.20}$$

In the case $\lambda_\mu \neq \lambda_\nu$ the states are mutually orthogonal and may be normalized. Useful results can be obtained when it is assumed that both right and left eigenstates form complete basis sets. When their corresponding eigenvalues are nondegenerate, this tends to be true. In general, they are complex which implies that oscillations and damping may be exhibited.

Given these observations, spectral representations for the Lindblad operator have the form

$$\mathcal{L} = \sum_\nu |x_\nu\rangle\rangle\lambda_\nu\,\langle\langle y_\nu|, \qquad \mathcal{L}^\dagger = \sum_\nu |y_\nu\rangle\rangle\lambda_\nu^*\langle\langle x_\nu|. \tag{2.21}$$

In the case in which both sets of eigenstates are complete, there is a resolution of the identity operator given as

$$\mathcal{I} = \sum_\nu |x_\nu\rangle\rangle\langle\langle y_\nu| = \sum_\nu |y_\nu\rangle\rangle\langle\langle x_\nu|. \tag{2.22}$$

Let us introduce the following mapping Φ which acts on the states $\{|x_\nu\rangle\rangle\}$ with range $\{|y_\nu\rangle\rangle\}$ as

$$\Phi = \sum_\nu |y_\nu\rangle\rangle\langle\langle y_\nu|. \tag{2.23}$$

The inverse of this mapping is given by

$$\Phi^{-1} = \sum_\nu |x_\nu\rangle\rangle\langle\langle x_\nu|. \tag{2.24}$$

These are positive Hermitian operators. It is possible to define a new metric denoted M founded on the following bilinear form,

$$M_\Phi(\rho_1,\rho_2) = \langle\langle\rho_1|\Phi|\rho_2\rangle\rangle = \mathrm{Tr}(\rho_1^\dagger\Phi\rho_2). \tag{2.25}$$

The metric defines a topology on the space of all quantum states such that

$$M_\Phi(|x_\nu\rangle\rangle\langle\langle x_\mu|) = \langle\langle\Phi|x_\mu\rangle\rangle = \langle\langle x_\nu|y_\mu\rangle\rangle = \delta_{\nu\mu}. \tag{2.26}$$

By using Φ^{-1} a similar construction can be carried out on the $|y_\nu\rangle\rangle$ states, hence Φ may be considered to be a metric on the manifold of physical states. Using (**??**) and (**??**), it is found that

$$\Phi\mathcal{L}\Phi^{-1} = \sum_\nu |y_\nu\rangle\rangle\langle\langle y_\nu|\mathcal{L}\sum_\mu |x_\mu\rangle\rangle\langle\langle x_\mu| = \sum_{\mu,\nu} |y_\nu\rangle\rangle\langle\langle y_\nu|\mathcal{L}|x_\mu\rangle\rangle\langle\langle x_\mu|$$

$$= \sum_{\mu,\nu} |y_\nu\rangle\rangle \lambda_\nu^* \langle\langle y_\nu | x_\mu \rangle\rangle \langle\langle x_\mu| = \sum_{\mu,\nu} |y_\nu\rangle\rangle \delta_{\mu\nu} \langle\langle x_\mu| \, \lambda_\nu^* \tag{2.27}$$

$$= \sum_{\nu} |y_\nu\rangle\rangle \lambda_\nu^* \langle\langle x_\nu| = \mathcal{L}^{*\,\dagger}.$$

It is assumed that complex conjugation affects only c-numbers and not states. Now (**??**) is an important result, since it implies the following set of relations

$$\Phi\mathcal{L} = \mathcal{L}^{*\dagger}\,\Phi, \qquad \mathcal{L}\,\Phi^{-1} = \Phi^{-1}\mathcal{L}^{*\dagger},$$
$$\Phi\mathcal{L}^* = \mathcal{L}^{\dagger}\,\Phi, \qquad \mathcal{L}^*\,\Phi^{-1} = \Phi^{-1}\,\mathcal{L}^{\dagger}. \tag{2.28}$$

If $\mathcal{L}$ has real eigenvalues then $\mathcal{L}^* = \mathcal{L}$, so the operator is Hermitian with respect to M_Φ in (**??**). The results in (**??**) have many applications, as will be seen shortly. The time dependent state that is a solution to evolution equation (**??**) is given by

$$|\rho(t)\rangle\rangle = \sum_{\nu} \tau_\nu \, e^{\lambda_\nu t} |x_\nu\rangle\rangle. \tag{2.29}$$

The τ_ν take the form,

$$\tau_\nu = \langle\langle y_\nu | \rho(0) \rangle\rangle. \tag{2.30}$$

In physical terms, this means that for the time evolution operator, there exists a steady state $|x_o\rangle\rangle$ which corresponds to the eigenvalue $\lambda_o = 0$ which is stationary. This implies that

$$\mathcal{L}|x_o\rangle\rangle = 0. \tag{2.31}$$

Any solution must approach this in a smooth way so this means that $Re\,\lambda_\nu \leq 0$ for all ν.

2.3 A Model Example

At this point, a clear example is worth studying and kept in mind as we proceed. A simple 3-level system is considered in which there are two excited states $|1\rangle$ and $|2\rangle$ which decay to the ground state $|0\rangle$. The actual influence of the Hamiltonian will not be included in order that the model be soluble and independent of the Hamiltonian. The density matrix system of equations are given by

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho_{11} \\ \rho_{22} \\ \rho_{00} \end{pmatrix} = \begin{pmatrix} -\Gamma_1 & 0 & 0 \\ 0 & -\Gamma_2 & 0 \\ \Gamma_1 & \Gamma_2 & 0 \end{pmatrix} \begin{pmatrix} \rho_{11} \\ \rho_{22} \\ \rho_{00} \end{pmatrix}. \tag{2.32}$$

The off-diagonal elements decouple completely and need not be considered further. Although not complicated to follow, the model demonstrates how entropy is a monotonically increasing function, but may lose its relevance in the general case.

The solution to this first order system can be found if the initial state is taken to be

$$\rho(0) = \frac{1}{2}(|1\rangle\langle 1| + |2\rangle\langle 2|). \tag{2.33}$$

It has the form,

$$\rho_{11}(t) = \frac{1}{2}e^{-\Gamma_1 t}, \qquad \rho_{22}(t) = \frac{1}{2}e^{-\Gamma_1 t}, \qquad \rho_{00}(t) = 1 - \frac{1}{2}e^{-\Gamma_1 t} - \frac{1}{2}e^{-\Gamma_2 t}. \tag{2.34}$$

At the initial time, the von Neumann entropy and S_2 both produce the value $S = \log(2)$. The values of these two functions differ further on somewhat. For the case in which $\Gamma_1 = \Gamma_2$, we find

$$\begin{aligned} S_{vN} &= e^{-\Gamma t}(\log 2 + \Gamma t) - (1 - e^{\Gamma t})\log(1 - e^{-\Gamma t}), \\ S_2 &= -\log(1 - 2e^{-\Gamma t} + \tfrac{3}{2}e^{-2\Gamma t}). \end{aligned} \tag{2.35}$$

Both of these functions increase at the start, but after a maximum value, decay to zero. Both have a maximum value of $S_m = \log(3)$ assumed when all three levels have equal probabilities. This is the case when $t_c = \log(3/2)$. At this point the uncertainty in the level population is a maximum. The Lindblad operator tends to increase the entropy initially, so this is quite general. This comes from the mixing of the state $|0\rangle$ which will eventually take over and make the entropies go to zero.

Considering matrix (**??**) the eigenvalues of this matrix are $\lambda_0 = 0$, $\lambda_1 = \Gamma_1$ and $\lambda_2 = \Gamma_2$ and the eigenvectors are

$$\begin{aligned} \lambda_0 = 0, \quad |x_0\rangle\rangle &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \langle\langle y_0| = (1,1,1), \\ \lambda_1 = \Gamma_1, \quad |x_1\rangle\rangle &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \langle\langle y_1| = (1,0,0), \\ \lambda_2 = \Gamma_2, \quad |x_2\rangle\rangle &= \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \langle\langle y_2| = (0,1,0), \end{aligned} \tag{2.36}$$

$$\langle\langle y_m | x_n \rangle\rangle = \delta_{mn}.$$

The operator has a spectral resolution in terms of $|x_\nu\rangle\rangle$, $|y_\nu\rangle\rangle$,

$$\mathcal{L} = -\sum_n |x_n\rangle\rangle \, \lambda_n \, \langle\langle y_n|. \tag{2.37}$$

The matrix form of Φ is given by

$$\Phi = |y_0\rangle\rangle\langle\langle y_0| + |y_1\rangle\rangle\langle\langle y_1| + |y_2\rangle\rangle\langle\langle y_2| = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{2.38}$$

The eigenvalues of matrix $\mathcal{M}$ are all positive $\{1, 2 \pm \sqrt{3}\}$ so it can serve as a metric. The inverse matrix is

$$\Phi^{-1} = |x_0\rangle\rangle\langle\langle x_0| + |x_1\rangle\rangle\langle\langle x_1| + |x_2\rangle\rangle\langle\langle x_2| = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 3 \end{pmatrix} \tag{2.39}$$

The scalar product which is given by M is

$$M_S = \mathrm{Tr}(\rho M \rho) = 1 + \rho_{11}^2 + \rho_{22}^2. \tag{2.40}$$

For the case at hand, ρ_{11} and ρ_{22} are known and (**??**) gives

$$M_S = 1 + \frac{1}{4}e^{-2\Gamma_1 t} + \frac{1}{4}e^{-2\Gamma_2 t} \geq 1. \tag{2.41}$$

It approaches an asymptotic value monotonically, so at $t = 0$ we have $M_S = 3/2$ giving the measure entropy a value

$$S_M = \log(\frac{3}{2}). \tag{2.42}$$

This indicates that of these initially possible states, two are equally probable.

3 Generalized Entropy and Lyapunov Functions

Under irreversible time evolution a system singles out one particular direction of time and this defines the forward development [**?**, **?**]. In classical dynamics this is realized by introducing a monotonically changing functional called a Lyapunov function. A natural choice would be the object introduced by Fano,

$$P = \mathrm{Tr}\,\rho^2. \tag{3.1}$$

As noted previously, Fano uses minus the logarithm of (**??**) to serve as an entropy

$$S_2 = -\log\ P = -\log\langle\rho\rangle. \tag{3.2}$$

This is certainly not the only measure which has been proposed. The von Neumann entropy has also been mentioned and it is frequently used

$$S_{vN} = -\mathrm{Tr}(\rho\log\rho) = -\langle\log\rho\rangle. \tag{3.3}$$

These two entropies may be close to each other in some sense if ρ is localized according to some criterion. In fact the following bound obtains

$$S_2 \leq S_{vN}. \tag{3.4}$$

For pure states or even distributions, such as $\rho = N^{-1}$, the two agree. However, in general monotonic systems, they are not monotonic functions of time in general open systems. This is due to the fact that the system need not tend to increase mixing. The example of the spontaneous decay to a unique ground state has already been seen, so another quantity may be more suitable.

In fact, the bilinear function $M_\Phi(\rho,\rho)$ provides an example of a monotonically changing function of time. Differentiating and using (**??**), its conjugate and (**??**), we calculate

$$\frac{\partial}{\partial t} M_\Phi(\rho,\rho) = \mathrm{Tr}(\partial_t\rho^\dagger\Phi\rho) + \mathrm{Tr}(\rho^\dagger\Phi\partial_t\rho) = \mathrm{Tr}(\rho^\dagger\mathcal{L}^\dagger\Phi\rho) + \mathrm{Tr}(\rho^\dagger\Phi\mathcal{L}\rho)$$

$$= \mathrm{Tr}(\rho^\dagger\Phi(\mathcal{L}+\mathcal{L}^*\rho) = 2\mathrm{Re}(\mathrm{Tr}(\rho^\dagger\Phi\rho)). \tag{3.5}$$

Since the eigenvalues have non-positive real parts expansion (**??**)-(**??**) gives the inequality

$$\frac{\partial}{\partial t} M_\Phi(\rho,\rho) = \sum_\nu |\tau_\nu|^2\,(\lambda_\nu + \lambda_\nu^*)e^{(\lambda_\nu+\lambda_\nu^*)t} \leq 0. \tag{3.6}$$

This is stating the rate is decreasing with time and when a steady state has been attained, the evolution ceases. So a proper variational functional is proposed which determines a direction for time which is directed towards the eventual steady state. Thus, (**??**) represents the natural extension to the case of irreversible time evolution. Since the quantity decays monotonically and goes to zero, it should be replaced by its negative value. It actually makes more sense to examine S_2 in (**??**) and write

$$S_\Phi = -\log\big(\mathrm{Tr}(\rho^\dagger \Phi \rho)\big). \tag{3.7}$$

Then S_Φ is additive as well as monotonically increasing. Hence S_Φ becomes the sum of contributions of partial systems.

There is a certain amount of ambiguity in the metric as the state can be rescaled as

$$|x_\nu\rangle\rangle \to Q_\nu^{-1} |x_\nu\rangle\rangle, \qquad |y_\nu\rangle\rangle \to Q_\nu |y_\nu\rangle\rangle \tag{3.8}$$

preserving the normalization condition and real $\{Q_\nu\}$ can be taken arbitrarily. Under this transformation, the operator Φ is given by

$$\Phi = \sum_\nu |y_\nu\rangle\rangle Q_\nu^2 \langle\langle y_\nu|. \tag{3.9}$$

The scalar product $M_\Phi(\rho_1, \rho_2)$ takes the form,

$$M_\Phi(\rho_1, \rho_2) = \langle\langle \rho_1 | \Phi | \rho_2 \rangle\rangle = \sum_\nu \langle\langle \rho_1 | y_\nu \rangle\rangle Q_\nu^2 \langle\langle y_\nu | \rho_2 \rangle\rangle. \tag{3.10}$$

Here recall that $\tau_\nu = \langle\langle y_\nu | \rho \rangle\rangle$ is the expansion coefficient in $|\rho\rangle\rangle = \sum_\nu r_\nu |x_\nu\rangle\rangle$. The factor Q_ν^2 in (**??**) gives the freedom to gauge different states of different weights.

3.1 Some Thermal Relations

Dynamical relaxation is caused by coupling to a reservior which can be distinguished by a temperature T, so the ultimate steady state of the relaxing system is the equilibrium configuration

$$\rho_0 = \frac{1}{Z} e^{-\beta H}, \tag{3.11}$$

In (**??**), $\beta = 1/k_B T$, the partition function is $Z = \mathrm{Tr}(e^{-\beta H})$, and H is the Hamiltonian of the system. Also the equilibrium condition $\mathcal{L}\rho_0 = 0$ implies that since ρ_0 is real,

$$\mathcal{L}^* \rho_0^* = \mathcal{L}^* \rho_0 = 0. \tag{3.12}$$

It is also Hermitian, and this suggests the relation

$$\rho_0 \mathcal{L}^\dagger = \mathcal{L}^* \rho_0. \tag{3.13}$$

Using (**??**), we get the relation

$$\Phi^{-1}\mathcal{L}^{\dagger} = \mathcal{L}\Phi^{-1}. \tag{3.14}$$

This suggests we choose Φ, for finite temperature as,

$$\Phi = \rho_0^{-1} = Ze^{\beta H}. \tag{3.15}$$

This operator always exists for finite temperatures. The variational functional should be

$$M_0(\rho, \rho) = \langle\langle \rho | \rho_0^{-1} | \rho \rangle\rangle = \mathrm{Tr}(\rho^{\dagger} \rho_0^{-1} \rho). \tag{3.16}$$

Ordinary equilibrium thermophysics then applies at equilibrium where $\rho = \rho_0$.

Taking Φ as given in (**??**) implies that

$$0 \leq \mathrm{Tr}[(\rho - \rho_0)\rho_0^{-1}(\rho - \rho_0)] = \mathrm{Tr}[\rho\rho_0^{-1}\rho] - 1. \tag{3.17}$$

Hence M_0 in (**??**) has a lower bound given by

$$M_0(\rho, \rho) \geq 1. \tag{3.18}$$

Thus (**??**) implies that $M_0(\rho, \rho)$ approaches one from above. When ρ is close to the equilibrium ρ_0, approximately it holds that $\rho_0^{-1}\,\rho \doteq \mathbf{1}$. The following expansion may be developed

$$M_0 = \mathrm{Tr}(\rho\,\rho_0^{-1}\,\rho) = \mathrm{Tr}[\,\rho \exp(\log(\rho_0^{-1}\rho))] = \mathrm{Tr}[\rho(\mathbf{1} + \log(\rho_{\mathbf{0}}^{\mathbf{-1}}\rho))] \doteq \mathrm{Tr}\,\rho + \mathrm{Tr}\,[\rho\,\log(\rho_{\mathbf{0}}^{\mathbf{-1}}\rho)] \tag{3.19}$$

This allows us to write the following relation in a form which is always defined for operators,

$$M_0 - 1 \doteq \mathrm{Tr}(\rho \log \rho) - \mathrm{Tr}(\rho \log \rho_0)] = H(\rho|\rho_0). \tag{3.20}$$

This is often called the relative entropy and it gives a positive measure of the distance the state has to traverse in order to reach the final state. It is known to be positive and it decreases monotonically to its limiting value under dynamical evolution. Note that relative entropy is not symmetric in its arguments, so asymmetry emerges from physics in a natural way.

3.2 Entropy Inequality

Let us compare the relative entropy (**??**) to the entropy defined by the thermal functional

$$M_0 = \mathrm{Tr}[\rho\,\rho_0^{-1}\,\rho]. \tag{3.21}$$

Beginning with $\rho\log(\rho_0^{-1}\,\rho)$, its trace can be put in the form

$$\mathrm{Tr}[\rho\log(\rho_0^{-1}\,\rho)] = M_0\,\mathrm{Tr}[\rho_0(\frac{\rho_0^{-1}\rho}{M_0})(\log(M_0)+\log(\frac{\rho_0^{-1}\,\rho}{M_0})]. \tag{3.22}$$

There is the inequality for real numbers $x\log x \leq x(x-1)$, but it is applied to $\rho_0^{-1}\rho$ ignoring for now that even if the factors are positive, $\rho_0^{-1}\rho$ need not be, we finally arrive at the inequality

$$\mathrm{Tr}[\rho\,\log(\rho_0^{-1}\,\rho)] \leq \log(M_0) + M_0\mathrm{Tr}[\rho_0\big(\frac{\rho_0^{-1}\rho}{M_0}\big)\big((\frac{\rho_0^{-1}\rho}{M_0})-1)] = \log(M_0)+\mathrm{Tr}[\rho\,(\frac{\rho_0^{-1}\rho}{M_0}-1)]$$

$$= \log(\mathrm{Tr}(\rho\,\rho_0^{-1}\,\rho)). \tag{3.23}$$

Consequently, (**??**) implies the following inequality for $\log(M_0)$ as

$$\log[\mathrm{Tr}(\rho\,\rho_0^{-1}\,\rho)] \geq \mathrm{Tr}[\rho\log(\rho_0^{-1}\rho)]. \tag{3.24}$$

This actually sharpens (**??**). It relates the entropy S_Φ to the relative entropy generalizing (**??**). In terms of the relative von Neumann entropy, there is the inequality

$$S_{vN}^{relative} \geq S_{\rho_0^{-1}}. \tag{3.25}$$

Finally consider the principle of minimum entropy production to see if it can be located in this kind of theoretical structure. It is desirable to know if an expression for such a production can be derived. It is not necessary to force the state to deviate from its equilibrium condition, since subsequent time evolution will return it to equilibrium. The bilinear functional so defined will change monotonically from the value in its final state.

Return to the Lyapunov function (**??**) and write its rate of change

$$\frac{\partial}{\partial t}M_\Phi(\rho,\rho) = \mathrm{Tr}[\rho^\dagger\Phi(\mathcal{L}+\mathcal{L}^*)\rho]. \tag{3.26}$$

For the equilibrium state, this has to be zero. For virtual deviations, it is the case that

$$\frac{\partial}{\partial t} M[\rho_0 + \delta\rho, \rho_0 + \delta\rho] = \mathrm{Tr}[\delta\rho^\dagger\, \Phi(\mathcal{L} + \mathcal{L}^*)\delta\rho] \leq 0. \tag{3.27}$$

The principle begins to be relevant only near the final equlilibrium point, at which the entropy may be given approximately,

$$M_S = 1 + \mathrm{Tr}(\rho \log\ \rho) - \mathrm{Tr}(\rho \log \rho_0). \tag{3.28}$$

Differentiate (**??**) with respect to t with ρ_0 stationary to get $\dot{M}_S$,

$$\dot{M}_S = \mathrm{Tr}(\dot{\rho}) + \mathrm{Tr}(\dot{\rho} \log \rho) - \mathrm{Tr}(\dot{\rho} \log \rho_0). \tag{3.29}$$

This expression has to decrease to zero due to irreversible evolution. The first two terms in (**??**) can be related to the von Neumann entropy

$$\dot{S}_{vN} = -\mathrm{Tr}(\dot{\rho} \log\ \rho). \tag{3.30}$$

This can be related to the rate of change of the internal entropy of the system .

Consider the canonical ensemble (**??**) instead to understand better some of the implications of this along with a set of additionally constrained variables $\{B_i\}$ such that

$$\langle B_i \rangle = \mathrm{Tr}(\rho_0\, B_i) = \gamma_i. \tag{3.31}$$

These could represent the exchange of conserved quantities, atoms, molecules so forth. The minimum entropy principle establishes the equilibrium representation

$$\rho_0 = Z^{-1}\, \exp(-\beta H - \sum_i \lambda_i\, B_i). \tag{3.32}$$

The Lagrange multipliers $\{\lambda_i\}$ are determined by means of the relation

$$\gamma_i = \mathrm{Tr}(\rho_0\, B_i) = -\mathrm{Tr}(\frac{\partial \rho}{\partial \lambda_i}). \tag{3.33}$$

Placing this in (3.28) gives the rate of change

$$\dot{M}_s = \frac{\partial}{\partial t}\, \mathrm{Tr}(\rho \log\ \rho) + \beta \mathrm{Tr}(\dot{\rho} H) + \sum_i\ \lambda_i \mathrm{Tr}(\dot{\rho} B_i) \leq 0. \tag{3.34}$$

But the entropy of the system is $S_{int} = S_{vN}$ and it is found this implies the inequality

$$S_{int} \geq \frac{\dot{E}}{k_B T} + \sum_i \lambda_i \dot{\gamma}_i. \tag{3.35}$$

The change of entropy due to energy flux into the reservoir at temperature T is accounted for by the first term.

Inequality (**??**) states that for a fixed energy flux the system entropy must change at least this much . It allows us to state that the minimum entropy production rate takes place when the state is given by ρ_0 at which the variational functional takes the value zero. Given the additional constraints, virtual fixing of these parameters away from their equilibrium values gives their own contributions to the minimum value of the entropy production. Result (**??**) determines expressions for these contributions in the form $\lambda_i \dot{\gamma}_i$. Only in the case near equilibrium where (**??**) holds is the separation of entropy change between the system under consideration and its environment valid. The full expression (**??**) for the functional does not allow the identification of the two pieces. When away from equilibrium they emerge intermixed in an inseparable way.

4 Black Hole Entropy and Unitarity

It is the current belief that information in the real world has a physical nature and cannot simply disappear in any physical process. The information loss paradox however contradicts this claim and asserts instead that information can be lost in a black hole. This is to say many distinct physical states evolve into exactly the same final state. This corresponds to the case in which matter decays into a thermal state after collapsing into a black hole. This kind of picture leads to the paradoxical claim as to the violation of entropy conservation. In other words, it violates the principle of unitarity for quantum dynamics of an isolated system.

The ideas presented here may provide some help in sorting out this problem by considering the emission as related to a kind of tunneling process [**?**]. Up to now, the radiation from a black hole has been considered purely thermal as the background geometry has been considered fixed and energy conservation has not been enforced properly during the radiation process. It has been proposed that information could come out if the emitted radiation were not thermal. Hawking radiation can be thought of as a tunneling phenomenon. In this event, a pair of particles is spontaneously generated inside the event horizon. The positive energy particle tunnels out to infinity while the negative energy particle remains inside the horizon. It may also be thought that the pair is created outside the horizon and the negative energy particle tunnels inside the black hole.

When a system is composed of several parts, the reduced state for any part or subsystem may be mixed. Typically, the added information from all parts decreases. The mechanism for this loss of information is understood to be quantum entanglement. When two subsystems are entangled their respective entropies increase. This is equivalent to a decrease of information contained in the parts. The apparent loss to the total system information is due to information hidden inside correlations between the subsystems. In the case of a radiating black hole, it is the correlation between outgoing radiation states and the internal inaccessible states of the black hole. The original idea behind Hawking radiation is that it has a thermal spectrum. This is inconsistent with energy conservation due to the approximation of a fixed background geometry. There is now an approach to Hawking radiation in which it is thought of as a form of quantum tunneling. It is necessary to strictly enforce energy conservation of the tunneled s-wave outgoing particles. The tunneling barrier is created by the outgoing particle itself as ensured by energy conservation.

There are certainly some mechanisms for entropy production as outlined here. There is certainly the possibility of Lyapunov functions which have been associated with and studied in the area of quantum dynamical systems and chaos. One such paradigm will be mentioned that supports this. This is one that comes out of the conjecture by Verlinde [**?**] that gravity can be derived from an entropic force or that gravity is a kind of elasticity of space time. This related space-time elasticity could be considered as a modified reletivistic version of the Poincaré conjecture such that the Perelman W-entropy is used for deriving both gravitational equations and their thermodynamic properties.

4.1 Entropy Change for a Two-Manifold

A somewhat sophisticated example, although not entirely quantum model, of a flow is considered related to two manifolds or surfaces [**?**]. Consider the Ricci flow and define a quantity on a surface called the surface entropy. To this end, let R_{ij} be the Ricci tensor and R the scalar curvature function. We consider just the case $R(\cdot, 0) > 0$ and $R > 0$. Let r denote the average scalar curvature constant defined as

$$r = \frac{\int_{M^n} R\, d\mu}{\int_{M^n} d\mu}.$$

then the function f which is scalar is defined as $f = R - r$. As well, $d\mu$ is the integration measure on the manifold. The surface entropy is defined for a metric of strictly positive curvature as

$$S_e(g) = -\int_{M^n} R \log R\, d\mu. \tag{4.1}$$

This formally resembles an entropy because formally it resembles certain classical entropies, each of which is proportional to the integral of a positive function times its logarithm. Note the resemblance to the von Neumann entropy with this sign convention. The case of two manifolds will be of studied so we restrict n to be two.

Although more background knowledge might be supplied it will suffice to note the following relations which will be needed. The integral of Δf over M^2 is zero. If t is the evolution parameter or time the following results are used and given without proof,

$$\frac{\partial}{\partial t} R = \Delta R + R(R - r), \qquad \frac{\partial}{\partial t} d\mu = (r - R)\, d\mu. \tag{4.2}$$

Using (**??**) it is the case that

$$\frac{\partial}{\partial} R = (\frac{\partial R}{\partial t})\, d\mu + R(\frac{\partial}{\partial t}\, d\mu) = (\Delta R + R(R - r))\, d\mu + R(r - R)\, d\mu = \Delta R\, d\mu. \tag{4.3}$$

Theorem: If $(M^2, g(t))$ is a solution of the normalized Ricci flow on a compact surface such that $R(\cdot, 0) > 0$, then the entropy evolves by

$$\frac{dS_e}{dt} = \int_{M^2} \frac{|\nabla R|^2}{R}\, d\mu - \int_{M^2} (R - r)^2\, d\mu. \tag{4.4}$$

To prove this statement, differetiate (??) and substitute (??) and use integration by parts to calculate

$$\frac{dS_e}{dt} = \int_{M^2} (\frac{\partial}{\partial t} \log R)\, R\, d\mu - \int_{M^2} \log\, R \frac{\partial}{\partial R}(R\, d\mu)$$

$$= - \int_{M^2} \log[\Delta\, R + R(R-r)]\, d\mu + \int_{M^2} \log\, R\, \Delta R\, d\mu$$

$$= - \int_{M^2} R(R-r)\, d\mu + \int_{M^2} \frac{|\nabla R|^2}{R}\, d\mu = \int_{M^2} \frac{|\nabla R|^2}{R}\, d\mu - \int_{M^2} (R-r)^2\, d\mu.$$

□

Proposition: If $(M^2, g(t))$ is a soluion of the Ricci flow on a compact surface with $R(\cdot, 0) = 0$, then

$$\frac{dS_e}{dt} = \int_{M^2} \frac{1}{R} |\nabla R + R \nabla f|^2\, d\mu + 2 \int_{M^2} |W|^2\, d\mu \geq 0 \tag{4.5}$$

where W is the trace-free part of the Hessian of f defined as

$$W = \nabla\nabla\, f - \frac{1}{2}\, \Delta\, f \cdot g.$$

To prove this, recall that $\Delta f = R - r$ so expanding the first term on the right of (??) we get

$$\int_{M^2} \frac{1}{R}\, |\nabla R + R\, \nabla\, f|^2\, d\mu = \int_{M^2} \big(\frac{|\nabla R|^2}{R} - 2R(R-r) + R|\nabla f|^2\big)\, d\mu. \tag{4.6}$$

Commuting covariant derivatives and integrating by parts

$$\int_{M^2} (R-r)^2\, d\mu = \int_{M^2} (\Delta f)^2\, d\mu = - \int_{M^2} \langle \nabla f, \nabla \Delta f \rangle\, d\mu = - \int_{M^2} (\langle \nabla f, \Delta \nabla f \rangle - \mathrm{Ric}(\nabla f, \nabla f))\, d\mu$$

$$= \int_{M^2} (|\nabla\nabla f|^2 + \frac{1}{2} R |\nabla f|^2)\, d\mu.$$

This means that

$$\int_{M^2} |\nabla\nabla f|^2\, d\mu = \int_{M^2} (R-r)^2\, d\mu - \int_{M^2} \frac{1}{2} R |\nabla f|^2\, d\mu. \tag{4.7}$$

The last relation required for (??) is

$$2 \int_{M^2} |W|^2\, d\mu = - \int_{M^2} \big((\Delta f)^2 - 2|\nabla\nabla f|^2\big)\, d\mu = - \int_{M^2} -(R-r)^2 + R|\nabla f|^2\, d\mu. \tag{4.8}$$

Using (??), (??) and (??), we calculate that

$$2 \int_{M^2} |W|^2\, d\mu + \int_{M^2} \frac{1}{R} |\nabla R + R \nabla f|^2\, d\mu = \int_{M^2} \big((R-r)^2 - R|\nabla f|^2 + \frac{|\nabla R|^2}{R} - 2(R-r)^2 + R|\nabla f|^2\big)\, d\mu$$

$$= \int_{M^2} (-(R-r)^2 + \frac{|\nabla R|^2}{R})\, d\mu = \frac{dS_e}{dt}.$$

Since the two integrals in the very first term have positive integrands, the conclusion in (**??**) follows. □

4.2 Quantum Tunneling and Entropy Conservation

Recently tunneling has been proposed as an explanation for black hole radiation [**?**]. It occurs by means of a process similar to electron-positron pair creation in a constant electric field. The energy of the particle is supposed to change sign as it crosses the horizon, so a pair created just inside or outside the horizon may materialize with zero total energy after one part of the pair has tunneled to the opposite side of the horizon. This model can be used to provide a short, direct semiclassical derivation of black hole radiance. Energy conservation is taken into account, so the mass of the black hole must go down as it radiates and this ultimately drives the dynamics [**?**].

Consider then a hole in empty Schwarzschild space, but with dynamical geometry which enforces energy conservation. Let us choose coordinates which are not singular at the horizon, since the physics relates to the horizon. Introduce a time coordinate

$$t = t_s + 2\sqrt{2Mr} + 2M \ln \frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}},$$

where t_s is the Schwarzschild time. The line element of the space-time is

$$ds^2 = -(1 - \frac{2M}{r})\, dt^2 + 2\sqrt{\frac{2M}{r}}\, dtdr + dr^2 + r^2\, d\Omega^2.$$

These are called Painlevé coordinates. There is no singularity at $r = 2M$ as in the Schwarzschild case. These are stationary and nonsingular through the horizon. The advantage is that an effective vacuum state can be defined for a quantum field by having it annihilate modes which carry negative frequency with respect to t. It is interesting to note that such a state will look empty, or at least nonsingular, to a freely falling observer passing through the horizon. The radial null geodesics are given by

$$\dot{r} \equiv \frac{dr}{dt} = \pm 1 - \sqrt{\frac{2M}{r}}, \tag{4.9}$$

with the upper and lower signs corresponding to outgoing (ingoing) geodesics assuming t increases. Self gravitating shells in Hamiltonian gravity have been studied by Parikh and Wilcek [**?**]. It was found that when black hole mass is held fixed, the total ADM mass is allowed to vary, a shell of energy E moves in the geodesics of a spacetime with M replaced by $M + E$. Fixing the hole mass allows the hole mass to fluctuate a shell of energy E travels on a geodesic given by

$$ds^2 = -(1 - \frac{2(M-E)}{r})\, dt^2 + 2\sqrt{\frac{2(M-E)}{r}}\, dtdr + dr^2 + r^2\, d\Omega^2, \tag{4.10}$$

and (**??**) was used but with M replaced by $M - E$.

The imaginary part of the action for an s-wave outgoing, positive energy particle which crosses the horizon outwards from r_{in} to r_{out} can be expressed as

$$\mathrm{Im}\, S = \mathrm{Im} \int_{r_{in}}^{r_{out}} p\, dr = \mathrm{Im} \int_{r_{in}}^{r_{out}} \int_0^{p_r} dp\, dr. \tag{4.11}$$

To evaluate (**??**), multiply and divide the integral by the two sides of Hamilton's equation $\dot{r} = dH/dp_r|_r$, change the variable from momentum to energy, interchange the integrals and use the modified form of (**??**) with M replaced by $H = M - E$,

$$\mathrm{Im} S = \mathrm{Im} \int_M^{M-E} \int_{r_{in}}^{r_{out}} \frac{dr}{\dot{r}} dH = \mathrm{Im} \int_0^E \int_{r_{in}}^{r_{out}} \frac{dr}{1 - \sqrt{\frac{2(M-\mathcal{E})}{r}}} (-d\mathcal{E}).$$

The integral can be done by deforming the contour, so as to ensure that positive energy solutions decay in time, thus into the lower part of the E plane, with the result

$$\mathrm{Im}\, S = 4\pi E\, (M - \frac{E}{2})), \tag{4.12}$$

provided that $r_{in} > r_{out}$. Both particles or antiparticle tunneling can contribute to the rate for the Hawking process, so in a more complete treatment, their amplitudes would have to be added before squaring to get the semiclassical tunneling rate. This is going to concern only the prefactor. Consequently, the exponential part of the semiclassical emission rate is

$$\Gamma(M; E) \doteq e^{-2 \mathrm{Im}\, S} = e^{-8\pi E(M - E/2)} = e^{\Delta\, S_{BH}}. \tag{4.13}$$

It can be seen how energy conservation has been incorporated here.

4.3 Correlated Radiation and Information Conservation

At the end, the result is expressed in terms of the change in the Bekenstein-Hawking entropy of the black hole [?]. Instead Hawking radiation must be regarded as correlated radiation and such correlations may carry off information. In contrast, it is distinct from a thermal distribution which has the emmision rate

$$e^{-8\pi EM}.$$

This is not the same as (**??**). To go into this further, suppose two statistical events are considered, such as two emissions of Hawking radiation. Their joint probability is denoted as $p(A, B)$, and the probabilities for the two respective emissions are given by

$$p(A) = \int p(A, B)\, dB \qquad p(B) = \int p(A, B)\, dA.$$

Consider now whether $p(A, B) = p(A) \cdot p(B)$ holds; if equality holds there is no correlation between them and the events are independent. This is the case when the spectrum is taken to be thermal. For a non-thermal spectrum, equality does not hold in this. The emissions are dependent or correlated. The joint probablility $\Gamma(E_1, E_2)$ for two emissions of Hawking radiation, one at energy E_1 and another at E_2 is given by

$$\Gamma(E_1, E_2) \doteq \exp[-8\pi(E_1 + E_2)(M - \frac{E_1 + E_2}{2})]. \tag{4.14}$$

This has the form of (**??**) but with E replaced by $E_1 + E_2$, so the probability of two emissions at energies E_1 and E_2 is the same as the emission probability for a single Hawking radiation at energy $E_1 + E_2$, that is

$$\Gamma(E_1, E_2) = \Gamma(E_1 + E_2). \tag{4.15}$$

As seen in (**??**), energy conservation has been enforced by treating Hawking radiation as a form of quantum tunneling. The existence of correlations is then supported as $\Gamma(E_2|E_1) \neq \Gamma(E_2)$ where the conditional probability is determined to be

$$\Gamma(E_2|E_1) = \frac{\Gamma(E_1, E_2)}{\Gamma(E_1)} = \exp[-8\pi E_2(M - E_1 - \frac{E_2}{2})] \tag{4.16}$$

for an emission with energy E_2 *given* that an emission with energy E_1 has already taken place. As in information theory, the mutual information between two Hawking radiations with energies E_1 and E_2 is defined

$$S(E_1, E_2) = S(E_1) + S(E_2) - S(E_1, E_2) = S(E_2) - S(E_2|E_1), \tag{4.17}$$

where $S(E_1, E_2)$ is the entropy for the system and $S(E_2|E_1)$ is the conditional entropy. Substituting we find that between the two there results

$$S(E_2, E_1) = -\log\,\Gamma(E_2) + \log\,\Gamma(E_2|E_1) = 8\pi E_1\,E_2 \tag{4.18}$$

This implies hidden correlations with regard to the Hawking radiation.

For a list or collection of emissions each with energy E_i, the total correlation can be calculated along any one of the independent partitions. A partition is chosen according to the list of the subscripts such that the total correlation is the sum of the correlations between emissions E_1 and E_2, $E_1 \oplus E_2$ and E_3, $E_1 \oplus E_2 \oplus E_3$ and E_4. Continuing in the same way, $E_1 \oplus \cdots \oplus E_{n-1}$ and E_n, where $E_a \oplus E_b$ denotes the combined system of a and b. When the total correlation among all emissions is obtained, the amount of information it can encode is found again to exactly equal the amount previously found to be lost in the accounting of the paradox. So crucially black holes are in fact governed by the law of conservation of energy. Taking this into account is a major step towards resolving the paradox.

To see that the microscopic process of Hawking radiation with an energy dE is unitary then, the entropy carried away by an emission dE is

$$d\,S = -\log\,\Gamma(dE) = 8\pi\,(M - \frac{dE}{2})\,dE. \tag{4.19}$$

This may be compared with the increase of Beckenstein-Hawking entropy of a black hole

$$dS_{BH} = 4\pi\,[(M - dE)^2 - M^2] = -8\pi\;(M - \frac{dE}{2})\,dE. \tag{4.20}$$

This yields an entropy production ratio of $R = ||dS/dS_{BH}|| = 1$ for the non-thermal spectrum (**??**). The decrease of the entropy for a black hole is exactly balanced by the increase of entropy of its emitted radiations.

For a nonthermal spectrum, the entropy of the Hawking radiation can be calculated by counting the microstates $\{E_1, \ldots, E_n\}$ under the energy constraint $\sum_i E_i = M$. A fixed list of E_i specifies a microstate. The probability of the macrostate to occur is

$$\Gamma(E_1, \ldots, E_n) = \Gamma(M, E_1)\Gamma(M - E_1; E_2) \cdots \Gamma(M - \sum_{i=1}^{n-1} E_i; E_n). \tag{4.21}$$

Each of the terms on the right can be computed supposing the nonthermal spectrum. The last would be $\Gamma(M - \sum_{i=1}^{n-1} E_i; E_n) = \exp(-4\pi E_n^2)$. Collecting all terms $P(E_1, \ldots, E_n) = \exp(-4\pi M^2) = \exp(-S_{BH})$ and $\Omega = 1/P(E_1, \ldots, E_n) = \exp(S_{BH})$. As all microstates are equally likely, the total entropy of the radiation is the $S = \log(\Omega)$ which is just S_{BH} using Boltzmann's definition of entropy.

5 Final Remarks

Over the last century and a half, entropy and its role in thermodynamics has been one of the most widely discussed topics in physics. This is very evident in the analogy between the law of area increase and entropy increase. As in classical mechanics the precise definition of entropy depends on what observable is being measured. There have been many points of view. It is clear that many new developments will be needed in order to extend its range of applicability to new domains in physics, such as nonequiloibrium systems and black hole physics.

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