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Metric geometry and the determination of the Bohmian quantum potential

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Abstract

A geometric approach to quantum mechanics which is formulated in terms of Finsler geometry is developed. It is shown that quantum mechanics can be formulated in terms of Finsler configuration space trajectories which obey Newton-like evolution but in the presence of an additional kind of potential. This additional quantum potential which was obtained first by Bohm has the consequence of contributing to the forces driving the system. This geometric picture accounts for many aspects of quantum dynamics and leads to a more natural interpretation. It is found for example that dynamics can be accounted for by incorporating quantum effects into the geometry of space-time.

1. Introduction

It is still a task to understand and interpret quantum mechanics. It has become clear that quantum mechanics can be looked at from a geometric point of view. In fact, geometry can be applied to the study of the structure and interpretation of quantum mechanics. This is due largely to the fact that geometrical considerations have resulted in advances in various other areas of physics, most notably, general relativity. Quantum mechanics is not so easy to formulate in terms of a dynamical, geometric description in terms of trajectories in some configuration space for a number of reasons. It will be seen here that a particular framework for such a task can be developed.

A first reason for this is the important fact of quantum correlation and another is the complicated phenomenon of quantum entanglement. Both quantum coherence and decoherence effects, the state superposition principle and so forth have a natural description based on the Schrödinger equation. It has long been known that determinism is hard to reconcile with such a description of quantum dynamics. This problem has been discussed by both de Broglie and Bohm [1–3]. This view of things consists of material point trajectories carried along by a pilot wave that evolves according to the Schrödinger equation for the wavefunction of the system. Bohmian trajectories follow the flux lines of probability current which is a function of the position coordinate of the particle involved. This implies that there is a certain nonlocality inherent in the usual picture. Bohm realized that this kind of dynamics can be described by configuration space trajectories which follow a Newton's law evolution, but under the influence of a quantum potential in addition to the external potential. This quantum potential can be calculated in closed form. This quantum potential is added to the classical potential function and it makes a contribution to the evolution of the system.

It will be useful to have a brief description of Bohm's theory at hand [4], to be able to refer to it further on. Bohm began from a series of basic postulates. An individual system comprises a wave propagating in space and time along with a point particle which moves under the guidance of the wave. The wave is described mathematically by Schrödinger's equation $i\hbar\,\partial_t\psi(\mathbf{x},\,t) = H\psi(\mathbf{x},\,t)$, where H is the Hamiltonian. The motion of the particle is obtained as the solution to

$$\dot{\mathbf{x}} = \frac{1}{m}\nabla S(\mathbf{x},\,t)|_{\mathbf{x}=\mathbf{x}(t)} \tag{1.1}$$

where S is the phase of $\psi(\mathbf{x}, t)$ and m the mass. One initial condition for (1.1) suffices to solve it. These give a theory of motion which is consistent from a physical point of view. To get compatibility of the motion of an ensemble of particles with quantum mechanics, Bohm states that the probability that a particle in the ensemble resides between $\mathbf{x}$ and $\mathbf{x} + d\mathbf{x}$ at time t is $R^2(\mathbf{x}, t)\, d^3x$ where $R = |\psi|^2$.

In Bohm's approach, the wavefunction in polar form $\psi = R\, e^{iS/\hbar}$ is substituted into Schrödinger's equation. By isolating real and imaginary parts, he found the movement under the guidance of the wave happens in agreement with a law of motion of the form

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\nabla S^2 + \frac{\hbar^2}{2\, m\, R}\nabla^2 R + V = 0. \tag{1.2}$$

Equation (1.2) is equal to the classical Hamilton-Jacobi equation except for the appearance of the term

$$Q = \frac{\hbar^2}{2mR}\nabla^2 R. \tag{1.3}$$

It is referred to as the quantum potential. The equation of motion can be expressed as well in the form,

$$m\frac{d^2\mathbf{x}}{dt^2} = -\nabla(Q + V), \tag{1.4}$$

where $\mathbf{x} = \mathbf{x}(t)$ is the trajectory of the particle associated with its wavefunction.

One approach to obtain a geometric formulation and interpretation of quantum mechanics is to introduce a particular geometric structure on which to base everything. Finsler geometry is a metric geometry which is well suited to this purpose [5]. It is described by a metric tensor which is a function of both the position and momentum variables. The metric in Finsler geometry is in fact a generalization of a Riemannian metric, and the case in which the metric is developed from the quantum potential will be studied. The approach of Chern to Finsler geometry will be followed [6, 7].

The theory that results by proceeding in this way accounts for quantum effects as a consequence of the geometry of space-time [8, 9]. The curvature is self-induced as well as generated by the collection of particles in a nonlocal fashion resulting in a curved configuration space. The metric tensor and as a consequence the curvature of the manifold depends on the coordinates of the particle, as well as all the other particles of the system. The wavelike nature of quantum mechanics is then confined to accounting for how space is described by the particular geometry. The main result of proceeding in this fashion is the conclusion that quantum mechanical properties are a net result of the metric used for the underlying space itself [10–15].

2. Geometry of the projectivized tangent bundle and the Hilbert form

A mathematical presentation of the geometric setting will be given so that later the physical application can be adapted in a straightforward manner. Let M be an m-dimensional manifold. This manifold is said to be a Finsler manifold if the length s of any curve described by $t \to (u^1(t), \ldots, u^m(t))$ in coordinates $a \leqslant t \leqslant b$ is given by the integral

$$s = \int_a^b F\left(u^1, \ldots, u^m, \frac{du^1}{dt}, \ldots, \frac{du^m}{dt}\right) dt. \tag{2.1}$$

In (2.1), F is a smooth non-negative function in $2m$ variables and the function $F(\mathbf{x}, \mathbf{y})$ vanishes only when $\mathbf{y} = 0$. It is required to be symmetrically homogeneous of degree one in the $\mathbf{y}$ variables

$$F(x^1, \ldots, x^m;\ \lambda y^1, \ldots, \lambda\, y^m) = |\lambda|\, F(x^1, \ldots, x^m;\ y^1, \ldots, y^m), \qquad \lambda \in \mathbb{R}. \tag{2.2}$$

A Finsler manifold M has a tangent bundle π: $TM \to M$ and a cotangent bundle π^*: $T^*M \to M$. From TM the projectivized tangent bundle of M is obtained and denoted PTM, by identifying the non-zero vectors differing from each other by a real factor. Geometrically, PTM is the space of line elements on M. Let u^i, $1 \leqslant i \leqslant m$ be local coordinates on M. Then a non-zero tangent vector can be represented in the form

$$X = X^i\frac{\partial}{\partial u^i}, \tag{2.3}$$

with the X^i not all zero. The u^i, X^i are local coordinates on TM. They are also local coordinates on the space PTM with the X^i homogeneous coordinates which are determined up to a real factor.

A fundamental idea is to regard PTM as the base manifold of the vector bundle p^*TM which is pulled back by means of the canonical projection map p: $PTM \to M$ defined by

$$p(u^i, X^i) = (u^i). \tag{2.4}$$

Since the function $F(u^i, X^i)$ is homogeneous of degree one in the X^i variables, Euler's theorem for homogeneous functions implies that

$$X^i \frac{\partial F}{\partial X^i} = F. \tag{2.5}$$

Differentiating (2.5) with respect to X^j, we obtain

$$\frac{\partial F}{\partial X^j} + X^i \frac{\partial^2 F}{\partial X^i \partial X^j} = \frac{\partial F}{\partial X^j}. \tag{2.6}$$

Clearly (2.6) simplifies to

$$X^i \frac{\partial^2 F}{\partial X^i \partial X^j} = 0. \tag{2.7}$$

Result (2.7) implies that the first derivatives of F with respect to the X^i are homogeneous functions of degree zero in the X^i coordinates. Such functions are functions on PTM.

Suppose (v^i, Y^i) is another local coordinate system on PTM, then it is the case that

$$Y^i = \frac{\partial v^i}{\partial X^j} X^j, \tag{2.8}$$

hence

$$\frac{\partial F}{\partial X^i} = \frac{\partial F}{\partial Y^j} \frac{\partial v^j}{\partial u^i}. \tag{2.9}$$

Now an important one-form ω can be defined as

$$\omega = \frac{\partial F}{\partial X^i}\, du^i = \frac{\partial F}{\partial Y^i}\, dv^i. \tag{2.10}$$

It follows that ω in (2.10) is independent of the choice of local coordinates, hence it is defined intrinsically on PTM. The form (2.10) is usually referred to as the Hilbert form. By Euler's theorem, the arclength integral (2.1) with respect to M can be formulated as

$$s = \int_a^b \omega. \tag{2.11}$$

The integral (2.11) is called Hilbert's invariant integral. On exterior differentiation, the Hilbert form yields a connection with some remarkable properties.

Let

$$e_i = p_i^j \frac{\partial}{\partial u^j} \tag{2.12}$$

be an orthonormal frame field on the bundle P^*TM and

$$\omega^j = q_k^j\, du^k \tag{2.13}$$

its dual coframe field, as we have the relations

$$(e_i, e_j) = p_i^j g_{sk} p_j^k = \delta_{ij}, \qquad \langle e_i, \omega^j \rangle = \delta_i^j. \tag{2.14}$$

The former in (2.14) is referred to as the orthonormality relation and the latter is the duality condition, which is equivalent to

$$p_i^j q_j^k = \delta_i^k.$$

This means that the matrices (p_i^j) and (q_j^k) are matrix inverses of each other. The orthonormality is then given with respect to the following symmetric, covariant 2-tensor,

$$G = g_{ij}\, du^i \otimes du^j = \frac{\partial(\frac{1}{2}F^2)}{\partial X^i \partial X^j}\, du^i \otimes du^j = \left(F \frac{\partial^2 F}{\partial X^i \partial X^j} + \frac{\partial F}{\partial X^i} \frac{\partial F}{\partial X^j} \right) du^i \otimes du^j. \tag{2.15}$$

This is defined intrinsically on PTM.

This is supposed to be positive definite, the strong convexity hypothesis. Then (2.15) will be referred to as a Finsler metric. In the case of Riemannian geometry, F^2 reduces to the form

$$F^2(u^i, X^j) = g_{ij}(u)\, X^i X^j,$$

where the components g_{ij} are functions of the u^i only. In the Finsler picture, the g_{ij} are in general functions of both u^i and X^i. They are also homogeneous of degree zero in the X^i variables. Thus, the g_{ij} are functions on the space PTM.

3. Finsler geometry and quantum mechanics

A specific type of Lagrangian is introduced and we turn to the physical problem that is related to Finsler geometry. The manifold M will be defined as the configuration space manifold and TM is the corresponding tangent space. If $\mathcal{L}: \mathbb{R} \times TM \to \mathbb{R}$ is a time-dependent Lagrangian function, a generalized homogeneous Lagrangian can be defined by

$$\Lambda(t, \mathbf{x}, \dot{t}, \dot{\mathbf{x}}) = \dot{t}\mathcal{L}\left(t, \mathbf{x}, \frac{\dot{\mathbf{x}}}{\dot{t}}\right), \tag{3.1}$$

and $\dot{x}^i = dx_i/d\tau$ for $i = 1, \dots, n$. To get (3.1), a new parameter $\tau(t)$ is introduced. It indicates the progress of the system in the extended configuration space. The Euler–Lagrange equations which correspond to the Lagrangian Λ in (3.1) are invariant with respect to any regular transformation of the parameter $\tau(t)$. The idea is to consider time as an additional generalized coordinate $q_0 = t \in \mathbb{R}$. The dynamics may be described with respect to a $(3n + 1)$-dimensional space which is coordinatized in terms of the generalized coordinates and velocities $\{(q^a)_{a=0}^n, (\dot{q}^a)_{a=0}^n\} = \{\mathbf{q}, \dot{\mathbf{q}}\}$. Thus, $q^0 = t$, $\dot{q}^n = dt/d\tau$, $q^i = x_i$. The accompanying velocities are defined as $\{\dot{q}^a = dq_a/d\tau\}_{a=1}^n$. For a Lagrangian with time-dependent potential $V(\mathbf{x}, t)$ of the form $\mathcal{L}(t, \mathbf{x}, \mathbf{x}') = T(\mathbf{x}') - V(\mathbf{x}, t)$, (3.1) is

$$\Lambda(\mathbf{q}, \dot{\mathbf{q}}) = \mathcal{T}(\dot{q}^i)\,\frac{1}{\dot{q}^0} - Q(\mathbf{q})\dot{q}^0, \tag{3.2}$$

such that $\mathcal{T}(\dot{\mathbf{q}}^i) = T(\mathbf{x}')(\dot{q}^0)^2$ and $T(\mathbf{x}') = \frac{1}{2}\sum_{i=1}^n m_i(x_i')^2$. To make the correspondence between this and the previous section clear, the function Λ in (3.2) is identified with the function F in (2.1) and in (2.15) for the Finsler metric. Moreover, (u^i, X^j) will be identified with $(\mathbf{q}, \dot{\mathbf{q}})$. This idea could be extended to any Lagrangian that satisfies the properties required for a Finsler metric.

The dynamics is then described in a Finsler space M^{3n+1} with coordinates u^i identified as the $\mathbf{q} = (q^0, \dots, q^n)$ described by a homogeneous Lagrangian $\Lambda(\mathbf{q}, \dot{\mathbf{q}})$. The line element between two adjacent points in space, with summation over a, b implied, is given in terms of Λ as

$$d\,s = (g_{ab}(\mathbf{q}, \dot{\mathbf{q}})\,dq^a dq^b)^{1/2} = \Lambda(\mathbf{q}, \dot{\mathbf{q}})\,d\tau. \tag{3.3}$$

The Finsler metric in (3.3) defines a dynamical system through the minimization of the action functional

$$I(\Gamma) = \int_{t_1}^{t_2} \Lambda(\mathbf{q}, \dot{\mathbf{q}})\,d\tau \tag{3.4}$$

evaluated with respect to a path Γ with given initial and final conditions. Thus, we have positive homogeneity of degree one in the second argument. Moreover, Λ is nonvanishing when the second argument is nonvanishing and finally

$$g_{ab} = \frac{1}{2}\frac{\partial^2\Lambda(\mathbf{q}, \dot{\mathbf{q}})}{\partial\dot{q}^a\partial\dot{q}^b}\xi^a\xi^b > 0, \tag{3.5}$$

for all $\boldsymbol{\xi} \neq \lambda\dot{\mathbf{q}}$. This is the third condition for a metric to be Finsler. Thus the requirements for a Finsler metric are satisfied in this case.

Let us outline without a lot of calculation how the components of the metric tensor $g_{\alpha\beta}(\mathbf{q}, \dot{\mathbf{q}})$ for the space can be calculated. Starting with (3.2),

$$g_{00}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\,\frac{\partial^2\Lambda^2(\mathbf{q}, \dot{\mathbf{q}})}{\partial\dot{q}^0\partial\dot{q}^0} = (\Lambda'(\mathbf{q}, \dot{\mathbf{q}}))^2 + \Lambda(\mathbf{q}, \dot{\mathbf{q}})\Lambda''(\mathbf{q}, \dot{\mathbf{q}}),$$

where

$$\Lambda'(\mathbf{q}, \dot{\mathbf{q}}) = -\mathcal{T}\frac{1}{(\dot{q}^0)^2} - Q_0(\mathbf{q}, t)\dot{q}^0 - Q(\mathbf{q}, t), \qquad \Lambda''(\mathbf{q}, \dot{\mathbf{q}}) = 2\mathcal{T}\frac{1}{(\dot{q}^0)^3} - Q_{00}(\mathbf{q}, t)\dot{q}^0 - 2Q_0(\mathbf{q}, t),$$

$$g_{ij}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\frac{\partial^2\Lambda^2(\mathbf{q}, \dot{\mathbf{q}})}{\partial\dot{q}^i\partial\dot{q}^j} = \frac{1}{2}\frac{\partial}{\partial\dot{q}^i}\left(2\Lambda(\mathbf{q}, \dot{\mathbf{q}})\frac{\partial\Lambda(\mathbf{q}, \dot{\mathbf{q}})}{\partial\dot{\mathbf{q}}^i)}\right) = \left(m_i\delta_{il}\frac{\dot{q}^l}{\dot{q}^0}\right)\left(m_j\delta_{jk}\frac{\dot{q}^k}{\dot{q}^0}\right) + \Lambda(\mathbf{q}, \dot{\mathbf{q}})\delta_{ij}m_i\frac{1}{\dot{q}^0},$$

and lastly,

$$g_{0i}(\mathbf{q},\,\dot{\mathbf{q}}) = -2\frac{\mathcal{T}(\dot{\mathbf{q}})\mathcal{T}_i(\dot{\mathbf{q}})}{(\dot{q}^0)^3} - Q_0(\mathbf{q},\,t)\mathcal{T}_i(\dot{\mathbf{q}}).$$

4. Quantum dynamics

The Lagrange system is obtained by minimizing the action functional (3.4) and leads directly to $3n+1$ equations with $3n$ independent degrees of freedom. The equations of motion are then generated through

$$\frac{d}{dt}\left(\frac{\partial\Lambda(\mathbf{q},\,\dot{\mathbf{q}})}{\partial\dot{q}^a}\right) - \frac{\partial\Lambda(\mathbf{q},\,\dot{\mathbf{q}})}{\partial q^a} = 0, \qquad a = 0,\,1,\ldots,n. \tag{4.1}$$

The condition of homogeneity of degree one in $\dot{\mathbf{q}}$ yields according to Euler's theorem

$$\frac{\partial\Lambda(\mathbf{q},\,\dot{\mathbf{q}})}{\partial\dot{q}^a}\dot{q}^a = \Lambda(\mathbf{q},\,\dot{\mathbf{q}}), \qquad \frac{\partial^2\Lambda(\mathbf{q},\,\dot{\mathbf{q}})}{\partial\dot{q}^a\partial\dot{q}^b}\dot{q}^a = 0. \tag{4.2}$$

It follows from (4.1) that the following constraint holds

$$\begin{aligned}\sum_a \dot{q}^a\left(\frac{d}{d\tau}\left(\frac{\partial\Lambda(\mathbf{q},\,\dot{\mathbf{q}})}{\partial\dot{q}^a}\right) - \frac{\partial\Lambda(\mathbf{q},\,\dot{\mathbf{q}})}{\partial q^a}\right) &= \frac{d}{d\tau}\sum_a\left(\dot{q}^a\frac{\partial\Lambda(\mathbf{q},\,\dot{\mathbf{q}})}{\partial q^a}\right) - \sum_a\left(\ddot{q}^a\frac{\partial\Lambda(\mathbf{q},\,\dot{\mathbf{q}})}{\partial\dot{q}^a} + \dot{q}^a\frac{\partial\Lambda(\mathbf{q},\,\dot{\mathbf{q}})}{\partial q^q}\right)\\ &= \frac{d}{d\tau}\Lambda(\mathbf{q},\,\dot{\mathbf{q}}) - \frac{d}{d\tau}\Lambda(\mathbf{q},\,\dot{\mathbf{q}}) = 0.\end{aligned} \tag{4.3}$$

The remaining equation sets the freedom for the choice of $\tau\,(q_a)$ and to establish this, one may take $\tau = t$ so that $q^0 = 1$ and $\Lambda(\mathbf{q},\,\dot{\mathbf{q}}) = \mathcal{L}(t,\,\mathbf{x},\,\mathbf{x}')$, so the standard Euler–Lagrange equations in $\mathcal{L}$ result. Thus there is a correspondence between the Euler–Lagrange equations for the function $\Lambda(\mathbf{q},\,\dot{\mathbf{q}})$ and the geodesic equation which is characterized by the metric tensor (3.5) in the curved manifold.

Let us then start with the Euler–Lagrange equations given in terms of the arclength parameter. When the arclength parameter s is chosen for τ, by definition $\Lambda(\mathbf{q},\,\dot{\mathbf{q}}) = 1$ along the trajectory. This fact will be used in what follows. Taking into account the definition of the metric in (3.5), we may write

$$\frac{1}{2}\frac{\partial^2\Lambda^2(\mathbf{q},\,\dot{\mathbf{q}})}{\partial\dot{q}^a\partial\dot{q}^b}\dot{q}^b = \frac{\partial}{\partial\dot{q}^a}\left(\Lambda(\mathbf{q},\,\dot{\mathbf{q}})\frac{\partial\Lambda(\mathbf{q},\,\dot{\mathbf{q}})}{\partial\dot{q}^b}\right)\dot{q}^b = \frac{\partial\Lambda}{\partial\dot{q}^a}\frac{\partial\Lambda}{\partial\dot{q}^b}\dot{q}^b + \Lambda(\mathbf{q},\,\dot{\mathbf{q}})\frac{\partial^2\Lambda}{\partial\dot{q}^a\partial\dot{q}^b}\dot{q}^b$$

Using (2.6) this equation simplifies to the form

$$\frac{1}{2}\frac{\partial^2\Lambda^2(\mathbf{q},\,\dot{\mathbf{q}})}{\partial\dot{q}^a\partial\dot{q}^b}\dot{q}^b = \frac{\partial\Lambda}{\partial q^a}(\mathbf{q},\,\dot{\mathbf{q}})\Lambda(\mathbf{q},\,\dot{\mathbf{q}}). \tag{4.4}$$

Equivalently, in terms of metric g_{ab}, (4.4) is

$$g_{ab}\,\dot{q}^b = \Lambda(\mathbf{q},\,\dot{\mathbf{q}})\frac{\partial\Lambda}{\partial\dot{q}^a}. \tag{4.5}$$

Differentiating (4.5) with respect to s, it follows that

$$\frac{d}{ds}(g_{ab}\dot{q}^b) = \frac{d}{ds}\left(\Lambda(\mathbf{q},\,\dot{\mathbf{q}})\frac{\partial\Lambda}{\partial\dot{q}^a}\right) = \frac{d\Lambda}{ds}\frac{\partial\Lambda}{\partial\dot{q}^a} + \Lambda(\mathbf{q},\,\dot{\mathbf{q}})\frac{d}{ds}\left(\frac{\partial\Lambda}{\partial q^a}\right). \tag{4.6}$$

Restricting (4.6) to reside on the path, as indicated earlier $\Lambda(\mathbf{q},\,\dot{\mathbf{q}}) = 1$ and it follows from (4.6) that

$$\frac{d}{ds}(g_{ab}\dot{q}_b) = \frac{d}{ds}\left(\frac{\partial\Lambda}{\partial\dot{q}^a}\right) = \frac{\partial\Lambda}{\partial q^a}. \tag{4.7}$$

The right-hand side of (4.7) can be put in the form

$$\frac{\partial}{\partial q^l}\Lambda(\mathbf{q},\,\dot{\mathbf{q}}) = \frac{\partial}{\partial q^l}\sqrt{g_{ab}(\mathbf{q},\,\dot{\mathbf{q}})\dot{q}^a\dot{q}^b} = \frac{1}{2\Lambda(\mathbf{q},\,\dot{\mathbf{q}})}\frac{\partial g_{ab}}{\partial q^l}\dot{q}^a\dot{q}^b = \frac{1}{2}\frac{\partial g_{ab}(\mathbf{q},\,\dot{\mathbf{q}})}{\partial q^l}\dot{q}^a\dot{q}^b. \tag{4.8}$$

The last equality holds when the arclength parameter s is chosen to be the variable τ so as remarked, $\Lambda(\mathbf{q},\,\dot{\mathbf{q}}) = 1$ along the path. Then (4.7) can be put in the form,

$$\frac{d}{ds}\,(g_{lb}(\mathbf{q},\,\dot{\mathbf{q}})\dot{q}^b) - \frac{1}{2}\frac{\partial g_{ab}(\mathbf{q},\,\dot{\mathbf{q}})}{\partial q^l}\dot{q}^a\dot{q}^b = 0. \tag{4.9}$$

Expanding out the s derivative, (4.9) becomes

$$g_{lb}(\mathbf{q},\,\dot{\mathbf{q}})\ddot{q}^b + \frac{\partial g_{la}}{\partial q^b}\dot{q}^a\dot{q}^b - \frac{1}{2}\frac{\partial g_{ab}(\mathbf{q},\,\dot{\mathbf{q}})}{\partial q^l}\dot{q}^a\dot{q}^b = 0. \tag{4.10}$$

Introducing the Christoffel symbols which are defined in the usual way, we write

$$\Gamma_{blc} = \frac{1}{2}\left(\frac{\partial g_{bl}}{\partial q^c} + \frac{\partial g_{lc}}{\partial q^b} - \frac{\partial g_{cb}}{\partial q^l}\right), \qquad \Gamma^a_{bc} = g^{la}\Gamma_{blc}. \tag{4.11}$$

It is apparent that the following relation holds,

$$\Gamma^a_{bc}\,\dot{q}^b\dot{q}^c = \frac{1}{2}g^{la}\left(2\frac{\partial g_{bl}}{\partial q^c}\dot{q}^b\dot{q}^c - \frac{1}{2}\frac{\partial g_{cb}}{\partial q^l}\dot{q}^b\dot{q}^c\right) = g^{la}\left(\frac{\partial g_{bl}}{\partial q^l} - \frac{1}{2}\frac{\partial g_{cb}}{\partial q^l}\right)\dot{q}^a\dot{q}^b. \tag{4.12}$$

Relations (4.11) and (4.12) permit us to write (4.10) in the following form,

$$\ddot{q}^a + g^{ac}\left(\frac{\partial g_{cs}}{\partial q^b} - \frac{1}{2}\frac{\partial g_{sb}}{\partial q^c}\right)\dot{q}^s\dot{q}^b = 0. \tag{4.13}$$

Comparing (4.13) with (4.12), it is clear that (4.13) is of exactly the required form,

$$\ddot{q}^a + \Gamma^a_{sb}\,\dot{q}^s\dot{q}^b = 0. \tag{4.14}$$

Finsler metric (3.5) is used to compute the set of Γ^a_{sb} and solutions to (4.14) will give geodesic curves on the manifold.

5. Quantum dynamics and the quantum potential

Consider now the concrete case of a Lagrangian which is related to the Schrödinger equation and satisfies the conditions for a Finsler space. A Lagrangian density relevant to this type of quantum dynamics and depends on the scalar field $\varphi(\mathbf{x}, t)$ is

$$\mathcal{L} = \frac{1}{2}(\varphi^*\dot{\varphi} - \dot{\varphi}^*\varphi) - \frac{\hbar^2}{2m}\nabla_k\varphi\;\nabla_k\varphi^* - V\varphi\varphi^*. \tag{5.1}$$

In (5.1), ∇_{i} is the gradient or derivative operator in the designated coordinates and $\Delta = (\nabla_k)^2$. The time dependent Schrödinger equation is obtained from (5.1) by using it in conjunction with the principle of least action. A quantum dynamics can therefore be obtained by regarding the wavefunction as the classical complex scalar function $\varphi(\mathbf{x}, t)$. This should have a representation in terms of trajectories, and these are derived from the field conservation law. This is the starting point. This law is obtained from the stress-energy momentum tensor and it takes the form,

$$\partial_a T^a_b = \frac{\partial\mathcal{L}}{\partial q_b}, \tag{5.2}$$

In (5.2), T^a_b is given by

$$T^a_b = -\left[\frac{\partial\mathcal{L}}{\partial(\partial_a\rho)}\partial_b\rho + \frac{\partial\mathcal{L}}{\partial(\partial_a S)}\partial_a S\right] + \delta^a_b\mathcal{L}. \tag{5.3}$$

The functions ρ and S in (5.3) are related to the polar representation of the field φ [12],

$$\varphi(\mathbf{x},\,t) = R(\mathbf{x},\,t)\;e^{iS(\mathbf{x},t)/\hbar}. \tag{5.4}$$

We may refer to $S(\mathbf{x}, t)$ as the phase of $\varphi(\mathbf{x}, t)$ and the density ρ is determined from

$$\rho(\mathbf{x},\,t) = R^2(\mathbf{x},\,t). \tag{5.5}$$

Using (5.4), we may calculate the required derivatives

$$\begin{aligned} &\varphi^*\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi^*}{\partial t}\varphi = 2\frac{i}{\hbar}\frac{\partial S}{\partial t},\\ &\nabla_k\varphi\;\nabla_k\varphi^* = \left(\nabla_k R + \frac{i}{\hbar}R\nabla_k S\right)\left(\nabla_k R - \frac{i}{\hbar}R\nabla_k S\right) = \frac{1}{4\rho}(\nabla_k\rho)^2 + \frac{\rho}{\hbar^2}(\nabla_k S)^2. \end{aligned} \tag{5.6}$$

Substituting (5.6) into Lagrangian density (5.1), this puts it in the form,

$$\mathcal{L} = -\rho\left(\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla_k S)^2 + V(\mathbf{x})\right) - \frac{\hbar^2}{2m}\left(\frac{(\nabla_k \rho)^2}{4\rho}\right). \tag{5.7}$$

Lagrangian (5.7) will be used to calculate the form of the conservation law given as

$$\frac{\partial T_0^0(\mathbf{x}, t)}{\partial t} + T_{0,k}^k(\mathbf{x}, t) = -\rho\,\frac{\partial V(\mathbf{x})}{\partial t}. \tag{5.8}$$

The result will be expressed in terms of variables ρ and S. To carry this out, the components of the energy-momentum tensor are needed. The first, $T_0^0(\mathbf{x}, t)$ is given by

$$\begin{aligned} T_0^0(\mathbf{x}, t) &= -\left[\frac{\partial\mathcal{L}}{\partial\dot\rho}\dot\rho + \frac{\partial\mathcal{L}}{\partial\dot S}\dot S\right] + \mathcal{L} \\ &= \rho\dot S - \rho\left[\dot S + \frac{(\nabla S)^2}{2m} + V(\mathbf{x})\right] - \frac{\hbar^2}{2m\rho}(\nabla\rho)^2 = -\rho\left[\frac{1}{2m}(\nabla S)^2 + V(\mathbf{x})\right] - \frac{\hbar^2}{2m\rho}(\nabla\rho)^2. \end{aligned} \tag{5.9}$$

Differentiating this result for T_0^0 with respect to t, there results

$$\begin{aligned} -\frac{\partial T_0^0}{\partial t} &= \frac{\partial\rho}{\partial t}\left[\frac{1}{2m}(\nabla S)^2 + V(\mathbf{x})\right] + \rho\left[\frac{\partial}{\partial t}(\frac{(\nabla S)^2}{2m} + \frac{\partial V}{\partial t}\right] + \frac{\hbar^2}{2m}\frac{\partial}{\partial t}\frac{(\nabla\rho)^2}{\rho}. \\ &= \frac{\partial\rho}{\partial t}\left[\frac{1}{2m}(\nabla S)^2 + V(\mathbf{x})\right] + \rho\left[\frac{1}{m}\nabla S\cdot\nabla\dot S + \frac{\partial V}{\partial t}\right] \\ &\quad + \frac{\hbar^2}{8mR^4}\left[4R\nabla_k R\left(\frac{\partial}{\partial t}\nabla_k\rho\right)R^2 - (2R\nabla R)^2\cdot 2R\cdot\dot R\right] \end{aligned} \tag{5.10}$$

It has been found that

$$-\frac{\partial T_0^0}{\partial t} = \rho\frac{\partial\rho}{\partial t}\left[\frac{1}{2m}(\nabla S)^2 + V\right] + \rho\left[\frac{1}{m}\nabla S\cdot\nabla\dot S + \frac{\partial V}{\partial t}\right] + \frac{\hbar^2}{m}\nabla R\cdot\nabla\dot R. \tag{5.11}$$

It remains to calculate the component T_0^k given by

$$T_0^k = \frac{\hbar^2}{4m\rho}(\nabla_k\rho)\frac{\partial\rho}{\partial t} + \frac{\rho}{m}(\nabla_k S)\frac{\partial S}{\partial t}. \tag{5.12}$$

Differentiate T_0^k with respect to the spatial variables q_k, that is x_k, $(k > 0)$, to give

$$\begin{aligned} \frac{\partial T_0^k}{\partial q_k} &= \frac{\partial}{\partial q_k}\left(\frac{\hbar^2}{4m\rho}(\nabla_k\,\rho)\frac{\partial\rho}{\partial t} + \frac{\rho}{m}(\nabla_k\,S)\frac{\partial S}{\partial t}\right) \\ &= \nabla_k\rho\,\frac{\partial S}{\partial t}\frac{\nabla_k S}{m} + \frac{\rho}{m}\nabla_k\left(\frac{\partial S}{\partial t}\nabla_k S\right) + \hbar^2\left[\nabla_k\frac{\partial\rho}{\partial t}\frac{\nabla_k\rho}{4m\rho} + \frac{\rho}{4m}\left(\frac{\Delta\rho}{\rho} - \frac{(\nabla_k\rho)^2}{\rho^2}\right)\right] \\ &= \frac{2}{m}R\,(\nabla_k R)\,\nabla_k\left(\frac{\partial S}{\partial t}\right) + \frac{1}{m}R^2\nabla_k\left(\frac{\partial S}{\partial t}\nabla_k S\right) + \hbar^2\left[\frac{1}{2m\rho}\left(\frac{\partial}{\partial t}\nabla_k\rho\right)(\nabla_k\rho)\right. \\ &\quad \left. + \frac{1}{2m}R\frac{\partial R}{\partial t}\left(\frac{2}{\rho}(\partial_k R)^2 + \frac{2}{R}\Delta R\right) - \frac{1}{4m\rho^2}(2R\nabla_k R)^2\right] \\ &= \frac{2}{m}R(\nabla_k R)\,\frac{\partial S}{\partial t}\nabla_k S + \frac{1}{m}R^2\left(\nabla_k\frac{\partial S}{\partial t}\nabla_k S + \frac{\partial S}{\partial t}\Delta S\right) + \frac{\hbar^2}{m}\left[\frac{\partial}{\partial t}\nabla_k\,R + \frac{\partial R}{\partial t}\Delta R\right]. \end{aligned} \tag{5.13}$$

Substituting the results for derivatives (5.10) and (5.13) into (5.8), the following expression is found

$$\begin{aligned} &-\rho\left[\frac{1}{2m}(\nabla_k\rho)^2 + V\right] - \rho\left[\frac{1}{m}\nabla_k S\nabla_k\dot S + \frac{\partial V}{\partial t}\right] - \frac{\hbar^2}{m}\nabla_k R\nabla_k\dot R + \frac{2}{m}R\nabla_k R\frac{\partial S}{\partial t}\nabla_k S + \frac{R^2}{m}(\nabla_k(\dot S\nabla_k S) + \dot S\Delta S) \\ &+\frac{\hbar^2}{m}[\nabla_k\dot R\nabla_k R + \dot R\Delta R] = -R^2\frac{\partial V}{\partial t}. \end{aligned} \tag{5.14}$$

Simplifying (5.14), it reduces to

$$-\frac{\partial\rho}{\partial t}\left[\frac{1}{2m}(\nabla_k S)^2 + V\right] + \frac{2}{m}R(\nabla_k R)\left(\frac{\partial S}{\partial t}\right)\nabla_k S + \frac{\rho}{m}\left(\frac{\partial S}{\partial t}\right)\Delta S + \frac{\hbar^2}{m}\frac{\partial R}{\partial t}\Delta R = 0. \tag{5.15}$$

With $\dot{S}$ and $\dot{R}$ the time derivatives of S and R assumed not identically zero, multiply (5.15) by $1/(\dot{S}\,\dot{R})$ so that (5.15) after grouping terms ends up in the form

$$\frac{1}{\dot{S}}\left[[\frac{1}{2m}(\nabla_k S)^2 + V] - \frac{\hbar^2}{2m}\frac{\Delta R}{R}\right] - \frac{1}{\dot{R}}\left[\frac{1}{m}\nabla_k R\,\nabla_k S - R\frac{\Delta S}{2m}\right] = 0. \tag{5.16}$$

The original equation has been expressed in the form (5.16) which, upon identifying F and G in the obvious way, is equivalent to

$$\frac{1}{\dot{S}}F(S,\,R) - \frac{1}{\dot{R}}G(S,\,R) = 0. \tag{5.17}$$

A system of the form (5.17)–(5.16), is separable and hence can be expressed as the following pair of equations

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla_k S)^2 - \frac{\hbar^2}{2mR}(\Delta R) + V = 0, \qquad \frac{\partial \rho}{\partial t} + \frac{1}{m}(\nabla_k R)\cdot(\nabla_k S) + \frac{1}{2m}R\Delta\, S = 0. \tag{5.18}$$

Using (5.5), the equations in (5.18) are just the following set,

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla_k S)^2 - \frac{\hbar^2}{2mR}(\Delta R) + V = 0, \qquad \frac{\partial \rho}{\partial t} + \nabla_k\left(\frac{\rho\nabla_k S}{m}\right) = 0. \tag{5.19}$$

The conservation law has been reformulated in terms of these two coupled partial differential equations for the amplitude and phase of the function $\varphi(\mathbf{x}, t)$. The first equation in (5.19) defines the quantum potential $Q(\mathbf{x}, t)$ to be

$$Q(\mathbf{x},\,t) = -\frac{\hbar^2}{2m}\frac{\Delta R(\mathbf{x},\,t)}{R(\mathbf{x},\,t)}. \tag{5.20}$$

The first equation in (5.19) has the form of a classical Hamilton-Jacobi equation, except for the presence of the quantum potential (5.20) present in it. It is the case then that (5.20) can be regarded as the Hamilton-Jacobi equation for the classical dynamics of a particle in a potential $V(\mathbf{x}) + Q(\mathbf{x}, t)$. The solution by characteristics for some specified initial condition corresponds to the trajectory of the equation of motion which resembles a Newton equation of motion

$$m\frac{d}{dt}\mathbf{x}'(t) = -\nabla_{\mathbf{x}}(V(\mathbf{x}) + Q(\mathbf{x},\,t)), \tag{5.21}$$

with Lagrangian $\mathcal{L}(\mathbf{x},\,\mathbf{x}',\,t) = \mathcal{T}(\mathbf{x}') - (V(\mathbf{x}) + Q(\mathbf{x},\,t))$ and $d/dt = \partial/\partial t + \mathbf{x}'\cdot\nabla$. At this point, (5.18)–(5.21) can be compared with the results Bohm obtained(1.2)–(1.4).

6. Summary and conclusions

For a system of particles with coordinates $x_i = q^i \in \mathbb{R}^3$, $i = 1, \ldots, n$, it has been shown the dynamics can be formulated as if it occurred in an extended configuration space with additional coordinate $q^0 = t$. The evolution of the dynamics is measured by a proper time parameter τ for any initial condition. The dynamics follows a deterministic trajectory in a $3n + 1$ dimensional Finsler manifold with curvature.

The trajectories follow according to the geodesic equation (4.14) which, including a classical time-independent potential, is

$$\ddot{q}^\mu + \Gamma^\mu_{\nu\alpha}(\mathbf{q},\,\dot{\mathbf{q}})\dot{q}^\nu\dot{q}^\alpha = -g^{\mu\nu}\frac{\partial V(\mathbf{q})}{\partial q^\nu}, \tag{6.1}$$

where $\Gamma^\mu_{\nu\alpha}$ are given by (4.11) in terms of metric (3.5) and $\Lambda(\mathbf{q},\,\dot{\mathbf{q}})$ by (3.2). In these equations $\tau = s$ is used, where s is arclength (3.3), $V(\mathbf{q})$ is the classical potential and $Q(\mathbf{q}, t)$ is the quantum potential. What precedes has proved the following. Starting with Lagrangian density such as (5.1), the quantum dynamics can be described using trajectories following the Newton-like equation of motion (5.21). Equivalently, the same result can be obtained following the derivation due to Bohm in (1.2)–(1.4). The time-dependent Lagrangian associated to the dynamics (3.2) can be expressed geometrically according to Finsler geometry described in section 2 and (3.3)–(3.5). A geodesic dynamics results from this given through (6.1).

Thus the time dependent Lagrangian associated with this dynamics has been expressed in terms of Finsler geometry, and this leads to the dynamics implied by (5.21). All of the particles together generate in a nonlocal fashion the curvature of configuration space which is experienced by all.

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