Complex short pulse equation and its integrable discretizations

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COMPLEX SHORT PULSE EQUATION AND ITS INTEGRABLE DISCRETIZATIONS

A Thesis

by

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ABSTRACT

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In this thesis, we are mainly concerned with the complex short pulse (CSP) equation which was proposed in Physica D [1]. We present the Lax pair of the CSP equation and show the compatibility condition gives the CSP equation. Therefore, the integrability is confirmed. Then, a set of bilinear equations is proposed which yields the CSP equation through the hodograph transformation. Based on the bilinear form, general N-soliton solution is given in determinant form. Regarding two-soliton solution, a bound state can be formed if the velocities of two solitons are equal. Furthermore, a semi- and fully discrete analogues are constructed by using Hirota's bilinear method and defining discrete hodograph transformations. General N-soliton solutions are also presented for both the semi-discrete and fully discrete versions of the CSP equation. The Lax pair for the semi-discrete CSP equation is found and therefore the Lax integrability is confirmed.

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CHAPTER I

INTRODUCTION

The nonlinear Schrödinger(NLS) equation (1.1) is a universal nonlinear model able to describe many physical nonlinear systems. It describes the evolution of slow moving packets of quasi-monochromatic waves in a nonlinear dispersive media. Applications include, but are not limited to hydrodynamics, nonlinear optics, nonlinear acoustics water waves, etc. The nonlinear Schrödinger equation is integrable and the solution may be obtained via the inverse scattering transform method.

$$i\partial_t \Psi = -\frac{1}{2}\partial_x^2 \Psi + \kappa|\Psi|^2\Psi \tag{1.1}$$

However, once we enter the domain of ultra-short pulses within the order of femtosecond $(10^{-15}s)$, the nonlinear Schrödinger equation loses accuracy. One way to remedy this problem is to add several higher-order dispersive terms in order to create a higher-order NLS equation. Another way is to create an applicable fit to the frequency-dependent dielectric constant $\varepsilon(\omega)$ in the required spectral range. The short pulse equation is one such proposed model[2]. Proposed by Schäfer and Wayne [3, 2] the short pulse equation describes the propagation of ultra-short optical pulses through a nonlinear media.

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \tag{1.2}$$

Here, $u(x,t)$ is a real-valued function that represents the magnitude of the electrical field. The subscripts x and t are used to define partial derivatives. The short pulse equation has been shown to be completely integrable [4, 5, 6]. Recently the solitary wave solutions [7], two-loop soliton solutions [8], and periodic solutions [9] have been discovered. There is an established connection between the short pulse equation and the sine-Gordon equation via hodograph transformation [10, 11]. N-soliton, multi-loop and multi-breather solutions were also established by using Hirota's bilinear method [12]. Recently the complex short pulse equation

$$q_{xt} + q + \frac{1}{2}(|q|^2 q_x)_x = 0 \tag{1.3}$$

along with its two-component generalization

$$q_{1,xt} + q_1 + \frac{1}{2}((|q_1|^2 + |q_2|^2) q_{1,x})_x = 0, \tag{1.4}$$

$$q_{2,xt} + q_2 + \frac{1}{2}((|q_1|^2 + |q_2|^2) q_{2,x})_x = 0. \tag{1.5}$$

was proposed in [13]. With it we are able to describe optical waves using a complex-valued function. There are many advantages that arise from utilizing complex representation. Both the real value and the imaginary value show the amplitude and phase respectively of the pulse wave, two fundamental characteristics for a wave packet. This way we can combine them both into one neatly packed complex-valued function. Now, why do we want a complex-valued function to represent the wave function? The reason is that wave functions by observation closely resemble that of a rotating rope. Each point in the rotating rope makes a circle whose magnitude is easier to represent using a complex variable. The advantages have been seen before in analytical results related to the nonlinear Schrödinger equation and the coupled nolinear Schrödinger equation. As shown in [13], both the complex short pulse equation and the coupled complex short pulse equations admit explicit expressions for one-soliton and multi-soliton solutions along with physical interpretations. Unlike the short pulse equation, which has no physical interpretation for its one-soliton solution, the one-soliton solution for the complex short pulse equation is an envelope soliton with a few optical cycles. This is great for describing ultra-short pulses. It is guaranteed to be integrable due to the existence of Lax pairs and infinite conservation laws [4, 5, 6].

Our goal in this paper is to propose a discretization of both the dependent and independent variables of the complex short pulse equation along with their corresponding proofs. In chapter 2 we introduce the concept of Lax pairs and derive the complex short pulse equation from its compatibility condition. The integrability of the CSP equation is confirmed due to the existence of this Lax pair. Hirota's bilinear method is utilized in chapter 3 to derive the complex short pulse equation via its bilinear form. In chapter 4 we show it's determinant solution and include both the one-soliton solution and two-soliton soultion. We also show MATlab simulations of both solutions along with pointing out interesting properties that have been discovered. The semi-discretization of the CSP equation begins with discretizing the dependent variable in chapter 5. By semi-discrete transform and hodograph transformations we show that the bilinear form of the semi-discretized CSP equation is equivalent to our CSP equation. The multi-soliton solution to the semi-discrete CSP equation is shown and the Lax pairs are proven along with it's compatibility condition. This process confirms that it is also integrable like the CSP equation. In the following chapter we propose a fully-discretiziation of the CSP equation. This means that we also discretize the independent variable to provide a more accurate numerical model for the complex short pulse equation. Future work includes deriving a Lax pair and proving that the compatibility condition holds to show the integrability of the fully discrete CSP equation.

CHAPTER II

LAX PAIRS AND INTEGRABILITY

It is obvious that the CSP equation is integrable since it admits the Lax pair(2.1).

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi, \tag{2.1}$$

with

$$U = \lambda \begin{pmatrix} 1 & q_x \\ q_x^* & -1 \end{pmatrix}, \quad V = \begin{pmatrix} -\frac{\lambda}{2}|q|^2 - \frac{1}{4\lambda} & -\frac{\lambda}{2}|q|^2 q_x + \frac{q}{2} \\ -\frac{\lambda}{2}|q|^2 q_x^* - \frac{q^*}{2} & \frac{\lambda}{2}|q|^2 + \frac{1}{4\lambda} \end{pmatrix} \tag{2.2}$$

Based on the Lax pair proposed, one can solve the initial value problem of the CSP equation by using the inverse scattering transform. In addition, one can derive one-, two-, and even multi-soliton solutions by constructing the Darboux transformation from the Lax pair. However, we are going to seek for the multi-soliton solutions of the CSP equation by virtue of Hirota's bilinear method[12] in the subsequent section. We can recover the complex short pulse equation by computing the compatibility condition $U_t - V_x + [U,V] = \mathbf{0}$ where $[U,V] = UV - VU$.

For the partial derivatives we get

$$U_t = \begin{pmatrix} 0 & \lambda q_{xt} \\ \lambda q_{xt}^* & 0 \end{pmatrix}, \quad V_x = \begin{pmatrix} -\frac{\lambda}{2}(qq_x^* + q_x q^*) & -\frac{\lambda}{2}(|q|^2 q_x)_x + \frac{q_x}{2} \\ -\frac{\lambda}{2}(|q|^2 q_x^*)_x - \frac{q_x^*}{2} & \frac{\lambda}{2}(qq_x^* + q_x q^*) \end{pmatrix} \tag{2.3}$$

as for the matrix multiplications UV and VU

$$\lambda \begin{pmatrix} 1 & q_x \\ q_x^* & -1 \end{pmatrix} \begin{pmatrix} -\frac{\lambda}{2}|q|^2 - \frac{1}{4\lambda} & -\frac{\lambda}{2}|q|^2 q_x + \frac{q}{2} \\ -\frac{\lambda}{2}|q|^2 q_x^* - \frac{q^*}{2} & \frac{\lambda}{2}|q|^2 + \frac{1}{4\lambda} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\lambda}{2}(\lambda|q|^2 q_x q_x^* + \lambda|q|^2 + q^* q_x) - \frac{1}{4} & \frac{1}{4}q_x - \frac{\lambda}{2}q \\ \frac{\lambda}{2}q^* - \frac{1}{4} & -\frac{\lambda}{2}(\lambda|q|^2 q_x q_x^* + \lambda|q|^2 - q q_x^*) - \frac{1}{4} \end{pmatrix} \tag{2.4}$$

On the other hand for VU we get

$$\lambda \begin{pmatrix} -\frac{\lambda}{2}|q|^2 - \frac{1}{4\lambda} & -\frac{\lambda}{2}|q|^2 q_x + \frac{q}{2} \\ -\frac{\lambda}{2}|q|^2 q_x^* - \frac{q^*}{2} & \frac{\lambda}{2}|q|^2 + \frac{1}{4\lambda} \end{pmatrix} \begin{pmatrix} 1 & q_x \\ q_x^* & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\lambda}{2}(\lambda|q|^2 q_x q_x^* + \lambda|q|^2 - q q_x^*) - \frac{1}{4} & -\frac{\lambda}{2}q - \frac{1}{4}q_x \\ -\frac{\lambda}{2}q^* + \frac{1}{4}q_x^* & -\frac{\lambda}{2}(\lambda|q|^2 q_x q_x^* + \lambda|q|^2 + q^* q_x) - \frac{1}{4} \end{pmatrix} \tag{2.5}$$

Now, we take the difference of (2.4) and (2.5) for $[U,V]$

$$[U,V] = \begin{pmatrix} -\frac{\lambda}{2}(q^* q_x + q q_x^*) & \frac{1}{2}q_x + \lambda q \\ \lambda q^* - \frac{1}{4}q_x^* - \frac{1}{4} & \frac{\lambda}{2}(q q_x^* + q^* q_x) \end{pmatrix} \tag{2.6}$$

Finally $U_t - V_x + [U,V] = \mathbf{0}$ becomes

$$\begin{pmatrix} 0 & q_{xt} + (\frac{1}{2}|q|^2 q_x)_x + q \\ q_{xt}^* + (\frac{1}{2}|q|^2 q_x^*)_x + q^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.7}$$

This equation is basically the CSP equation so the Lax pair holds true and the integrability is confirmed.

CHAPTER III

BILINEAR FORM AND DETERMINANT SOLUTION

3.1 Bilinear Form

Proposition 1. *The complex short pulse equation is shown to come from the following bilinear equations.*

$$D_sD_y f\cdot g = fg \tag{3.1}$$

$$D_s^2 f\cdot f = \frac{1}{2}|g|^2 \tag{3.2}$$

by dependent variable transformation

$$q = \frac{g}{f} \tag{3.3}$$

and hodograph transformation

$$x = y - 2(\ln f)_s, \quad t = -s \tag{3.4}$$

where D is the Hirota D-operator defined by

$$D_s^nD_y^m f\cdot g = \left(\frac{\partial}{\partial s}-\frac{\partial}{\partial s'}\right)^n\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^m f(y,s)g(y',s')|_{y=y',s=s'}$$

Proof. Note that f is considered to be positive definite since we cannot divide by zero. If we divide both sides of each equation by f^2, the bilinear equation (3.1) becomes

$$\frac{f_{sy}g - f_s g_y - f_y g_s + f g_{sy}}{f^2} = \frac{g}{f} \tag{3.5}$$

this is crucial because from there we can rewrite the left hand side of the function as the sum of the following two functions,

$$\left(\frac{g}{f}\right)_{sy} = \frac{f g_{sy}}{f^2} - \frac{f_y g_s}{f^2} - \frac{f_{sy} g}{f^2} - \frac{f_s g_y}{f^2} + 2\frac{g f_s f_y}{f^3}, \quad 2\frac{g}{f}(\ln f)_{sy} = 2\frac{f_{sy} g}{f^2} - 2\frac{g f_s f_y}{f^3} \tag{3.6}$$

We do this in order to rewrite the equation in this form,

$$\left(\frac{g}{f}\right)_{sy} + 2\frac{g}{f}(\ln f)_{sy} = \frac{g}{f} \tag{3.7}$$

Following the same process we can audit the second bilinear equation (3.2) by also dividing both sides by f^2

$$\frac{2f_{ss}f - 2f_s^2}{f^2} = \frac{1}{2}\frac{|g|^2}{f^2} \tag{3.8}$$

the left hand side is just $2(\ln f)_{ss}$, which reduces (3.8) to

$$(\ln f)_{ss} = \frac{1}{4}\frac{|q|^2}{f^2} \tag{3.9}$$

So now, we have changed our bilinear equations (3.1) and (3.2) into (3.7) and (3.9) respectively. From the hodograph (3.4) and dependent variable (3.3) transformations, we have

$$\frac{\partial x}{\partial s} = -2(\ln f)_{ss} = -\frac{1}{2}|q|^2, \quad \frac{\partial x}{\partial y} = 1 - 2(\ln f)_{ss}, \tag{3.10}$$

(3.10) implies that

$$\partial_y = \rho^{-1}\partial_x, \quad \partial_s = -\partial_t - \frac{1}{2}|q|^2\partial_x \tag{3.11}$$

if we let $1 - 2(\ln f)_{sy} = \rho^{-1}$.

Note that (3.7) can be rewritten like so,

$$\left(\frac{g}{f}\right)_{sy} = (1 - 2(\ln f)_{sy})\frac{g}{f} \tag{3.12}$$

or by using ρ

$$\rho\left(\frac{g}{f}\right)_{sy} = \frac{g}{f} \tag{3.13}$$

which by the equations in (3.10) yields,

$$\partial_x(-\partial_t - \frac{1}{2}|q|^2\partial_x)q = q$$

which is nothing but the CSP equation, completing our proof of the bilinear form. □

3.2 Determinant solution

Theorem 1 (Determinant solutions to the bilinear equation)**.** *The bilinear equations* (3.1)-(3.2) *admit the following determinant solution*

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad g = \begin{vmatrix} A & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & C & \mathbf{0} \end{vmatrix}, \tag{3.14}$$

where I is an $N \times N$ identity matrix, $\mathbf{0}$ is a N-component zero row vector, A and B are $N \times N$ matrices, Φ, C are N-component row vectors whose elements can be defined by the following equations

$$a_{ij} = \frac{1}{2(p_i^{-1} + p_j^{*-1})} e^{\xi_i + \xi_j^*}, \quad b_{ij} = \frac{\alpha_i^* \alpha_j}{2(p_i^{*-1} + p_j^{-1})}. \tag{3.15}$$

$$\Phi = (e^{\xi_1}, e^{\xi_2}, \dots, e^{\xi_N}), \quad C = -(\alpha_1, \alpha_2, \dots, \alpha_N), \tag{3.16}$$

where $\xi_i = p_i y + p_i^{-1} s + \xi_{i0}$, $\xi_i^* = p_i^* y + p_i^{*-1} s + \xi_{i0}^*, p_i, \alpha_i$ and ξ_{i0} for all $(i = 1, 2, ..., N)$ are complex constants. Note that p^* is the complex conjugate of p and follows for ξ^*.

$$f = \begin{vmatrix} \frac{1}{2(p_1^{-1}+p_1^{*-1})}e^{\xi_1+\xi_1^*} & 1 \\ -1 & \frac{\alpha_1^*\alpha_1}{2(p_1^{*-1}+p_1^{-1})} \end{vmatrix} = 1 + c_{1\bar{1}}e^{\xi_1+\xi_1^*} \tag{3.17}$$

$$g = \begin{vmatrix} \frac{1}{2(p_1^{-1}+p_1^{*-1})}e^{\xi_1+\xi_1^*} & 1 & e^{\xi_1} \\ -1 & \frac{\alpha_1^*\alpha_1}{2(p_1^{*-1}+p_1^{-1})} & 0 \\ 0 & -\alpha_1 & 0 \end{vmatrix} = \alpha_1 e^{\xi_1}, \tag{3.18}$$

where $c_{1\bar{1}} = \frac{|\alpha_1|^2|p_1|^4}{4(p_1+p_1^*)^2}$

Using the variable transformation and the Hodograph tranform we arrive at the following one-soliton solution.

$$q = \frac{\alpha_1 e^{\xi_1}}{1+c_{1\bar{1}}e^{\xi_1+\xi_1^*}} = \frac{\alpha_1}{|\alpha_1|}\frac{2p_{1R}}{|p_1|^2}e^{i\xi_1 I}sech(\xi_{1R}+\xi_{10}), \tag{3.19}$$

with

$$\rho = 1 - \frac{2p_{1R}^2}{|p_1|^2}sech^2(\xi_{1R}+\xi_{10}), \tag{3.20}$$

and

$$x = y - \frac{2p_{1R}^2}{|p_1|^2}(tanh(\xi_{1R}+\xi_{10})+1), \quad t = -s, \tag{3.21}$$

where $p_1 = p_{1R} + ip_{1I}$*, and*

$$\xi_{1R} = p_{1R}s + \frac{p_{1R}}{|p_1|^2}\tau, \quad \eta_{1I} = p_{1I}s - \frac{p_{1I}}{|p_1|^2}\tau, \quad \eta_{10} = \ln\frac{|\alpha_1||p_1|^2}{4p_{1R}} \tag{3.22}$$

As for the two-soliton solution

$$f = \begin{vmatrix} \frac{e^{\xi_1+\xi_1^*}}{2(p_1^{-1}+p_1^{*-1})} & \frac{e^{\xi_1+\xi_2^*}}{2(p_1^{-1}+p_2^{*-1})} & 1 & 0 \\ \frac{e^{\xi_2+\xi_1^*}}{2(p_2^{-1}+p_1^{*-1})} & \frac{e^{\xi_2+\xi_2^*}}{2(p_2^{-1}+p_2^{*-1})} & 0 & 1 \\ -1 & 0 & \frac{\alpha_1\alpha_1^*}{2(p_1^{-1}+p_1^{*-1})} & \frac{\alpha_1\alpha_2^*}{2(p_2^{-1}+p_1^{*-1})} \\ 0 & -1 & \frac{\alpha_2\alpha_1^*}{2(p_1^{-1}+p_2^{*-1})} & \frac{\alpha_2\alpha_2^*}{2(p_2^{-1}+p_2^{*-1})} \end{vmatrix}$$

$$= 1 + c_{1\bar{1}} e^{\xi_1+\xi_1^*} + c_{2\bar{1}} e^{\xi_2+\xi_1^*} + c_{1\bar{2}} e^{\xi_1+\xi_2^*} + c_{2\bar{2}} e^{\xi_2+\xi_2^*} + c_{12\bar{1}\bar{2}} e^{\xi_1+\xi_2+\xi_1^*+\xi_2^*} \tag{3.23}$$

$$g = \begin{vmatrix} \frac{e^{\xi_1+\xi_1^*}}{2(p_1^{-1}+p_1^{*-1})} & \frac{e^{\xi_1+\xi_2^*}}{2(p_1^{-1}+p_2^{*-1})} & 1 & 0 & e^{\xi_1} \\ \frac{e^{\xi_2+\xi_1^*}}{2(p_2^{-1}+p_1^{*-1})} & \frac{e^{\xi_2+\xi_2^*}}{2(p_2^{-1}+p_2^{*-1})} & 0 & 1 & e^{\xi_2} \\ -1 & 0 & \frac{\alpha_1\alpha_1^*}{2(p_1^{-1}+p_1^{*-1})} & \frac{\alpha_1\alpha_2^*}{2(p_2^{-1}+p_1^{*-1})} & 0 \\ 0 & -1 & \frac{\alpha_2\alpha_1^*}{2(p_1^{-1}+p_2^{*-1})} & \frac{\alpha_2\alpha_2^*}{2(p_2^{-1}+p_2^{*-1})} & 0 \\ 0 & 0 & -\alpha_1 & -\alpha_2 & 0 \end{vmatrix}$$

$$= \alpha_1 e^{\xi_1} + \alpha_2 e^{\xi_2} + c_{12\bar{1}} e^{\xi_1+\xi_2+\xi_1^*} + c_{12\bar{2}} e^{\xi_1+\xi_2+\xi_2^*}, \tag{3.24}$$

where the variables

$$c_{i\bar{j}} = \frac{p_i^2 p_j^{*2}}{4(p_i + p_j^*)^2}, \quad c_{12\bar{i}}(p_2 - p_1) = \left(\frac{\alpha_2 c_{1\bar{i}}}{p_2 + p_i^*} - \frac{\alpha_1 c_{2\bar{i}}}{p_1 + p_i^*} \right),$$

$$c_{12\bar{1}\bar{2}} = |p_2 - p_1|^2 \left(\frac{c_{1\bar{1}} c_{2\bar{2}}}{(p_1 + p_2^*)(p_2 + p_1^*)} - \frac{c_{1\bar{2}} c_{2\bar{1}}}{(p_1 + p_1^*)(p_2 + p_2^*)} \right),$$

This is basically the two-soliton solution we will look at in the next chapter.

CHAPTER IV

SOLITON SOLUTIONS TO THE BILINEAR EQUATIONS

4.1 One-soliton solution

Using the previous sections determinant solution

$$f = 1 + \frac{|\alpha_1|^2 |p_1|^4}{4(p_1 + p_1^*)^2} e^{\xi_1 + \xi_1^*} = 1 + c_{1\bar{1}} e^{\xi_1 + \xi_1^*} \tag{4.1}$$

and

$$g = \alpha_1 e^{\xi_1}, \tag{4.2}$$

we know that

$$q = \frac{g}{f} = \frac{\alpha_1 e^{\xi_1}}{1 + c_{1\bar{1}} e^{\xi_1 + \xi_1^*}} \tag{4.3}$$

We can rewrite (4.3), knowing that $\xi_1 = \xi_{1R} + i\xi_{1I}$, $\xi_1 + \xi_1^* = 2\xi_{1R}$ and $c_{1\bar{1}} = e^{2\xi_{10}}$ to get

$$q = \frac{\alpha_1 e^{\xi_{1R}} e^{i\xi_{1I}}}{1 + e^{2\xi_{1R} + 2\xi_{10}}}. \tag{4.4}$$

From here we can divide both the numerator and denominator by $e^{\xi_{1R} + \xi_{10}}$ and we have

$$q = \frac{\alpha_1 e^{\xi_{1I}} e^{-\xi_{10}}}{e^{\xi_{1R} + \xi_{10}} + e^{-(\xi_{1R} + \xi_{10})}}. \tag{4.5}$$

This equation can be rewritten using the definition of the cosh function like so

$$q = \alpha_1 e^{i\xi_{1I}} \frac{4p_{1R}}{|\alpha_1||p_1|^2} \frac{1}{2\cosh(\xi_{1R}+\xi_{10})} \tag{4.6}$$

$$= \frac{\alpha_1}{|\alpha_1|} \frac{2p_{1R}}{|p_1|^2} e^{i\xi_{1I}} \mathrm{sech}(\xi_{1R}+\xi_{10}) \tag{4.7}$$

Looking back at the hodograph transformation of the variable x, we calculate the derivative $(\ln f)_s$

$$\begin{aligned} x &= y - 2(\ln f)_s \\ &= y - 2\frac{f_s}{f} \\ &= y - 2\frac{c_{1\bar{1}} \frac{\partial}{\partial s}[e^{\xi_1+\xi_1^*}]}{1 + c_{1\bar{1}} e^{\xi_1+\xi_1^*}} \\ &= y - 2\frac{p_{1R}}{|p_1|^2}(\tanh(\xi_{1R}+\xi_{10})+1)) \end{aligned}$$

From this we can very easily calculate the partial derivative $\frac{\partial x}{\partial y}$

$$\frac{\partial x}{\partial y} = 1 - 2\frac{p_{1R}^2}{|p_1|^2}\mathrm{sech}^2(\xi_{1R}+\xi_{10}) \tag{4.8}$$

There are three distinct cases each dictated by the value of $\frac{p_{1R}^2}{|p_1|^2}$. They are smooth, loop and cuspon solitons.

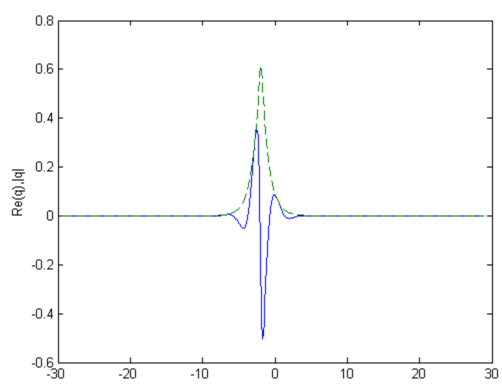

Figure 4.1: Smooth Soliton

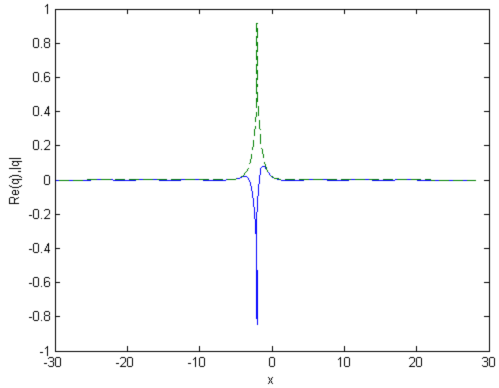

Figure 4.2: Loop Soliton

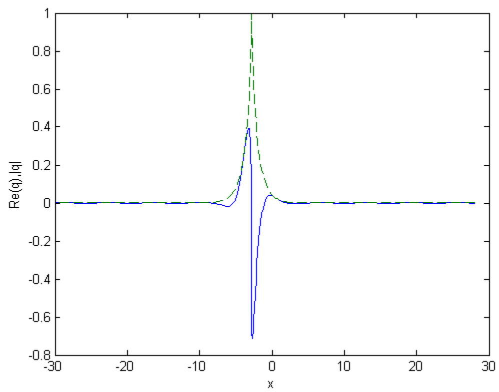

Figure 4.3: Cuspon Soliton

- Smooth Soliton: When $|p_{1R}| < |p_{1I}|$, $\frac{\partial x}{\partial y}$ is always positive. In this case we see an envelope soliton similar to the one seen for the NLS equation. In Figure 4.1 we show an example where $p_1 = 1.0 + 1.5i$

- Loop Soliton: When $|p_{1R}| > |p_{1I}|$, the minimum value of $\frac{\partial x}{\partial y}$ becomes negative. Also $\frac{\partial x}{\partial y} \to 1$ as $y \to \pm\infty$, and it has two zeros besides the peak of the envelope soliton. In addition $\frac{\partial x}{\partial y} < 0$ between these two zeroes. In Figure 4.2 we have an example where $p_1 = 1.5 + 1.0i$

- Cuspon Soliton: When $|p_{1R}| = |p_{1I}|$, $\frac{\partial x}{\partial y}$ has a minimum value of zero at $\eta_{1R} + \eta_{10}$. This makes the derivative of the envelope $|q|$ with respect to x infinity at its peak point. In Figure 4.3 a cuspon soliton can be seen using the following parameters $p_1 = 1.0 + 1.0i$

4.2 Two-soliton solution

Using the determinant solutions, (3.23) and (3.24), we can construct the following equations.

$$
\begin{aligned}
f =& 1 + a_{1\bar{1}} e^{\eta_1+\bar{\eta}_1} + a_{1\bar{2}} e^{\eta_1+\bar{\eta}_2} + a_{2\bar{1}} e^{\eta_2+\bar{\eta}_1} + a_{2\bar{2}} e^{\eta_2+\bar{\eta}_2} \\
&+ |P_{12}|^2 (a_{1\bar{1}} a_{2\bar{2}} P_{1\bar{2}} P_{2\bar{1}} - a_{1\bar{2}} a_{2\bar{1}} P_{1\bar{1}} P_{2\bar{2}}) e^{\eta_1+\eta_2+\bar{\eta}_1+\bar{\eta}_2}, \\
g =& \alpha_1 e^{\eta_1} + alpha_2 e^{\eta_2} + P_{12}(\alpha_1 P_{1\bar{1}} a_{2\bar{1}} - \alpha_2 P_{2\bar{1}} a_{1\bar{1}}) e^{\eta_1+\eta_2+\bar{\eta}_1} \\
&+ P_{12}(\alpha_1 P_{1\bar{2}} a_{2\bar{2}} - \alpha_2 P_{2\bar{2}} a_{1\bar{2}}) e^{\eta_1+\eta_2+\bar{\eta}_2},
\end{aligned}
$$

where

$$
P_{ij} = \frac{p_i - p_j}{p_i + p_j}, \quad P_{i\bar{j}} = \frac{p_i - \bar{p}_j}{p_i + \bar{p}_j}, \quad a_{i\bar{j}} = \frac{\alpha_i \bar{\alpha}_j (p_i \bar{p}_j)^2}{4(p_i + \bar{p}_j)^2}, \tag{4.9}
$$

and $\eta_j = p_j y + p_j^{-1} s$, $\bar{\eta}_j = \bar{p}_j y + \bar{p}_j^{-1} s$.

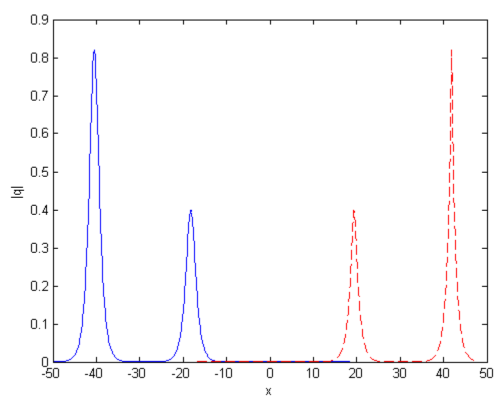

Figure 4.4: Ellastic collision between two solitons of different velocity

In order to avoid the singularity of the envelope solitons, the conditions $|p_{1R}| < |p_{1I}|$ and $|p_{2R}| < |p_{2I}|$ need to be satisfied. When both solitons stay apart from each other, the amplitude is calculated to be $2\frac{|p_{iR}|}{|p_i|^2}$, and its velocity is $\frac{-1}{|p_i|^2}$ in the ys-coordinate system. Accordingly, if we have a soliton of larger velocity come from the negative side towards another soliton of smaller velocity we have a collision. Furthermore this collision is elastic, the solitons shape and amplitude remains unchanged while we only have a small variation in phase shift. Below we show before and after images of the collision. In Figure 4.4 we show a simulation while taking the following parameters $\alpha_1 = \alpha_2 = 1.0$, $p_1 = 1 + 1.2i$, and $p_2 = 1 + 2i$.

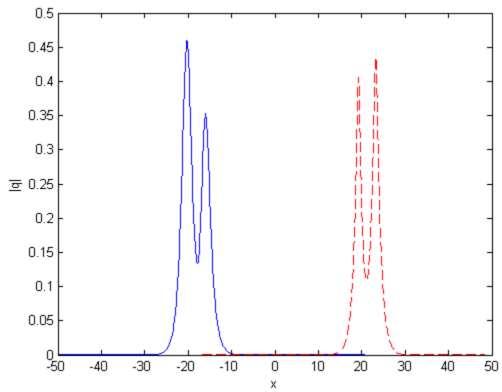

Figure 4.5: Bound state of two-solitons with equal velocities.

Recall that the velocity of the envelope soliton is $-1/|p_i|^2$ in the ys-coordinate system, if we have that $|p_1|^2 = |p_2|^2$ we form a bound state in which the solitons stay in the immediate vicinity of each other and proceed with the same velocity. We show an example of a bound state in Figure 4.5 with the following parameters $\alpha_1 = \alpha_2 = 1.0$, $p_1 = 1.2 + 1.886796i$, and $p_2 = 1 + 2i$.

CHAPTER V

SEMI-DISCRETIZATION

First of all, why do we want to create an integrable discretization of the complex short pulse equation? An integrable discretization of the CSP equation is useful in designing a numerical algorithm. With this we can create simulations of the CSP equation. We will begin by discretizing the y-variable from the bilinear equations, which will in turn discretize our x-variable. We know that both f and g are functions of y and s. By setting $y_k = k * 2a$ with $2a$ as the step size on the y-axis, we can attempt to discretize the dependent variable. This leads us to the following equations.

$$f_k = f(y_k, s), \quad g_k = g(y_k, s)$$

With that we also have the bilinear form of the semi-discretized complex short pulse equation (5.1).

$$\begin{cases} \frac{1}{2a} D_s(g_{k+1} \cdot f_k - g_k \cdot f_{k+1}) = \frac{1}{2} g_{k+1} \cdot f_k + g_k \cdot f_{k+1} \\ D_s^2(f_k \cdot f_k) = \frac{1}{2}|g_k|^2 \end{cases} \tag{5.1}$$

We still need to prove that these bilinear equations are equivalent to the CSP equation. We will begin by looking at (5.1). First we must divide both sides of the equations by $f_k \cdot f_{k+1}$

$$\begin{aligned} \frac{D_s g_{k+1} \cdot f_k}{f_k \cdot f_{k+1}} &= \frac{\frac{d}{ds} g_{k+1} f_k - g_{k+1} \frac{d}{ds} f_k}{f_k \cdot f_{k+1}} \\ &= \frac{\frac{d}{ds} g_{k+1}}{f_{k+1}} - \frac{g_{k+1}}{f_{k+1}} \frac{\frac{d}{ds} f_k}{f_k} \\ &= \frac{\frac{d}{ds} g_{k+1}}{f_{k+1}} - \frac{g_{k+1}}{f_{k+1}} (\ln f_k)_s \end{aligned}$$

Similarly we derive the other part of the left hand side as

$$\frac{D_s g_k \cdot f_{k+1}}{f_k \cdot f_{k+1}} = \frac{\frac{d}{ds} g_k}{f_k} - \frac{g_k}{f_k} (\ln f_{k+1})_s$$

So we put them together to get

$$\frac{D_s g_{k+1} \cdot f_k}{f_k \cdot f_{k+1}} - \frac{D_s g_k \cdot f_{k+1}}{f_k \cdot f_{k+1}} = \frac{\frac{d}{ds} g_{k+1}}{f_{k+1}} - \frac{g_{k+1}}{f_{k+1}} (\ln f_k)_s - \frac{\frac{d}{ds} g_k}{f_k} - \frac{g_k}{f_k} (\ln f_{k+1})_s$$

Recall that

$$q_k = \frac{g_k}{f_k}$$

$$x_k = 2ka - 2(\ln f_k)_s, \quad t = -s$$

We will use them to define q_k and x_k respectively.

So now the right hand side reduces to (5.2) using the semi-discrete transformations above.

$$\frac{g_{k+1} f_k + g_k f_{k+1}}{f_k f_{k+1}} = \frac{g_{k+1}}{f_{k+1}} + \frac{g_k}{f_k} = q_{k+1} + q_k \tag{5.2}$$

Finally we set both sides equal to each other.

$$\frac{d}{ds}\left(\frac{g_{k+1}}{f_{k+1}} - \frac{g_k}{f_k}\right) + \left(\frac{g_{k+1}}{f_{k+1}} + \frac{g_k}{f_k}\right)((\ln f_{k+1})_s - (\ln f_k)_s) = a\left(\frac{g_{k+1}}{f_{k+1}} + \frac{g_k}{f_k}\right)$$

We can rewrite the logarithm using the properties of logarithms.

$$\frac{d}{ds}\left(\frac{g_{k+1}}{f_{k+1}} - \frac{g_k}{f_k}\right) + \left(\frac{g_{k+1}}{f_{k+1}} + \frac{g_k}{f_k}\right)\left(\ln \frac{f_{k+1}}{f_k}\right)_s = a\left(\frac{g_{k+1}}{f_{k+1}} + \frac{g_k}{f_k}\right)$$

From here if we move the $\left(\frac{g_{k+1}}{f_{k+1}} + \frac{g_k}{f_k}\right)$ term to the right side and combine like terms . The equation turns out to be

$$\frac{d}{ds}\left(\frac{g_{k+1}}{f_{k+1}}-\frac{g_k}{f_k}\right)=\left(a-\left(\ln\frac{f_{k+1}}{f_k}\right)_s\right)\left(\frac{g_{k+1}}{f_{k+1}}+\frac{g_k}{f_k}\right)$$

Using the aforementioned transformations and the definition of δ_k we get

$$\frac{d}{dt}(q_{k+1}-q_k)+\frac{\delta_k}{2}(q_{k+1}+q_k)=0 \tag{5.3}$$

Now in order to show that the second bilinear equation (5.1) gets us back the the CSP equation we take both sides and divide them by f_k^2 instead. On the left hand side we get

$$\begin{aligned}\frac{D_s^2 f_k\cdot f_k}{f_k^2}&=\frac{2(f_k\frac{\partial^2}{\partial s^2}(f_k)-\frac{\partial}{\partial s}(f_k))}{f_k^2}\\&=2(\ln f_k)_{ss}\\&=\frac{1}{2}\frac{g_kg_k^*}{f_kf_k}\\&=\frac{1}{2}q_kq_k^*\\&=\frac{1}{2}|q_k|^2\end{aligned}$$

As for the right hand side,

$$\frac{1}{2}\frac{|g_k|^2}{f_k^2}=\frac{1}{2}|q_k|^2$$

If we set $\delta_k=x_{k+1}-x_k$,

$$\delta_k=2a-2(\ln\frac{f_{k+1}}{f_k})_s$$

We can then take the partial derivatives with respect to s to get

$$\frac{d\delta_k}{ds}=-2(\ln\frac{f_{k+1}}{f_k})_{ss}$$

which is the same as

$$\frac{d\delta_k}{dt} = \frac{1}{2}(|q_{k+1}|^2 - |q_k|^2) \tag{5.4}$$

by the variable transformation $t = -s$.

Combining (5.4) and (5.3) will give us the semi-discrete CSP equation.

$$\begin{cases} \frac{d}{dt}(q_{k+1} - q_k) + \frac{\delta_k}{2}(q_{k+1} + q_k) = 0 \\ \frac{\partial \delta_k}{\partial t} = \frac{1}{2}(|q_{k+1}|^2 - |q_k|^2) \end{cases} \tag{5.5}$$

Now that we've established the semi discrete CSP equation, we can conclude that it admits the following multi-soliton solution.

$$f_k = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad g_k = \begin{vmatrix} A & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & C_1 & \mathbf{0} \end{vmatrix}, \tag{5.6}$$

where the elements of A and B are defined by the following equations.

$$a_{ij} = \frac{1}{2(p_i^{-1} + p_j^{*-1})} e^{\xi_i + \bar{\xi}_j}, \quad b_{ij} = \frac{\alpha_i^* \alpha_j}{2(p_j^{-1} + p_i^{*-1})}$$

and

$$e^{\xi_i} = \left(\frac{1 + ap_i}{1 - ap_i}\right)^k \exp\left(\frac{1}{p_i}s + \xi_{i0}\right)$$

while

$$e^{\xi_j^*} = \left(\frac{1 + ap_j^*}{1 - ap_j^*}\right)^k \exp\left(\frac{1}{p_j^*}s + \bar{\xi}_{j0}\right)$$

The semi-discrete CSP equation admits the Lax pair (5.7) and we will show that the CSP equation comes from its compatibility condition.

$$\Psi_{k+1} = U_k \Psi_k, \quad \Psi_{k,t} = V_k \Psi_k \tag{5.7}$$

where

$$U_k = \begin{pmatrix} 1-i\lambda\delta_k & -i\lambda(q_{k+1}-q_k) \\ -i\lambda(q^*_{k+1}-q^*_k) & 1-i\lambda\delta_k \end{pmatrix}, \quad V_k = \begin{pmatrix} \frac{i}{4\lambda} & -\frac{1}{2}q_k \\ \frac{1}{2}q^*_k & -\frac{i}{4\lambda} \end{pmatrix} \tag{5.8}$$

First we need to derive the compatibility condition from its Lax pair.

$$\begin{aligned} \frac{d\Psi_{k+1}}{dt} &= \frac{d}{dt}(U_k\Psi_k) \\ &= \frac{dU_k}{dt}\Psi_k + U_k\frac{d\Psi_k}{dt} \\ &= \frac{dU_k}{dt}\Psi_k + U_k(V_k\Psi_k) \end{aligned}$$

Similarly

$$\frac{d\Psi_{k+1}}{dt} = V_{k+1}U_k\Psi_k \tag{5.9}$$

Now we set them equal to each other and get the compatibility condition (5.10)

$$\frac{dU_k}{dt} + U_kV_k - V_{k+1}U_k = \mathbf{0} \tag{5.10}$$

Now we need to check the condition to see if it gives us the semi-discrete CSP equation. The partial derivative gives us

$$\frac{dU_k}{dt} = \begin{pmatrix} -i\lambda\frac{d\delta_k}{dt} & -i\lambda\frac{d}{dt}(q_{k+1}-q_k) \\ -i\lambda\frac{d}{dt}(q^*_{k+1}-q^*_k) & -i\lambda\frac{d\delta_k}{dt} \end{pmatrix} \tag{5.11}$$

As for the matrix multiplications we calculate them separately then combine them later.

$$\begin{aligned} U_kV_k &= \begin{pmatrix} (1-i\lambda\delta_k)(\frac{i}{4\lambda}) - (i\lambda(q_{k+1}-q_k))(\frac{1}{2}q^*_k) & (1-i\lambda\delta_k)(-\frac{1}{2}q_k) - i\lambda(q_{k+1}-q_k)(-\frac{i}{4\lambda}) \\ -i\lambda(q^*_{k+1}-q^*_k)(\frac{i}{4\lambda}) + (1-i\lambda\delta_k)(\frac{1}{2}q^*_k) & -i\lambda(q^*_{k+1}-q^*_k)(-\frac{1}{2}q_k) + (1-i\lambda\delta_k)(-\frac{i}{4\lambda}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{i}{4\lambda} + \frac{1}{4}\delta_k - \frac{1}{2}q^*_k i\lambda(q_{k+1}-q_k) & -\frac{1}{2}q_k + \frac{1}{2}q_k i\lambda\delta_k - \frac{1}{4}(q_{k+1}-q_k) \\ \frac{1}{4}(q^*_{k+1}-q^*_k) + \frac{1}{2}q^*_k - \frac{1}{2}q^*_k i\lambda\delta_k & \frac{1}{2}q_k i\lambda(q^*_{k+1}-q^*_k) - \frac{i}{4\lambda} - \frac{1}{4}\delta_k \end{pmatrix} \end{aligned}$$

and

$$V_{k+1}U_k = \begin{pmatrix} \frac{i}{4\lambda}(1-i\lambda\delta_k)+\frac{1}{2}q_{k+1}(i\lambda(q^*_{k+1}-q^*_k)) & -\frac{i}{4\lambda}(i\lambda(q_{k+1}-q_k))-\frac{1}{2}q_{k+1}(1-i\lambda\delta_k) \\ \frac{1}{2}q^*_{k+1}(1-i\lambda\delta_k)+\frac{i}{4\lambda}(i\lambda(q^*_{k+1}-q^*_k)) & -\frac{1}{2}q^*_{k+1}(i\lambda(q_{k+1}-q_k))-\frac{i}{4\lambda}(1-i\lambda\delta_k) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{i}{4\lambda}+\frac{1}{4}\delta_k+\frac{1}{2}q_{k+1}i\lambda(q^*_{k+1}-q^*_k) & \frac{1}{4}(q_{k+1}-q_k)-\frac{1}{2}q_{k+1}+\frac{1}{2}q_{k+1}i\lambda\delta_k \\ \frac{1}{2}q^*_{k+1}-\frac{1}{2}q^*_{k+1}i\lambda\delta_k-\frac{1}{4}(q^*_{k+1}-q^*_k) & -\frac{1}{2}q^*_{k+1}i\lambda(q_{k+1}-q_k)-\frac{i}{4\lambda}-\frac{1}{4}\delta_k \end{pmatrix}$$

So for this we get

$$U_kV_k - V_{k+1}U_k = \begin{pmatrix} -\frac{1}{2}i\lambda(q_{k+1}q^*_{k+1}-q_kq^*_k) & -\frac{1}{2}i\lambda\delta_k(q_{k+1}-q_k) \\ -\frac{1}{2}i\lambda\delta_k(q^*_{k+1}-q^*_k) & \frac{1}{2}i\lambda(q_kq^*_k+q_{k+1}q^*_{k+1}) \end{pmatrix} \tag{5.12}$$

Finally when we combine (5.11) and (5.12) we get the semi discrete short pulse equation (5.13).

$$\begin{pmatrix} \frac{d\delta_k}{dt}+\frac{1}{2}(q_{k+1}q^*_{k+1}-q_kq^*_k) & \frac{d}{dt}(q_{k+1}-q_k)+\frac{1}{2}\delta_k(q_{k+1}-q_k) \\ \frac{d}{dt}(q^*_{k+1}-q^*_k)+\frac{1}{2}\delta_k(q^*_{k+1}-q^*_k) & -\frac{d\delta_k}{dt}+\frac{1}{2}(q_kq^*_k+q_{k+1}q^*_{k+1}) \end{pmatrix} \tag{5.13}$$

Now we need show that the bilinear equation for the semi-discrete CSP equation holds true on both sides.

$$\frac{1}{2a}D_s(g_{k+1}\cdot f_k - g_k\cdot f_{k+1})$$

We distribute Hirota's D operator and say that

$$\frac{1}{2a}\left(\frac{\partial g_{k+1}}{\partial s}f_k - g_{k+1}\frac{\partial f_k}{\partial s} + g_k\frac{\partial f_{k+1}}{\partial s} - f_{k+1}\frac{\partial g_k}{\partial s}\right)$$

We can use the Taylor expansion for f_{k+1} and g_{k+1} to prove the equality.

$$f_{k+1} = f + 2af_y + \cdots$$

$$g_{k+1} = g + 2ag_y + \cdots$$

We can ignore the higher order terms because they are increasingly getting smaller. This gives us

$$\frac{1}{2a}\left[(g_s + 2ag_{ys})f - (g + 2ag_y)f_s + (f_s + 2af_{ys})g - (f + 2af_y)g_s\right]$$

We distribute the multiplications for clarity

$$\frac{1}{2a}\left[g_s f + 2ag_{ys} f - gf_s - 2ag_y f_s + gf_s + 2agf_{ys} - g_s f - 2ag_s f_y\right]$$

Now as $a \to 0$ we get

$$\left[g_{ys} f - g_y f_s + gf_{ys} - g_s f_y\right] \tag{5.14}$$

which is the right hand side of the bilinear equations essentially proving the bilinear equation.

CHAPTER VI

FULLY DISCRETE COMPLEX SHORT PULSE EQUATION

The following is the bilinear form for the fully discrete CSP equation.

$$g_{k+1}^{l+1}f_k^l - g_k^{l+1}f_{k+1}^l - g_{k+1}^l f_k^{l+1} + g_k^l f_{k+1}^{l+1} = ab(g_{k+1}^{l+1}f_k^l + g_k^{l+1}f_{k+1}^l + g_{k+1}^l f_k^{l+1} + g_k^l f_{k+1}^{l+1}) \quad (6.1)$$

$$f_k^{l+1}f_k^{l-1} - f_k^l f_k^l = b^2 g_k^l g_k^{-l} \quad (6.2)$$

Note that the superscript is another index and $g_k^l = g_{k,l}$ as well as $f_k^l = f_{k,l}$ The reason it was written as a superscript is for aesthetic purposes. The bilinear functions admit the following multi-soliton solution:

$$f_k^l = \begin{vmatrix} A & I \\ -I & B \end{vmatrix},$$

$$g_k^l = \begin{vmatrix} A & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & C_1 & 0 \end{vmatrix}, \quad \bar{g}_k^l = \begin{vmatrix} A & I & \mathbf{0}^T \\ -I & B & -(C_1^*)^T \\ \bar{\Phi} & \mathbf{0} & 0 \end{vmatrix}$$

where the elements of the matrix A and B are defined respectively by

$$a_{ij} = \frac{1}{2(\mu_i^{-1} + \bar{\mu}_j^{-1})} e^{\xi_i + \xi_j^*}, \quad b_{ij} = \frac{\alpha_i^* \alpha_j}{2(\mu_j^{-1} + \bar{\mu}_i^{-1})}$$

where

$$\mu_i = p_i - b, \quad \bar{\mu}_i = p_i^* + b$$

and

$$e^{\xi_i} = \left(\frac{1+ap_i}{1-ap_i}\right)^k \left(\frac{1+bp_i^{-1}}{1-bp_i^{-1}}\right)^l$$

From here we can construct the fully discrete CSP equation: First we define

$$q_k^l = \frac{g_k^l}{f_k^l}, \quad q_k^l = \frac{\bar{g}_k^l}{f_k^l}$$

and

$$\Gamma_k^l = \frac{f_k^{l+1} f_{k+1}^l}{f_{k+1}^{l+1} f_k^l}, \quad \Gamma_k^l = 1 + (\delta_k^l - 2a)b.$$

$$(1-ab)(q_k^l + q_{k+1}^{l+1}) = (1+ab)(q_k^{l+1} + q_{k+1}^l)(1 + (\delta_k^l - 2a)b) \tag{6.3}$$

$$\frac{1+(\delta_k^l - 2a)b}{1+(\delta_k^{l-1} - 2a)b} = \frac{1+b^2 q_k^l \bar{g}_k^l}{1+b^2 q_{k+1}^l \bar{g}_{k+1}^l} \tag{6.4}$$

If we let $b \to 0$, $q_k^l \to q_k$, and $\bar{q}_k^l \to q_k^*$ we can show that the two equations converge to the semi-discrete complex short pulse equation.

$$\frac{d}{dt}(q_{k+1} - q_k) = \frac{1}{2}\delta_k (q_{k+1} - q_k),$$

$$\frac{d}{dt}\delta_k = -\frac{1}{2}(|q_{k+1}|^2 - |q_k|^2).$$

Continuous limit: We can rewrite the first equation as

$$\begin{aligned} &\frac{1}{2b}(q_k^l + q_{k+1}^{l+1} - q_k^{l+1} - q_{k+1}^l) \\ &\quad = \frac{a}{2}(q_k^l + q_{k+1}^{l+1} + q_k^{l+1} + q_{k+1}^l) + \frac{1+ab}{2}(q_k^{l+1} + q_{k+1}^l)(\delta_k^l - 2a) \end{aligned}$$

If we let $b \to 0$, the above equation converges to

$$\frac{d}{dt}(q_{k+1}-q_k) = \frac{1}{2}\delta_k(q_{k+1}+q_k),$$

On the other hand, take the Taylor expansion on both sides of the second bilinear equation

$$b\delta_k^l - b\delta_k^{l-1} = b^2(q_k^l \bar{q}_k^l) - b^2(q_{k+1}^l \bar{q}_{k+1}^l)$$

or

$$\frac{\delta_k^l - \delta_k^{l-1}}{2b} = -\frac{1}{2}(q_k^l \bar{q}_k^l - q_{k+1}^l \bar{q}_{k+1}^l)$$

Now as $b \to 0$ our functions $q_k^l \to q_k$ and $\bar{q}_k^l \to q_k^*$. Therefore we end up having convergence to

$$\frac{d}{dt}\delta_k = -\frac{1}{2}(|q_{k+1}|^2 - |q_k|^2).$$

Showing that the full discretization recovers the complex short pulse equation.

CHAPTER VII

CONCLUSION

The Lax pair and multi-soliton solution to the complex short pulse equation were proposed and proved. Then we constructed a semi-discrete complex short pulse equation along with its N-soliton solutions. Finally we constructed a fully-discrete complex short pulse equation and introduced its determinant solution.

CHAPTER VIII

FUTURE WORK

The full-discretization still needs its Lax pair. Another thing to note is that we have only looked at the focusing case in other words the bright solitons of the complex short pulse equation. The defocusing case has yet to be studied along with its dark solitons which are yet to be discovered. Of course this is outside of the scope of this paper and can be considered future work in regards to the complex short pulse equation.

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BIOGRAPHICAL SKETCH

Raul Guajardo, born in Dallas, TX on May 5, 1990, now lives at 1803 Uranium St. in PeÃśitas, TX. He received his MS in Applied Math from The University of Texas - Pan American in May 2015. Acting president of the Society of Industrial and Applied Mathematics(SIAM) UTPA student chapter, he has hosted many mathematical events in the past semester. He looks forward to continuing his career as a researcher in the area of mathematics. Before moving on to his higher education, Raul was involved in many scholastic organizations including the University Interscholastic League(UIL) and Business Professionals of America(BPA). These organizations evolved his competitive spirit and introduced him to many new concepts and ideas in relation to mathematics. While a member to both organizations he was able to make it to their respective State competitions. In BPA he made it to nationals and competed in Reno, Nevada vs. over a hundred students. These events provided him with valuable experience and insight about Mathematics and pushed him to make it his career.