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An action for a classical string, the equation of motion and group invariant classical solutions

Research Article

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Abstract: A string action which is essentially a Willmore functional is presented and studied. This action determines the physics of a surface in Euclidean three space which can be used to model classical string configurations. By varying this action an equation of motion for the mean curvature of the surface is obtained which is shown to govern certain classical string configurations. Several classes of classical solutions for this equation are discussed from the symmetry group point of view and an application is presented.

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Keywords: classical strings • Gaussian curvature • mean curvature • Weierstrass representation • Euler–Lagrange equation

1. Introduction-action and equation of motion

String models have been of interest for many reasons recently. One reason in particular is the work that has been done with regard to generalized Weierstrass representations. An important application of this work is the study of string models from the point of view that strings can be represented by surfaces embedded in some higher dimensional space. Even in four-dimensional Euclidean and Minkowski spaces, generalized Weierstrass formulas for surfaces have been used to study various types of string configurations [1–3].

The intention here is to take a particular kind of classical string model which has been of use in the areas of particle physics [4] and cosmology and seems to have originated with the work of Polyakov and Kleinert [5, 6]. The action will take the form of a Willmore functional. It will be shown that by varying the functional, an equation emerges which relates the basic invariants of the string world surface, in particular the constant Gauss curvature K and mean curvature H of the surface. Once a mean curvature function has been obtained, it is known that a surface can be induced using a generalized Weierstrass representation [7]. The equation we then study is similar to one which has physical applications to such areas as phase transitions [8], meson field theories [9] and superconductors [10].
The action of the string which is considered here will take the form

$$\mathcal{A} = \gamma \iint dS + \alpha \iint H^2 \, dS. \qquad (1)$$

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In (1) both α and γ are constants and integration takes place over the string world surface S which has extrinsic mean curvature H.

Let $x^{\mu}(u_1, u_2)$ be a parametric representation of the surface M in three dimensional Euclidean space E^3 and n^{μ} the unit normal to the surface, where μ goes from 1 to 3. The induced metric is then given by $g_{ij}(u_1, u_2) = x^{\mu}_{,i} x^{\mu}_{,j}$ where the commas indicate partial differentiation with respect to u_i. If Γ^k_{ij} denote the Christoffel symbols for the metric g_{ij}, we will need to make use of the Gauss and Weingarten equation

$$x^{\mu}_{,ij} = \Gamma^k_{ij} x^{\mu}_{,k} + b_{ij} n^{\mu}, \qquad n^{\mu}_{,i} = -b_{ij} g^{jk} x^{\mu}_{,k}. \tag{2}$$

Here b_{ij} are the coefficients of the second quadratic form of the surface. The compatibility of the equations in (2) give the more familiar form of the Gauss equation

$$R_{ijkl} = b_{ik} b_{jl} - b_{il} b_{jk}, \tag{3}$$

and the Codazzi equations

$$b_{ij;k} - b_{ik;j} = 0, \tag{4}$$

where $i, j, k = 1, 2$ and the semicolon indicates covariant differentiation. When (3) and (4) are satisfied, (2) can be integrated such that the solution x^{μ} determines the surface up to motions in E^3. There are two geometrical invariants of the solution which will appear, namely, its Gaussian curvature

$$K = -\frac{1}{2} R, \qquad R = g^{il} g^{jk} R_{ijkl}, \tag{5}$$

and its mean curvature H which is given by

$$H = \frac{1}{2} b_{ij} g^{ij} = \frac{1}{2} b^i_i. \tag{6}$$

It is sufficient to consider normal variations of the surface in physical applications. For a given surface M with position vector $x^{\mu}(u_1, u_2)$, we form the surface $\bar{M}$ parallel to M by defining

$$\bar{x}^{\mu} = x^{\mu} + t f n^{\mu}, \quad -\epsilon < t < \epsilon, \tag{7}$$

for f a sufficiently smooth function given on M. Let δ be the operator $\partial/\partial t|_{t=0}$, so it follows for example that $\delta \bar{x}^{\mu} = f n^{\mu}$. From the definition of δ, we obtain

$$\delta \bar{x}^{\mu}_{,i} = f_{,i} n^{\mu} + f n^{\mu}_{,i}.$$

Moreover,

$$\bar{x}^{\mu}_{,ij} = x^{\mu}_{,ij} + t(f_{,ij} n^{\mu} + f_{,i} n^{\mu}_{,j} + f_{,j} n^{\mu}_{,i} + f n^{\mu}_{,ij}),$$

and so it follows that

$$\delta x^{\mu}_{,ij} = f_{,ij} n^{\mu} + f_{,i} n^{\mu}_{,j} + f_{,j} n^{\mu}_{,i} + f n^{\mu}_{,ij}. \tag{8}$$

Using these results, the variation of the metric tensor $g_{ij} = x^{\mu}_{,i} x^{\mu}_{,j}$ is calculated to be

$$\delta g_{ij} = \delta x^{\mu}_{,i} x^{\mu}_{,j} + x^{\mu}_{,i} \delta x^{\mu}_{,j} = f(n^{\mu}_{,i} x^{\mu}_{,j} + n^{\mu}_{,j} x^{\mu}_{,i}),$$

and $n^{\mu} x^{\mu}_{,i} = 0$ has been used. Using Weingarten equation (2)b, this can be further simplified

$$\delta g_{ij} = f(b_{is} g^{sk} x^{\mu}_{,k} x^{\mu}_{,j} - b_{js} g^{sk} x^{\mu}_{,k} x^{\mu}_{,i}) = -f b_{ij} - f b_{ji} = -2 f b_{ij}. \tag{9}$$

By varying the expression $g_{ij} g^{jk} = \delta^k_i$ and using (9), we obtain $g_{ij} \delta g^{jk} = -\delta g_{ij} g^{jk}$. Therefore,

$$\delta g^{ij} = -g^{il} g^{kj} \, \delta g_{lk} = 2 f b^{ij}.$$

As usual, denote $g = \det(g_{ij})$, so we can write

$$\delta \sqrt{g} = \left(\frac{\partial \sqrt{g}}{\partial x^{\mu}_{,i}} \right) \delta x^{\mu}_{,i}. \tag{10}$$

Therefore making use of (6), we determine that

$$\delta \sqrt{g} = \left(\frac{\partial}{\partial x^{\mu}_{,i}} \sqrt{g} \right) = \sqrt{g} g^{im} x^{\mu}_{,m} n^{\mu}_{,i} f$$

$$= -\sqrt{g} g^{im} x^{\mu}_{,m} f b_{ij} g^{jk} x^{\mu}_{,k} = -\sqrt{g} g^{im} g_{mk} g^{kj} f b_{ij} \tag{11}$$

$$= -\sqrt{g} g^{ij} f b_{ij} = -2H \sqrt{g} f.$$

Contracting the Gauss equations (2)a with n^{μ} and using $n^{\mu} x^{\mu}_{,i} = 0$, it follows that $b_{ij} = n^{\mu} x^{\mu}_{,ij}$. From b_{ij} in this form of b_{ij} we calculate

$$\delta b_{ij} = \delta n^{\mu} x^{\mu}_{,ij} + n^{\mu} \delta x^{\mu}_{,ij} = \Gamma^k_{ij} x^{\mu}_{,k} \delta n^{\mu} + b_{ij} n^{\mu} \delta n^{\mu} + n^{\mu} \, \delta x^{\mu}_{,ij}. \tag{12}$$

Since n^{μ} is a normal, it satisfies $n^{\mu} n^{\mu} = 1$ and $n^{\mu} x^{\mu}_{,i} = 0$. It follows from these that $\delta n^{\mu} \, n^{\mu} = 0$ and $\delta n^{\mu} x^{\mu}_{,i} + n^{\mu} \delta x^{\mu}_{,i} = 0$, hence

$$\delta n^{\mu} \, x^{\mu}_{,i} = -n^{\mu}(f_{,i} n^{\mu} + f n^{\mu}_{,i}) = -f_{,i}. \tag{13}$$

Differentiating the condition $n^\mu n^\mu_{,i} = 0$ with respect to u_j, it follows that

$$n^\mu n^\mu_{,ij} = -n^\mu_{,i} n^\mu_{,j} = -b_{is} g^{sk} x^\mu_{,k} b_{jt} g^{tl} x^\mu_{,l} \\ = -b_{is} b_{jt} g^{sk} g^{tl} g_{kl} = -b_{is} b_{jt} g^{st}.$$

Finally, from the expression for the variation of the second quadratic form in (12) and using (8) and (13), the variation of b_{ij} in (12) is given by

$$\delta b_{ij} = \Gamma^k_{ij}(-f_{,k}) + n^\mu (f_{,ij} n^\mu + f_{,i} n^\mu_{,j} + f_{,j} n^\mu_{,i} + f n^\mu_{,ij}) \\ = -\Gamma^k_{ij} f_{,k} + f_{,ij} + f n^\mu n^\mu_{,ij} \\ = f_{,ij} - \Gamma^k_{ij} f_{,k} - f b_{ik} b_{jl} g^{kl}. \tag{14}$$

Since the covariant derivative of $f_{,i}$ is given by

$$f_{,i;j} = \frac{\partial f_{,i}}{\partial u_j} - \Gamma^k_{ij} f_{,k},$$

the expression (14) can be written as

$$\delta b_{ij} = f_{,i;j} - f b_{ik} b_{jl} g^{kl}. \tag{15}$$

To calculate the variation of the action, we require the variation of H^2. Starting with (6), we find using (15) that

$$\delta H^2 = H \delta(b_{ij} g^{ij}) = H g^{ij} \delta b_{ij} + H b_{ij}\, \delta g^{ij} \\ = H(g^{ij}[f_{,i;j} - f b_{ik} b_{jl} g^{kl}] + 2 f b_{ij} b^{ij}) \\ = H(g^{ij} f_{,i;j} - f b^j_k b^k_j + 2 f b^j_i b^i_j).$$

Introducing the Laplace-Beltrami operator given by

$$\Delta = g^{ij}(f_{,ij} - \Gamma^l_{ij} f_{,l}),$$

this can be written

$$\delta H^2 = H(\Delta f + f b^i_k b^k_i). \tag{16}$$

From expression (3) for R_{ijkl}, we calculate R to be

$$R = b^i_k b^k_i - b^i_i b^k_k = b^i_k b^k_i - 4H^2,$$

upon using $H = \frac{1}{2} b^i_i$. Solving this for $b^i_k b^k_i$ and substituting into (16), the final result is obtained

$$\delta H^2 = H(\Delta f + f b^i_k b^k_i) = H(\Delta f + f(R + 4H^2)). \tag{17}$$

By writing the action in the form

$$\mathcal{A} = \iint (\gamma + \alpha H^2)\sqrt{g}\, du_1 du_2, \tag{18}$$

the normal variation of $\mathcal{A}$ can be easily determined with the results which have been determined here. The Euler-Lagrange equation follows from the vanishing of the normal variation of string action (18),

$$\delta\mathcal{A} = \iint \delta\sqrt{g}(\gamma + \alpha H^2)\, du_1\, du_2 \\ + \iint \alpha \delta H^2 \sqrt{g}\, du_1 du_2 \\ = \iint [-2\gamma f H - 2\alpha f H^3 + \alpha H(\Delta f + f(R + 4H^2))]\, dS \\ = \iint [(-2\gamma H + \alpha(2H^3 + HR)) f + \alpha H \Delta f]\, dS. \tag{19}$$

On closed surfaces, the Laplace-Beltrami operator Δ is a self-adjoint operator hence

$$\iint \varphi \Delta f\, dS = \iint f \Delta\varphi\, dS. \tag{20}$$

Therefore, using (20) in (19), the normal variation of the action $\delta\mathcal{A}$ can be written in the following form

$$\delta\mathcal{A} = \iint [-2\gamma H + \alpha(\Delta H + 2H^3 + RH)] f(u_1, u_2)\, dS = 0. \tag{21}$$

Due to the arbitrariness of the function $f(u_1, u_2)$, the equation of motion now follows by equating the part of the integrand inside the square brackets to zero. This procedure yields the following equation for the function H

$$\Delta H + 2H^3 + RH - \frac{2\gamma}{\alpha} H = 0. \tag{22}$$

Using (5)a, this can be put in the alternate form

$$\Delta H + 2H^3 - 2\left(K + \frac{2\gamma}{\alpha}\right) H = 0. \tag{23}$$

There are a few other forms of equation (23) we would like to note. In terms of the conformal metric, (23) can be written as

$$u^{-2} \partial\bar\partial H + 2H(H^2 - K) - \frac{2\gamma}{\alpha} H = 0.$$

Here $\partial = \partial/\partial z$ and $\bar\partial = \partial/\partial \bar z$. With $K = -u^{-2}\partial\bar\partial \ln u$, $\varphi = H^{-1}$ and $p = u/\varphi$, by expanding out the derivatives, the Euler-Lagrange equation takes the form

$$\partial\bar\partial\varphi + [2p^2 + \partial\bar\partial \ln p^2]\varphi - \frac{2\gamma}{\alpha} p^2 \varphi^3 = 0. \tag{24}$$

The same equation (23) has appeared and been used by Konopelchenko [11] in investigating and modeling strings by surfaces as well.

2. Group invariant solutions for H

For the purpose of writing solutions, since K is constant the factor $2\gamma/\alpha$ can be incorporated into K and so it suffices to study the equation

$$\Delta H + 2H(H^2 - K) = 0. \tag{25}$$

Equation (25) can be taken as the defining property of a Willmore surface, whether or not the surface is compact or even orientable. In order to solve exactly (25) we adopt an approach based on the symmetry reduction method for partial differential equations as elaborated in [12]. Thus we determine group invariant solutions here, although others may exist. For $K \neq 0$, the symmetry group of Euler equation (25) is simply the Euclidean Lie group $E(2)$. A basis for its Lie algebra $e(2)$ is provided by two translations P_i and L, realized as

$$P_i = \frac{\partial}{\partial x_i}, \qquad L = x_2\partial_{x_1} - x_1\partial_{x_2}. \tag{26}$$

For $K = 0$, the symmetry group of (25) is the similitude group $Sim(2)$. Its Lie algebra $Sim(2)$, in addition to the translations and rotation in (26), includes also dilation realized as

$$D = x_1\partial_{x_1} + x_2\partial_{x_2} - H\partial_H. \tag{27}$$

The sub-algebras of $e(2)$ corresponding to subgroups that are maximal among those with orbits of codimension one in Euclidean 2-dimensional space can be represented by

$$\{P_2\}, \qquad \{L\}. \tag{28}$$

For $Sim(3)$, we have, in addition to (25), sub-algebras represented by

$$\{D\}, \qquad \{D + bL, b \neq 0, b \in \mathbb{R}\}. \tag{29}$$

If $K \neq 0$, the symmetry reduction for (25) is reduced to ODE of the form

$$H = y(\xi), \quad \ddot{y} + \frac{k}{\xi}\dot{y} - 2Ky + 2y^3 = 0, \quad k = 0, 1, \tag{30}$$

by means of subgroups of the isometry groups as shown in Table 1.
If $K = 0$, then the symmetry reduction for the equation (25) which is

$$\Delta H + 2H^3 = 0, \tag{31}$$

reduces to an ODE using subgroups of the similitude group involving dilation. We introduce the notation

$$H(\mathbf{x}) = \sigma(\mathbf{x})y(\xi). \tag{32}$$

and the results are summarized in Table 2 [13].
Passing through all subgroups of the symmetry group corresponding to the algebras (26) and (27), we obtain all symmetry variables ξ. Applying the symmetry reduction as described in [12] we can reduce the equation of motion (25) to many possible ordinary differential equations. Next, we perform the singularity analysis in order to determine whether these ODEs possess the Painlevé property [14].
We start our analysis with subgroups of the isometry group $E(2)$. The translationally invariant solutions of (25) have the form

$$H(\mathbf{x}) = y(\xi), \quad \xi = x_2. \tag{33}$$

The function $y(\xi)$ has to satisfy the following ODE

$$\ddot{y} = 2Ky - 2y^3. \tag{34}$$

After integration, we can write this as follows

$$\dot{y}^2 = -(y^2 - y_1^2)(y^2 - y_2^2),$$

$$y_1^2 + y_2^2 = -2K, \qquad y_1^2 \cdot y_2^2 = 2C, \quad C \in \mathbb{R}. \tag{35}$$

The roots of (35) are given by

$$y_{1/2} = K \pm (K^2 - 2C)^{1/2}. \tag{36}$$

Here C is a real integration constant, related to the energy of the system. We concentrate here only on real solutions for which $y \in \mathbb{R}$ and $\dot{y}^2 \geq 0$. If all the roots of

$$P(y) = -(y^2 - y_1^2)(y^2 - y_2),$$

are different, then the solution y of (35) can be expressed in terms of Jacobi elliptic functions. If any of the roots coincide, then we obtain elementary solutions. Let us consider each of the cases individually.

1. Constant solutions

$$y = 0, \quad y = \pm K^{1/2}.$$

Table 1. Symmetry Reduction for (25) when $K \neq 0$.

	Algebra	Invariance Group	Symmetry Variable ξ	Reduced ODE	Painlevé Property
1.	P_1	$T(1)$	x_2	$\ddot{y} - 2Ky + 2y^3 = 0$	Yes
2.	L	$O(2)$	$(x_1^2 + x_2^2)^{1/2}$	$\ddot{y} + \frac{1}{\xi}\dot{y} - 2Ky + 2y^3 = 0$	No

Table 2. Symmetry Reduction for (25) when $K = 0$.

	Algebra	σ	Symmetry Variable ξ	Reduced ODE	Painlevé Property
1.	D	$\frac{1}{\sqrt{2}x_1}$	$\frac{x_2}{x_1}$	$(1+\xi^2)\ddot{y} + 4\xi\dot{y}$ $+2y + y^3 = 0$	Yes
2.	$D + bL$ $b \neq 0$	$4b((b^2+4)$ $\cdot(x_1^2 + x_2^2))^{-1/2}$	$\xi = \frac{4b}{b^2+4}\left[-\frac{b}{2}\ln(x_1^2+x_2^2)^{1/2} + \arctan\left(\frac{x_2}{x_1}\right)\right]$	$\ddot{y} + \dot{y}$ $+\frac{b^2+4}{4b^2}y + 8y^3 = 0$	Yes $b = \pm 6i$

2. One double root, two simple ones occur when

$$y_1 = (-2K)^{1/2} > 0, \quad y_2 = 0, \quad K < 0.$$

The corresponding solution is in this case a finite solitary wave of the form

$$y = \epsilon y_1[\cosh(y_1(\xi - \xi_0))]^{-1}, \tag{37}$$

and the asymptotic behavior of (37) is

$$\lim_{\xi \to \pm\infty} y(\xi) = 0, \quad \lim_{\xi \to \xi_0} y(\xi) = \epsilon y_1.$$

where $0 \leq y \leq y_1$ for $\epsilon = 1$ or $-y_1 \leq y < 0$ for $\epsilon = -1$.

3. Four real simple roots obtain when

$$0 < y_1 < y_2, \quad 0 < C < -K^2, \quad K < 0.$$

The solution is a cnoidal wave

$$y = \epsilon y_2 dn[y_2(\xi - \xi_0), k], \quad k = \left(1 - \frac{y_1^2}{y_2^2}\right)^{1/2}. \tag{38}$$

We have to distinguish two separate cases when $y_1 \leq y \leq y_2$, $\epsilon = 1$, and $-y_2 \leq y \leq -y_1$, $\epsilon = -1$.

The period, or wavelength, of the solution (38) is real

$$T = \frac{2I(k)}{y_2},$$

where $I(k)$ is an elliptic integral as defined in Byrd and Friedman [15].

4. Solutions corresponding to two simple real and two pure imaginary roots take place when

$$C \leq 0, \quad y_1 \equiv r \geq 0, \quad y_2 \equiv iq, \quad q > 0, \quad K \leq 0.$$

If $-y_1 \leq y(\xi) \leq y_1$ then a periodic finite solution exists

$$y = rcn\{(r^2+q^2)^{1/2}(\xi - \xi_0), k\},$$
$$\frac{1}{\sqrt{2}} < k^2 = \frac{r^2}{r^2+q^2} < 1. \tag{39}$$

The period is given by

$$T = \frac{I(k)}{(r^2+q^2)^{1/2}}.$$

(i) Cylindrically Invariant Solutions.
The algebra L leads to the following reduction

$$H = y(\xi),$$
$$\ddot{y} + \frac{1}{\xi}\dot{y} - 2Ky + 2y^3 = 0,$$
$$\xi = (x_1^2 + x_2^2)^{1/2}. \tag{40}$$

The ODE (40) does not have the Painlevé property.

(ii) Solutions invariant under subgroups involving dilations.

Subalgebra $\{D, P_3\}$ There are solutions invariant under the subgroup corresponding to the algebra $\{D, P_3\}$ which have the form

$$H = \frac{1}{\sqrt{2}x_1} P(\xi), \tag{41}$$

where $\xi = x_2/x_1$ and $P(\xi)$ satisfies

$$(1 + \xi^2)\ddot{y} + 4\xi\dot{y} + 2y + y^3 = 0. \tag{42}$$

This ODE has the Painlevé property can is put in standard form PVIII by putting

$$y = \left(\frac{2}{1 + \xi^2}\right)^{1/2} W(i\varphi), \quad \varphi = \arctan \xi, \tag{43}$$

where φ is the polar angle such that $x_1 = \rho \cos \varphi$, and $x_2 = \rho \sin \varphi$. Setting $\eta = i\varphi$ in (43) we find that $W(\eta)$ satisfies

$$\begin{aligned} \ddot{W} &= 2W^3 + W, \\ \dot{W}^2 &= (W^2 - W_1^2)(W^2 - W_2^2), \\ W_{1,2}^2 &= \frac{-1 \pm C}{2}, \end{aligned} \tag{44}$$

where C is an integration constant.

For $H(\mathbf{x})$ to be real, either $W(\xi)$ is real or pure imaginary and we set

$$W(\eta) = Q(\varphi), \quad or \quad W(\eta) = iN(\varphi) \tag{45}$$

for W real or imaginary, respectively. These functions satisfy

$$\begin{aligned} (Q_\varphi)^2 &= -(Q^2 - W_1^2)(Q^2 - W_2^2), \quad H \in \mathbb{R}, \\ (N_\varphi)^2 &= (N^2 + W_1^2)(N^2 + W_2^2), \quad N \in \mathbb{R}. \end{aligned} \tag{46}$$

Returning to the original $H(\mathbf{x})$ we have

$$H(\mathbf{x}) = \frac{1}{\rho} Q(\varphi). \tag{47}$$

(a) There are constant solutions of (44), $W = 0$ and the non-zero solutions $W = \epsilon i/\sqrt{2}$ lead to a solution.

(b) Two double roots: $W_1 = W_2 = i/\sqrt{2}$. Hence we obtain two types of solutions

$$\begin{aligned} H(\mathbf{x}) &= \frac{\epsilon}{\sqrt{2}} \frac{1}{(x_1^2 + x_2^2)^{1/2}} \tanh\left(\frac{\varphi - \varphi_0}{\sqrt{2}}\right), \\ H(\mathbf{x}) &= \frac{\epsilon}{\sqrt{2}} \frac{1}{(x_1^2 + x_2^2)^{1/2}} \coth\left(\frac{\varphi - \varphi_0}{\sqrt{2}}\right), \\ &\epsilon = \pm 1. \end{aligned} \tag{48}$$

The solution (48)a is finite as a function of φ and (48)b has a singularity for $\varphi - \varphi_0 = 0$. Both of these solutions are singular along the z axis and are multi-valued, since they depend on the angle φ and $H(\rho, \varphi, z) \neq H(\rho, \varphi + 2l\pi, z)$, where l is an integer.

(c) One real double root, two imaginary simple ones: $W_1 = i$, $W_2 = 0$. The solution is pure imaginary and generates the singular solution

$$H(\mathbf{x}) = \frac{1}{\rho \cos(\varphi - \varphi_0)}, \tag{49}$$

which is a single valued solution.

All other solutions correspond to simple roots in (46). Four real simple roots cannot occur. The cases that do occur are the following ones.

(d) Two real, two imaginary simple roots. We have $C^2 > 1$ in (44) and put

$$W_1^2 = p^2, \quad W_2^2 = -p^2 - 1, \quad p^2 > 0.$$

The real solution is given by

$$H = \frac{p}{\rho} cn \left((2p^2 + 1)^{1/2}(\varphi - \varphi_0), \frac{p}{(2p^2 + 1)^{1/2}} \right) \tag{50}$$

This is multi-valued and singular along the z axis and periodic in the azimuthal angle φ.

(e) Four pure imaginary simple roots. We have $-1 < C < 1$ and put

$$W_1^2 = -q^2, \quad W_2^2 = -1 + q^2, \quad 0 < q^2 < 1.$$

We obtain two types of solutions

$$H = \frac{q}{\rho} sn \left((1 - q^2)^{1/2}(\varphi - \varphi_0), \frac{q}{(1 - q^2)^{1/2}} \right), \tag{51}$$

$$H = \frac{(1-q^2)^{1/2}}{\rho}\left[sn\left((1-q^2)^{1/2}(\varphi-\varphi_0), \frac{q}{(1-q^2)^{1/2}}\right)\right]^{-1}. \tag{52}$$

Both of these are real, multi-valued and singular along the z axis. The solution (51) is finite as a function of φ whereas (52) has singularities for $(1-q^2)^{1/2}(\varphi-\varphi_0) = 2nK$ where K is the real period of $sn(x,k)$.

(f) Four complex roots. We put

$$W_1 = p + iq, \quad W_2 = p - iq,$$
$$p^2 - q^2 = -\frac{1}{2}, \quad p > 0 \quad q > 1/\sqrt{2}.$$

The solutions for W are pure imaginary. For H we obtain

$$H(\mathbf{x}) = \left(\frac{4p^2+1}{2}\right)^{1/2}\frac{1}{\rho}\left\{tn\left[\left(\frac{4p^2+1}{2}\right)^{1/2}(\varphi-\varphi_0), k\right]\right.$$
$$\left.\times\, dn\left[\left(\frac{4p^2+1}{2}\right)^{1/2}(\varphi-\varphi_0), k\right]\right\}^{-1}, \tag{53}$$

where

$$k^2 = \frac{2p^2+1}{4p^2+1}.$$

This solution is real and singular along the z axis and as well for $[2(4p^2+1)]^{1/2}(\varphi-\varphi_0) = 4K(k)$ where $4K$ is the real period of $cn(x,k)$. It is multi-valued with respect to the azimuthal angle φ as well.

Subalgebra $\{D + bL_3, P_3\}$. The reduction is obtained by putting

$$H(\mathbf{x}) = \frac{2b}{(b^2+4)^{1/2}}\frac{1}{\sqrt{2}\rho}y(\xi), \tag{54}$$

where

$$\xi = \frac{4b}{b^2+4}\left(-\frac{b}{2}\ln\rho + \varphi\right) \tag{55}$$

and $x_1 = \rho\cos\varphi$, $x_2 = \rho\sin\varphi$. The function $y(\xi)$ must satisfy the ODE

$$\ddot{y} + \dot{y} + \frac{b^2+4}{4b^2}y + 8y^3 = 0. \tag{56}$$

Equation (56) has the Painlevé property only for $b = \pm 6i$ and b is restricted to be real, as long as we are considering real solutions. Reducing (56) to its standard form, we obtain

$$y(\xi) = \lambda_0 e^{-\xi/3}W(\eta), \quad \eta = -\frac{3i\lambda_0}{\sqrt{2}}e^{-\xi/3}, \tag{57}$$

with λ_0 =constant and $W(\eta)$ satisfying

$$W_{\eta\eta} = \frac{2}{\lambda_0^2}e^{2\xi/3}\frac{b^2+36}{36b^2}W + 16W^3. \tag{58}$$

For $b^2+36 = 0$ we obtain an equation that we can immediately integrate to give

$$(W_\eta)^2 = W^4 - K \tag{59}$$

where A is an integration constant. For $K = 0$ we obtain the complex solution

$$H(\mathbf{x}) = \frac{\epsilon}{(2\rho)^{1/2}}(\rho e^{i\varphi})^{1/4}\frac{1}{\rho^{1/2}(\rho e^{i\epsilon\varphi})^{1/4} + \tilde{\gamma}}. \tag{60}$$

Here $\tilde{\gamma}$ is a constant. For $K \neq 0$ we obtain complex solutions in terms of Jacobi elliptic functions. These can be written as

$$H(\mathbf{x}) = \frac{1}{\sqrt{2\rho}}(K\rho e^{i\epsilon\varphi})^{1/4}cn[\rho^{1/2}(K\rho e^{i\epsilon\varphi})^{1/4} - \eta_0, 1/\sqrt{2}]. \tag{61}$$

In equation (61), the argument of the cn function, as well as all the constants and the solution itself, are complex.

Some qualitative understanding as to how these solutions of (56) behave for any $b \in \mathbb{R}$ can be obtained by reducing it to a first-order ODE. To do this we put $q = y$ and $w = y_\xi$ and view w as a function of q. We then have

$$w\frac{dw}{dy} + w + \frac{b^2+4}{4b^2}q + q^3 = 0. \tag{62}$$

A solution

$$w = g(q, C_1) \tag{63}$$

of (62) will yield a solution of (56) by a further quadrature,

$$\int\frac{dy}{g(y, C_1)} = (\xi - \xi_0). \tag{64}$$

The functions thus obtained could theoretically be used to generate surfaces in Euclidean three-space. The generalized Weierstrass representation for inducing surfaces with nonconstant H has been discussed recently [16]. An application which is based on this will be given in the next section.

3. Application to classical string theory-modeling strings by surfaces

Given a mean curvature function H which satisfies (23), there is a procedure which will allow the determination of a surface with mean curvature H. The following nonlinear Dirac-type system of differential equations determine two complex-valued functions ψ_1 and ψ_2 corresponding to H [1]

$$\partial\psi_1 = pH\psi_2, \qquad \bar\partial\psi_2 = -pH\psi_1,$$

$$\bar\partial\bar\psi_1 = pH\bar\psi_2, \qquad \partial\bar\psi_2 = -pH\bar\psi_1, \tag{65}$$

$$p = |\psi_1|^2 + |\psi_2|^2.$$

In (65), ψ_1 and ψ_2 are two complex functions of the complex variables $(z, \bar z)$. We have used the following notation for the derivatives $\partial = \partial/\partial z$ and $\bar\partial = \partial/\partial\bar z$, where bar denotes complex conjugation and $H = H(z, \bar z)$ denotes the mean curvature of the surface. The system (65) actually determines a set of constant mean curvature surfaces which are obtained by means of the following parametrization $(z, \bar z) \to (X_1(z, \bar z), X_2(z, \bar z), X_3(z, \bar z))$. The X_j can be determined explicitly [16] from the solutions to (65) by means of the following inducing

$$\begin{aligned} X_1 + iX_2 &= 2i\int_{z_0}^{z} (\bar\psi_1^2\, dz' - \bar\psi_2^2\, d\bar z'), \\ X_1 - iX_2 &= 2i\int_{z_0}^{z} (\psi_2^2\, dz' - \psi_1^2\, d\bar z'), \\ X_3 &= -2\int_{z_0}^{z} (\bar\psi_1\psi_2\, dz' + \psi_1\bar\psi_2\, d\bar z'). \end{aligned} \tag{66}$$

If the solutions for H obtained above were interpreted as string-particle states, or approximations to them, the results of Section 2 could be thought of as providing a set of states or particles in some classical sense about which quantum corrections could be calculated [11, 17]. Many properties such as spin have been left out of this model and saying this gives a spectroscopy may be saying too much. However, other properties could be included at a later phase. Once the coordinates have been calculated from (66), other physical quantities can be defined [4]. For example, taking $z = (\xi, \eta)$, the canonical momenta conjugate to X_μ can be defined to be

$$P_\mu(\xi, \eta) = C\frac{\partial X_\mu}{\partial \xi}. \tag{67}$$

where C is an appropriate constant. Other quantities such as mass could be defined based on this.

Let us consider (24) in two cases (i) $\varphi = \varphi_0$ is constant, and (ii) $p = p_0$ is constant. Surfaces can be calculated for these cases more easily.

For (i), when $\varphi = \varphi_0$, equation (24) reduces to a second order linear equation for p^2

$$\partial\bar\partial p^2 + 2(1 - \frac{2\gamma}{\alpha}\varphi_0^2)p^2 = 0.$$

Setting $\beta = 2(1 - 2\gamma/\alpha\varphi_0^2)$ and $\theta = \ln p^2$, this is seen to be become the Liouville equation

$$\partial\bar\partial\theta + \beta e^\theta = 0.$$

Solutions can be obtained of the form

$$\theta = \log\partial G + \log\bar\partial\bar G - 2\log\left(|G|^2 + \frac{\beta}{2}\right).$$

This satisfies the equation for $\beta \neq 0$ and $G(z)$ is an arbitrary analytic function.

For solutions which correspond to $p = F(z)\bar F(\bar z)$, where F is analytic, it is seen that $\partial\bar\partial\theta = 0$, and the Liouville equation holds only for $\beta = 0$.

Two solutions of (65) which have constant φ are given by

$$\psi_1 = \epsilon n^{1/2}\frac{z^n \bar z^{(n-1)/2}}{1 + |z|^{-2n}}, \quad \psi_2 = \epsilon n^{1/2}\frac{z^{(n-1)/2}}{1 + |z|^{2n}}.$$

as well as

$$\psi_1 = \epsilon\bar\lambda^{1/2}\frac{e^{\bar\lambda\bar z/2}}{e^{\lambda z} + e^{-\lambda z}}, \quad \psi_2 = \epsilon\lambda^{1/2}\frac{e^{-\lambda z/2}}{e^{\lambda z} + e^{\bar\lambda\bar z}},$$

where $\epsilon = \pm 1$. It is found that the Liouville equation is satisfied for both of these solutions provided that $\beta = 2$. Coordinates of surfaces have been determined for both of these solutions in [7].

Let us consider the case in which $p = p_0$ is constant. The Euler-Lagrange equation (24) becomes

$$\partial\bar\partial\varphi + 2p_0^2\varphi - \frac{2\gamma}{\alpha}p_0^2\varphi^2 = 0.$$

In real variables with $\gamma = 0$, this reduces to

$$\varphi_{xx} + \varphi_{yy} + 8p_0^2\varphi = 0.$$

For the square domain $0 \leq x, y \leq \pi$ with nonvanishing φ at the boundary, there exists a family of solutions given by $\varphi_{nm} = a_{nm}\sin nx \sin my$, $p_0^2 = (1/8)(n^2 + m^2)$, where $n, m \in I$ and A_{nm} are constants. When $\gamma \neq 0$, there exists a kink solution.

4. Summary

The main goal of string theory is to modify point particle interactions, and in particular the gravitational interaction, at short distances by the introduction of string states and by exchange of massive string states to transmit forces. The theory also introduces a new coupling constant, the string tension. As the string propagates through space-time, it sweeps out a surface referred to as its world sheet.

In the simple model under consideration here, the Nambu-Goto action depends on the area of the world sheet of the string in this simpler Euclidean space. The results obtained here could be interpreted as a class of topological soliton solutions which represent a classical approximation to particles or the associated particles which transmit forces. Of course, the difference between bosons and fermions has not been accounted for in this simple model, so this correspondence is likely to be somewhat artificial. The interesting development which is presented here is that a given action can be used under an extremum principle to produce an equation which can be treated using certain techniques to generate large sets of nontrivial soliton solutions. This is interesting in itself in that the real situation presents large sets of particles which are often related to each other under certain types of groups of transformations. The multivaluedness of the functions corresponds to the possibility of different nontrivial topological charges corresponding to the particle states. An energy can be defined using the Willmore function itself. It might be imagined that the solutions to more complicated string theories could be obtained as has been done here, and it would hopefully be possible to put these solutions in correspondence with actual particles that exist or can be produced.

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