Quantum phases for a generalized harmonic oscillator

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Research Article

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Abstract: An effective Hamiltonian for the generalized harmonic oscillator is determined by using squeezed state wavefunctions. The equations of motion over an extended phase space are determined and then solved perturbatively for a specific choice of the oscillator parameters. These results are used to calculate the dynamic and geometric phases for the generalized oscillator with this choice of parameters.

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1. Introduction

The Berry phase has received a great deal of attention recently as originally it had much to do with exposing the underlying gauge structure which is associated with a specific phase shift produced during quantum adiabatic processes. Moreover, it has been demonstrated that this quantum phase can be linked with a corresponding classical angle which is called the Hannay angle [1, 2]. Simon as well as Aharonov and Anandan have also given attention to and generalized this phase which had been considered with regard to cyclic evolution, that is, the case in which the state returns to its initial condition [3–5]. The main conclusion of their work is that the phase is a geometrical property of the curve traversed in the projective Hilbert space which is relevant to the motion. Under adiabatic evolution, this phase factor is a gauge invariant generalization of the Berry phase. Of course, cyclic evolution is quite prevalent in wide classes of quantum systems. In the nonadiabatic case, the determination of the eigenvectors and calculating the nonadiabatic geometric phase for a time-dependent Hamiltonian may not be so simple.

Squeezed states have seen applications to many types of problems in theoretical physics, such as quantum optics, and more recently to applications pertaining to chaotic dynamical systems. There are also applications to the spin and statistics of solitons [6, 7]. A particular system which admits squeezed states as an exact solution is the generalized harmonic oscillator. In this sense, it represents an ideal system with which to study the time-dependent evolution of a squeezed state. Here we would like to introduce the Berry phase and then present a different approach to determining the phase for the generalized harmonic oscillator system. An effective way of carrying this out is to use squeezed states to study the general cyclic evolution of this oscillator. The general Hamiltonian for this oscillator can be written down in terms of a

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pair of quantum operators which correspond to the classical conjugate variables (q, p) of phase space, as well as a set of real time-dependent periodic parameters.

Although work has been done recently related to an exact solution approach to such questions [8, 9], the approach here is to begin from the fibre bundle setting [10] and then develop an approximate approach which might be amenable to extension to more field theoretic examples. If one regards the space of normalized states as a fibre bundle over the space of rays, then this bundle has a natural connection. This connection permits a comparison of the phases of states on two neighboring rays.

Matrix elements of the Hamiltonian between squeezed states generate an effective Hamiltonian in an extended space such that it breaks down into two contributions. One corresponds to the classical part and can be thought to generate the motion of the expectation values. In addition, there is a term which will be referred to as a fluctuation term. This part describes the evolution of the quantum fluctuations. Squeezed states will be introduced and their dynamics will be reviewed. The time evolution of a squeezed state can be described by an effective Hamiltonian on an extended phase space. There is a related differential equation which describes the phase change and this will be developed here. Clearly, the periodic and quasiperiodic solution of the effective Hamiltonian corresponds to the cyclic and quasicyclic evolution of a given squeezed state. Thus cyclic as well as quasicyclic evolution of the squeezed state can be studied. The quantal phase which is calculated can be interpreted as an area in the extended phase space which is swept out by a periodic orbit in both the adiabatic and nonadiabatic cases.

2. The classical oscillator

The classical Hamiltonian for the generalized harmonic oscillator with one degree of freedom is given by

$$H = \frac{1}{2}(a(t)q^2 + b(t)p^2 + 2c(t)qp). \tag{1}$$

The parameters in the Hamiltonian are the quantities a, b and c. When these are held fixed, (1) describes oscillatory motion around elliptic contours in the phase plane provided that $ab > c^2$. From (2.1), Hamilton's equations have the form

$$\dot{q} = \frac{\partial H}{\partial p} = cq + bp, \qquad \dot{p} = -\frac{\partial H}{\partial q} = -aq - cp. \tag{2}$$

Solving (2) for p, we obtain that

$$p = \frac{1}{b}(\dot{q} - cq). \tag{3}$$

A second order equation in q can be obtained by differentiating $\dot{q}$ in (2) once more,

$$\ddot{q} = \dot{c}q + c\dot{q} + \dot{b}p + b\dot{p} = \dot{c}q + c^2 q + \dot{b}p - abq. \tag{4}$$

Therefore, a second order equation for q in terms of a, b and c is

$$\ddot{q} - \frac{\dot{b}}{b}\dot{q} + \left[ab - c^2 + \frac{1}{b}(\dot{b}c - \dot{c}b)\right] q = 0. \tag{5}$$

This equation can be solved in a straightforward way when a, b, c are held fixed, since it reduces to

$$\ddot{q} + (ab - c^2)q = 0. \tag{6}$$

When the contours are closed, we can define $\omega^2 = ab - c^2 > 0$, and a solution for (2.5) is given by

$$q(t) = A\cos(\varphi(t)), \qquad \varphi(t) = \omega t + \varphi_0, \tag{7}$$

and A is an arbitrary constant to be determined. Since $\dot{q}(t) = -A\omega \sin(\varphi(t))$, the variable $p(t)$ is determined from (2) to be

$$p(t) = -\frac{A}{b}(\omega \sin\varphi(t) + c\cos\varphi(t)). \tag{8}$$

If a, b and c are held fixed, by differentiating H with respect to t and replacing $(\dot{q}, \dot{p})$ from (2), it is found that $\dot{H} = 0$. In fact, putting (7) and (8) into (1), it is found to simplify to

$$H = \frac{\omega^2}{2b}A^2. \tag{9}$$

The constant A is related to the total energy of the oscillator. Thus, a new coordinate can be introduced in the form $I = H/\omega$. Since $\dot{H} = 0$, it follows that $\dot{I} = 0$ is the equation of motion for I. In fact, A can now be determined in terms of I to be $A = \sqrt{2bI/\omega}$. If $\mathbf{R}$ represents the parameters $\{a, b, c\}$, the variables (q, p) are then given by

$$q(I, \varphi; \mathbf{R}) = \left(\frac{2bI}{\omega}\right)^{1/2} \cos\varphi(t), \tag{10}$$

$$p(I, \varphi; \mathbf{R}) = -\left(\frac{2bI}{\omega}\right)^{1/2} \left(\frac{c}{b}\cos\varphi(t) + \frac{\omega}{b}\sin\varphi(t)\right). \tag{11}$$

From (10) and (11), the coordinate (φ, I) can be obtained in terms of the pair (q, p) as follows

$$\varphi(p, q; \mathbf{R}) = \tan^{-1}\left[-\left(\frac{bp}{\omega q} + \frac{c}{\omega} \right) \right], \qquad (12)$$

$$I(p, q; \mathbf{R}) = \frac{1}{2b\omega}\left(q^2\omega^2 + (pb + cq)^2 \right). \qquad (13)$$

Differentiating either of (φ, I) in (12), (13) with respect to t and using $(\dot q, \dot p)$ from (2), the equations of motion $\dot\varphi = \omega$ and $\dot I = 0$ are again generated.

3. Quantum oscillator and squeezed states

The results which will be obtained here will be found by using squeezed state wavefunctions, and so some information relevant to some of their properties will be introduced [11, 12]. A squeezed state is defined by the ordinary harmonic oscillator displacement operator acting on the vacuum state

$$\begin{aligned} |\Psi\rangle &= \exp(\alpha \hat a^\dagger - \alpha^* \hat a)|\phi\rangle, \\ |\phi\rangle &= \exp(\tfrac{1}{2}(\beta \hat a^{\dagger 2} - \beta^* \hat a^2))|0\rangle, \end{aligned} \qquad (14)$$

where $\hat a$ and $\hat a^\dagger$ are boson creation and annihilation operators, which satisfy the canonical commutation relation $[\hat a, \hat a^\dagger] = 1$. Introducing the definitions $\hat p = i\sqrt{\hbar/2}(\hat a^\dagger - \hat a)$ and $\hat q = \sqrt{\hbar/2}(\hat a^\dagger + \hat a)$, we have $p = \langle \Psi, t|\hat p|\Psi, t\rangle$ and $q = \langle \Psi, t|\hat q|\Psi, t\rangle$. The remaining matrix elements which will be required take the form

$$\langle \Psi, t|(\hat p - p)^2|\Psi, t\rangle = \hbar\left(\frac{1}{4G} + 4\Pi^2 G \right),$$
$$\langle \Psi, t|(\hat q - q)^2|\Psi, t\rangle = \hbar G, \qquad (15)$$

$$\langle \Psi, t|\hat q\hat p + \hat p\hat q|\Psi, t\rangle = 2qp + 4G\Pi. \qquad (16)$$

The canonical pair (G, Π) has been introduced by Jackiw and Kerman [13] to account for quantum fluctuations. The Hamiltonian of the generalized harmonic oscillator in terms of the operators $(\hat q, \hat p)$ then takes the form,

$$\hat H = \frac{1}{2}\Big(a(t)\hat q^2 + b(t)\hat p^2 + c(t)(\hat q\hat p + \hat p\hat q) \Big). \qquad (17)$$

The pair $(\hat p, \hat q)$ in $\hat H$ in (17) are the operators corresponding to the expectation values (p, q), which can be regarded as conjugate variables in a classical phase space. The functions which are not operators but parameters are denoted $a(t)$, $b(t)$, $c(t)$ and will be periodic functions of time with common period T. The matrix element of $\hat H$ when evaluated between states $|\Psi, t\rangle$ and its conjugate is given by

$$\begin{aligned} &\langle \Psi, t|\hat H|\Psi, t\rangle = \\ &= \frac{1}{2}\langle \Psi, t|a(t)\hat q^2 + b(t)\hat p^2 + c(t)(\hat q\hat p + +\hat p\hat q)|\Psi, t\rangle \\ &= \frac{1}{2}\Big[a(t)(q^2 + \hbar G) + b(t)\Big(p^2 + \frac{\hbar}{4G} + 4\hbar\Pi^2 G \Big) + \\ &+ c(t)(2qp + 4\hbar G\Pi) \Big] = \frac{1}{2}\Big(a(t)q^2 + b(t)p^2 + 2c(t)qp \Big) + \\ &+ \frac{\hbar}{2}\left[a(t)G + b(t)\left(\frac{1}{4G} + 4\Pi^2 G \right) + 2c(t)G\Pi \right]. \end{aligned} \qquad (18)$$

Thus by applying $\hat H$ to the squeezed states leads to an effective Hamiltonian over the extended phase space $(q, p; G, \Pi)$ of the form

$$H_e(q, p; G, \Pi, t) = H_c(q, p, t) + \hbar H_f(G, \Pi, t). \qquad (19)$$

The first term is given by

$$H_c = \frac{1}{2}\Big(a(t)q^2 + b(t)p^2 + 2c(t)qp \Big), \qquad (20)$$

and matches the classical form of the Hamiltonian given in (2.1). The second term is given by

$$H_f = \frac{1}{2}\Big(a(t)G + b(t)\Big(\frac{1}{4G} + 4\Pi^2 G \Big) + 4c(t)G\Pi \Big). \qquad (21)$$

The term H_f can be thought of as describing the evolution of the quantum fluctuations. The dynamical equations for the expectation values of the operators $\hat p$, $\hat q$ and the variables pertaining to the quantum fluctuations can be obtained from the time-dependent variational principle in the form

$$\begin{aligned} \dot q &= \frac{\partial H_e}{\partial p}, \qquad \dot p = -\frac{\partial H_e}{\partial q} \\ \hbar\dot G &= \frac{\partial H_e}{\partial \Pi}, \quad \hbar\dot\Pi = -\frac{\partial H_e}{\partial G}. \end{aligned} \qquad (22)$$

Differentiation with respect to time is indicated by a dot. Using (19) in (22), the following system of four equations results

$$\dot{q} = bp + cq,$$
$$\dot{p} = -aq - cp,$$
$$\dot{G} = 4b\Pi G + 2cG,$$
$$\dot{\Pi} = -\frac{1}{2}a + \frac{1}{2}b\left(\frac{1}{4G^2} - 4\Pi^2\right) - 2c\Pi. \tag{23}$$

The first pair of equations in (10) coincides exactly with the equations in (2). The remaining pair in (19) describe quantum fluctuations. Thus these equations describe the motion of the expectation values as well as the evolution of the quantum fluctuation terms.

4. Quantum phases

The mathematical language of fibre bundles provides a powerful tool for the study of geometric phases. The splitting of the total phase into geometrical and dynamical parts is determined by a choice of connection, and a natural connection is the scalar product which provides the standard Berry phase. Let us consider a class of evolution which transforms a state $|\Psi, t\rangle$ back onto its own fibre

$$|\Psi_\lambda, t\rangle = e^{i\lambda(t)}|\Psi, t\rangle \tag{24}$$

under a quantum evolution which is determined by a Hamiltonian operator $\hat{H}$. The time-dependent variational principle leaves an ambiguity in terms of the time-dependent phase $\lambda(t)$. The operator $(i\hbar\partial/\partial t - \hat{H})$ can be applied to the state (24) to give

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{H}\right)|\Psi_\lambda, t\rangle =$$
$$= -\hbar\dot{\lambda}(t)e^{i\lambda}|\Psi, t\rangle + i\hbar e^{i\lambda}\frac{\partial}{\partial t}|\Psi, t\rangle - e^{i\lambda}\hat{H}|\Psi, t\rangle. \tag{25}$$

Contracting (25) on the left with $|\hat{\Psi}, t\rangle$ and invoking the Schrödinger equation to equate this to zero, an expression for $\dot{\lambda}(t)$ is obtained

$$\dot{\lambda}(t) = \langle\Psi, t|i\frac{\partial}{\partial t}|\Psi, t\rangle - \frac{1}{\hbar}\langle\Psi, t|\hat{H}|\Psi, t\rangle. \tag{26}$$

This phase will be well-defined for general nonadiabatic and noncyclic evolution of a squeezed state wavefunction. It can be thought to represent a specific phase change generated during a time evolution of the squeezed state. It is clear from (26) that the phase consists of two parts. The second term measures the time evolution and will be referred to as the dynamical phase given by

$$\lambda_D(t) = -\frac{1}{\hbar}\int_0^t \langle\Psi, s|\hat{H}|\Psi, s\rangle\, ds. \tag{27}$$

The term which remains in the first part of (26) can be viewed as a difference between the total phase and the dynamical phase, which should be real and positive. It is usually referred to as the geometric phase since it corresponds with the Aharonov-Anandan phase for the case in which the evolution is cyclic.

To obtain a useful formula for the geometric phase, it is best to proceed as follows. To work out the first term in $\dot{\lambda}$ in (26), we begin with an expression for the squeezed state $|\Psi, t\rangle$, that is, we begin with the following Gaussian-type state

$$|\Psi, t\rangle = \frac{1}{(2G)^{1/4}}\exp\left[\frac{i}{\hbar}\left(p\hat{q} - q\hat{p}\right)\right]\times$$
$$\times\exp\left[\frac{1}{2\hbar}\left(1 - \frac{1}{2G} + 2i\Pi\right)\hat{q}^2\right]|0\rangle. \tag{28}$$

Differentiating the state (28) with respect to t, there results

$$\frac{\partial}{\partial t}|\Psi, t\rangle = -\frac{\dot{G}}{4G}|\Psi, t\rangle + \frac{i}{\hbar}(\dot{p}\hat{q} - \dot{q}\hat{p})|\Psi, t\rangle$$
$$+\frac{1}{2\hbar}\left(\frac{\dot{G}}{2G^2} + 2i\dot{\Pi}\right)\exp\left[\frac{i}{\hbar}(p\hat{q} - q\hat{p})\right]\times$$
$$\times\hat{q}^2\exp\left[-\frac{i}{\hbar}(p\hat{q} - q\hat{p})\right]|\Psi, t\rangle. \tag{29}$$

Applying the Baker-Hausdorff lemma to the last term in (29), it becomes

$$\frac{\partial}{\partial t}|\Psi, t\rangle = -\frac{\dot{G}}{4G}|\Psi, t\rangle + \frac{i}{\hbar}(\dot{p}\hat{q} - \dot{q}\hat{p})|\Psi, t\rangle +$$
$$+\frac{1}{2\hbar}\left(\frac{\dot{G}}{2G^2} + 2i\dot{\Pi}\right)(\hat{q} - q)^2|\Psi, t\rangle. \tag{30}$$

From (30), the required matrix element follows

$$\langle\Psi, t|\frac{\partial}{\partial t}|\Psi, t\rangle = -\frac{\dot{G}}{4G} + \frac{i}{\hbar}(\dot{p}q - \dot{q}p) + \tag{31}$$
$$+\frac{1}{2\hbar}\left(\frac{\dot{G}}{2G^2} + 2i\dot{\Pi}\right)\hbar G = \frac{i}{\hbar}(\dot{p}q - \dot{q}p) + i\dot{\Pi}G.$$

Multiply both sides of this by i and then integrating both sides with respect to time, the geometric phase is found to be given by

$$\lambda_G(t) = \int_0^t\left(\frac{1}{\hbar}(\dot{q}p - \dot{p}q) - \dot{\Pi}G\right)ds. \tag{32}$$

Therefore, both the evolution of the expectation value (q, p) as well as the evolution of the quantum fluctuation terms (G, Π) contribute to the geometric phase. It might be remarked that the former term depends on $\hbar$, while the contribution due to the quantum fluctuation terms is actually independent of $\hbar$.

5. Calculation of the phases for the generalized oscillator

The phases λ_D and λ_G will be calculated for the generalized harmonic oscillator under the following choices of the periodic parameters

$$\begin{aligned} a(t) &= 1 + \epsilon \cos(\omega t), \\ b(t) &= 1 - \epsilon \cos(\omega t), \\ c(t) &= \epsilon \sin(\omega t). \end{aligned} \tag{33}$$

The elliptic case is the one for which $a(t)b(t) > c^2(t)$. In terms of ϵ, this takes the form $(1 + \epsilon \cos(\omega t))(1 - \epsilon \cos(\omega t)) > \epsilon^2 \sin^2(\omega t)$, which implies that $\epsilon < 1$. For one period of evolution, the parameters follow a trajectory such that $a + b = 2$, $(a-1)^2 + c^2 = \epsilon^2$.

Now an approximate solution in powers of ϵ to the set of equations in system (23) will be determined by expanding the variables in powers of ϵ. To begin with, the first pair of equations in (23) are coupled and can be solved together by expanding p and q in the form

$$q(t) = \sum_{k=0}^{\infty} q_k(t)\epsilon^k, \quad p(t) = \sum_{k=0}^{\infty} p_k(t)\epsilon^k. \tag{34}$$

Substituting (34) into (23), we obtain the following pair of coupled equations

$$\begin{aligned} &\dot{q}_0 + \dot{q}_1\epsilon + \dot{q}_2\epsilon^2 + \cdots = p_0 + (p_1 - p_0\cos(\omega t) + \\ &+ q_0 \sin(\omega t))\epsilon + (p_2 - p_1\cos(\omega t) + q_1\sin(\omega t))\epsilon^2 + \cdots, \\ &\dot{p}_0 + \dot{p}_1\epsilon + \dot{p}_2\epsilon^2 + \cdots = -q_0 - (q_1 + q_0\cos(\omega t) + \\ &+ p_0\sin(\omega t))\epsilon - (q_2 + q_1\cos(\omega t) + p_1\sin(\omega t))\epsilon^2 + \cdots \end{aligned} \tag{35}$$

These will be solved only to order ϵ. Equating coefficients of the powers of ϵ on both sides of the expressions in (35), a coupled system is obtained which can be solved order by order. The pair at order zero has the solution

$$q_0 = \sin t, \qquad p_0 = \cos t. \tag{36}$$

Substituting this solution in the pair from first order and setting the integration constants to zero, the first order contributions are found next,

$$\begin{aligned} q_1(t) &= \frac{\omega+1}{\omega(\omega+2)}\Big(\cos\Big((\omega+1)t\Big) - \sin\Big((\omega+1)t\Big), \\ p_1(t) &= \Big(\frac{1}{\omega+1} - \frac{1}{\omega(\omega+2)}\Big)\cos\Big((\omega+1)t\Big) - \\ &- \frac{\sin\Big((\omega+1)t\Big)}{\omega(\omega+2)}. \end{aligned} \tag{37}$$

When $\epsilon = 0$, the system reduces to the harmonic oscillator form, and the effective Hamiltonian can be written in terms of the action angle variables defined by

$$I = \frac{1}{2\pi}\oint p\,dq, \qquad J = \frac{1}{2\pi}\oint \Pi\,dG. \tag{38}$$

The phase space contains periodic orbits of period $T_0 = 2\pi$ in (q, p) and period $T_0/2$ in (G, Π). Moreover, $q = p = 0$ and $G = 1/2$, $\Pi = 0$ specify the unique fixed point of the system. The phases λ_D and λ_G will be found by expanding about this point, and the subscript p will be used to denote this.

Now when $\epsilon \neq 0$, the (G, Π) variables are decoupled from the (q, p) variables. Taking expressions for the fluctuation variables (G, Π) of the form

$$G = \sum_{k=0}^{\infty} g_k(t)\epsilon^k, \qquad \Pi = \sum_{k=1}^{\infty} f_k(t)\epsilon^k, \tag{39}$$

and substituting (39) into (23), the following pair of equations result

$$\begin{aligned} &\dot{g}_0 + \dot{g}_1\epsilon + \dot{g}_2\epsilon^2 + \cdots = (4f_1g_0 + 2g_0\sin(\omega t))\epsilon + \\ &+ 2(-2g_0f_1\cos(\omega t) + 2f_1g_1 + 2f_2g_0 + g_1\sin(\omega t))\epsilon^2 + \cdots, \end{aligned}$$

$$\begin{aligned} \dot{f}_1\epsilon + \dot{f}_2\epsilon^2 &= \frac{1}{2}\Big(-1 + \frac{1}{4g_0^2}\Big) - \\ &- \frac{1}{2}\Big(\cos(\omega t) + \frac{g_1}{2g_0^3} + \frac{\cos(\omega t)}{4g_0^2}\Big)\epsilon \\ &+ \frac{1}{4g_0^4}\Big(g_0g_2 - \frac{1}{2}g_1^2 - 8f_1^2g_0^4 + \\ &+ g_0g_1\cos(\omega t) - 8f_1g_0^4\sin(\omega t)\Big)\epsilon^2 + \cdots. \end{aligned}$$

This system will be solved to order ϵ^2. Equating powers of ϵ on both sides of these equations, it is found that $g_0 = 1/2$ and the coupled system to order ϵ reduces to

$$\dot{g}_1 = 2f_1 + \sin(\omega t), \qquad \dot{f}_1 = -2g_1 - \cos(\omega t). \tag{40}$$

Differentiating $\dot{g}_1$ in (40) and substituting $\dot{f}_1$, the equation $\ddot{g}_1 = (\omega - 2)\cos(\omega t) - 4g_1$ is obtained. With the resulting integration constants set to zero, the solutions for g_1 and f_1 of the coupled pair (40) are given by

$$g_1 = -\frac{\cos(\omega t)}{\omega + 2}, \qquad f_1 = -\frac{\sin(\omega t)}{\omega + 2}.$$

The solutions for the order ϵ^2 terms f_2 g_2 are obtained next, and we simply give the results

$$f_2 = -\frac{\sin(2\omega t)}{(\omega + 2)^2}, \qquad g_2 = 0. \tag{41}$$

Therefore, to order ϵ^2, the quantities G_p and Π_p are given by

$$G_p = \frac{1}{2} - \frac{\cos(\omega t)}{\omega + 2}\epsilon,$$

$$\Pi_p = -\frac{\sin(\omega t)}{\omega + 2}\epsilon - \frac{\sin(2\omega t)}{(\omega + 2)^2}\epsilon^2. \tag{42}$$

Thus, a unique solution for the effective Hamiltonian (19) expanded about the fixed point $(q = 0, p = 0, G_p(t), \Pi_p(t))$ of period T for the system (23) has been generated. This is found from a cyclic squeezed state whose $\hat{q}$, $\hat{p}$ expectation values remain at the zero point, but the fluctuations which are related to the width of the wave packet vary with time. For such a cyclic solution, the phases λ_G and λ_D can be obtained by means of (27) and (32). The phase change is evaluated by integrating over a single period to give

$$\lambda_D = -\int_0^T H_f\Big(G_p(t), \Pi_p(t)\Big)\, dt = \left(-\frac{1}{2} + \frac{\omega + 1}{(\omega + 2)^2}\epsilon^2\right)\frac{2\pi}{\omega},$$

$$\lambda_G = -\int_0^T \dot{\Pi}_p G_p \, dt = \int_0^T \Pi_p \dot{G}_p \, dt = -\frac{\pi}{(\omega + 2)^2}\epsilon^2. \tag{43}$$

It is found that there are no contributions to order ϵ^3, and these give the total contribution to this order for each of the phases λ_D and λ_G.

The geometric phase is independent of $\hbar$, as the expectation values of the state stay fixed during a cyclic evolution. The expression for λ_G in (43) gives a beautiful result which combines the Berry phase in the adiabatic limit in which $\omega \to 0$ with the Aharonov-Bohm phase relevant to the nonadiabatic case. The geometric nature of the phase resides in the fact that it is related to a particular area encircled by the periodic orbit in the fluctuation domain.

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