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Miguel Rodriguez
The University of Texas Rio Grande Valley

Jing Li

Zhijun Qiao
The University of Texas Rio Grande Valley, zhijun.qiao@utrgv.edu

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Article

Negative Order KdV Equation with No Solitary Traveling Waves

Miguel Rodriguez [1], **Jing Li** [2] and **Zhijun Qiao** [1,*]

[1] School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, Edinburg, TX 78539, USA; miguel.a.rodriguez01@utrgv.edu
[2] Interdisciplinary Research Institute, Faculty of Science, Beijing University of Technology, Beijing 100124, China; leejing@bjut.edu.cn
* Correspondence: zhijun.qiao@utrgv.edu

Abstract: We consider the negative order KdV (NKdV) hierarchy which generates nonlinear integrable equations. Selecting different seed functions produces different evolution equations. We apply the traveling wave setting to study one of these equations. Assuming a particular type of solution leads us to solve a cubic equation. New solutions are found, but none of these are classical solitary traveling wave solutions.

Keywords: integrable systems; KdV hierarchy; traveling waves

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1. Introduction

The study of water waves is an important area of research in mathematical physics. Several integrable systems in the form of nonlinear partial differential equations are used to describe their motion. There has been an effort to discover new solutions to these systems, particularly soliton solutions. The study of solitons and integrable systems has many applications beyond fluid mechanics, including nonlinear optics, classical and quantum field theories, etc. Recently, the focus has been placed on non-smooth soliton solutions such as peakons and cuspons. It is expected that models with these solutions can explain non-linear effects such as wave breaking.

Solitary waves were first described by John Scott Russell in his 1838 report [1]. While riding on horseback by the Union Canal in Scotland, he witnessed " ... a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed". He followed this "Wave of Translation" for approximately two miles, eventually losing it in the windings of the channel. Later, in laboratory experiments, Russell replicated this phenomenon. He confirmed the existence of solitary waves, described as long, shallow, water waves of permanent form. This was controversial since the current mathematical theories did not support their existence [2].

The issue was solved when Korteweg and de Vries derived a new equation [3]. The Korteweg-de Vries (KdV) equation is given by

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

where u is the dependent variable, t is the temporal variable, x is the spatial variable, and subscripts denote partial derivatives. This nonlinear evolution equation governs the one-dimensional motion of small amplitude, surface gravity waves propagating in shallow water. It was known to have the solitary wave solution

$$u(x,t) = 2k^2\operatorname{sech}^2\left(k(x - 4k^2t - x_0)\right) \tag{2}$$

where k, and x_0 are constants [2]. Zabusky and Kruskal discovered that these solutions interact elastically with each other, thereby calling them solitons [4]. These solitons retain their shape and velocity after passing through each other, differing only by a phase shift.

A new development came when Miura gave the transformation

$$u = -(v^2 + v_x) \tag{3}$$

which produces a solution to the KdV equation from a solution to the modified KdV (mKdV) equation [5]. Miura's transformation helped in proving that the KdV equation has an infinite number of conservation laws. It also motivated the development of the Inverse Scattering Transform by Gardner, Greene, Kruskal, and Miura, which could solve the inital-value problem of the KdV equation [6]. The Inverse Scattering Method would linearize the KdV equation by relating it to the time-independent Schrodinger scattering problem

$$L\psi = \psi_{xx} + u\psi = \lambda\psi \tag{4}$$

which is well known from quantum mechanics [2]. Here, the eigenvalues and the eigenfunctions produce the scattering data. Then, the inverse scattering problem reconstructs the potential u, which is also the solution to the KdV equation.

Lax later gave a general method to associate nonlinear evolution equations with linear operators, where the eigenvalues of the linear operator correspond to the integrals of the evolution equation [7]. Consider the linear system

$$L\psi = \lambda\psi \tag{5}$$

$$\psi_t = M\psi \tag{6}$$

where L is the spectral problem operator, and M is the time evolution operator. The compatibility condition for this system is given by Lax's equation

$$L_t + [L, M] = 0 \tag{7}$$

where $[L, M]$ is the commutator of L and M. Taking

$$L = \partial^2 + u \tag{8}$$

$$M = -4\partial^3 - 3(\partial u + u\partial) \tag{9}$$

Equation (7) becomes the KdV equation, where L and M are called its Lax pair. In fact, many evolution equations can arise from the compatibility condition (7) by guessing the correct operators.

Following this discovery, Lax used a recursion procedure to introduce the KdV hierarchy [8]. The positive order KdV hierarchy includes as its member the known KdV equation. The negative order KdV (NKdV) hierarchy includes the NKdV equation

$$v_t = w_x \tag{10}$$

$$w_{xxx} + 4vw_x + 2v_x w = 0 \tag{11}$$

first given by Verosky [9]. Around the same time, the popular Camassa-Holm (CH) equation given by

$$m_t + m_x u + 2mu_x = 0 \tag{12}$$

$$m = u - u_{xx} \tag{13}$$

was derived by Roberto Camassa and Darryl Holm [10]. The CH equation has attracted a lot of attention since it contains peaked soliton (peakon) solutions, which are solitons with a discontinuous first derivative at the peak. It has already been verified that the well-known CH equation possessing peakons is actually in the negative order of integrable hierarchy [11] where the CH equation was shown to have the algebro-geometric solutions on a symplectic submanifold. Other similar integrable peakon models, including the DP equation and the FORQ equation, are in fact also in the negative order of integrable hierarchy [12–14]. Therefore, the study of negative order models from an integrable hierarchy becomes more and more important in seeking peakon and weak solutions.

Those CH peakon solutions develop singularities in finite time, capturing the features of breaking waves [15]. It was then pointed out by Fuchssteiner that the NKdV equation is actually gauge-equivalent to the CH equation [16]. Furthermore, Qiao and Li unified the Lax representations for the positive and negative KdV hierarchies [17]. They also studied the first member in the NKdV hierarchy and provided all possible traveling wave solutions, including soliton, kink wave, and periodic solutions. Soon after, Qiao and Fan gave multi-soliton and multi-kink wave solutions to NKdV equations through bilinear Backlund transformations [18]. Tracking the negative order models in the integrable hierarchy may be traced back to earlier work through using the commutator structure and inverse of Lenard operators [19–22].

There were few developments in the following years. For instance, Wazwaz [23] generalized negative-order KdV equations [18] to (3+1) dimensions via the KdV recursion operator. Additionally, Chen deduced quasi-periodic solutions for the NKdV hierarchy by using backward Neumann systems [24].

Little work has been done on the NKdV hierarchy produced through other seed functions since then. We know that the NKdV equation contains soliton solutions, including the interesting peakons. It remains to be seen whether other equations in the NKdV hierarchy possess solitary traveling wave solutions. In this paper we explore this possibility.

We will show how to obtain the KdV hierarchy starting from the Schrodinger-KdV spectral problem. First, we find Lenard's operators and their inverse. Then, we construct the recursion operators for the positive and negative order hierarchies. Lenard's sequence is defined by powers of the recursion operator acting on the chosen seed functions. Finally, the entire KdV hierarchy is given, which includes the NKdV hierarchy. Depending on our choice of seed function, the hierarchy produces different nonlinear integrable equations. We show all the possible seed functions for the NKdV hierarchy, as well as the equations these produce with the setting $k = -1$. We attempt to solve one of these equations given by

$$\left(-\frac{u_{xx}}{u} \right)_{t_{-1}} = g(t_n)\left(2uu_x\partial^{-1}u^{-2} + 1\right) \tag{14}$$

which hadn't been studied in detail. First we apply a substitution to eliminate the integral in the equation. Then we apply the traveling wave setting to convert the partial differential equation into an ordinary differential equation. Using some algebra and calculus we obtain the new equation

$$3cu'u'' - cuu''' - 2Auu' + u^2 = 0. \tag{15}$$

We assume solutions of the form

$$u(\xi) = ae^{\lambda\xi}. \tag{16}$$

Plugging in u, its derivatives and reducing gives a cubic equation in λ. So, if we select a λ that solves the cubic equation, then (16) will be a solution to (15). The methods presented by Smith in [25] are used to solve the cubic equation. Formulas to find λ are derived for the several different cases. In the cases where λ is real, the solution (16) is just an exponential function. In the cases where λ is complex, we can separate the solution (16) into its real and imaginary parts. This leads to some interesting solutions, however, none of these are classical solitary traveling waves. Plots of the solutions for some special cases are created using the Mathematica software. We also considered solutions of the form

$$u(\xi) = ae^{\alpha\xi} + be^{\beta\xi}. \tag{17}$$

After plugging in u, its derivatives, and reducing leads to three separate cubic equations. The only way to solve all of them is by setting $\alpha = \beta$, which results in the same solution (16) considered previously. Other solution types considered are $u(\xi) = a\,\mathrm{sech}(b\xi)$, and $u(\xi) = a\tanh(b\xi)$. Direct substitution shows that they cannot be solutions. Lastly, we considered solutions with the peakon form

$$u(\xi) = ae^{-\lambda|\xi|}. \tag{18}$$

Pluggin in u, its derivatives, and reducing leads to an undefined equation. To summarize, after considering the different solution types we did not find any classical solitary traveling wave solutions.

2. Materials and Methods

2.1. The KdV Hierarchy

The Schrodinger-KdV spectral problem is given by

$$L\psi = \psi_{xx} + v\psi = \lambda\psi \tag{19}$$

where λ is an eigenvalue, ψ is the eigenfunction, and v is a potential function. To produce the KdV hierarchy, we must find the operators K and J which satisfy the Lenard operator relation

$$K \cdot \nabla\lambda = \lambda J \cdot \nabla\lambda \tag{20}$$

where $\nabla\lambda = \psi^2$ is the functional gradient of the spectral problem with respect to v. First, we calculate

$$(\nabla\lambda)_x = 2\psi\psi_x \tag{21}$$

$$(\nabla\lambda)_{xx} = 2(\psi_x^2 + (\lambda - v)\psi^2) \tag{22}$$

$$(\nabla\lambda)_{xxx} = 2\big(2(\lambda - v)(\nabla\lambda)_x - v_x\nabla\lambda\big). \tag{23}$$

It follows from (23) that

$$\frac{1}{4}(\nabla\lambda)_{xxx} + v(\nabla\lambda)_x + \frac{1}{2}v_x\nabla\lambda = \lambda(\nabla\lambda)_x. \tag{24}$$

This can be written as

$$\left(\frac{1}{4}\partial^3 + \frac{1}{2}(v\partial + \partial v)\right) \cdot \nabla\lambda = \lambda\partial \cdot \nabla\lambda \tag{25}$$

where ∂ is the differential operator with respect to x. Comparing (20) and (25), we conclude that Lenard's operators are given by

$$K = \frac{1}{4}\partial^3 + \frac{1}{2}(v\partial + \partial v) \tag{26}$$

$$J = \partial. \tag{27}$$

Introducing the setting $v = -\frac{u_{xx}}{u}$, turns K into the product form

$$K = \frac{1}{4}u^{-2}\partial u^2\partial u^2\partial u^{-2} \tag{28}$$

Ref. [17]. Now, the inverse operators are given by

$$K^{-1} = 4u^2\partial^{-1}u^{-2}\partial^{-1}u^{-2}\partial^{-1}u^2 \tag{29}$$

$$J^{-1} = \partial^{-1}. \tag{30}$$

So, the recursion operator and its inverse are given by

$$\mathcal{L} = J^{-1}K = \frac{1}{4}\partial^{-1}u^{-2}\partial u^2\partial u^2\partial u^{-2} \tag{31}$$

$$\mathcal{L}^{-1} = K^{-1}J = 4u^2\partial^{-1}u^{-2}\partial^{-1}u^{-2}\partial^{-1}u^2\partial. \tag{32}$$

Next, Lenard's sequence is defined as

$$G_j = \begin{cases} \mathcal{L}^j \cdot G_0, & j \geq 0 \\ \mathcal{L}^{j+1} \cdot G_{-1}, & j < 0 \end{cases} \tag{33}$$

where $j \in \mathbb{Z}$, and the seed functions are given by

$$G_0 \in Ker(J) = \{G \in C^\infty(\mathbb{R}) | JG = 0\} \tag{34}$$

$$G_{-1} \in Ker(K) = \{G \in C^\infty(\mathbb{R}) | KG = 0\}. \tag{35}$$

Finally, the entire KdV hierarchy is given by

$$v_{t_k} = JG_k = KG_{k-1} \tag{36}$$

where t is the time variable, and $k \in \mathbb{Z}$. Using $k \geq 0$ gives the positive order KdV hierarchy, while $k < 0$ gives the negative order KdV (NKdV) hierarchy.

Depending on our choice of seed function G_{-1}, we can obtain different equations in the NKdV hierarchy. Since $KG_{-1} = 0$, then $G_{-1} = K^{-1}0$. This results in three possible seed functions:

$$G^1_{-1} = f(t_n)u^2 \tag{37}$$

$$G^2_{-1} = g(t_n)u^2\partial^{-1}u^{-2} \tag{38}$$

$$G^3_{-1} = h(t_n)u^2\partial^{-1}u^{-2}\partial^{-1}u^{-2} \tag{39}$$

where $f(t_n), g(t_n), h(t_n)$ are functions of t but independent of x, and ∂^{-1} represents an integral with respect to x. Therefore, selecting $k = -1$ in the hierarchy (36) produces three distinct equations:

$$\left(-\frac{u_{xx}}{u}\right)_{t_{-1}} = 2f(t_n)uu_x \tag{40}$$

$$\left(-\frac{u_{xx}}{u}\right)_{t_{-1}} = g(t_n)\left(2uu_x\partial^{-1}u^{-2} + 1\right) \tag{41}$$

$$\left(-\frac{u_{xx}}{u}\right)_{t_{-1}} = h(t_n)\left(2uu_x\partial^{-1}u^{-2}\partial^{-1}u^{-2} + \partial^{-1}u^{-2}\right). \tag{42}$$

These equations have the Lax representation

$$L_{t_{-1}} = [V^l_{-1}, L] = \left[\left(\frac{1}{4}G^l_{-1,x} - \frac{1}{2}G^l_{-1}\partial\right)L^{-1}, L\right] \tag{43}$$

with

$$L = \partial^2 + v = u^{-1}\partial u^2\partial u^{-1} \tag{44}$$

$$L^{-1} = u\partial^{-1}u^{-2}\partial^{-1}u \tag{45}$$

and $l = 1, 2, 3$. The first Equation (40) has been studied before and solutions were given [17]. We will focus our attention on the second Equation (41) and attempt to find new traveling wave solutions.

2.2. Traveling Wave Setting

With the setting $g(t_n) = 1$, Equation (41) becomes

$$\left(-\frac{u_{xx}}{u}\right)_t = 2uu_x\partial^{-1}u^{-2} + 1. \tag{46}$$

We apply the substitution

$$u^{-2} = w_x \tag{47}$$

to eliminate the integral in (46). We get

$$\left(-\frac{u_{xx}}{u}\right)_t = 2uu_x\partial^{-1}w_x + 1 \tag{48}$$

$$\Rightarrow \left(-\frac{u_{xx}}{u}\right)_t = 2uu_x w + 1 \tag{49}$$

$$\Rightarrow \left(-\frac{u_{xx}}{u}\right)_t = (u^2)_x w + 1 \tag{50}$$

From (47) we have

$$u = w_x^{-\frac{1}{2}}. \tag{51}$$

Substituting into (50) gives

$$-\left(w_x^{\frac{1}{2}}(w_x^{-\frac{1}{2}})_{xx}\right)_t = \left(w_x^{-1}\right)_x w + 1 \tag{52}$$

$$\Rightarrow -\left(w_x^{\frac{1}{2}}(w_x^{-\frac{1}{2}})_{xx}\right)_t = 1 - \frac{ww_{xx}}{(w_x)^2} \tag{53}$$

$$\Rightarrow -\left(w_x^{\frac{1}{2}}(w_x^{-\frac{1}{2}})_{xx}\right)_t = \frac{w_xw_x - ww_{xx}}{(w_x)^2} \tag{54}$$

$$\Rightarrow -\left(w_x^{\frac{1}{2}}(w_x^{-\frac{1}{2}})_{xx}\right)_t = \left(\frac{w}{w_x}\right)_x. \tag{55}$$

Now we introduce the traveling wave setting

$$u(x,t) = u(x - ct) = u(\xi). \tag{56}$$

The goal is to go from a partial differential equation with two variables x and t, into an ordinary differential equation with the variable ξ. The derivatives can be changed by

$$\frac{d}{dt} = \frac{d\xi}{dt}\frac{d}{d\xi} = -c\frac{d}{d\xi} \tag{57}$$

$$\frac{d}{dx} = \frac{d\xi}{dx}\frac{d}{d\xi} = \frac{d}{d\xi}. \tag{58}$$

Applying the traveling wave setting to (55) gives

$$c\left((w')^{\frac{1}{2}}\left((w')^{-\frac{1}{2}}\right)''\right)' = \left(\frac{w}{w'}\right)' \tag{59}$$

where the derivatives are with respect to ξ. We can integrate both sides of (59) to get

$$c(w')^{\frac{1}{2}}\left((w')^{-\frac{1}{2}}\right)'' = \frac{w}{w'} + A \tag{60}$$

where A is an integration constant. Multiplying both sides of (60) by w' gives

$$w + Aw' = c(w')^{\frac{3}{2}}\left((w')^{-\frac{1}{2}}\right)'' \tag{61}$$

$$\Rightarrow w + Aw' = c(w')^{\frac{3}{2}}\left(-\frac{1}{2}(w')^{-\frac{3}{2}}(w'')\right)' \tag{62}$$

$$\Rightarrow w + Aw' = c(w')^{\frac{3}{2}}\left(\frac{3}{4}(w')^{-\frac{5}{2}}(w'')^2 - \frac{1}{2}(w')^{-\frac{3}{2}}(w''')\right) \tag{63}$$

$$\Rightarrow \frac{1}{c}(w + Aw') = \frac{3}{4}(w')^{-1}(w'')^2 - \frac{1}{2}w'''. \tag{64}$$

If we have derivatives of w instead of w itself, we can substitute back to u by using

$$w' = u^{-2}. \tag{65}$$

Differentiating both sides of (64) gives

$$\frac{1}{c}(w' + Aw'') = -\frac{3}{4}(w')^{-2}(w'')^3 + \frac{3}{2}(w')^{-1}w''w''' - \frac{1}{2}w''''. \tag{66}$$

Using (65) we calculate

$$w'' = -2u^{-3}u' \tag{67}$$

$$w''' = 6u^{-4}(u')^2 - 2u^{-3}u'' \tag{68}$$

$$w'''' = -24u^{-5}(u')^3 + 18u^{-4}u'u'' - 2u^{-3}u'''. \tag{69}$$

Plugging these into (66) gives

$$\frac{1}{c}(u^{-2} - 2Au^{-3}u') = -\frac{3}{4}u^4(-8u^{-9}(u')^3)+ \tag{70}$$

$$\frac{3}{2}u^2(-2u^{-3}u')(6u^{-4}(u')^2 - 2u^{-3}u'') - \frac{1}{2}(-24u^{-5}(u')^3 + 18u^{-4}u'u'' - 2u^{-3}u''')$$

$$\Rightarrow \frac{1}{c}(u^{-2} - 2Au^{-3}u') = 6u^{-5}(u')^3 \tag{71}$$

$$-18u^{-5}(u')^3 + 6u^{-4}u'u'' + 12u^{-5}(u')^3 - 9u^{-4}u'u'' + u^{-3}u'''$$

$$\Rightarrow \frac{1}{c}(u^{-2} - 2Au^{-3}u') = -3u^{-4}u'u'' + u^{-3}u''' \tag{72}$$

Multiplying both sides of (72) by u^4 gives

$$\Rightarrow \frac{1}{c}(u^2 - 2Auu') = -3u'u'' + uu''' \tag{73}$$

which can be turned into

$$3cu'u'' - cuu''' - 2Auu' + u^2 = 0. \tag{74}$$

Equation (74) is a new equation for which we will find new solutions.

2.3. Solving Cubic Equations

Finding solutions to the integrable equation will require us to solve a depressed cubic equation, so we review how to find the solutions. Consider the depressed cubic equation

$$x^3 + Px = Q \tag{75}$$

where P and Q are real constants. Since it is a cubic equation, (75) must have three solutions. We begin by assuming solutions of the form

$$x = a - b. \tag{76}$$

Plugging in (76) into (75) gives

$$(a-b)^3 + P(a-b) = Q. \tag{77}$$

Using the identity

$$(a-b)^3 + 3ab(a-b) = a^3 - b^3 \tag{78}$$

and comparing to (77) leads to the system of equations

$$\begin{cases} ab = \frac{P}{3} \\ a^3 - b^3 = Q \end{cases} \tag{79}$$

Ref. [25]. Using substitution, we get

$$\left(\frac{P}{3b}\right)^3 - b^3 = Q. \tag{80}$$

Multiplying by b^3, we have

$$\left(b^3\right)^2 + Qb^3 - \left(\frac{P}{3}\right)^3 = 0 \tag{81}$$

which has the form of a quadratic equation. Any quadratic equation

$$ax^2 + bx + c = 0 \tag{82}$$

where a, b, c are constants can be solved by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{83}$$

Applying Formula (83) to Equation (81), we find that

$$b^3 = -\frac{Q}{2} \pm \sqrt{\Delta} \tag{84}$$

$$\Delta = \left(\frac{P}{3}\right)^3 + \left(\frac{Q}{2}\right)^2 \tag{85}$$

Ref. [25]. Here, Δ is the discriminant, and the nature of the solutions to (75) will depend on the sign of Δ.

Let's first consider the simplest case, $\Delta = 0$. Here, we have

$$b^3 = -\frac{Q}{2}. \tag{86}$$

We can see that

$$b_1 = \sqrt[3]{-\frac{Q}{2}} \tag{87}$$

is a solution for b. Multiplying (87) by any cubic root of 1 would also be a solution for b. The three cubic roots of 1 are given by

$$\omega = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \tag{88}$$

$$\omega^2 = e^{\frac{4\pi i}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \tag{89}$$

$$\omega^3 = e^{2\pi i} = 1. \tag{90}$$

It follows that

$$b_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{Q}{2}} \tag{91}$$

$$b_3 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{Q}{2}} \tag{92}$$

are the other solutions for b. For each b, we find the corresponding a by the relation

$$a = \frac{P}{3b} \tag{93}$$

which is derived from (79). We find that

$$a_1 = \sqrt[3]{\frac{Q}{2}} \tag{94}$$

$$a_2 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[3]{\frac{Q}{2}} \tag{95}$$

$$a_3 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[3]{\frac{Q}{2}}. \tag{96}$$

Finally, the solutions to (75) for the case $\Delta = 0$ are given by

$$x_1 = a_1 - b_1 = \sqrt[3]{4Q} \tag{97}$$

$$x_2 = a_2 - b_2 = -\sqrt[3]{\frac{Q}{2}} \tag{98}$$

$$x_3 = a_3 - b_3 = -\sqrt[3]{\frac{Q}{2}}. \tag{99}$$

They are all real solutions with two of them repeated.

Next, we consider the case $\Delta > 0$. From (84), it follows that one solution for b is

$$b_1 = \sqrt[3]{-\frac{Q}{2} \pm \sqrt{\Delta}}. \tag{100}$$

Multiplying by (88) and (89) gives

$$b_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{Q}{2} \pm \sqrt{\Delta}} \tag{101}$$

$$b_3 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{Q}{2} \pm \sqrt{\Delta}}. \tag{102}$$

From using (93), we find that

$$a_1 = \sqrt[3]{\frac{Q}{2} \pm \sqrt{\Delta}} \tag{103}$$

$$a_2 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[3]{\frac{Q}{2} \pm \sqrt{\Delta}} \tag{104}$$

$$a_3 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[3]{\frac{Q}{2} \pm \sqrt{\Delta}}. \tag{105}$$

Then, by using (76), we find

$$x_1 = \sqrt[3]{\frac{Q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{Q}{2} - \sqrt{\Delta}} \tag{106}$$

$$x_2 = -\frac{1}{2}\left(\sqrt[3]{\frac{Q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{Q}{2} - \sqrt{\Delta}}\right) - \frac{\sqrt{3}}{2}i\left(\sqrt[3]{\frac{Q}{2} + \sqrt{\Delta}} - \sqrt[3]{\frac{Q}{2} - \sqrt{\Delta}}\right) \tag{107}$$

$$x_3 = -\frac{1}{2}\left(\sqrt[3]{\frac{Q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{Q}{2} - \sqrt{\Delta}}\right) + \frac{\sqrt{3}}{2}i\left(\sqrt[3]{\frac{Q}{2} + \sqrt{\Delta}} - \sqrt[3]{\frac{Q}{2} - \sqrt{\Delta}}\right) \tag{108}$$

with Δ given by (85). These are the solutions to Equation (75) in the case $\Delta > 0$. There is one real solution and two complex conjugate solutions.

Lastly, there is the case $\Delta < 0$. Here, we have

$$b^3 = -\frac{Q}{2} \pm i\sqrt{-\Delta}. \tag{109}$$

We recall that a complex number $x + iy$ can be converted to the polar form $re^{i\theta}$, where

$$r = \sqrt{x^2 + y^2} \tag{110}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right). \tag{111}$$

So, we give (109) in the polar form

$$b^3 = \sqrt{\left(\frac{Q}{2}\right)^2 - \Delta} \cdot e^{\pm i \tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)}. \tag{112}$$

Taking the cubic root gives

$$b_1 = \sqrt[6]{\left(\frac{Q}{2}\right)^2 - \Delta} \cdot e^{\pm \frac{i}{3}\tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)}. \tag{113}$$

Multiplying by (88) and (89) gives

$$b_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[6]{\left(\frac{Q}{2}\right)^2 - \Delta} \cdot e^{\pm\frac{i}{3}\tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)} \tag{114}$$

$$b_3 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[6]{\left(\frac{Q}{2}\right)^2 - \Delta} \cdot e^{\pm\frac{i}{3}\tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)}. \tag{115}$$

Using (93), we find that

$$a_1 = -\sqrt[6]{\left(\frac{Q}{2}\right)^2 - \Delta} \cdot e^{\mp\frac{i}{3}\tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)} \tag{116}$$

$$a_2 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[6]{\left(\frac{Q}{2}\right)^2 - \Delta} \cdot e^{\mp\frac{i}{3}\tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)} \tag{117}$$

$$a_3 = \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[6]{\left(\frac{Q}{2}\right)^2 - \Delta} \cdot e^{\mp\frac{i}{3}\tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)}. \tag{118}$$

We use (76) and convert back to Cartesian form through Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{119}$$

Simplifying, we get

$$x_1 = -2\sqrt{\frac{-P}{3}}\cos\left(\frac{1}{3}\tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)\right) \tag{120}$$

$$x_2 = \sqrt{\frac{-P}{3}}\left(\cos\left(\frac{1}{3}\tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)\right) + \sqrt{3}\sin\left(\frac{1}{3}\tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)\right)\right) \tag{121}$$

$$x_3 = \sqrt{\frac{-P}{3}}\left(\cos\left(\frac{1}{3}\tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)\right) - \sqrt{3}\sin\left(\frac{1}{3}\tan^{-1}\left(\frac{2\sqrt{-\Delta}}{Q}\right)\right)\right) \tag{122}$$

where Δ is given by (85). These are the solutions to Equation (75) in the case $\Delta < 0$. All are real solutions.

3. Results

3.1. New Solutions

We consider solutions of the form

$$u(\xi) = ae^{\lambda\xi} \tag{123}$$

where a is a real constant and $\lambda \in \mathbb{C}$ is a number to be determined. First, we calculate

$$u' = a\lambda e^{\lambda\xi} \tag{124}$$

$$u'' = a\lambda^2 e^{\lambda\xi} \tag{125}$$

$$u''' = a\lambda^3 e^{\lambda\xi}. \tag{126}$$

Plugging in u and its derivatives into Equation (74) gives

$$3ca^2\lambda^3 e^{2\lambda\xi} - ca^2\lambda^3 e^{2\lambda\xi} - 2Aa^2\lambda e^{2\lambda\xi} + a^2 e^{2\lambda\xi} = 0 \tag{127}$$

$$\Rightarrow 2c\lambda^3 - 2A\lambda + 1 = 0 \tag{128}$$

This gives us the cubic equation

$$\lambda^3 - \frac{A}{c}\lambda + \frac{1}{2c} = 0, \qquad c \neq 0 \tag{129}$$

If we select a λ that solves the cubic Equation (129), then (123) will be a solution to Equation (74). We will use the formulas derived in Section 2.3 to find λ for all the cases. Comparing Equation (129) to the general cubic Equation (75), we have

$$P = -\frac{A}{c} \tag{130}$$

$$Q = -\frac{1}{2c}. \tag{131}$$

Inserting P and Q into (85) we obtain the discriminant

$$\Delta = -\left(\frac{A}{3c}\right)^3 + \left(\frac{1}{4c}\right)^2 \tag{132}$$

$$\Rightarrow \Delta = \frac{1}{16c^2}\left(1 - \frac{16A^3}{27c}\right). \tag{133}$$

Case 1: The case $\Delta = 0$ corresponds to

$$1 - \frac{16A^3}{27c} = 0 \tag{134}$$

$$\Rightarrow c = \frac{16A^3}{27} \neq 0. \tag{135}$$

From the Formulas (97)–(99), we find that

$$\lambda_1 = -\sqrt[3]{\frac{2}{c}} \tag{136}$$

$$\lambda_2 = \lambda_3 = \sqrt[3]{\frac{1}{4c}}. \tag{137}$$

These λ are all real numbers. The solutions to Equation (74) in the case $c = \frac{16A^3}{27} \neq 0$ are given by

$$u_1 = ae^{-\sqrt[3]{\frac{2}{c}}\xi} \tag{138}$$

$$u_2 = ae^{\sqrt[3]{\frac{1}{4c}}\xi}. \tag{139}$$

Case 2: The case $\Delta < 0$ corresponds to

$$1 - \frac{16A^3}{27c} < 0 \tag{140}$$

$$\Rightarrow 1 < \frac{16A^3}{27c} \tag{141}$$

This occurs when

$$0 < c < \frac{16A^3}{27} \tag{142}$$

or when

$$0 > c > \frac{16A^3}{27}. \tag{143}$$

From the Formulas (120)–(122), we find that

$$\lambda_1 = -2\sqrt{\frac{A}{3c}}\cos\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right) \tag{144}$$

$$\lambda_2 = \sqrt{\frac{A}{3c}}\left(\cos\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right)+\sqrt{3}\sin\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right)\right) \tag{145}$$

$$\lambda_3 = \sqrt{\frac{A}{3c}}\left(\cos\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right)-\sqrt{3}\sin\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right)\right) \tag{146}$$

These λ are all real numbers. The solutions to Equation (74), in the cases $0 < c < \frac{16A^3}{27}$ or $0 > c > \frac{16A^3}{27}$, are given by

$$u_1 = ae^{-2\sqrt{\frac{A}{3c}}\cos\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right)\xi} \tag{147}$$

$$u_2 = ae^{\sqrt{\frac{A}{3c}}\left(\cos\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right)+\sqrt{3}\sin\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right)\right)\xi} \tag{148}$$

$$u_3 = ae^{\sqrt{\frac{A}{3c}}\left(\cos\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right)-\sqrt{3}\sin\left(\frac{1}{3}\tan^{-1}\left(\sqrt{\frac{16A^3}{27c}-1}\right)\right)\right)\xi}. \tag{149}$$

Case 3: The case $\Delta > 0$ corresponds to

$$1 - \frac{16A^3}{27c} > 0 \tag{150}$$

$$\Rightarrow 1 > \frac{16A^3}{27c} \tag{151}$$

This occurs when

$$c > \frac{16A^3}{27}, \qquad c > 0 \tag{152}$$

or when

$$c < \frac{16A^3}{27}, \qquad c < 0. \tag{153}$$

From the Formulas (106)–(108), we find that

$$\lambda_1 = -\sqrt[3]{\frac{1}{4c}\left(1+\sqrt{1-\frac{16A^3}{27c}}\right)} - \sqrt[3]{\frac{1}{4c}\left(1-\sqrt{1-\frac{16A^3}{27c}}\right)} \tag{154}$$

$$\lambda_2 = \frac{1}{2}\left(\sqrt[3]{\frac{1}{4c}}(1+\sqrt{\sigma}) + \sqrt[3]{\frac{1}{4c}}(1-\sqrt{\sigma})\right) + \frac{\sqrt{3}}{2}i\left(\sqrt[3]{\frac{1}{4c}}(1+\sqrt{\sigma}) - \sqrt[3]{\frac{1}{4c}}(1-\sqrt{\sigma})\right) \tag{155}$$

$$\lambda_3 = \frac{1}{2}\left(\sqrt[3]{\frac{1}{4c}}(1+\sqrt{\sigma}) + \sqrt[3]{\frac{1}{4c}}(1-\sqrt{\sigma})\right) - \frac{\sqrt{3}}{2}i\left(\sqrt[3]{\frac{1}{4c}}(1+\sqrt{\sigma}) - \sqrt[3]{\frac{1}{4c}}(1-\sqrt{\sigma})\right) \tag{156}$$

where

$$\sigma = 1 - \frac{16A^3}{27c}. \tag{157}$$

Here λ_1 is real, while λ_2 and λ_3 are complex. Pluggin in λ into (123) gives

$$u_1 = ae^{\left(-\sqrt[3]{\frac{1}{4c}\left(1+\sqrt{1-\frac{16A^3}{27c}}\right)}-\sqrt[3]{\frac{1}{4c}\left(1-\sqrt{1-\frac{16A^3}{27c}}\right)}\right)\xi} \tag{158}$$

$$u_2 = ae^{\left(\frac{1}{2}\left(\sqrt[3]{\frac{1}{4c}\left(1+\sqrt{1-\frac{16A^3}{27c}}\right)}+\sqrt[3]{\frac{1}{4c}\left(1-\sqrt{1-\frac{16A^3}{27c}}\right)}\right)+\frac{\sqrt{3}}{2}i\left(\sqrt[3]{\frac{1}{4c}\left(1+\sqrt{1-\frac{16A^3}{27c}}\right)}-\sqrt[3]{\frac{1}{4c}\left(1-\sqrt{1-\frac{16A^3}{27c}}\right)}\right)\right)\xi} \tag{159}$$

$$u_3 = ae^{\left(\frac{1}{2}\left(\sqrt[3]{\frac{1}{4c}\left(1+\sqrt{1-\frac{16A^3}{27c}}\right)}+\sqrt[3]{\frac{1}{4c}\left(1-\sqrt{1-\frac{16A^3}{27c}}\right)}\right)-\frac{\sqrt{3}}{2}i\left(\sqrt[3]{\frac{1}{4c}\left(1+\sqrt{1-\frac{16A^3}{27c}}\right)}-\sqrt[3]{\frac{1}{4c}\left(1-\sqrt{1-\frac{16A^3}{27c}}\right)}\right)\right)\xi}. \tag{160}$$

Using Euler's Formula (119) we can separate u_2 and u_3 into their real and imaginary parts. For example, setting $a = 1$, $A = -3$, $c = 2$, we get

$$Re(u_2) = Re(u_3) = e^{\frac{1}{2}\left(\sqrt[3]{\frac{1}{2}}-\sqrt[3]{\frac{1}{4}}\right)\xi}\cos\left(\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{1}{2}}+\sqrt[3]{\frac{1}{4}}\right)\xi\right) \tag{161}$$

$$Im(u_2) = -Im(u_3) = e^{\frac{1}{2}\left(\sqrt[3]{\frac{1}{2}}-\sqrt[3]{\frac{1}{4}}\right)\xi}\sin\left(\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{1}{2}}+\sqrt[3]{\frac{1}{4}}\right)\xi\right). \tag{162}$$

We plot each of these parts separately. These solutions represent oscillating waves that decay at negative infinity, see Figures 1 and 2.

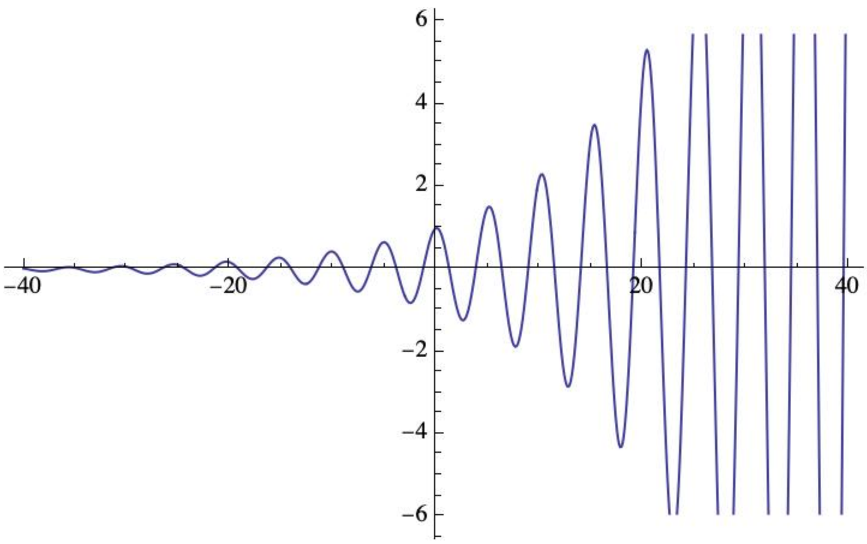

Figure 1. Mathematica plot of $Re(u_2)$ with $a = 1$, $A = -3$, $c = 2$.

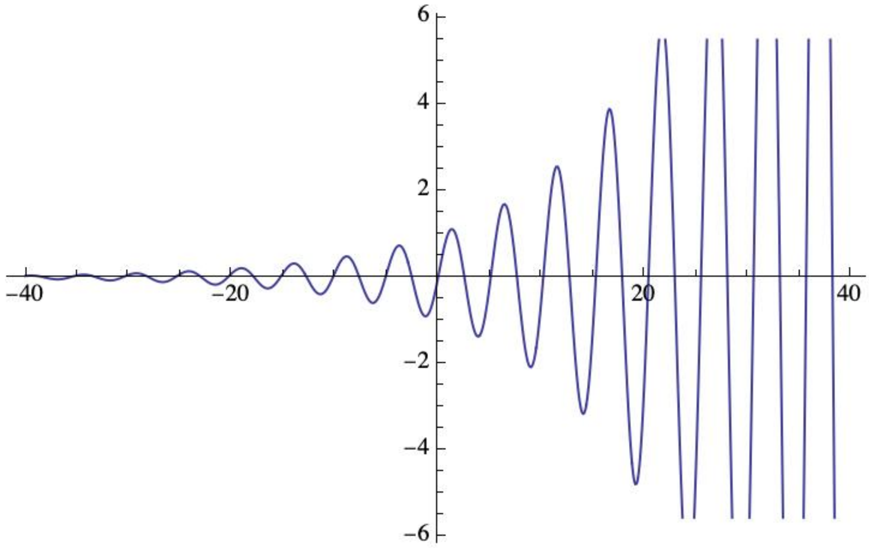

Figure 2. Mathematica plot of $Im(u_2)$ with $a = 1$, $A = -3$, $c = 2$.

3.2. Other Solution Types

We consider the possibility of solutions with the form

$$u(\xi) = ae^{\alpha\xi} + be^{\beta\xi} \tag{163}$$

where a, b are constants and α, β are numbers to be determined. First, we calculate

$$u' = a\alpha e^{\alpha\xi} + b\beta e^{\beta\xi} \tag{164}$$

$$u'' = a\alpha^2 e^{\alpha\xi} + b\beta^2 e^{\beta\xi} \tag{165}$$

$$u''' = a\alpha^3 e^{\alpha\xi} + b\beta^3 e^{\beta\xi} \tag{166}$$

$$u'u'' = a^2\alpha^3 e^{2\alpha\xi} + ab(\alpha^2\beta + \alpha\beta^2)e^{(\alpha+\beta)\xi} + b^2\beta^3 e^{2\beta\xi} \tag{167}$$

$$uu''' = a^2\alpha^3 e^{2\alpha\xi} + ab(\alpha^3 + \beta^3)e^{(\alpha+\beta)\xi} + b^2\beta^3 e^{2\beta\xi} \tag{168}$$

$$uu' = a^2\alpha e^{2\alpha\xi} + ab(\alpha + \beta)e^{(\alpha+\beta)\xi} + b^2\beta e^{2\beta\xi} \tag{169}$$

$$u^2 = a^2 e^{2\alpha\xi} + 2ab e^{(\alpha+\beta)\xi} + b^2 e^{2\beta\xi}. \tag{170}$$

Plugging these into Equation (74) gives

$$\begin{aligned} &a^2 e^{2\alpha\xi}\left(2c\alpha^3 - 2A\alpha + 1\right) + abe^{(\alpha+\beta)\xi}\Big(3c\left(\alpha^2\beta + \beta^2\alpha\right) - c\left(\alpha^3 + \beta^3\right) - 2A\left(\alpha + \beta\right) + 2\Big) \\ &+b^2 e^{2\beta\xi}\left(2c\beta^3 - 2A\beta + 1\right) = 0. \end{aligned} \tag{171}$$

This leads to three seperate equations:

$$\alpha^3 - \frac{A}{c}\alpha + \frac{1}{2c} = 0 \tag{172}$$

$$\beta^3 - \frac{A}{c}\beta + \frac{1}{2c} = 0 \tag{173}$$

$$3c\left(\alpha^2\beta + \beta^2\alpha\right) - c\left(\alpha^3 + \beta^3\right) - 2A\left(\alpha + \beta\right) + 2 = 0 \tag{174}$$

where $c \neq 0$. We must find the relationship between α and β, so that all three equations are solved. Consider

$$\alpha = s\beta \tag{175}$$

where s is a constant. Plugging in (175) into the third Equation (174) gives

$$3c(s^2\beta^3 + s\beta^3) - c(s^3\beta^3 + \beta^3) - 2A(s\beta + \beta) + 2 = 0 \tag{176}$$

$$\Rightarrow c\beta^3(-s^3 + 3s^2 + 3s - 1) - 2A\beta(s+1) + 2 = 0 \tag{177}$$

$$\Rightarrow c\beta^3(s^2 - 4s + 1)(s+1) + 2A\beta(s+1) - 2 = 0 \tag{178}$$

$$\Rightarrow \beta^3 + \frac{A}{c}\beta\left(\frac{2}{s^2 - 4s + 1}\right) - \frac{2}{c(s+1)(s^2 - 4s + 1)} = 0. \tag{179}$$

In order for Equation (179) to be consistent with Equation (173) we require

$$\frac{2}{s^2 - 4s + 1} = -1 \tag{180}$$

$$\Rightarrow s^2 - 4s + 3 = 0 \tag{181}$$

$$\Rightarrow s = 1, \quad s = 3 \tag{182}$$

We also require

$$-\frac{2}{(s+1)(s^2 - 4s + 1)} = \frac{1}{2} \tag{183}$$

$$\Rightarrow s^3 - 3s^2 - 3s + 5 = 0 \tag{184}$$

$$\Rightarrow s = 1, \quad s = 1 + \sqrt{6}, \quad s = 1 - \sqrt{6} \tag{185}$$

The only value of s that meets both requirements is $s = 1$. From (175) this means that $\alpha = \beta$. Therefore, the solution type (163) becomes

$$u(\xi) = (a + b)e^{\alpha\xi}. \tag{186}$$

This form is identical to the form (123) which was already considered, so we don't obtain any new solutions.

Next, we consider solutions of the form

$$u(\xi) = a\,\mathrm{sech}(b\xi) \tag{187}$$

where a and b are constants. First, we calculate

$$u' = -ab\tanh(b\xi)\,\mathrm{sech}(b\xi) \tag{188}$$

$$u'' = ab^2\big(\tanh^2(b\xi)\,\mathrm{sech}(b\xi) - \mathrm{sech}^3(b\xi)\big) \tag{189}$$

$$u''' = ab^3\big(5\tanh(b\xi)\,\mathrm{sech}^3(b\xi) - \tanh^3(b\xi)\,\mathrm{sech}(b\xi)\big) \tag{190}$$

$$u'u'' = a^2b^3\big(\tanh(b\xi)\,\mathrm{sech}^4(b\xi) - \tanh^3(b\xi)\,\mathrm{sech}^2(b\xi)\big) \tag{191}$$

$$uu''' = a^2b^3\big(5\tanh(b\xi)\,\mathrm{sech}^4(b\xi) - \tanh^3(b\xi)\,\mathrm{sech}^2(b\xi)\big) \tag{192}$$

$$uu' = -a^2b\tanh(b\xi)\,\mathrm{sech}^2(b\xi) \tag{193}$$

$$u^2 = a^2\,\mathrm{sech}^2(b\xi) \tag{194}$$

Plugging these into Equation (74) gives

$$-2ca^2b^3\tanh(b\xi)\operatorname{sech}^4(b\xi) - 2ca^2b^3\tanh^3(b\xi)\operatorname{sech}^2(b\xi) \tag{195}$$
$$+2Aa^2b\tanh(b\xi)\operatorname{sech}^2(b\xi) + a^2\operatorname{sech}^2(b\xi) = 0.$$

Dividing by $-a^2\tanh(b\xi)\operatorname{sech}^2(b\xi)$ with $b \neq 0$, we get

$$2cb^3\big(\operatorname{sech}^2(b\xi) + \tanh^2(b\xi)\big) - 2Ab - \coth(b\xi) = 0. \tag{196}$$

Using $\operatorname{sech}^2(b\xi) + \tanh^2(b\xi) = 1$, we get

$$2cb^3 - 2Ab = \coth(b\xi). \tag{197}$$

Since the right hand side of (197) is a function of ξ, there is no value of b that solves this equation. Therefore (187) is not a solution to Equation (74).

Next, we consider solutions of the form

$$u(\xi) = a\tanh(b\xi) \tag{198}$$

where a and b are constants. First, we calculate

$$u' = ab\operatorname{sech}^2(b\xi) \tag{199}$$
$$u'' = -2ab^2\tanh(b\xi)\operatorname{sech}^2(b\xi) \tag{200}$$
$$u''' = 2ab^3\big(2\tanh^2(b\xi)\operatorname{sech}^2(b\xi) - \operatorname{sech}^4(b\xi)\big) \tag{201}$$
$$u'u'' = -2a^2b^3\tanh(b\xi)\operatorname{sech}^4(b\xi) \tag{202}$$
$$uu''' = 2a^2b^3\big(2\tanh^3(b\xi)\operatorname{sech}^2(b\xi) - \tanh(b\xi)\operatorname{sech}^4(b\xi)\big) \tag{203}$$
$$uu' = a^2b\tanh(b\xi)\operatorname{sech}^2(b\xi) \tag{204}$$
$$u^2 = a^2\tanh^2(b\xi) \tag{205}$$

Plugging these into (74) gives

$$-4ca^2b^3\tanh(b\xi)\operatorname{sech}^4(b\xi) - 4ca^2b^3\tanh^3(b\xi)\operatorname{sech}^2(b\xi) \tag{206}$$
$$-2Aa^2b\tanh(b\xi)\operatorname{sech}^2(b\xi) + a^2\tanh^2(b\xi) = 0. \tag{207}$$

Dividing by $-a^2\tanh(b\xi)\operatorname{sech}^2(b\xi)$ with $b \neq 0$, we get

$$4cb^3\big(\operatorname{sech}^2(b\xi) + \tanh^2(b\xi)\big) + 2Ab - \tanh(b\xi)\cosh^2(b\xi) = 0. \tag{208}$$

Using $\operatorname{sech}^2(b\xi) + \tanh^2(b\xi) = 1$, we get

$$4cb^3 + 2Ab - \sinh(b\xi)\cosh(b\xi) = 0. \tag{209}$$

Using $\sinh(b\xi)\cosh(b\xi) = \frac{1}{2}\sinh(2b\xi)$, we get

$$8cb^3 + 4Ab = \sinh(2b\xi). \tag{210}$$

Since the right hand side of (210) is a function of ξ there is no value of b that solves this equation. Therefore, (198) is not a solution to Equation (74).

Finally, we consider peakon solutions of the form

$$u(\xi) = ae^{-\lambda|\xi|} \tag{211}$$

where a is a constant and λ is a number to be determined. First, we define

$$|\xi|' = sgn(\xi) = \begin{cases} -1, & \xi < 0 \\ 0, & \xi = 0 \\ 1, & \xi > 0 \end{cases} \quad (212)$$

and

$$sgn(\xi)' = 2\delta(\xi) \quad (213)$$

with

$$\delta(\xi) = \begin{cases} 0, & \xi \neq 0 \\ \infty, & \xi = 0 \end{cases}. \quad (214)$$

Now, we can calculate

$$u' = -\lambda sgn(\xi) a e^{-\lambda|\xi|}. \quad (215)$$

Using $sgn^2(\xi) = 1$, we calculate

$$\begin{aligned} u'' &= \big(-2\lambda\delta(\xi) + \lambda^2 sgn^2(\xi)\big) a e^{-\lambda|\xi|} \\ &= \big(\lambda^2 - 2\lambda\delta(\xi)\big) a e^{-\lambda|\xi|}. \end{aligned} \quad (216)$$

Using $sgn(\xi)\delta(\xi) = 0$, we calculate

$$\begin{aligned} u''' &= \Big(-2\lambda\delta'(\xi) - \lambda sgn(\xi)\big(\lambda^2 - 2\lambda\delta(\xi)\big)\Big) a e^{-\lambda|\xi|} \\ &= \big(-2\lambda\delta'(\xi) - \lambda^3 sgn(\xi)\big) a e^{-\lambda|\xi|}. \end{aligned} \quad (217)$$

Plugging in u and its derivatives into (74), we get

$$-3c\lambda sgn(\xi)\big(\lambda^2 - 2\lambda\delta(\xi)\big) + c\big(2\lambda\delta'(\xi) + \lambda^3 sgn(\xi)\big) + 2A\lambda sgn(\xi) + 1 = 0. \quad (218)$$

Using $sgn(\xi)\delta(\xi) = 0$, we get

$$-2c\lambda^3 sgn(\xi) + 2c\lambda\delta'(\xi) + 2A\lambda sgn(\xi) + 1 = 0 \quad (219)$$

$$\Rightarrow \lambda^3 sgn(\xi) - \lambda\Big(\delta'(\xi) - \frac{A}{c} sgn(\xi)\Big) - \frac{1}{2c} = 0 \quad (220)$$

$$\Rightarrow \lambda^3 - \lambda\left(\frac{A}{c} + \delta'(\xi) sgn(\xi)\right) - \frac{1}{2c} sgn(\xi) = 0 \quad (221)$$

The term $\delta'(\xi) sgn(\xi)$ in (221) is undefined, so the existence of solutions of the type (211) is unknown. We leave this as an open problem.

4. Conclusions

Starting from the KdV spectral problem, we showed how to produce the KdV hierarchy. We then gave the different evolution equations that arise after using the different possible seed functions. After applying the traveling wave setting to Equation (41), we integrated to find the new ordinary differential equation

$$3cu'u'' - cuu''' - 2Auu' + u^2 = 0.$$

Assuming solutions of the form

$$u = ae^{\lambda\xi}$$

led us to a cubic Equation (129) in λ. After solving the cubic equation we gave new solutions for all the different cases. For the cases $\Delta = 0$ and $\Delta < 0$ we found that all λ are real, so the solutions are not particularly interesting. For $\Delta > 0$ we have complex λ, so we can separate u into its real and imaginary parts. Plotting each of them gives oscillatory wave solutions which decay at infinity.

We considered other solution forms which are common in these types of evolution equations. We found that

$$u = ae^{\alpha\xi} + be^{\beta\xi}$$

does not allow other solutions besides the trivial $\alpha = \beta$. We also found that

$$u = a\cosh(b\xi) \tag{222}$$

$$u = a\tanh(b\xi) \tag{223}$$

cannot be solutions to our equation. After considering peakon solutions of the form

$$u = ae^{-\lambda|\xi|}$$

we arrived at an equation with an undefined term. Therefore, we still don't know whether our equation allows for peakon solutions.

This research was able to find new solutions to an equation in the NKdV hierarchy. None of these solutions were classical solitary traveling wave solutions. In the future, we will continue to investigate the existence of peakon solutions in the NKdV hierachy, as well as other integrable systems.

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