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There continues to be great interest in the study of the heat equation on Riemannian manifolds. This may be due to the remarkable more recent work of Patodi [**1**]. It may also be due in part to the asymptotic expansion of Minakshisundaram and Pleijel. The heat equation involves a parabolic partial differential equation that describes the distribution of heat in a given region over time. This equation has also appears in probability theory to describe random walks. The heat equation is also of importance in Riemannian geometry, topology and applied mathematics.

The heat operator on a compact Riemannian manifold M is the operator $L = \nabla + \partial/\partial t$ which acts on $M \times \mathbb{R}_+$ of class C^2 in the first and of class C^1 in the second variable which operates on functions $f : M \times \mathbb{R}_+ \to \mathbb{R}$

$$Lf = 0, \qquad f(p, 0) = g(0), \qquad p \in M,$$

where $g : M \to \mathbb{R}$ is a given initial condition. Due to the appearance of the Laplace operator, among other reasons, the Laplacian on functions and its generalization to differential forms continues to be intensively studied

[2,3]. On a compact manifold, the spectrum of the Laplacian contains both topological and geometric information. By the Hodge theorem the dimension of the kernel of the Laplacian equals the corresponding Betti number. Therefore, the Laplacian determines the Euler characteristic. Although the theorem discussed is not new, the approach is different. The main objective then is to give what is hoped to be a new proof of a theorem concerning the spectral decomposition of a differential k-form with respect to this operator on a Riemannian manifold.

The Laplacian on k-forms on a Riemannian manifold is defined to be

$$\Delta^k = \delta^{k+1} d^k + d^{k-1} \delta^k \tag{1}$$

where d^k and δ^k are the exterior differentiation operator and its adjoint on k-forms [4,5]. The manifold M carries an inner product g, and this induces an inner product g on each tensor product $T_xM \otimes \cdots \otimes T_xM$, and hence on each exterior power $\Lambda^k T_x^* M$. This yields a global inner product

$$\langle \alpha, \beta \rangle = \int_M g(\alpha, \beta)\, dv, \tag{2}$$

for $\alpha, \beta \in C_0^\infty \Lambda^k T^* M$. The completion is denoted by $L^2 \Lambda^k T^* M$. In fact, the Sobolev space H_s for $n \in \mathbb{N}$ is the completion of C_0^∞ with respect to the norm

$$||f||_s = (\sum_{|\sigma| \leq s} ||D^\sigma f||_2^2)^{1/2}, \tag{3}$$

where $\sigma = (\sigma_1, \cdots, \sigma_n)$, and $\sigma_j \in \mathbb{Z}$ is a multi-index with $|\sigma| = \sum \sigma_j$ and

$$D^\sigma = \frac{\partial^{\sigma_1 + \cdots + \sigma_n}}{\partial_{x_1}^{\sigma_1} \cdots \partial_{x_n}^{\sigma_n}}.$$

We can identify $H_0 = L^2$. Given a cover $\{U_i, \varphi_i\}$ of (M, g) with subordinate partition of unity p_i, the Sobolov s-norm of a compactly supported k-form ω is defined to be the s-norm of the function $g(\omega, \omega)^{1/2}$ on M,

$$||\omega||_s = (\sum_i ||p_i g(\omega, \omega)^{1/2} \varphi_i||_s^2)^{1/2}.$$

The completion of $C_c^\infty \Lambda^k T^* M$ with respect to this norm is denoted $H_s \Lambda^k M$.

If the form ω is in the kernel of the Laplacian $\Delta = \Delta^k$ then

$$\langle \Delta\omega, \omega \rangle = \langle d\delta\omega, \omega \rangle + \langle \delta d\omega, \omega \rangle = \langle \delta\omega, \delta\omega \rangle + \langle d\omega, d\omega \rangle = 0. \tag{4}$$

In (4), the k has been omitted from the operators d and δ. So (4) implies that $d\omega = \delta\omega = 0$, and conversely, it is immediate that $d\omega = \delta\omega = 0$ implies that $\Delta\omega = 0$.

Now consider the operators $d : C^\infty \Lambda^{k-1} T^* M \to C^\infty \Lambda^k T^* M$ and $\delta : C^\infty \Lambda^{k+1} T^* M \to C^\infty \Lambda^k T^* M$ as operators on smooth forms. In fact, it is the case that

$$C^\infty \Lambda^k T^* M \supset \mathrm{Ker}\Delta \oplus \mathrm{Im}\, d \oplus \mathrm{Im}\, \delta, \tag{5}$$

where $\oplus$ indicates orthogonality with respect to the Hodge inner product. Clearly, if $\omega \in \mathrm{Ker}\Delta$, then it follows that $\langle \omega, d\alpha \rangle = \langle \delta\omega, \alpha \rangle = 0$. The other orthogonality relations follow similarly. In fact, it can be shown that equality holds between the two sides of (5), and this amounts to a statement of the Hodge Decomposition Theorem **[4,5,6]**.

The structure of (5) suggests that it would be useful to examine the operator defined by

$$D = d + \delta. \tag{6}$$

This is to be thought of as an operator on the space of smooth forms of mixed degree. The operator D is a first order differential operator and is related to the Laplacian on $C^\infty \Lambda^*$ by the equation

$$\Delta = D^2. \tag{7}$$

The objective is to develop a proof of the following theorem which can be useful in establishing equality in (5).

Theorem I. Let (M, g) be a compact, connected, oriented Riemannian manifold. There exists an orthonormal basis of $L^2 \Lambda^k (M, g)$ which consists of eigenforms of Laplacian (1) on k-forms. All the eigenvalues of (1) are nonnegative. Each eigenvalue has a finite multiplicity and the eigenvalues accumulate only at infinity.

To prove the theorem, a number of subsidiary results will be formulated and then used.

Lemma 1. Operator D is injective as a map $D: H_s^{\perp} = H_s \cap (\text{Ker}D)^{\perp} \to H_{s-1}$, where $(\text{Ker}D)^{\perp}$ is the orthogonal complement of $\text{Ker}D$ with respect to the Hodge metric on L^2 forms.

Let α, $\beta \in H_s$ be any two non-zero forms $\alpha \neq \beta$ such that $D\alpha = D\beta \neq 0$. By the definition of operator D, this implies that $d(\alpha - \beta) = \delta(\alpha - \beta)$. Hence, the degree of the resulting forms on each side of this equation must differ which implies that $\alpha - \beta = 0$.

On account of this lemma, an inverse operator D^{-1} corresponding to D can be defined where $D^{-1} : R(D) \to H_s^{\perp}$ and $R(D)$ denotes the range space of D which acts on H_s.

Lemma 2. The operator $D^{-1} : L^2 \cap (\text{Ker}D)^{\perp} \to H_1$ is bounded.

Let $\alpha \in L^2 \cap (\text{Ker}D)^{\perp}$ be any form on which D^{-1} is defined. Then both $||\alpha||_0$ and $||D^{-1}\alpha||_0$ are bounded quantities as $\alpha \in L^2$. By Garding's inequality, there exists a constant $c > 0$ such that

$$||D^{-1}\alpha||_1 \leq c(||D^{-1}\alpha||_0 + ||\alpha||_0).$$

This result implies that D^{-1} is bounded in H_1.

Lemma 3. The operator $D^{-1} : R(D) \to H_s$ is bounded for all s and $R(D) \subset H_{s-1}$.

As an induction hypothesis, suppose that $D^{-1}\omega$ is bounded for forms from the spaces H_m with $m = 1, \cdots, s-1$ and so $D^{-1} : R(D) \subset H_{s-1} \to H_s^{\perp}$. By applying Garding's inequality again, for $\omega \in R(D) \subset H_{s-1}$, it follows that since $||\omega||_{s-1} < \infty$, there exists a constant $C_s > 0$ such that $||D^{-1}\omega||_s \leq C_s(||D^{-1}\omega|_{s-1} + ||\omega||_{s-1}) < \infty$. By induction on s, it follows that the operator D^{-1} is bounded for all s.

An embedding operator $j : X \to Y$, it may be recalled, acting between spaces X and Y is defined to be $j(x) = x$ for all $x \in X$. The embedding $X \subset Y$ is called continuous if and only if j is continuous so $||x||_Y \leq k||x||_X$ for all $x \in X$ and k constant. This is applied to the operator D^{-1}. As a consequence of Lemma 3, the canonical embedding operator $j : H_t \to H_s$ is compact for $t > s$, by the Rellich-Kondarachev compactness theorem. Therefore, with forming the composition of D^{-1} with j, it follows that $j \circ D^{-1} : L^2 \cap (\mathrm{Ker} D)^{\perp}$ is a compact operator on the Hilbert space $R(D)$. This conclusion permits the spectral theorem for compact operators to be applied to the operator D^{-1}. This is the next step.

By Theorem 6.11.2 in **[7]**, there exists an orthonormal basis of eigenforms $\{\omega_n\}$ and their corresponding eigenvalues $\{\mu_n\}$ of Laplace operator (1) such that ω can be expanded in terms of the eigenforms $\{\omega_n\}$ as

$$\omega = \sum_n \langle \omega, \omega_n \rangle \, \omega_n \tag{8}$$

and moreover D^{-1} can be expressed in terms of this set of eigenvalues as follows

$$D^{-1}\omega = \sum_n \mu_n \langle \omega, \omega_n \rangle \omega_n. \tag{9}$$

The only finite or infinite accumulation point for the set of eigenvalues μ_n is the value zero and the eigenvalues for the operator D are the reciprocals of the set $\{\mu_n\}$. This is the main conclusion given in Theorem I.

It can also be shown that the spectrum of D restricted to $(\mathrm{Ker} D)^{\perp}$ is bounded away from zero in the following way. Suppose there exists a set of forms ω_n such that $||\omega_n||_0 = 1$ is satisfied for all n and $||D\omega_n||_0 \to 0$ as n increases. As already noted $\mathrm{Im}(d)$ is orthogonal to $\mathrm{Im}(\delta)$, so it follows that

$$||D\omega_n||_0^2 = ||(d+\delta)\omega_n||_0^2 = \int_M \langle d\omega_n, d\omega_n \rangle \, dv + \int_M \langle \delta\omega_n, \delta\omega_n \rangle \, dv = ||d\omega_n||_0^2 + ||\delta\omega_n||_0^2.$$

Given it is assumed that $||D\omega_n||_0^2 \to 0$, each of the norms on the right-hand side of this result must approach zero as n increases since both $||d\omega_n||_0^2$

and $||\delta\omega_n||_0^2$ are positive or zero. Consequently, it follows from () that both $d\omega_n \to 0$ and $\delta\omega_n \to 0$ in L^2.

Applying Garding's inequality, it is seen that the forms ω_n must satisfy the following bound

$$||\omega_n||_1 \leq C(||\omega_n||_0 + ||D\omega_n||_0) \leq C(1 + ||D\omega_n||_0).$$

It can then be concluded that the set of forms $\{\omega_n\}$ is a bounded sequence in H_1. Therefore, there exists an ω such that $\omega_n \to \omega$ is L^2 with $\omega \in (\mathrm{Ker} D)^{\perp}$. However, it follows that for any form $\vartheta \in L^2$ that

$$\langle D\omega_n, \vartheta \rangle = \langle d\omega_n + \delta\omega_n, \vartheta \rangle = \langle \omega_n, D\vartheta \rangle$$

holds by the adjoint property of the operators d and δ. Since $D\omega_n \to 0$, it follows that $\langle D\omega_n, \vartheta \rangle \to 0$. From this fact, it may be concluded that $\langle \omega_n, D\vartheta \rangle \to 0$ as well for any $\vartheta \in L^2$. This implies that $\omega \in (\mathrm{Im}\, D)^{\perp} = \mathrm{Ker} D$ or finally $\omega \in (\mathrm{Ker}\, D) \cap (\mathrm{Ker} D)^{\perp} = \{0\}$. However, this conclusion contradicts the statement that $||\omega||_0 = 1$, and we are done.

Now apply this to the system of eigenforms which is guaranteed to exist by the spectral theorem. If $\{\omega_n\}$ is an infinite system of eigenforms for D which has been normalized to one, then the set of λ_n cannot approach zero. Otherwise it would follow that $||D\omega_n||_0 = \lambda_n ||\omega_n||_0 = \lambda_n \to 0$ as n becomes large. However, this immediately contradicts what was demonstrated above. Consequently, the eigenvalues of D are bounded away from zero and the λ_n increase towards $+\infty$. This means that the eigenvalues for the operator D^{-1} approach zero. Since the eigenvalues approach infinity, the operator Δ is always an unbounded operator on $L^2(M)$.

These results have importance for the study of the Laplacian and its eigenforms. As an example, it is possible to show that the eigenforms of the Laplacian (1) on k-forms are smooth for all k. Let $\omega \in C^2 \Lambda^k T^* M$ satisfy $\Delta\omega = \lambda\omega$, then $\omega \in C^{\infty} \Lambda^k T^* M$. To see this, if ω satisfies this equation, then

$$e^{-t\Delta}\omega = e^{-t\lambda}\omega.$$

Since the heat operator on forms has a smooth kernel, $e^{-t\Delta}\omega$ is smooth, so this last equation implies that ω itself is smooth.

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