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Yasuhiro Ohta

Bao-Feng Feng

The University of Texas Rio Grande Valley

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General rogue wave solutions to the discrete nonlinear Schrödinger equation

Yasuhiro Ohta

*Department of Mathematics, Kobe University
Rokko, Kobe 657-8501, Japan*

Bao-Feng Feng

*School of Mathematical and Statistical Sciences
The University of Texas Rio Grande Valley*

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Abstract

In the present paper, we attempt to construct both the general rogue wave solutions to the fully discrete nonlinear Schrödinger (fd-NLS) equation via the KP-Toda reduction method. First, we deduce the general breather solution of the fd-NLS equation starting from a pair of bilinear equations. We then derive the general rogue wave solution by taking a limit to the breather solution.

1 Introduction

Rogue waves (RWs) or freak waves are spontaneously excited local nonlinear waves with large amplitudes which appear from nowhere and disappear with no trace [1]. The simplest form of such waves were firstly discovered Peregrine in the nonlinear Schrödinger (NLS) equation [2], and their higher order forms were found 20 years later in [3, 4, 5, 6, 7, 8, 9]. Such extreme wave have been observed in various different contexts such as oceanography [10], hydrodynamic [11, 12], Bose-Einstein condensate [13], plasma [14] and nonlinear optic [11, 15, 16].

Motivated by these physical applications, rogue wave solutions have been found in many other nonlinear wave equations such as the derivative Schrödinger (NLS) equation [17, 18, 19, 20, 21], the Manakov system [22, 23, 24], Davey-Stewartson I and II equation [25, 26], the three-wave equation [27], the Boussinesq equation [28], the Yajima-Oikawa equation [29, 30].

On the other hand, the study of rogue waves in discrete integrable systems is much less. As far as we are aware, only the rogue waves in semi-discrete NLS equation, or the so-called Ablowitz-Ladik equation have been reported in the literature [31, 32]. In the present paper, we attempt to construct the rogue wave solutions to the fully discrete nonlinear Schrödinger (fd-NLS) equation, which was originally discovered by Ablowitz and Ladik [33, 34] and rediscovered by Hirota and Ohta through Hirota's bilinear approach [35]. It was also discussed in [36, 37]. In a recent paper by Hirose et al., the integrable discretization of the local induction equation which is gauge equivalent to the NLS equation was presented in [38].

The remainder of the present paper is organized as follows. In section 2, we will construct general breather solution to the fd-NLS equation. We derive general rogue wave solutions to the fd-NLS equation by taking the limit to the general breather solution in section 3. The paper is concluded in section 4.

2 Breather solution for fully discrete NLS equation

Based on the bilinear formulation, we are able to derive the breather solution of the fully discrete NLS equation, which is given by the following theorem.

Theorem 1. *The fully discrete NLS equation,*

$$\begin{cases} i(q_k^{t+1} - q_k^t) = (q_{k+1}^t + q_{k-1}^{t+1})(1 + \epsilon|q_k^t|^2)\Gamma_k^t, \\ \Gamma_{k+1}^t = \frac{1 + \epsilon|q_k^t|^2}{1 + \epsilon|q_k^{t+1}|^2}\Gamma_k^t, \end{cases} \quad (1)$$

where $\epsilon = \pm 1$, admits the N -breather solution,

$$q_k^t = \frac{g_k^t}{f_k^t} \frac{1-r}{2\sqrt{|r|}} e^{ik\theta} \left(\frac{1-c}{1-\bar{c}} \frac{1+r\bar{c}}{1+rc} \right)^t, \quad \Gamma_k^t = \frac{4rR}{(1+r)^2} \frac{f_{k-1}^{t+1}f_k^t}{f_k^{t+1}f_{k-1}^t}, \quad (2)$$

where $\bar{}$ means complex conjugate, r is a real constant whose sign coincides with ϵ , $\epsilon = \text{sign } r$, c is a complex constant, R and θ are determined by

$$\frac{1}{i} \frac{1+c}{1-\bar{c}} \frac{1-r\bar{c}}{1+rc} = Re^{i\theta}, \quad (3)$$

and f_k^t and g_k^t are given by

$$f_k^t = \tau_k^t(0), \quad g_k^t = \tau_k^t(1), \quad (4)$$

with

$$\tau_k^t(n) = \det_{1 \leq i, j \leq N} \left(A_{ij}^{(n)}(k, t) \right), \quad (5)$$

$$A_{ij}^{(n)} = \frac{1}{1 - rp_i \bar{p}_j} \left(a_i \varphi_n(p_i) \overline{a_j \varphi_{-n}(p_j)} + b_i \varphi_n(-p_i) \overline{b_j \varphi_{-n}(-p_j)} \right) - \frac{1}{1 + rp_i \bar{p}_j} \left(a_i \varphi_n(p_i) \overline{b_j \varphi_{-n}(-p_j)} + b_i \varphi_n(-p_i) \overline{a_j \varphi_{-n}(p_j)} \right), \quad (6)$$

$$\varphi_n(p) = \left(\frac{1 - rp}{1 + p} \right)^n \left(\frac{1 + p}{1 + rp} \right)^k \left(\frac{1 + p}{1 + rp} \frac{1 + p/\bar{c}}{1 + rc p} \right)^t, \quad (7)$$

where p_i, a_i, b_i are complex constants.

Proof. By using the Gramian technique, we can directly prove that the τ function of discrete KP/Toda hierarchy,

$$\tau_n(k, K, l, L) = \det_{1 \leq i, j \leq N} \left(m_{ij}^{(n)}(k, K, l, L) \right),$$

satisfies the discrete bilinear equations,

$$\left\{ \begin{array}{l} (1 - aA)\tau_n(k + 1, K + 1, l, L)\tau_n(k, K, l, L) \\ - \tau_n(k + 1, K, l, L)\tau_n(k, K + 1, l, L) \\ + aA\tau_{n+1}(k + 1, K, l, L)\tau_{n-1}(k, K + 1, l, L) = 0, \\ A(a - b)(1 - aB)\tau_{n+1}(k + 1, K, l + 1, L + 1)\tau_n(k, K + 1, l, L) \\ - a(A - B)(1 - Ab)\tau_{n+1}(k + 1, K, l, L)\tau_n(k, K + 1, l + 1, L + 1) \\ + Ab(1 - aA)\tau_{n+1}(k, K, l + 1, L + 1)\tau_n(k + 1, K + 1, l, L) \\ - aB(1 - aA)\tau_{n+1}(k + 1, K + 1, l, L)\tau_n(k, K, l + 1, L + 1) = 0, \end{array} \right. \quad (8)$$

if the matrix element $m_{ij}^{(n)}$ satisfies

$$\begin{aligned} m_{ij}^{(n+1)}(k, K, l, L) - m_{ij}^{(n)}(k, K, l, L) &= (-1)^{n+1} \phi_i^{(n)}(k, K, l, L) \psi_j^{(-n-1)}(k, K, l, L), \\ m_{ij}^{(n)}(k + 1, K, l, L) - m_{ij}^{(n)}(k, K, l, L) &= (-1)^n a \phi_i^{(n)}(k + 1, K, l, L) \psi_j^{(-n)}(k, K, l, L), \\ m_{ij}^{(n)}(k, K + 1, l, L) - m_{ij}^{(n)}(k, K, l, L) &= (-1)^n A \phi_i^{(n-1)}(k, K + 1, l, L) \psi_j^{(-n-1)}(k, K, l, L), \\ m_{ij}^{(n)}(k, K, l + 1, L) - m_{ij}^{(n)}(k, K, l, L) &= (-1)^n b \phi_i^{(n)}(k, K, l + 1, L) \psi_j^{(-n)}(k, K, l, L), \\ m_{ij}^{(n)}(k, K, l, L + 1) - m_{ij}^{(n)}(k, K, l, L) &= (-1)^n B \phi_i^{(n-1)}(k, K, l, L + 1) \psi_j^{(-n-1)}(k, K, l, L), \end{aligned}$$

where $\phi_i^{(n)}$ and $\psi_j^{(n)}$ are arbitrary functions satisfying the linear dispersion relations,

$$\begin{aligned}
\phi_i^{(n)}(k, K, l, L) - \phi_i^{(n)}(k-1, K, l, L) &= a\phi_i^{(n+1)}(k, K, l, L), \\
\phi_i^{(n)}(k, K, l, L) - \phi_i^{(n)}(k, K-1, l, L) &= A\phi_i^{(n-1)}(k, K, l, L), \\
\phi_i^{(n)}(k, K, l, L) - \phi_i^{(n)}(k, K, l-1, L) &= b\phi_i^{(n+1)}(k, K, l, L), \\
\phi_i^{(n)}(k, K, l, L) - \phi_i^{(n)}(k, K, l, L-1) &= B\phi_i^{(n-1)}(k, K, l, L), \\
\psi_j^{(n)}(k+1, K, l, L) - \psi_j^{(n)}(k, K, l, L) &= a\psi_j^{(n+1)}(k, K, l, L), \\
\psi_j^{(n)}(k, K+1, l, L) - \psi_j^{(n)}(k, K, l, L) &= A\psi_j^{(n-1)}(k, K, l, L), \\
\psi_j^{(n)}(k, K, l+1, L) - \psi_j^{(n)}(k, K, l, L) &= b\psi_j^{(n+1)}(k, K, l, L), \\
\psi_j^{(n)}(k, K, l, L+1) - \psi_j^{(n)}(k, K, l, L) &= B\psi_j^{(n-1)}(k, K, l, L),
\end{aligned}$$

where k, K, l, L are discrete independent variables and a, A, b, B are difference intervals. In order to derive the breather solution, we assume

$$\begin{aligned}
m_{ij}^{(n)}(k, K, l, L) &= \sum_{\nu=1}^2 \sum_{\mu=1}^2 \frac{a_{i\nu} b_{j\mu}}{p_{i\nu} + q_{j\mu}} \left(-\frac{p_{i\nu}}{q_{j\mu}} \right)^n \left(\frac{1 + a q_{j\mu}}{1 - a p_{i\nu}} \right)^k \left(\frac{1 + A/q_{j\mu}}{1 - A/p_{i\nu}} \right)^K \left(\frac{1 + b q_{j\mu}}{1 - b p_{i\nu}} \right)^l \left(\frac{1 + B/q_{j\mu}}{1 - B/p_{i\nu}} \right)^L, \\
\phi_i^{(n)}(k, K, l, L) &= \sum_{\nu=1}^2 a_{i\nu} p_{i\nu}^n (1 - a p_{i\nu})^{-k} (1 - A/p_{i\nu})^{-K} (1 - b p_{i\nu})^{-l} (1 - B/p_{i\nu})^{-L}, \\
\psi_j^{(n)}(k, K, l, L) &= \sum_{\mu=1}^2 b_{j\mu} q_{j\mu}^n (1 + a q_{j\mu})^k (1 + A/q_{j\mu})^K (1 + b q_{j\mu})^l (1 + B/q_{j\mu})^L,
\end{aligned}$$

where $p_{i\nu}, q_{j\mu}, a_{i\nu}, b_{j\mu}$ are arbitrary constants. Obviously, the $\phi_i^{(n)}$ and $\psi_j^{(n)}$ defined satisfy the above linear dispersion relations, therefore the determinant with above defined element $m_{ij}^{(n)}(k, K, l, L)$ satisfy the bilinear equations (8).

Next, we proceed to the reductions. By imposing

$$p_{i2} = -\frac{A}{a} \frac{1 - a p_{i1}}{p_{i1} - A}, \quad q_{j2} = -\frac{A}{a} \frac{1 + a q_{j1}}{q_{j1} + A},$$

we can show that τ_n satisfies the following reduction condition,

$$\tau_n(k+1, K-1, l, L) = \tau_n(k, K, l, L) \prod_{i=1}^N \frac{q_{i1} q_{i2}}{p_{i1} p_{i2}},$$

it follows that the bilinear equations (8) are reduced to

$$\begin{cases} (1 - aA)\sigma_{k+1}^t(n)\sigma_{k-1}^t(n) - \sigma_k^t(n)\sigma_k^t(n) + aA\sigma_k^t(n+1)\sigma_k^t(n-1) = 0, \\ A(a-b)(1-aB)\sigma_k^{t+1}(n+1)\sigma_k^t(n) - a(A-B)(1-Ab)\sigma_k^t(n+1)\sigma_k^{t+1}(n) \\ \quad + Ab(1-aA)\sigma_{k-1}^{t+1}(n+1)\sigma_{k+1}^t(n) - aB(1-aA)\sigma_{k+1}^t(n+1)\sigma_{k-1}^{t+1}(n) = 0, \end{cases} \quad (9)$$

by taking $\sigma_k^t(n) = \tau_n(k, 0, t, t)$. Furthermore by taking

$$A = -\epsilon\bar{a}, \quad B = -\epsilon\bar{b}, \quad q_{j1} = \frac{\bar{a}\bar{p}_{j1} + \epsilon a}{a(1 - \bar{a}\bar{p}_{j1})}, \quad b_{j1} = \frac{\bar{a}_{j1}}{a(1 - \bar{a}\bar{p}_{j1})}, \quad b_{j2} = \frac{\bar{a}_{j2}}{a(1 - \bar{a}\bar{p}_{j2})},$$

where $\epsilon = +1$ or -1 , $\sigma_k^t(n)$ satisfies the complex conjugate condition,

$$\sigma_k^t(-n)G = \overline{\sigma_k^t(n)G},$$

with a gauge factor G which is given below in eq.(11). In order to simplify the final expression, let us parametrise by

$$|a|^2 = \epsilon \frac{r - 2 + 1/r}{4}, \quad b = \frac{2a}{1-r} \frac{1-rc}{1+c}, \quad p_{i1} = \frac{1-1/r}{2a} \frac{1-rp_i}{1+p_i},$$

$$a_{i1} = \epsilon \frac{r-1/r}{4} a_i, \quad a_{i2} = -a(\bar{a} + \epsilon p_{i2}) b_i,$$

where r is a real constant, c is a complex constant, p_i represents the complex wave number of the i th breather, a_i and b_i stand for the complex phase constants of the i th breather. We comment that ϵ is equal to the sign of r because of the positivity of $|a|^2$. Then we have

$$\begin{aligned} & \tilde{m}_{ij}^n(k, K, l, L) \\ & := m_{ij}^n(k, K, l, L) \frac{-4}{(1-r)(1+p_i)(1+\bar{p}_j)} \left(-r \frac{1-\bar{p}_j^2}{1-r^2\bar{p}_j^2} \right)^n r^{K-k} (rc)^{-l} \left(r\bar{c} \frac{1-p_i^2/\bar{c}^2}{1-r^2p_i^2} \frac{1-\bar{p}_j^2}{1-r^2\bar{c}^2\bar{p}_j^2} \right)^L \\ & = \frac{a_i\bar{a}_j}{1-rp_i\bar{p}_j} \left(\frac{1-rp_i}{1+p_i} \frac{1+\bar{p}_j}{1-r\bar{p}_j} \right)^n \left(\frac{1+p_i}{1+rp_i} \frac{1+\bar{p}_j}{1+r\bar{p}_j} \right)^k \left(\frac{1-rp_i}{1-p_i} \frac{1-r\bar{p}_j}{1-\bar{p}_j} \right)^K \\ & \quad \times \left(\frac{1+p_i}{1+rcp_i} \frac{1+\bar{p}_j/c}{1+r\bar{p}_j} \right)^l \left(\frac{1+p_i/\bar{c}}{1+rp_i} \frac{1+\bar{p}_j}{1+r\bar{c}\bar{p}_j} \right)^L \\ & - \frac{a_i\bar{b}_j}{1+rp_i\bar{p}_j} \left(\frac{1-rp_i}{1+p_i} \frac{1-\bar{p}_j}{1+r\bar{p}_j} \right)^n \left(\frac{1+p_i}{1+rp_i} \frac{1-\bar{p}_j}{1-r\bar{p}_j} \right)^k \left(\frac{1-rp_i}{1-p_i} \frac{1+r\bar{p}_j}{1+\bar{p}_j} \right)^K \\ & \quad \times \left(\frac{1+p_i}{1+rcp_i} \frac{1-\bar{p}_j/c}{1-r\bar{p}_j} \right)^l \left(\frac{1+p_i/\bar{c}}{1+rp_i} \frac{1-\bar{p}_j}{1-r\bar{c}\bar{p}_j} \right)^L \\ & - \frac{b_i\bar{a}_j}{1+rp_i\bar{p}_j} \left(\frac{1+rp_i}{1-p_i} \frac{1+\bar{p}_j}{1-r\bar{p}_j} \right)^n \left(\frac{1-p_i}{1-rp_i} \frac{1+\bar{p}_j}{1+r\bar{p}_j} \right)^k \left(\frac{1+rp_i}{1+p_i} \frac{1-r\bar{p}_j}{1-\bar{p}_j} \right)^K \\ & \quad \times \left(\frac{1-p_i}{1-rp_i} \frac{1+\bar{p}_j/c}{1+r\bar{p}_j} \right)^l \left(\frac{1-p_i/\bar{c}}{1-rp_i} \frac{1+\bar{p}_j}{1+r\bar{c}\bar{p}_j} \right)^L \\ & + \frac{b_i\bar{b}_j}{1-rp_i\bar{p}_j} \left(\frac{1+rp_i}{1-p_i} \frac{1-\bar{p}_j}{1+r\bar{p}_j} \right)^n \left(\frac{1-p_i}{1-rp_i} \frac{1-\bar{p}_j}{1-r\bar{p}_j} \right)^k \left(\frac{1+rp_i}{1+p_i} \frac{1+r\bar{p}_j}{1+\bar{p}_j} \right)^K \\ & \quad \times \left(\frac{1-p_i}{1-rp_i} \frac{1-\bar{p}_j/c}{1-r\bar{p}_j} \right)^l \left(\frac{1-p_i/\bar{c}}{1-rp_i} \frac{1-\bar{p}_j}{1-r\bar{c}\bar{p}_j} \right)^L, \end{aligned}$$

and the bilinear equations (9) are reduced to

$$\begin{cases} (1+r)^2\tau_{k+1}^t(n)\tau_{k-1}^t(n) - 4r\tau_k^t(n)\tau_k^t(n) - (1-r)^2\tau_k^t(n+1)\tau_k^t(n-1) = 0, \\ (1-c)(1+r\bar{c})\tau_k^{t+1}(n+1)\tau_k^t(n) - (1-\bar{c})(1+rc)\tau_k^t(n+1)\tau_k^{t+1}(n) \\ -(1+\bar{c})(1-rc)\tau_{k-1}^{t+1}(n+1)\tau_{k+1}^t(n) + (1+c)(1-r\bar{c})\tau_{k+1}^t(n+1)\tau_{k-1}^{t+1}(n) = 0, \end{cases} \quad (10)$$

where $\tau_k^t(n)$ is given by

$$\tau_k^t(n) = \det_{1 \leq i, j \leq N} (\tilde{m}_{ij}^n(k, 0, t, t)) = \sigma_k^t(n)G, \quad (11)$$

$$G = \prod_{i=1}^N \frac{-4}{(1-r)(1+p_i)(1+\bar{p}_i)} \left(-r \frac{1-\bar{p}_i^2}{1-r^2\bar{p}_i^2} \right)^n r^{-k} \left(\frac{\bar{c}1-p_i^2/\bar{c}^2}{c1-r^2p_i^2} \frac{1-\bar{p}_i^2}{1-r^2\bar{c}^2\bar{p}_i^2} \right)^t,$$

which is nothing but $\tau_k^t(n)$ in Theorem 1. For $f_k^t = \tau_k^t(0)$, $g_k^t = \tau_k^t(1)$, $\bar{g}_k^t = \tau_k^t(-1)$, the bilinear equations are written as

$$\begin{cases} (1+r)^2f_{k+1}^t f_{k-1}^t - 4rf_k^t f_k^t - (1-r)^2g_k^t \bar{g}_k^t = 0, \\ (1-c)(1+r\bar{c})g_k^{t+1} f_k^t - (1-\bar{c})(1+rc)g_k^t f_k^{t+1} \\ -(1+\bar{c})(1-rc)g_{k-1}^{t+1} f_{k+1}^t + (1+c)(1-r\bar{c})g_{k+1}^t f_{k-1}^{t+1} = 0, \end{cases}$$

from which the fully discrete NLS eq.(1) is straightforwardly derived through the variable transformation (2). This completes the proof of Theorem 1. \square

In the case of $N = 1$, we have the 1-breather solution,

$$\tau_k^t(n) = \frac{1}{1-r|p|^2} \left(|a|^2 \varphi_n(p) \overline{\varphi_{-n}(p)} + |b|^2 \varphi_n(-p) \overline{\varphi_{-n}(-p)} \right) - \frac{1}{1+r|p|^2} \left(a\bar{b} \varphi_n(p) \overline{\varphi_{-n}(-p)} + \bar{a}b \varphi_n(-p) \overline{\varphi_{-n}(p)} \right),$$

with (2)-(7) where p, a, b are complex constants (the index 1 of p_1, a_1, b_1 are omitted for notational simplicity). The Akhmediev breather which is the breather solution localized in time t can be derived by taking the wave number p pure imaginary. If we require the regularity of the solution only on the lattice points, i.e., $f_k^t \neq 0$ for integers k and t , then r can be either positive or negative and the discrete NLS equations (1) of both focusing type and defocusing type admit regular breather solutions by locating the singularities (if exist) off the lattice points. On the other hand if we require the solution to be regular in whole real space, i.e., f_k^t is non-zero for all real numbers k and t , then r has to be positive and only the focusing discrete NLS admits the regular breathers for generic parameters. Some exceptional regular solutions for $r < 0$ can be derived from the above $\tau_k^t(n)$ but usually they are not called breathers and we don't discuss about such solutions.

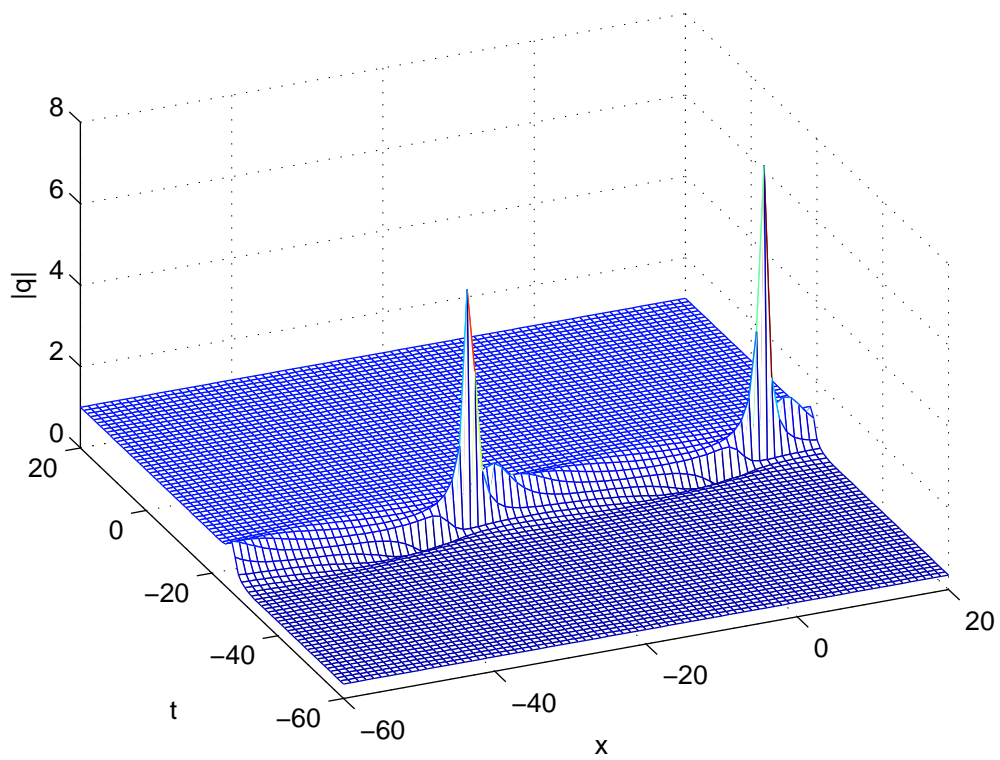


Figure 1: One-breather solution with parameters $r = 2.0$, $c = 3 + 2i$, $p = 0.5i$

3 Rogue wave solution for fully discrete NLS equation

The rogue wave solution of rational function type can be derived as a limit of the breather solution. In Theorem 1, we take

$$a_i = \frac{1}{2} \left(1 + \sum_{\nu=1}^i c_\nu p_i^{2\nu-1} \right), \quad b_i = \frac{1}{2} \left(1 - \sum_{\nu=1}^i c_\nu p_i^{2\nu-1} \right),$$

scale the τ function by $\tau_k^t(n) / \prod_{i=1}^N (p_i \bar{p}_i)^{2i-1}$ and finally take the limit $p_i \rightarrow 0$ successively for $i = 1, 2, \dots, N$. Then the leading order of $\tau_k^t(n)$ in p_i 's gives a polynomial of k and t which turns to be the rogue wave solution. This result is summarized in the following theorem.

Theorem 2. *The N th order rogue wave solution for the fully discrete NLS equation (1) is given by*

$$\tau_k^t(n) = \det_{1 \leq i, j \leq N} \left(B_{ij}^{(n)}(k, t) \right), \quad B_{ij}^{(n)} = \sum_{\nu=1}^{2 \min(i, j)} r^{\nu-1} \Phi_{2i-\nu}^{(n)} \overline{\Phi_{2j-\nu}^{(-n)}},$$

$$\Phi_i^{(n)} = S_i(x(n)) + \sum_{\mu=1}^{\lfloor \frac{i+1}{2} \rfloor} c_\mu S_{i+1-2\mu}(x(n)), \quad x(n) = (x_1(n), x_2(n), \dots, x_h(n), \dots),$$

$$x_h(n) = \frac{(-1)^h}{h} \left((1 - (-r)^h)n - (1 - r^h)k - (1 - r^h + (1/\bar{c})^h - (rc)^h)t \right), \quad \text{for } h = 1, 2, \dots,$$

with the variable transformations (2)-(4), where $\lfloor \cdot \rfloor$ means the Gauss symbol and $S_\mu(x)$ is the so-called elementary Schur function defined by $\sum_{\mu=0}^{\infty} S_\mu(x) \lambda^\mu = \exp \sum_{h=1}^{\infty} x_h \lambda^h$. This solution has N complex parameters c_μ , $\mu = 1, 2, \dots, N$.

Proof. Firstly $\varphi_n(p)$ in (7) is written as $\varphi_n(p) = \sum_{\mu=0}^{\infty} S_\mu(x(n)) p^\mu$. We rewrite $A_{ij}^{(n)}$ in Theorem 1 as

$$A_{ij}^{(n)} = \frac{1}{1 - (rp_i \bar{p}_j)^2} \left(a_i \varphi_n(p_i) - b_i \varphi_n(-p_i) \right) \left(\overline{a_j \varphi_{-n}(p_j) - b_j \varphi_{-n}(-p_j)} \right) \\ + \frac{rp_i \bar{p}_j}{1 - (rp_i \bar{p}_j)^2} \left(a_i \varphi_n(p_i) + b_i \varphi_n(-p_i) \right) \left(\overline{a_j \varphi_{-n}(p_j) + b_j \varphi_{-n}(-p_j)} \right).$$

Denoting $a_i - b_i = p_i d_i$ and $a_i + b_i = s_i$, the four factors in the above expression are written as

$$\begin{aligned}
a_i \varphi_n(p_i) - b_i \varphi_n(-p_i) &= \sum_{\mu=0}^{\infty} S_{\mu}(x(n)) p_i^{\mu} (a_i - (-1)^{\mu} b_i) \\
&= (p_i d_i, p_i s_i, p_i^3 d_i, p_i^3 s_i, \dots)^t (S_0(x(n)), S_1(x(n)), S_2(x(n)), S_3(x(n)), \dots), \\
\overline{a_j \varphi_{-n}(p_j) - b_j \varphi_{-n}(-p_j)} &= \sum_{\mu=0}^{\infty} \overline{S_{\mu}(x(-n)) p_j^{\mu} (a_j - (-1)^{\mu} b_j)} \\
&= (S_0(\overline{x(-n)}), S_1(\overline{x(-n)}), S_2(\overline{x(-n)}), S_3(\overline{x(-n)}), \dots)^t (\bar{p}_j \bar{d}_j, \bar{p}_j \bar{s}_j, \bar{p}_j^3 \bar{d}_j, \bar{p}_j^3 \bar{s}_j, \dots), \\
a_i \varphi_n(p_i) + b_i \varphi_n(-p_i) &= \sum_{\mu=0}^{\infty} S_{\mu}(x(n)) p_i^{\mu} (a_i + (-1)^{\mu} b_i) \\
&= (s_i, p_i^2 d_i, p_i^2 s_i, p_i^4 d_i, \dots)^t (S_0(x(n)), S_1(x(n)), S_2(x(n)), S_3(x(n)), \dots), \\
\overline{a_j \varphi_{-n}(p_j) + b_j \varphi_{-n}(-p_j)} &= \sum_{\mu=0}^{\infty} \overline{S_{\mu}(x(-n)) p_j^{\mu} (a_j + (-1)^{\mu} b_j)} \\
&= (S_0(\overline{x(-n)}), S_1(\overline{x(-n)}), S_2(\overline{x(-n)}), S_3(\overline{x(-n)}), \dots)^t (\bar{s}_j, \bar{p}_j^2 \bar{d}_j, \bar{p}_j^2 \bar{s}_j, \bar{p}_j^4 \bar{d}_j, \dots),
\end{aligned}$$

where ${}^t v$ means transpose of v . Thus we obtain

$$\begin{aligned}
A_{ij}^{(n)} &= \sum_{\lambda=0}^{\infty} (r p_i \bar{p}_j)^{2\lambda} (p_i d_i, p_i s_i, p_i^3 d_i, p_i^3 s_i, \dots)^t (S_0(x(n)), S_1(x(n)), S_2(x(n)), S_3(x(n)), \dots) \\
&\quad \times (S_0(\overline{x(-n)}), S_1(\overline{x(-n)}), S_2(\overline{x(-n)}), S_3(\overline{x(-n)}), \dots)^t (\bar{p}_j \bar{d}_j, \bar{p}_j \bar{s}_j, \bar{p}_j^3 \bar{d}_j, \bar{p}_j^3 \bar{s}_j, \dots) \\
&\quad + \sum_{\lambda=0}^{\infty} (r p_i \bar{p}_j)^{2\lambda+1} (s_i, p_i^2 d_i, p_i^2 s_i, p_i^4 d_i, \dots)^t (S_0(x(n)), S_1(x(n)), S_2(x(n)), S_3(x(n)), \dots) \\
&\quad \times (S_0(\overline{x(-n)}), S_1(\overline{x(-n)}), S_2(\overline{x(-n)}), S_3(\overline{x(-n)}), \dots)^t (\bar{s}_j, \bar{p}_j^2 \bar{d}_j, \bar{p}_j^2 \bar{s}_j, \bar{p}_j^4 \bar{d}_j, \dots) \\
&= (p_i d_i \quad p_i s_i \quad p_i^3 d_i \quad p_i^3 s_i \quad \dots) \begin{pmatrix} S_0(x(n)) & & & & 0 \\ S_1(x(n)) & S_0(x(n)) & & & \\ S_2(x(n)) & S_1(x(n)) & S_0(x(n)) & & \\ S_3(x(n)) & S_2(x(n)) & S_1(x(n)) & S_0(x(n)) & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & & & & 0 \\ & r & & & \\ & & r^2 & & \\ & & & r^3 & \\ 0 & & & & \ddots \end{pmatrix} \begin{pmatrix} S_0(\overline{x(-n)}) & S_1(\overline{x(-n)}) & S_2(\overline{x(-n)}) & S_3(\overline{x(-n)}) & \dots \\ & S_0(\overline{x(-n)}) & S_1(\overline{x(-n)}) & S_2(\overline{x(-n)}) & \dots \\ & & S_0(\overline{x(-n)}) & S_1(\overline{x(-n)}) & \dots \\ & & & S_0(\overline{x(-n)}) & \dots \\ 0 & & & & \ddots \end{pmatrix} \begin{pmatrix} \bar{p}_j \bar{d}_j \\ \bar{p}_j \bar{s}_j \\ \bar{p}_j^3 \bar{d}_j \\ \bar{p}_j^3 \bar{s}_j \\ \vdots \end{pmatrix}.
\end{aligned}$$

Therefore $\tau_k^t(n)$ in Theorem 1 is given in the form of the following determinant,

$$\tau_k^t(n) = \begin{vmatrix} \begin{pmatrix} p_1 d_1 & p_1 s_1 & p_1^3 d_1 & p_1^3 s_1 & \cdots \\ p_2 d_2 & p_2 s_2 & p_2^3 d_2 & p_2^3 s_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_N d_N & p_N s_N & p_N^3 d_N & p_N^3 s_N & \cdots \end{pmatrix} & \begin{pmatrix} S_0(x(n)) \\ S_1(x(n)) & S_0(x(n)) \\ S_2(x(n)) & S_1(x(n)) & S_0(x(n)) \\ S_3(x(n)) & S_2(x(n)) & S_1(x(n)) & S_0(x(n)) \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} & 0 \\ \times \begin{pmatrix} 1 & & & & 0 \\ & r & & & \\ & & r^2 & & \\ & & & r^3 & \\ 0 & & & & \ddots \end{pmatrix} & \begin{pmatrix} S_0(\overline{x(-n)}) & S_1(\overline{x(-n)}) & S_2(\overline{x(-n)}) & S_3(\overline{x(-n)}) & \cdots \\ S_0(\overline{x(-n)}) & S_1(\overline{x(-n)}) & S_2(\overline{x(-n)}) & S_3(\overline{x(-n)}) & \cdots \\ & S_0(\overline{x(-n)}) & S_1(\overline{x(-n)}) & S_2(\overline{x(-n)}) & \cdots \\ & & S_0(\overline{x(-n)}) & S_1(\overline{x(-n)}) & \cdots \\ & & & S_0(\overline{x(-n)}) & \cdots \end{pmatrix} & 0 \\ \times \begin{pmatrix} \bar{p}_1 \bar{d}_1 & \bar{p}_2 \bar{d}_2 & \cdots & \bar{p}_N \bar{d}_N \\ \bar{p}_1 \bar{s}_1 & \bar{p}_2 \bar{s}_2 & \cdots & \bar{p}_N \bar{s}_N \\ \bar{p}_1^3 \bar{d}_1 & \bar{p}_2^3 \bar{d}_2 & \cdots & \bar{p}_N^3 \bar{d}_N \\ \bar{p}_1^3 \bar{s}_1 & \bar{p}_2^3 \bar{s}_2 & \cdots & \bar{p}_N^3 \bar{s}_N \\ \vdots & \vdots & & \vdots \end{pmatrix} & \end{vmatrix}.$$

Now let us take

$$a_i = \frac{1}{2} \left(1 + \sum_{\nu=1}^i c_\nu p_i^{2\nu-1} \right), \quad b_i = \frac{1}{2} \left(1 - \sum_{\nu=1}^i c_\nu p_i^{2\nu-1} \right).$$

Then we have $d_i = \sum_{\nu=1}^i c_\nu p_i^{2\nu-2}$ and $s_i = 1$, and the above $\tau_k^t(n)$ is $O(p_1 \bar{p}_1 p_2 \bar{p}_2 \cdots p_N \bar{p}_N)$ as $p_i \rightarrow 0$ for $1 \leq i \leq N$. In order to take the lowest order in p_1 , we consider the limit, $\tilde{\tau}_k^t(n) := \lim_{p_1 \rightarrow 0} \tau_k^t(n) / (p_1 \bar{p}_1)$. In this limit, the leading order becomes $O((p_2 \bar{p}_2 \cdots p_N \bar{p}_N)^3)$, thus for picking up the lowest order in p_2 , we take the limit, $\lim_{p_2 \rightarrow 0} \tilde{\tau}_k^t(n) / (p_2 \bar{p}_2)^3$. So the leading order becomes $O((p_3 \bar{p}_3 \cdots p_N \bar{p}_N)^5)$. Repeating

this procedure, finally we obtain the τ function of rogue wave solution from that of breather, $\tau_k^t(n)$,

$$\begin{aligned}
& \lim_{p_N \rightarrow 0} \cdots \lim_{p_2 \rightarrow 0} \lim_{p_1 \rightarrow 0} \frac{\tau_k^t(n)}{p_1 \bar{p}_1 p_2^3 \bar{p}_2^3 \cdots p_N^{2N-1} \bar{p}_N^{2N-1}} \\
&= \left(\begin{array}{cccccccc} c_1 & 1 & & & & & & \\ c_2 & 0 & c_1 & 1 & & & & \\ c_3 & 0 & c_2 & 0 & c_1 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & \\ c_N & 0 & c_{N-1} & 0 & \cdots & \cdots & \cdots & c_1 & 1 \end{array} \right) \left(\begin{array}{cccccccc} S_0(x(n)) & & & & & & & 0 \\ S_1(x(n)) & S_0(x(n)) & & & & & & \\ S_2(x(n)) & S_1(x(n)) & S_0(x(n)) & & & & & \\ S_3(x(n)) & S_2(x(n)) & S_1(x(n)) & S_0(x(n)) & & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \end{array} \right) \\
& \quad \times \left(\begin{array}{cccc} 1 & & & 0 \\ & r & & \\ & & r^2 & \\ & & & r^3 \\ 0 & & & \ddots \end{array} \right) \left(\begin{array}{cccccc} S_0(\overline{x(-n)}) & S_1(\overline{x(-n)}) & S_2(\overline{x(-n)}) & S_3(\overline{x(-n)}) & \cdots \\ S_0(\overline{x(-n)}) & S_1(\overline{x(-n)}) & S_2(\overline{x(-n)}) & S_2(\overline{x(-n)}) & \cdots \\ & S_0(\overline{x(-n)}) & S_1(\overline{x(-n)}) & S_1(\overline{x(-n)}) & \cdots \\ & & S_0(\overline{x(-n)}) & S_0(\overline{x(-n)}) & \cdots \\ 0 & & & & \ddots \end{array} \right) \\
& \quad \times \left(\begin{array}{ccccc} \bar{c}_1 & \bar{c}_2 & \bar{c}_3 & \cdots & \bar{c}_N \\ 1 & 0 & 0 & \cdots & 0 \\ & \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_{N-1} \\ & 1 & 0 & \cdots & 0 \\ & & \bar{c}_1 & & \vdots \\ & & 1 & & \vdots \\ & & & \ddots & \vdots \\ & & & & \bar{c}_1 \\ 0 & & & & 1 \end{array} \right) \\
&= \left(\begin{array}{cccccccc} \Phi_1^{(n)} & \Phi_0^{(n)} & & & & & & \\ \Phi_3^{(n)} & \Phi_2^{(n)} & \Phi_1^{(n)} & \Phi_0^{(n)} & & & & \\ \Phi_5^{(n)} & \Phi_4^{(n)} & \Phi_3^{(n)} & \Phi_2^{(n)} & \Phi_1^{(n)} & \Phi_0^{(n)} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\ \Phi_{2N-1}^{(n)} & \Phi_{2N-2}^{(n)} & \Phi_{2N-3}^{(n)} & \Phi_{2N-4}^{(n)} & \cdots & \cdots & \cdots & \Phi_1^{(n)} & \Phi_0^{(n)} \end{array} \right) \left(\begin{array}{ccc} 1 & & 0 \\ & r & \\ & & r^2 \\ & & & r^3 \\ 0 & & & \ddots \end{array} \right) \\
& \quad \times \left(\begin{array}{cccc} \frac{\Phi_1^{(-n)}}{\Phi_0^{(-n)}} & \frac{\Phi_3^{(-n)}}{\Phi_2^{(-n)}} & \frac{\Phi_5^{(-n)}}{\Phi_4^{(-n)}} & \cdots & \frac{\Phi_{2N-1}^{(-n)}}{\Phi_{2N-2}^{(-n)}} \\ & \frac{\Phi_1^{(-n)}}{\Phi_0^{(-n)}} & \frac{\Phi_3^{(-n)}}{\Phi_2^{(-n)}} & \cdots & \frac{\Phi_{2N-3}^{(-n)}}{\Phi_{2N-4}^{(-n)}} \\ & & \frac{\Phi_1^{(-n)}}{\Phi_0^{(-n)}} & & \vdots \\ & & & \ddots & \vdots \\ & & & & \frac{\Phi_1^{(-n)}}{\Phi_0^{(-n)}} \\ & & & & \frac{\Phi_1^{(-n)}}{\Phi_0^{(-n)}} \\ & & & & \vdots \\ & & & & \vdots \\ & & & & \frac{\Phi_1^{(-n)}}{\Phi_0^{(-n)}} \\ 0 & & & & \frac{\Phi_1^{(-n)}}{\Phi_0^{(-n)}} \end{array} \right) \Big|_{11}
\end{aligned}$$

By calculating the matrix element, it is easy to see that the above determinant is equal to $\det_{1 \leq i, j \leq N} (B_{ij}^{(n)}(k, t))$. We completed the proof of Theorem 2. \square

By taking $N = 1$ we obtain the fully discrete Peregrine rogue wave solution,

$$\begin{aligned} \tau_k^t(n) = & (-(1+r)n + (1-r)k + (1-r + 1/\bar{c} - rc)t + c_1) \\ & \times ((1+r)n + (1-r)k + (1-r + 1/c - r\bar{c})t + \bar{c}_1) + r, \end{aligned} \quad (12)$$

where c_1 is a complex constant. Similarly to the breather solution, there are rogue wave solutions regular on the lattice for both focusing case ($r > 0$) and defocusing case ($r < 0$), since if there are zeros of f_k^t we can avoid explosion of solution by displacing the zeros off the lattice points. However for regularity of the solution on the real two dimensional space of (k, t) , we have to take r positive. First order and second-order rogue wave solutions are shown in Figs. 2 and 3, respectively.

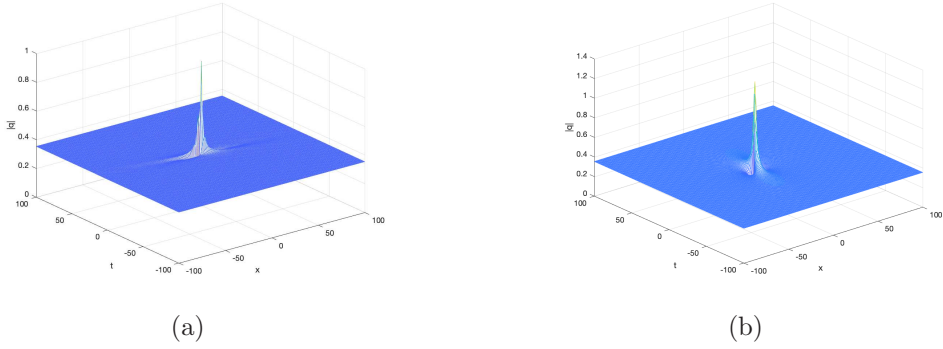


Figure 2: First order rogue wave solution (a) $r = 2.0$, $c = 1 + 2i$, $c_1 = 2$ (b) $r = 0.5$, $c = 1 + 2i$, $c_1 = 2$.

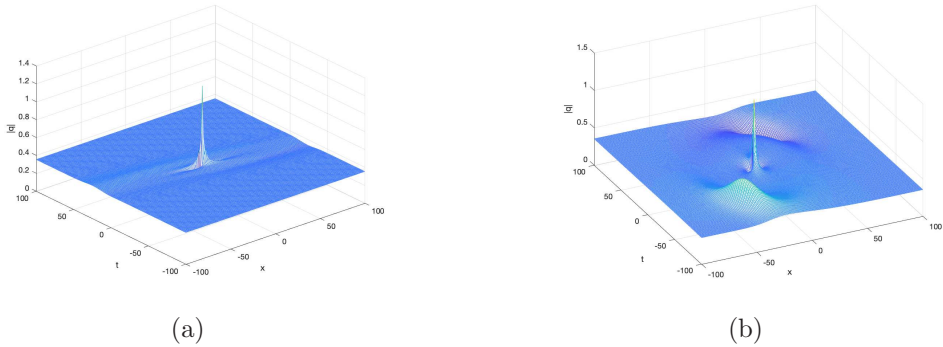


Figure 3: Second order rogue wave solution (a) $r = 2.0$, $c = 1 + 2i$, $c_1 = 2.0$, $c_2 = 2 + 2i$; (b) $r = 0.5$, $c = 2 + i$, $c_1 = 2$, $c_2 = 2 + 2i$.

There is an exceptional regular solution for $r < 0$ which is obtained by taking c real and $(\text{Im } c_1)^2 > -r$ in (12), but this is not a rogue wave solution but a traveling wave solution. An example is shown in Fig. 4.

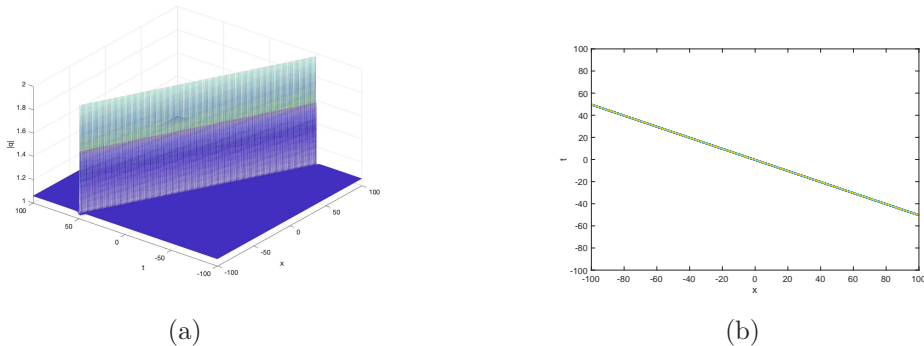


Figure 4: Traveling wave solution with $r = -0.5$, $c = 1.0$, $c_1 = 1 + i$: (a) profile, (b) contour plot.

4 Concluding Remarks

Even though the study of rogue waves has attracted much attention in more than one decade, the rogue wave solution in fully discrete integrable systems has not been reported yet. In this paper, by taking the fully discrete NLS equation, we firstly constructed its general breather solution via the KP-Toda reduction method. Then by taking a list to the parameters succeeded in constructing its general rogue wave solution by taking the limit of $p_i \rightarrow 0$ successively for $i = 1, \dots, N$.

We expect to construct rogue wave solutions in other discrete systems such as the discrete complex sine-Gordon equation in the future.

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