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Global in time self-interacting Dirac fields in the de Sitter space

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Abstract

In this paper the semilinear equation of the spin-$\frac{1}{2}$ fields in the de Sitter space is investigated. We prove the existence of the global in time small data solution in the expanding de Sitter universe. Then, under the Lochak-Majorana condition, we prove the existence of the global in time solution with large data. The sufficient conditions for the solutions to blow up in finite time are given for large data in the expanding and contracting de Sitter spacetimes. The influence of the Hubble constant on the lifespan is estimated.

0 Introduction

In this article we study the solvability and solutions of the semilinear Dirac equation in the de Sitter space, which is a member of the curved spacetimes of the Friedmann-Lemaître-Robertson-Walker (FLRW) models of cosmology (see, e.g., [11, 17]). More precisely, we derive conditions on the mass term, Hubble constant, nonlinear term, and initial function, which guarantee either global in time existence or an estimate of the lifespan of the solution to the Dirac equation in the expanding or contracting de Sitter universe.

Although the linear Dirac equation in the curved spacetime was known for a long time (see, e.g., [4, 10, 18, 19, 25]) and was well-investigated, and the semilinear Dirac equation in the Minkowski space was the focus of many publications (see, e.g., [1, 2, 7, 8, 15] and references therein), the semilinear Dirac equation in curved spacetime is, to the best of our knowledge, lacking any study. The latest astronomical discovery that the expansion of the universe is accelerating underscores the need to consider the equations of the quantum field theory in the curved spacetimes, especially in the inflationary theories of the early universe. We are motivated by the importance of the spin-$\frac{1}{2}$ particles in the expanding universe.

The metric tensor of the spatially flat de Sitter space has the scale factor $a(t) = e^{Ht}$ (see, e.g., [16]), which models the expanding or contracting universe if $H > 0$ or $H < 0$, respectively. The curvature of this space is $-12H^2$. The Dirac equation in the de Sitter space is (see, e.g., [6])

$$\left(i\gamma^0\partial_0 + ie^{-Ht}\gamma^1\partial_1 + ie^{-Ht}\gamma^2\partial_2 + ie^{-Ht}\gamma^3\partial_3 + i\frac{3}{2}H\gamma^0 - m\mathbb{I}_4\right)\psi = f\,,$$

where the contravariant gamma matrices are (see, e.g., [5, p. 61, Sec.6.2])

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} \mathbb{O}_2 & \sigma^k \\ -\sigma^k & \mathbb{O}_2 \end{pmatrix}, \quad k = 1, 2, 3,$$

$$\gamma^5 := -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{O}_2 & -\mathbb{I}_2 \\ -\mathbb{I}_2 & \mathbb{O}_2 \end{pmatrix} \text{ chirality matrix}\,.$$

Here $\partial_0 = \frac{\partial}{\partial t}$, $\partial_k = \frac{\partial}{\partial x_k}$, $k = 1,2,3$, while σ^1, σ^2, and σ^3 are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\mathbb{I}_n$, $\mathbb{O}_n$ denote the $n \times n$ identity and zero matrices, respectively.

Consider the Dirac equation

$$\left(i\gamma^0 \partial_0 + ie^{-Ht} \sum_{\ell=1,2,3} \gamma^\ell \partial_\ell + i\frac{3}{2} H \gamma^0 - m\mathbb{I}_4 + \gamma^0 V(x,t) \right) \psi = 0\,, \tag{0.1}$$

where the matrix-valued potential $V(x,t) : \mathbb{R}^4 \longrightarrow M_4(\mathbb{C})$ has the following structure

$$V(x,t) = \sum_{\ell=1,2,3} A_\ell(x,t)\alpha^j + A_0(x,t)\gamma^0 + V_0(x,t)\,.$$

In particular, it can be generated by the electromagnetic potential $(A_0(x,t), A_\ell(x,t))$ (the magnetic potential $\vec{A}$, the pseudoscalar potential A_0) $\vec{A} = (A_1(x,t), A_2(x,t), A_2(x,t)) : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$, $\quad A_0(x,t) : \mathbb{R}^4 \longrightarrow \mathbb{R}$, $\quad V_0 = V_0^* : \mathbb{R}^4 \longrightarrow M_4(\mathbb{C})$. We use the notation V^* for the complex conjugate transpose of a matrix: $V^*(x,t) = \overline{V(x,t)^T}$. Hence, the Dirac equation can be written as follows,

$$\left(\gamma^0 \left(i\partial_0 + A_0(x,t) \right) + \sum_{\ell=1,2,3} \gamma^\ell \left(ie^{-Ht}\partial_\ell + A_\ell(x,t) \right) + i\frac{3}{2} H\gamma^0 - m\mathbb{I}_4 + \gamma^0 V_0(x,t) \right) \psi = 0\,,$$

with the notation

$$\alpha^k = \gamma^0\gamma^k, \quad (\alpha^k)^* = \alpha^k, \quad k = 1,2,3,$$

$$\alpha^1 = \begin{pmatrix} \mathbb{O}_2 & \sigma_1 \\ \sigma_1 & \mathbb{O}_2 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} \mathbb{O}_2 & \sigma_2 \\ \sigma_2 & \mathbb{O}_2 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} \mathbb{O}_2 & \sigma_3 \\ \sigma_3 & \mathbb{O}_2 \end{pmatrix},$$

it can be also written as follows

$$\left(\partial_0 + e^{-Ht} \sum_{\ell=1,2,3} \alpha^\ell \partial_\ell + \frac{3}{2} H\mathbb{I}_4 + im\gamma^0 - iV(x,t) \right) \psi = 0\,.$$

In this paper, we study the semilinear Dirac equation, which describes the self-interacting field due to the nonlinear term $F = F(\psi)$ satisfying the following condition.

Condition ($\mathcal{L}$) The function $F = F(\psi) \in C^3(\mathbb{C}^4; \mathbb{C}^4)$ is Lipschitz continuous with exponent α in the space $H_{(s)}(\mathbb{R}^3)$, that is, there exists a constant $C > 0$ such that

$$\|F(\psi_1) - F(\psi_2)\|_{H_{(s)}(\mathbb{R}^3)} \le C \|\psi_1 - \psi_2\|_{H_{(s)}(\mathbb{R}^3)} \left(\|\psi_1\|^\alpha_{H_{(s)}(\mathbb{R}^3)} + \|\psi_2\|^\alpha_{H_{(s)}(\mathbb{R}^3)} \right).$$

The polynomial in ψ vector-valued functions and the functions $F(\psi) = (\gamma^0\psi, \psi)\gamma^0\psi$, $F(\psi) = |\gamma^0\gamma^5\psi|^\alpha\psi$, $F(\psi) = |\gamma^0\gamma^5\psi|^\alpha\gamma^0\psi$, $F(\psi) = \pm|\psi|^{1+\alpha}$, $F(\psi) = \pm|\psi|^\alpha\psi$ are important examples of the Lipschitz continuous with exponent $\alpha > 0$ in the Sobolev space $H_{(s)}(\mathbb{R}^3)$, $s \ge 3$, functions.

In the next theorem the mass m is allowed to assume a complex value $m \in \mathbb{C}$, taking into account that in the important cases of $m = 0, \pm iH$ the Dirac equation obeys Huygens' principle [27, 31]. Define the space

$$X(R,s,\gamma) := \left\{ \varphi(x,t) \in C([0,\infty); H_{(s)}(\mathbb{R}^3)) \,\middle|\, \|\varphi\|_X := \sup_{[0,\infty)} e^{\gamma t} \|\varphi(x,t)\|_{H_{(s)}(\mathbb{R}^3)} \le R \right\}.$$

For the potential V we write $V \in \mathscr{B}_{(k;\ell)}$ that implies that all entries of the matrix V belong to the space

$$\mathscr{B}^{(\ell,k)} := \left\{ v \in C^{\ell,k}_{t,x}([0,\infty) \times \mathbb{R}^3) \,\middle|\, \partial_t^j \partial_x^\alpha v(x,t) \in L^\infty([0,\infty) \times \mathbb{R}^3),\ \forall\alpha,\ |\alpha| \le k,\ \forall j \le \ell \right\}.$$

Theorem 0.1 *Let $F = F(\psi) \in C^3(\mathbb{C}^4;\mathbb{C}^4)$ be the Lipschitz continuous with exponent $\alpha > 0$ in the space $H_{(s)}(\mathbb{R}^3)$, $s \geq 3$, function, and the potential $V \in \mathscr{B}^{(s,0)}$ is self-adjoint, $V(x,t) = V^*(x,t)$. Assume also that $2|\Im(m)| < 3H$.*

Then there is $\varepsilon_0 > 0$ such that for all ε and $\psi_0 \in H_{(s)}$, $\|\psi_0\|_{(s)} \leq \varepsilon < \varepsilon_0$, the problem

$$\begin{cases} \left(\partial_0 + e^{-Ht} \sum\limits_{\ell=1,2,3} \alpha^\ell \partial_\ell + \dfrac{3}{2} H \mathbb{I}_4 + im\gamma^0 - iV(x,t) \right) \psi = F(\psi)\,, \\ \psi(x,0) = \psi_0(x) \end{cases} \tag{0.2}$$

has a global solution $\psi \in X(2\varepsilon, s, \frac{3}{2}H - |\Im(m)|)$. The solution scatters to a free solution as $t \to +\infty$.

We do not know if the condition $3H > 2|\Im(m)|$ is necessary for the global existence of the solution to the equation of (0.2). Since the Dirac equation (0.1) is non-invariant with respect to time inversion and its solutions have different properties in different directions of time, we claim in Theorem 1.3 (Subsection 1.3) only the asymptotic behavior of the solution at large positive time. In fact, the solution is asymptotically free under the following assumption:

$$4|\Im(m)| + 2|\Im(m)|\alpha < 3H\alpha\,.$$

In the case of $V(x,t) \equiv 0$, the asymptotics at $t \to +\infty$ is written via the explicit representation formulas, which are derived in [30] for the solution of the Dirac equation in the de Sitter spacetime.

Concerning small amplitude global solutions of the nonlinear Dirac equation in the Minkowski space ($H = 0$), one can consult [15], where one can also find previous references on that topic. Machihara, Nakamura, Nakanishi, and Ozawa [15] considered the nonlinear Dirac equation with the real mass and cubic nonlinearity $F(\psi) = -i\lambda(\gamma^0\psi,\psi)\gamma^0\psi$. The global well-posedness for small $H_{(1)}(\mathbb{R}^3)$ data with a slight regularity for angular variables is proven in [15] by using the endpoint Strichartz estimates.

Next we consider the Cauchy problem with large data

$$\begin{cases} \left(i\gamma^0\partial_0 + ie^{-Ht} \sum\limits_{\ell=1,2,3} \gamma^\ell \partial_\ell + i\dfrac{3}{2} H \gamma^0 - m\mathbb{I}_4 + \gamma^0 V(x,t) \right) \psi(x,t) \\ \qquad\qquad\qquad = F\left(\psi^*\gamma^0\psi\,,\, \psi^*\gamma^0\gamma^5\psi\right)\psi(x,t), \quad t > 0, \\ \psi(x,0) = \Psi_0(x) + \varepsilon\chi_0(x). \end{cases} \tag{0.3}$$

Here the function $F = F(\xi,\eta)$, $F \in C^\infty(\mathbb{R}^2;\mathbb{C}^4)$, has the form

$$F\left(\psi^*\gamma^0\psi\,,\, \psi^*\gamma^0\gamma^5\psi\right) = \alpha\left(\psi^*\gamma^0\psi\,,\, \psi^*\gamma^0\gamma^5\psi\right)\mathbb{I}_4 + i\beta\left(\psi^*\gamma^0\psi\,,\, \psi^*\gamma^0\gamma^5\psi\right)\gamma^5, \tag{0.4}$$

where α and β are real-valued functions and

$$\alpha(\xi,\eta) = O(|\xi| + |\eta|), \quad \beta(\xi,\eta) = O(|\xi| + |\eta|), \quad |\xi| + |\eta| \to 0\,. \tag{0.5}$$

The equation is a symmetric hyperbolic system, and the local existence of the solution is known (see, e.g., [24]). The local Cauchy problem for (0.3) is well posed in $C^0([0,T]; (H_{(s)}(\mathbb{R}^3))^4)$, $s \geq 3$, for some $T > 0$ (see, e.g., [14]).

The main result of this paper is the following statement.

Theorem 0.2 *Assume that $m \in \mathbb{R}$, $H > 0$, and the potential $V \in \mathscr{B}^{(\infty,\infty)}$ is self-adjoint, $V^*(x,t) = V(x,t)$. Moreover, assume that*

$$V^T(x,t)\gamma^2 + \gamma^2 V(x,t) = 0\,, \tag{0.6}$$

while F takes the form (0.4) with (0.5). Assume also that the function $\Psi_0 = \Psi_0(x) \in C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$ satisfies the Lochak-Majorana condition

$$\rho^2(\Psi_0(x)) := |\Psi_0^*(x)\gamma^0\Psi_0(x)|^2 + |\Psi_0^*(x)\gamma^0\gamma^5\Psi_0(x)|^2 = 0 \quad \textit{for all} \quad x \in \mathbb{R}^3\,. \tag{0.7}$$

Then for $\chi_0 \in C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$ there exists an $\varepsilon_0 > 0$ such that the Cauchy problem (0.3) with $0 < \varepsilon < \varepsilon_0$ has a unique solution $\psi = \psi(x,t)$ such that as a function of time $\psi(t) \in C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$ for all $t \in (0,\infty)$. The solution scatters to the solution of the free Dirac equation as $t \to +\infty$.

Thus, the global in time solutions exist for the initial data from the small neighborhood of the unbounded conic set of functions satisfying the Lochak-Majorana condition. The last condition is independent of the mass m, the value of the positive Hubble constant $H > 0$, and, consequently, the curvature of spacetime. Another aim of the present article is to gain insight into the asymptotic behavior of an open set of large data solutions. In order to write an asymptotic at large time for the solutions given by Theorems 0.1,0.2, we use in Theorems 1.3, 4.1 the explicit representation formulas for the solution of the Dirac equation in the de Sitter spacetime, which are obtained in [30].

Remark 0.3 *If the potential V is independent of x, then the condition of the boundedness of the V can be dropped.*

The global existence of large amplitude solutions for the nonlinear massless Dirac equation in the Minkowski space was considered by Bachelot [2]. The only smallness assumption in [2] was on the chiral invariant related to the Lochak-Majorana condition. In [2] the asymptotic behavior of the solution, particularly the equipartition of energy and the decay of Lorentz-invariant products, were also studied. Bachelot in [2] developed the approach that appeals to the estimates obtained by the replacing the generators of the Poincaré group with the Fermi operators. That method requires certain regularity of the data in order to be applied.

D'Ancona and Okamoto [8] studied a massless cubic Dirac equation in the Minkowski space

$$\left(i\partial_0 + i \sum_{\ell=1,2,3} \alpha^\ell \partial_\ell + V(x) \right) \psi = \langle \gamma^0 u, u \rangle \gamma^0 u \,,$$

perturbed by a large potential with almost critical regularity. They proved global existence and scattering for small initial data in $H_{(1)}$ with additional angular regularity. The main tool was the endpoint Strichartz estimate for the perturbed Dirac flow that was proved by the arguments of [9]. In particular, the result of [8] covers the case of spherically symmetric data with small $H_{(1)}$ norm. In the absence of the magnetic potential $\vec{A}$, and when 0 is not a resonance for $i\sum_{\ell=1,2,3} \alpha^\ell \partial_\ell + V(x)$, and the potential $V = A_0\gamma^0 + V_0$ satisfies (0.6), D'Ancona and Okamoto proved the global existence and scattering for large initial data having a small chiral component, related to the Lochak-Majorana condition.

In [7], Candy and Herr proved that the Cauchy problem for the cubic Dirac equation in the Minkowski space is globally well-posed and that its solutions scatter to free solutions as $t \to \pm\infty$.

The rest of this paper is organized as follows. In Section 1, we prove Theorem 0.1 by derivation of the energy estimate (subsection 1.1) and fixed point argument (subsection 1.2). The asymptotics on the positive half-line of time and the representation of the solution of the free Dirac equation in de Sitter space are given in Theorem 1.3 (subsection 1.3). In Section 2, we analyze the Lochak-Majorana condition in the de Sitter space and obtain its time-evolution. In Section 3, we prove Theorem 0.2 except the asymptotics part that follows from Theorem 4.1 of Section 4. There, we also give the asymptotics at infinity for the solutions of Theorem 0.2. In Section 5, we show the blow-up result for the large data solution of the Dirac equation in expanding (Theorem 5.2, $H > 0$) and contracting (Theorem 5.3, $H < 0$) universes. Finally, in Section 6, we prove that the classical solutions of the semilinear Dirac equation obey the finite propagation speed property.

1 Small data global existence

1.1 Proof of Theorem 0.1. Energy estimate

First we develop the energy estimates for the solution of the equation. Denote the Dirac operator in the de Sitter space by

$$\mathscr{D}_{dS}(x,t,\partial) := \partial_0 + e^{-Ht} \sum_{\ell=1,2,3} \alpha^\ell \partial_\ell + \frac{3}{2} H \mathbb{I}_4 + im\gamma^0 - iV(x,t)$$

and

$$\delta_+ := -3H + 2|\Im(m)|, \qquad \delta_- := -3H - 2|\Im(m)| \,. \tag{1.1}$$

Lemma 1.1 *Assume that $H \in \mathbb{R}$, $f \in C([0,\infty);(H_{(k)}(\mathbb{R}^3))^4)$, and that the potential $V \in \mathscr{B}_{(0,k)}$ is self-adjoint, $V(x,t) = V(x,t)^*$. Then for the solution $\psi = \psi(x,t)$ of the equation*

$$\mathscr{D}_{dS}(x,t,\partial)\psi(x,t) = f(x,t) \tag{1.2}$$

in the Sobolev space $H_{(k)}(\mathbb{R}^3)$ one has

$$\begin{aligned} \|\psi(t)\|_k &\le C_k e^{\frac{1}{2}\delta_+(t-s)}\|\psi(s)\|_k + C_k e^{\frac{1}{2}\delta_+ t}\int_s^t e^{-\frac{1}{2}\delta_+\tau}\|f(\tau)\|_k\,d\tau \quad \text{for all} \quad s \le t\,,\\ \|\psi(t)\|_k &\le C_k e^{\frac{1}{2}\delta_-(t-s)}\|\psi(s)\|_k + C_k e^{\frac{1}{2}\delta_- t}\int_t^s e^{-\frac{1}{2}\delta_-\tau}\|f(\tau)\|_k\,d\tau \quad \text{for all} \quad t \le s\,. \end{aligned}$$

Proof. Consider the energy integral

$$E(t) = \int_{\mathbb{R}^3} |\psi(x,t)|^2\,dx\,.$$

Then, by the finite speed propagation property (see Section 6)

$$\frac{d}{dt}E(t) = \int_{\mathbb{R}^3}\left[-3H|\psi(x,t)|^2 + 2\Im(m)\psi^*(x,t)\gamma^0\psi(x,t)\right]\,dx + \int_{\mathbb{R}^3} 2\Re\left[f^*(x,t)\psi(x,t)\right]\,dx\,.$$

Here the following identities have been used

$$(\partial_k\psi^*(x,t))\alpha^k\psi(x,t) + \psi^*(x,t)\alpha^k\partial_k\psi(x,t) = \partial_k(\psi^*(x,t)\alpha^k\psi(x,t))\,,$$

$$\int_{\mathbb{R}^3}\left[(\partial_k\psi^*(x,t))\alpha^k\psi(x,t) + \psi^*(x,t)\alpha^k\partial_k\psi(x,t)\right]\,dx = 0\,,$$

since $(\alpha^k)^* = \alpha^k$ $(k = 1,2,3)$. The assumption $H \in \mathbb{R}$ implies

$$\begin{aligned} (-3H - 2|\Im(m)|)\,|\psi(x,t)|^2 &\le -3H|\psi(x,t)|^2 + 2\Im(m)\psi^*(x,t)\gamma^0\psi(x,t)\\ &\le (-3H + 2|\Im(m)|)\,|\psi(x,t)|^2\,. \end{aligned}$$

Then

$$\delta_-|\psi(x,t)|^2 \le -3H|\psi(x,t)|^2 + 2\Im(m)\psi^*(x,t)\gamma^0\psi(x,t) \le \delta_+|\psi(x,t)|^2$$

and

$$\delta_- E(t) + \int_{\mathbb{R}^3} 2\Re\left[f^*(x,t)\psi(x,t)\right]\,dx \le \frac{d}{dt}E(t) \le \delta_+ E(t) + \int_{\mathbb{R}^3} 2\Re\left[f^*(x,t)\psi(x,t)\right]\,dx\,. \tag{1.3}$$

In particular, the right-hand side gives

$$\frac{d}{dt}E(t) \le \delta_+ E(t) + \int_{\mathbb{R}^3} 2\Re\left[f^*(x,t)\psi(x,t)\right]\,dx$$

that leads to

$$\frac{d}{dt}\left(E(t)e^{-\delta_+ t}\right) \le 2e^{-\delta_+ t}\|f(x,t)\|\|\psi(x,t)\|\,.$$

We integrate it from s to t:

$$E(t)e^{-\delta_+ t} \le E(s)e^{-\delta_+ s} + 2\int_s^t e^{-\delta_+\tau}\|f(\tau)\|\|\psi(\tau)\|\,d\tau, \quad s \le t\,.$$

If we fix s and denote

$$y(t) := \max_{\tau\in[s,t]} e^{-\frac{1}{2}\delta_+\tau}\|\psi(\tau)\|_k, \quad y^2(t) = \max_{\tau\in[s,t]} e^{-\delta_+\tau}\|\psi(\tau)\|_k^2\,,$$

then for $k=0$ the inequality

$$y^2(t) \le y^2(s) + 2y(t)\int_s^t e^{-\frac{1}{2}\delta_+\tau}\|f(\tau)\|_k\,d\tau$$

yields

$$y^2(t) \le c^2y^2(s) + c^2\left(\int_s^t e^{-\frac{1}{2}\delta_+\tau}\|f(\tau)\|_k\,d\tau\right)^2 .$$

Consequently, for $k=0$ we have obtained

$$e^{-\frac{1}{2}\delta_+t}\|\psi(t)\|_k \le ce^{-\frac{1}{2}\delta_+s}\|\psi(s)\|_k + c\int_s^t e^{-\frac{1}{2}\delta_+\tau}\|f(\tau)\|_k\,d\tau\,, \quad s\le t\,,$$

that is, the first inequality of the lemma.

Next, we choose the left-hand side of (1.3)

$$\delta_-E(t) + \int_{\mathbb{R}^3} 2\Re\left[f^*(x,t)\psi(x,t)\right]\,dx \le \frac{d}{dt}E(t)$$

and rewrite it as follows:

$$e^{-\delta_-t}\int_{\mathbb{R}^3} 2\Re\left[f^*(x,t)\psi(x,t)\right]\,dx \le \frac{d}{dt}\left(E(t)e^{-\delta_-t}\right) .$$

We integrate the last inequality from t to s, $t<s$:

$$\int_t^s e^{-\delta_-t}\int_{\mathbb{R}^3} 2\Re\left[f^*(x,t)\psi(x,t)\right]\,dx \le E(s)e^{-\delta_-s} - E(t)e^{-\delta_-t} .$$

It follows

$$-2\int_t^s e^{-\delta_+\tau}\|f(\tau)\|\|\psi(\tau)\|\,d\tau \le E(s)e^{-\delta_-s} - E(t)e^{-\delta_-t}$$

and

$$E(t)e^{-\delta_-t} \le E(s)e^{-\delta_-s} + 2\int_t^s e^{-\delta_-\tau}\|f(\tau)\|\|\psi(\tau)\|\,d\tau\,, \quad t\le s\,.$$

If we fix s and denote

$$y_-(t) := \max_{\tau\in[t,s]} e^{-\frac{1}{2}\delta_-\tau}\|\psi(\tau)\|_k, \quad y_-^2(t) = \max_{\tau\in[t,s]} e^{-\delta_-\tau}\|\psi(\tau)\|_k^2\,,$$

then for $k=0$ the estimate

$$y_-^2(t) \le y_-^2(s) + 2y_-(t)\int_t^s e^{-\frac{1}{2}\delta_-\tau}\|f(\tau)\|\,d\tau\,, \quad t\le s\,,$$

implies

$$y_-(t) \le cy_-(s) + c\int_t^s e^{-\frac{1}{2}\delta_-\tau}\|f(\tau)\|\,d\tau\,, \quad t\le s\,,$$

that yields for $k=0$

$$e^{-\frac{1}{2}\delta_-t}\|\psi(t)\|_k \le ce^{-\frac{1}{2}\delta_-s}\|\psi(s)\|_k + c\int_t^s e^{-\frac{1}{2}\delta_-\tau}\|f(\tau)\|_k\,d\tau\,, \quad t\le s\,,$$

that is, the second inequality of the lemma. For every $k>0$ the inequalities follow from the case of $k=0$ by differentiation. The lemma is proved. $\square$

1.2 Proof of Theorem 0.1

We are going to apply the Banach fixed-point theorem. Denote by $S(t,s)$ the propagator (fundamental solution for the Cauchy problem), that is, an operator-valued solution of the problem

$$\left\{ \begin{array}{l} \mathscr{D}_{dS}(x,t,\partial)S(t,s) = 0\,, \quad t,s \in \mathbb{R}\,, \\ S(s,s) = I \,(\text{identity operator})\,. \end{array} \right. \tag{1.4}$$

Then the solution of the problem

$$\left\{ \begin{array}{l} \mathscr{D}_{dS}(x,t,\partial)\psi(x,t) = f(x,t)\,, \quad t \geq s\,, \\ \psi(x,s) = \psi_0(x)\,, \end{array} \right.$$

is given by Duhamel's principle

$$\psi(x,t) = S(t,s)\psi_0(x) + \int_s^t S(t,\tau)f(x,\tau)\,d\tau\,.$$

It is known (see, e.g., [14, 23]) that for $\psi_0 \in H_{(k)}(\mathbb{R}^3)$ and $f \in C([0,\infty); H_{(k)}(\mathbb{R}^3))$, $k > 5/2$, the unique solution $\psi \in C([0,\infty); H_{(k)}(\mathbb{R}^3)) \cap C^1([0,\infty); H_{(k-1)}(\mathbb{R}^3))$ exists.

Next, we define the operator $\mathcal{S}$ by

$$\mathcal{S}\psi(x,t) := S(t,0)\psi_0(t) + \int_0^t S(t,\tau)F(\psi(x,\tau))\,d\tau\,.$$

We are going to prove that $\mathcal{S}$ is a contraction such that

$$\mathcal{S} : X(R,s,\gamma) \longrightarrow X(R,s,\gamma)$$

for sufficiently small ε_0 and R.

According to Lemma 1.1, with $k > 3/2$ we have

$$\|\psi(t)\|_k \leq ce^{\frac{1}{2}\delta_+(t-s)}\|\psi(s)\|_k + ce^{\frac{1}{2}\delta_+t}\int_s^t e^{-\frac{1}{2}\delta_+\tau}\|f(\tau)\|_k\,d\tau\,, \quad s < t\,.$$

Set $s = 0$ and $\delta := -\frac{1}{2}\delta_+ = \frac{1}{2}(3H - 2|\Im(m)|) > 0$ in the last inequality, then by using the condition $(\mathcal{L})$ we obtain

$$\begin{aligned} \|\psi(t)\|_k &\leq ce^{-\delta t}\|\psi(0)\|_k + ce^{-\delta t}\int_0^t e^{\delta\tau}\|F(\psi(\tau))\|_k\,d\tau \\ &\leq ce^{-\delta t}\|\psi(0)\|_k + ce^{-\delta t}\int_0^t e^{\delta\tau}\|\psi(\tau)\|_k^{1+\alpha}\,d\tau, \quad t > 0\,. \end{aligned}$$

Hence,

$$\begin{aligned} \sup_{t\in[0,\infty)} e^{\delta t}\|\psi(t)\|_k &\leq c\|\psi(0)\|_{(k)} + c\left(\sup_{t\in[0,\infty)} e^{\delta t}\|\psi(t)\|_k\right)^{1+\alpha}\int_0^t e^{-\alpha\delta\tau}\,d\tau \\ &\leq c\|\psi(0)\|_{(k)} + c(\alpha\delta)^{-1}(1-e^{-\alpha\delta t})\left(\sup_{t\in[0,\infty)} e^{\delta t}\|\psi(t)\|_k\right)^{1+\alpha} \\ &\leq C\|\psi(0)\|_{(k)} + C(\alpha\delta)^{-1}\left(\sup_{t\in[0,\infty)} e^{\delta t}\|\psi(t)\|_k\right)^{1+\alpha}\,. \end{aligned}$$

Thus, the last inequality proves that the operator $\mathcal{S}$ maps

$$X(R,k,\delta) := \left\{\varphi(x,t) \in C([0,\infty); H_{(k)})\,\Big|\, \|\varphi\|_X := \sup_{[0,\infty)} e^{\delta t}\|\varphi(x,t)\|_{H_{(k)}} \leq R\right\}$$

into itself provided that the initial function $\psi_0(x) \in H_{(k)}$, $\|\psi_0\|_{H_{(k)}} < \varepsilon$, and that the numbers ε and R are sufficiently small, namely, if

$$C\varepsilon + C\frac{1}{\alpha\delta}R^{1+\alpha} < R\,.$$

In order to verify a contraction property, we write

$$\mathcal{S}\psi_1(x,t) - \mathcal{S}\psi_2(x,t) = \int_0^t S(t,\tau)\left(F(\psi_1(x,\tau)) - F(\psi_2(x,\tau))\right)\,d\tau$$

and use Lemma 1.1 and the condition $(\mathcal{L})$ to estimate the norm

$$\begin{aligned}
\|\mathcal{S}\psi_1(x,t) - \mathcal{S}\psi_2(x,t)\|_k &\le ce^{-\delta t}\int_0^t e^{\delta\tau}\|F(\psi_1(\tau)) - F(\psi_2(\tau))\|_k\,d\tau \\
&\le Ce^{-\delta t}\int_0^t e^{\delta\tau}\|\psi_1(\tau) - \psi_2(\tau)\|_k\left(\|\psi_1(\tau)\|_k^\alpha + \|\psi_2(\tau)\|_k^\alpha\right)\,d\tau\,.
\end{aligned}$$

It follows

$$\begin{aligned}
&\sup_{t\in[0,\infty)} e^{\delta t}\|\mathcal{S}\psi_1(x,t) - \mathcal{S}\psi_2(x,t)\|_k \\
\le\ & C\left(\sup_{\tau\in[0,\infty)} e^{\delta\tau}\|\psi_1(\tau) - \psi_2(\tau)\|_k\right)\int_0^t e^{-\alpha\delta\tau}e^{\alpha\delta\tau}\left(\|\psi_1(\tau)\|_k^\alpha + \|\psi_2(\tau)\|_k^\alpha\right)\,d\tau \\
\le\ & C\left(\sup_{\tau\in[0,\infty)} e^{\delta\tau}\|\psi_1(\tau) - \psi_2(\tau)\|_k\right)\left(\sup_{\tau\in[0,\infty)} e^{\delta\tau}\left(\|\psi_1(\tau)\|_k + \|\psi_2(\tau)\|_k\right)\right)^\alpha\int_0^t e^{-\alpha\delta\tau}\,d\tau \\
\le\ & C_1\left(\sup_{\tau\in[0,\infty)} e^{\delta\tau}\|\psi_1(\tau) - \psi_2(\tau)\|_k\right)\left(\sup_{\tau\in[0,\infty)} e^{\delta\tau}\left(\|\psi_1(\tau)\|_k + \|\psi_2(\tau)\|_k\right)\right)^\alpha \\
\le\ & C_1 d(\psi_1,\psi_2)2^\alpha R^\alpha\,.
\end{aligned}$$

Then we choose R such that $C_1 2^\alpha R^\alpha < 1$. Banach fixed-point theorem completes the proof of Theorem 0.1. □

1.3 Large time asymptotics

For the given initial function $\psi_0(x)$ we want to show that there are the solution $\psi(x,t)$ of the problem

$$\begin{cases} \mathscr{D}_{dS}(x,t,\partial)\psi(x,t) = F(\psi(x,t))\,, \\ \psi(x,0) = \psi_0(x)\,, \end{cases} \tag{1.5}$$

and the solution $\psi^+(x,t)$ of the free Dirac equation

$$\begin{cases} \mathscr{D}_{dS}(x,t,\partial)\psi^+(x,t) = 0\,, \\ \psi^+(x,0) = \psi_0^+(x)\,, \end{cases} \tag{1.6}$$

such that

$$\lim_{t\to+\infty}\left\|\psi(x,t) - \psi^+(x,t)\right\|_{(H_{(k)}(\mathbb{R}^3))^4} = 0\,. \tag{1.7}$$

From the viewpoint of scattering theory the function $\psi(x,t)$ has free asymptotics as $t \to +\infty$. The mapping $\mathscr{S}_+ : \psi_0 \longmapsto \psi_0^+$ is related to the *wave operator* W_+ of the scattering theory if (1.7) is fulfilled (see, e.g., [21, 28]).

We are going to prove the existence of the operator $\mathscr{S}_+$. Let $S(t,s)$ be the propagator, that is, the solution of the problem (1.4), then

$$\psi(x,t) = S(t,s)\psi(x,s)\,, \qquad t,s \in \mathbb{R}, \quad x \in \mathbb{R}^3\,.$$

The operator $S(t,0)$ is written in the explicit form in [30]. To find for the operator $S(t,s)$ a similar explicit form one can use the symmetry property of the Dirac operator and of the de Sitter spacetime (see, [26, Sec.142]) .

With the aid of the operator $S(t,s)$, we look for the solution of (1.5) via the following integral equation

$$\psi(x,t) = \psi^+(x,t) - \int_t^\infty S(t,\tau)F(\psi(x,\tau))\,d\tau\,,$$

provided that the integral is convergent. For the initial condition we have

$$\psi_0^+(x) = \psi_0(x) + \int_0^\infty S(0,\tau)F(\psi(x,\tau))\,d\tau\,. \tag{1.8}$$

To prove the existence of the operator $\mathscr{S}_+$ it suffices a convergence of the integral of (1.8).

Lemma 1.2 *Let $F = F(\psi) \in C^3(\mathbb{C}^4;\mathbb{C}^4)$ be the Lipschitz continuous with exponent $\alpha > 0$ in the space $H_{(k)}(\mathbb{R}^3)$, $k \geq 3$, function. Assume that the potential $V \in \mathscr{B}_{(0,k)}$ is self-adjoint, $V(x,t) = V(x,t)^*$, $4|\Im(m)| + 2|\Im(m)|\alpha < 3H\alpha$, and the function $\psi \in C([0,\infty);(H_{(k)}(\mathbb{R}^3))^4)$ solves the problem (0.2).*

Then the limit

$$\lim_{t\to+\infty}\int_0^t S(0,\tau)F(\psi(x,\tau))\,d\tau \tag{1.9}$$

exists in the space $(H_{(k)}(\mathbb{R}^3))^4$.

Proof. According to Lemma 1.1 we have

$$\begin{aligned} \|S(0,\tau)F(\psi(x,\tau))\|_k &\leq ce^{\frac{1}{2}\delta_-(-\tau)}\|F(\psi(x,\tau))\|_k \\ &\leq ce^{\frac{1}{2}\delta_-(-\tau)}\|\psi(x,\tau)\|_k^{1+\alpha}\,, \quad 0<\tau. \end{aligned}$$

Then the existence of the limit follows from

$$\begin{aligned} \|S(0,\tau)F(\psi(x,\tau))\|_k &\leq ce^{\frac{1}{2}\delta_-(-\tau)}\|\psi(x,\tau)\|_k^{1+\alpha} \\ &\leq c_1e^{-\frac{1}{2}(-3H-2|\Im(m)|)\tau+(-\frac{1}{2}(3H-2|\Im(m)|)\tau)(1+\alpha)}\|\psi(x,0)\|_k^{1+\alpha}\,, \end{aligned}$$

with δ_+ and δ_- of (1.1). According to the condition of the lemma

$$-\frac{1}{2}(-3H-2|\Im(m)|) + (-\frac{1}{2}(3H-2|\Im(m)|))(1+\alpha) < 0$$

and, consequently, the limit (1.9) exists. The lemma is proved. $\square$

Next we give an explicit representation formula for the solution $\psi^+(x,t)$ that will be used in the theorem on the existence of the operator $\mathscr{S}_+$. We define forward and backward light cones as the boundaries of $D_+(x_0,t_0)$ and $D_-(x_0,t_0)$, respectively, where

$$D_\pm\,(x_0,t_0) := \left\{(x,t)\in\mathbb{R}^{3+1}\;|\;|x-x_0| \leq \pm\,(\phi(t)-\phi(t_0))\right\}\,, \tag{1.10}$$

and $\phi(t) := (1-e^{-Ht})/H$ is a distance function. For $(x_0,t_0)\in\mathbb{R}^3\times\mathbb{R}$, $M\in\mathbb{C}$, $r = |x-x_0|/H$, we follow [29, 30] and define the function

$$\begin{aligned} E(r,t;0,t_0;M) \;&:=\; 4^{-\frac{M}{H}}e^{M(t_0+t)}\left(\left(e^{-Ht_0}+e^{-Ht}\right)^2-(Hr)^2\right)^{\frac{M}{H}-\frac{1}{2}} \\ &\quad\times F\left(\frac{1}{2}-\frac{M}{H},\frac{1}{2}-\frac{M}{H};1;\frac{\left(e^{-Ht}-e^{-Ht_0}\right)^2-(rH)^2}{\left(e^{-Ht}+e^{-Ht_0}\right)^2-(rH)^2}\right)\,, \end{aligned}$$

where $(x,t)\in D_+(x_0,t_0)\cup D_-(x_0,t_0)$ and $F\big(a,b;c;\zeta\big)$ is the hypergeometric function (see, e.g., [3]). Denote

$$M_+ = \frac{1}{2}H + im, \quad M_- = \frac{1}{2}H - im\,.$$

Let $e^{H\cdot}$ be the operator of multiplication by the function e^{Ht}. Theorem 0.2 [30] gives the representation formula for the solutions of the Cauchy problem. In order to formulate it, we need the operator $\mathcal{G}(x,t,D_x;M)$ defined by

$$
\begin{aligned}
& \mathcal{G}(x,t,D_x;M)[f](x,t) \\
= \quad & 2\int_0^t db \int_0^{\phi(t)-\phi(b)} E(r,t;0,b;M) \int_{\mathbb{R}^n} \mathcal{E}^w(x-y,r) f(y,b)\, dy\, dr, \quad f \in C_0^\infty(\mathbb{R}^4),
\end{aligned}
$$

where $\mathcal{E}^w(x,r)$ is a fundamental solution of the Cauchy problem

$$
\begin{cases}
v_{tt} - \Delta v = 0, \quad x \in \mathbb{R}^3\,, \quad t \in \mathbb{R}\,, \\
v(x,0) = \varphi(x), \quad v_t(x,0) = 0\,, \quad x \in \mathbb{R}^3\,,
\end{cases}
$$

in the Minkowski space, that is, for $n = 3$ (see, e.g., [20])

$$
\mathcal{E}^w(x,t) := \frac{1}{4\pi}\frac{\partial}{\partial t}\frac{1}{t}\delta(|x|-t)\,.
$$

The distribution $\delta(|x|-t)$ is defined by

$$
\langle \delta(|\cdot|-t), \psi(\cdot)\rangle = \int_{|x|=t} \psi(x)\, dx \quad \text{for} \quad \psi \in C_0^\infty\left(\mathbb{R}^3\right).
$$

The Kirchhoff's formula gives

$$
v(x,t) = \frac{\partial}{\partial t} V(x,t), \quad \text{where} \quad V(x,t) = \frac{t}{4\pi}\int_{|y|=1} \varphi(x+ty)\, dS_y\,.
$$

We also need the kernel function

$$
K_1(r,t;M) := E(r,t;0,0;M)
$$

and the operator $\mathcal{K}_1(x,t,D_x;M)$, which is defined as follows:

$$
\mathcal{K}_1(x,t,D_x;M)[\varphi](x,t) = 2\int_0^{\phi(t)} K_1(s,t;M) \int_{\mathbb{R}^n} \mathcal{E}^w(x-y,s)\varphi(y)\, dy\, ds\,, \quad \varphi \in C_0^\infty(\mathbb{R}^3), \tag{1.11}
$$

or

$$
\mathcal{K}_1(x,t,D_x;M)[\varphi](x,t) = 2\int_0^{\phi(t)} K_1(s,t;M)\, ds \frac{\partial}{\partial s}\frac{s}{4\pi}\int_{|y|=1} \varphi(x+sy)\, dS_y\,, \quad \varphi \in C_0^\infty(\mathbb{R}^3).
$$

According to Theorem 0.2 [30], the solution to the Cauchy problem

$$
\begin{cases}
\left(i\gamma^0\partial_0 + ie^{-Ht}\sum\limits_{k=1,2,3}\gamma^k\partial_k + i\frac{3}{2}H\gamma^0 - m\mathbb{I}_4\right)\Psi(x,t) = F(x,t)\,, \\
\Psi(x,0) = \Phi(x)\,,
\end{cases}
$$

where $m \in \mathbb{C}$, is given by the following formula

$$
\begin{aligned}
\Psi(x,t) \quad = \quad & -e^{-Ht}\left(i\gamma^0\partial_0 + ie^{-Ht}\sum_{k=1,2,3}\gamma^k\partial_k - i\frac{H}{2}\gamma^0 + m\mathbb{I}_4\right) \\
& \times\Bigg[\begin{pmatrix} \mathcal{G}(x,t,D_x;M_+)\mathbb{I}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathcal{G}(x,t,D_x;M_-)\mathbb{I}_2 \end{pmatrix} [e^{H\cdot}F](x,t) \\
& + i\gamma^0 \begin{pmatrix} \mathcal{K}_1(x,t,D_x;M_+)\mathbb{I}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathcal{K}_1(x,t,D_x;M_-)\mathbb{I}_2 \end{pmatrix} [\Phi](x,t)\Bigg].
\end{aligned}
$$

The next theorem states the existence of the operator $\mathscr{S}^+ : \psi_0 \longmapsto \psi_0^+$.

Theorem 1.3 *Let $F = F(\psi) \in C^3(\mathbb{C}^4;\mathbb{C}^4)$ be the Lipschitz continuous with exponent $\alpha > 0$ in the space $H_{(k)}(\mathbb{R}^3)$, $k \geq 3$, function. Assume that the potential $V \in \mathscr{B}_{(0,k)}$ is self-adjoint, $V(x,t) = V(x,t)^*$,*

$$4|\Im(m)| + 2|\Im(m)|\alpha < 3H\alpha \,.$$

For every solution $\psi \in C([0,\infty);(H_{(k)}(\mathbb{R}^3))^4)$ of the semilinear Dirac equation in the de Sitter spacetime (1.5), let $\psi_0^+(x)$ be the function given by

$$\psi_0^+(x) = \mathscr{S}^+\psi_0(x) = \psi_0(x) + \int_0^\infty S(0,\tau)F(\psi(x,\tau))\,d\tau\,,$$

where $\psi_0(x) := \psi(x,0)$. Then the solution $\psi^+(x,t)$ of the Cauchy problem for the free Dirac equation (1.6) satisfies (1.7) and $\mathscr{S}^+ : \psi_0 \longmapsto \psi_0^+$ is the continuous operator in $H_{(k)}(\mathbb{R}^3)$.

Moreover, if $V(x,t) = 0$, then

$$\begin{aligned} \psi^+(x,t) &= -e^{-Ht}\left(i\gamma^0\partial_0 + ie^{-Ht}\sum_{k=1,2,3}\alpha^k\partial_k - i\frac{H}{2}\gamma^0 + m\mathbb{I}_4\right) \\ &\quad\times\left[i\gamma^0\begin{pmatrix} \mathcal{K}_1(x,t,D_x;M_+)\mathbb{I}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathcal{K}_1(x,t,D_x;M_-)\mathbb{I}_2\end{pmatrix}[\psi_0^+]\right]. \end{aligned} \tag{1.12}$$

Proof. According to Lemma 1.2 the function $\psi_0^+(x)$ exists. Then, in view of Theorem 0.2 [30], the function $\psi^+(x,t)$ of (1.12) solves initial value problem for the free Dirac equation (1.6). $\square$

In the case of the mass $m = 0, \pm iH$ the Dirac equation obeys Huygens' principle [27, 31] and the function $\psi^+(x,t)$ is simplified as follows. First, we note that according to (3.13) [31]

$$\begin{aligned} K_1\left(r,t;-\frac{1}{2}H\right) &= K_1\left(r,t;\frac{1}{2}H\right) = \frac{1}{2}e^{\frac{1}{2}Ht}\,, \\ K_1\left(r,t;\frac{3}{2}H\right) &= \frac{1}{4}e^{-\frac{1}{2}Ht}\left(\left(1-H^2r^2\right)e^{2Ht}+1\right). \end{aligned}$$

Consequently, by the definition (1.11) of the operator $\mathcal{K}_1$ we write

$$\mathcal{K}_1\left(x,t,D_x;\frac{1}{2}H\right)[\varphi(x)] = \mathcal{K}_1\left(x,t,D_x;-\frac{1}{2}H\right)[\varphi(x)] = e^{\frac{1}{2}Ht}V_\varphi(x,\phi(t))\,, \tag{1.13}$$

where $\phi(t) := (1-e^{-Ht})/H$ and

$$\begin{aligned} &\mathcal{K}_1\left(x,t,D_x;\frac{3}{2}H\right)[\varphi(x)] \\ &= \frac{1}{2}e^{\frac{3}{2}Ht}\left(1+e^{-2Ht}\right)V_{\varphi_1}(x,\phi(t)) - H^2\frac{1}{2}e^{\frac{3}{2}Ht}\phi(t)^2V_\varphi(x,\phi(t)) + H^2e^{\frac{3}{2}Ht}\int_0^{\phi(t)}V_\varphi(x,s)sds\,. \end{aligned} \tag{1.14}$$

Corollary 1.4 (i) *If $m = 0$, then for the solution of the Dirac equation we obtain from (2.11) [31]*

$$\psi^+(x,t) = e^{-Ht}\left(\partial_0\mathbb{I}_4 + e^{-Ht}\sum_{k=1,2,3}\gamma^k\gamma^0\partial_k - \frac{H}{2}\mathbb{I}_4\right)e^{Ht/2}\frac{\phi(t)}{4\pi}\int_{|y|=1}\psi_0^+(x+\phi(t)y)\,dS_y\,.$$

(ii) *For $m = iH$ we have $M_+ = -\frac{1}{2}H$ and $M_- = \frac{3}{2}H$. Then for the operators $\mathcal{K}_1\left(x,t,D_x;-\frac{1}{2}H\right)$ and $\mathcal{K}_1\left(x,t,D_x;\frac{3}{2}H\right)$ we have representation (3.14) [31] and (3.15) [31], respectively. Hence with $\psi_0^+(x) = (\psi_{00}^+(x),\psi_{01}^+(x),\psi_{02}^+(x),\psi_{03}^+(x))^T$*

$$\psi^+(x,t) = e^{-Ht}\left(\partial_0\mathbb{I}_4 + e^{-Ht}\sum_{k=1,2,3}\gamma^k\gamma^0\partial_k - \frac{H}{2}\mathbb{I}_4 + H\gamma^0\right)\begin{pmatrix} \mathcal{K}_1(x,t,D_x;-\frac{1}{2}H)[\psi_{00}^+(x)] \\ \mathcal{K}_1(x,t,D_x;-\frac{1}{2}H)[\psi_{01}^+(x)] \\ \mathcal{K}_1(x,t,D_x;\frac{3}{2}H)[\psi_{02}^+(x)] \\ \mathcal{K}_1(x,t,D_x;\frac{3}{2}H)[\psi_{03}^+(x)]\end{pmatrix}.$$

Here $\mathcal{K}_1(x,t,D_x;-\frac{1}{2}H)$ *and* $\mathcal{K}_1(x,t,D_x;\frac{3}{2}H)$ *are given by (1.13) and (1.14), respectively.*
(iii) *For the case of* $m=-iH$ *we have* $M_+=\frac{3}{2}H$, $M_-=-\frac{1}{2}H$, *and the formula is*

$$\psi^+(x,t)=e^{-Ht}\left(\partial_0+e^{-Ht}\sum_{k=1,2,3}\gamma^k\gamma^0\partial_k-\frac{H}{2}\mathbb{I}_4-H\gamma^0\right)\left(\begin{array}{c}\mathcal{K}_1(x,t,D_x;\frac{3}{2}H)[\psi^+_{00}(x)]\\ \mathcal{K}_1(x,t,D_x;\frac{3}{2}H)[\psi^+_{01}(x)]\\ \mathcal{K}_1(x,t,D_x;-\frac{1}{2}H)[\psi^+_{02}(x)]\\ \mathcal{K}_1(x,t,D_x;-\frac{1}{2}H)[\psi^+_{03}(x)]\end{array}\right).$$

2 Lochak-Majorana condition in de Sitter spacetime

Consider the matrix-valued potential function

$$A(x,t)=\alpha(x,t)\mathbb{I}_4+i\beta(x,t)\gamma^5,\ \alpha,\beta\in\mathbb{R},\quad \gamma^5:=-i\gamma^0\gamma^1\gamma^2\gamma^3\,, \tag{2.1}$$

that takes values in the space $\mathcal{M}=\left\{\alpha\mathbb{I}_4+i\beta\gamma^5,\ \alpha,\beta\in\mathbb{R}\right\}$. Then we consider the Dirac equation with the potential $A(x,t)$, where $\alpha,\beta\in C^0([0,\infty);\,L^2(\mathbb{R}^3))$.

Lemma 2.1 *For the solution* $\psi\in C^1([0,\infty);L^2(\mathbb{R}^3))\cap C^0([0,\infty);H_1(\mathbb{R}^3))$ *of the Dirac equation*

$$\left(i\gamma^0\partial_0+ie^{-Ht}\sum_{k=1,2,3}\gamma^k\partial_k+i\frac{3}{2}H\gamma^0-m\mathbb{I}_4+\gamma^0V(x,t)\right)\psi=-A\psi\,, \tag{2.2}$$

with the self-adjoint V *the following energy identity holds*

$$\|\psi(x,t)\|^2_{L^2(\mathbb{R}^3)}=e^{-3Ht}\|\psi(x,0)\|^2_{L^2(\mathbb{R}^3)}+2\Im(m)e^{-3Ht}\int_0^t e^{3Hs}\int_{\mathbb{R}^3}\psi^*(x,s)\gamma^0\psi\,(x,s)\,dx\,ds\,.$$

Proof. The equation (2.2) can be written as follows

$$\mathscr{D}_{dS}(x,t,\partial)\psi=i\gamma^0A\psi\,.$$

Hence

$$\begin{aligned}\frac{d}{dt}|\psi|^2 \quad&=\quad -3H|\psi|^2-\left(e^{-Ht}\sum_{\ell=1,2,3}(\partial_\ell\psi)^*\,(\gamma^\ell)^*\gamma^0-i\overline{m}\psi^*\gamma^0\right)\psi-i\psi^*A^*\gamma^0\psi\\ &\quad -\psi^*\left(e^{-Ht}\sum_{\ell=1,2,3}\gamma^0\gamma^\ell\partial_\ell\psi+im\gamma^0\psi\right)+i\psi^*\gamma^0A\psi\,.\end{aligned}$$

Now we simplify the terms with the potential A:

$$-i\psi^*A^*\gamma^0\psi+i\psi^*\gamma^0A\psi=0\,.$$

Thus,

$$\frac{d}{dt}|\psi|^2=-3H|\psi|^2+2\Im(m)\psi^*\gamma^0\psi-e^{-Ht}\sum_{\ell=1,2,3}\partial_\ell(\psi^*\,\gamma^0\gamma^\ell\psi)\,.$$

In view of the finite propagation speed property, it follows

$$\frac{d}{dt}\|\psi\|^2_{L^2(\mathbb{R}^3)}=-3H\|\psi\|^2_{L^2(\mathbb{R}^3)}+2\Im(m)\int_{\mathbb{R}^3}\psi^*\gamma^0\psi\,dx$$

and, consequently,

$$\frac{d}{dt}\left(e^{3Ht}\|\psi\|^2_{L^2(\mathbb{R}^3)}\right)=2e^{3Ht}\Im(m)\int_{\mathbb{R}^3}\psi^*\gamma^0\psi\,dx\,.$$

By integration we obtain

$$e^{3Ht}\|\psi(x,t)\|^2_{L^2(\mathbb{R}^3)}=\|\psi(x,0)\|^2_{L^2(\mathbb{R}^3)}+2\Im(m)\int_0^t e^{3Hs}\int_{\mathbb{R}^3}\psi^*(x,s)\gamma^0\psi\,(x,s)\,dx\,ds\,.$$

Lemma is proved. □

Lemma 2.2 *Assume that*

$$V^T(x,t)\gamma^2 + \gamma^2 V(x,t) = 0\,. \tag{2.3}$$

For the solution $\psi \in C^1([0,\infty); L^2(\mathbb{R}^3)) \cap C^0([0,\infty); H_1(\mathbb{R}^3))$ of the Dirac equation (2.2) one has

$$\int_{\mathbb{R}^3} \psi^T(x,t)\gamma^2\psi(x,t)\,dx = e^{-3Ht}\left(\int_{\mathbb{R}^3} \psi^T(x,0)\gamma^2\psi(x,0)\,dx\right).$$

In particular, for $z \in \mathbb{C}$, $|z| = 1$, one has

$$\int_{\mathbb{R}^3} 2\Re\Big(\overline{z}\psi^T(x,t)\gamma^2\psi(x,t)\Big)\,dx = e^{-3Ht}\int_{\mathbb{R}^3} 2\Re\Big(\overline{z}\psi^T(x,0)\gamma^2\psi(x,0)\Big)\,dx\,.$$

Proof. We multiply the equation by $\psi^T\gamma^2$:

$$\begin{aligned}
&\psi^T\gamma^2\partial_0\psi + e^{-Ht}\sum_{\ell=1,2,3}\psi^T\gamma^2\gamma^0\gamma^\ell\partial_\ell\psi + \frac{3}{2}H\psi^T\gamma^2\psi + im\psi^T\gamma^2\gamma^0\psi - i\psi^T\gamma^2V(x,t)\psi\\
&= A\psi^T\gamma^2 i\gamma^0\psi\,.
\end{aligned}$$

Here

$$\begin{aligned}
&\gamma^2\gamma^0 = \begin{pmatrix} \mathbb{O}_2 & -\sigma_2 \\ -\sigma_2 & \mathbb{O}_2 \end{pmatrix}, \quad \gamma^0\gamma^5 = \begin{pmatrix} \mathbb{O}_2 & -\mathbb{I}_2 \\ \mathbb{I}_2 & \mathbb{O}_2 \end{pmatrix} = -\gamma^5\gamma^0, \quad \gamma^5\gamma^2 + \gamma^2\gamma^5 = 0\,,\\
&\gamma^2\gamma^0\gamma^1 = \begin{pmatrix} -i\sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & i\sigma_3 \end{pmatrix}, \quad \gamma^2\gamma^0\gamma^2 = \gamma^0, \quad \gamma^2\gamma^0\gamma^3 = \begin{pmatrix} i\sigma_1 & \mathbb{O}_2 \\ \mathbb{O}_2 & -i\sigma_1 \end{pmatrix}.
\end{aligned}$$

Furthermore,

$$\partial_0\psi = -e^{-Ht}\sum_{\ell=1,2,3}\gamma^0\gamma^\ell\partial_\ell\psi - \frac{3}{2}H\psi - im\gamma^0\psi + iV(x,t)\psi + i\gamma^0A\psi\,,$$

and due to (2.3) we obtain

$$\begin{aligned}
\partial_t(\psi^T\gamma^2\psi) &= \left(-\sum_{\ell=1,2,3} e^{-Ht}(\partial_\ell\psi)^T(\gamma^\ell)^T\gamma^0 - \frac{3}{2}H\psi^T - im\psi^T\gamma^0 + i\psi^TA\gamma^0\right)\gamma^2\psi\\
&\quad +\psi^T\gamma^2\left(-e^{-Ht}\sum_{\ell=1,2,3}\gamma^0\gamma^\ell\partial_\ell\psi - \frac{3}{2}H\psi - im\gamma^0\psi + i\gamma^0A\psi\right).
\end{aligned}$$

Consider the terms with potential:

$$\begin{aligned}
\Big(i\psi^TA\gamma^0\Big)\gamma^2\psi + \psi^T\gamma^2\Big(i\gamma^0A\psi\Big) &= i\psi^T\left(\alpha\mathbb{I}_4 + i\beta\gamma^5\right)\gamma^0\gamma^2\psi + \psi^T\gamma^2 i\gamma^0\left(\alpha\mathbb{I}_4 + i\beta\gamma^5\right)\psi\\
&= i\psi^T\left(\alpha\mathbb{I}_4\gamma^0\gamma^2 + i\beta\gamma^5\gamma^0\gamma^2 + \alpha\mathbb{I}_4\gamma^2\gamma^0 + i\beta\gamma^2\gamma^0\gamma^5\right)\psi\\
&= i\psi^T\Big(i\beta\gamma^5\gamma^0\gamma^2 + i\beta\gamma^2\gamma^0\gamma^5\Big)\psi = 0\,.
\end{aligned}$$

Thus,

$$\partial_t(\psi^T\gamma^2\psi) = -3H\psi^T\gamma^2\psi - e^{-Ht}\sum_{\ell=1,2,3}\Big\{(\partial_\ell\psi)^T(\gamma^\ell)^T\gamma^0\gamma^2\psi + \psi^T\gamma^2\gamma^0\gamma^\ell\partial_\ell\psi\Big\}.$$

For the sum in the last equation, we have

$$(\gamma^1)^T\gamma^0\gamma^2 = \gamma^2\gamma^0\gamma^1\,, \quad (\gamma^2)^T\gamma^0\gamma^2 = \gamma^2\gamma^0\gamma^2\,, \quad (\gamma^3)^T\gamma^0\gamma^2 = \gamma^2\gamma^0\gamma^3\,.$$

It follows

$$\partial_t(\psi^T\gamma^2\psi) = -3H\psi^T\gamma^2\psi - e^{-Ht}\sum_{\ell=1,2,3}\partial_\ell\Big(\psi^T\gamma^2\gamma^0\gamma^\ell\psi\Big)$$

and, consequently,

$$\partial_t\int_{\mathbb{R}^3}(\psi^T\gamma^2\psi)\,dx = -3H\int_{\mathbb{R}^3}\psi^T\gamma^2\psi\,dx\,.$$

Thus, the first statement of the lemma is proved. To prove the second statement, we use $\overline{\gamma_2} = -\gamma_2$ and recall the formula from [2]:

$$|\psi - z\gamma^2\overline{\psi}|^2 = 2|\psi|^2 + 2\Re(\overline{z}\psi^T\gamma^2\psi)\,, \tag{2.4}$$

where $z\in\mathbb{C}$, $|z|=1$. The lemma is proved. □

Lemma 2.3 *For the solution $\psi\in C^1([0,\infty);L^2(\mathbb{R}^3))\cap C^0([0,\infty);H_{(1)}(\mathbb{R}^3))$ of the Dirac equation (2.2) one has*

$$\begin{aligned}\int_{\mathbb{R}^3}|\psi(x,t)-z\gamma^2\overline{\psi(x,t)}|^2\,dx &= e^{-3Ht}\Bigg\{\int_{\mathbb{R}^3}|\psi(x,0)-z\gamma^2\overline{\psi(x,0)}|^2\,dx\\ &\quad +4\Im(m)\int_0^t e^{3Hs}\int_{\mathbb{R}^3}\psi^*(x,s)\gamma^0\psi\,(x,s)\,dx\,ds\Bigg\}\,.\end{aligned}$$

Proof. From Lemma 2.1, Lemma 2.2, and (2.4)

$$\begin{aligned}&\int_{\mathbb{R}^3}|\psi(x,t)-z\gamma^2\overline{\psi(x,t)}|^2\,dx\\ =\;&\int_{\mathbb{R}^3}\Big(2|\psi|^2+2\Re(\overline{z}\psi^T\gamma^2\psi)\Big)\,dx\\ =\;&2\left(e^{-3Ht}\|\psi(x,0)\|^2+2\Im(m)e^{-3Ht}\int_0^t e^{3Hs}\int_{\mathbb{R}^3}\psi^*(x,s)\gamma^0\psi\,(x,s)dxds\right)\\ &+2\int_{\mathbb{R}^3}\Re\Big(\overline{z}\psi^T(x,0)\gamma^2\psi(x,0)\Big)\,dx\,.\end{aligned}$$

That is,

$$\begin{aligned}\int_{\mathbb{R}^3}|\psi(x,t)-z\gamma^2\overline{\psi(x,t)}|^2\,dx &= 2e^{-3Ht}\left(\|\psi(x,0)\|^2+\int_{\mathbb{R}^3}\Re\Big(\overline{z}\psi^T(x,0)\gamma^2\psi(x,0)\Big)\,dx\right)\\ &\quad +4\Im(m)e^{-3Ht}\int_0^t e^{3Hs}\int_{\mathbb{R}^3}\psi^*(x,s)\gamma^0\psi\,(x,s)\,dx\,ds\,.\end{aligned}$$

Lemma is proved. □

The following identity is easily seen for the function defined in (0.7):

$$\rho^2(\psi) = |\psi^*\gamma^0\psi|^2+|\psi^*\gamma^0\gamma^5\psi|^2 = (|\psi_1|^2+|\psi_2|^2-|\psi_3|^2-|\psi_4|^2)^2+(2\Im(\psi_1\overline{\psi_3})+2\Im(\psi_2\overline{\psi_4}))^2\,.$$

Corollary 2.4 *Assume that (2.3) is fulfilled. (i) If $\Im(m)=0$, then*

$$\int_{\mathbb{R}^3}|\psi(x,t)-z\gamma^2\overline{\psi(x,t)}|^2\,dx = e^{-3Ht}\int_{\mathbb{R}^3}|\psi(x,0)-z\gamma^2\overline{\psi(x,0)}|^2\,dx\,.$$

(ii) If $\psi(x,0)-z\gamma^2\overline{\psi(x,0)}=0$ and $\Im(m)\neq 0$, then

$$\int_{\mathbb{R}^3}|\psi(x,t)-z\gamma^2\overline{\psi(x,t)}|^2\,dx = 4\Im(m)e^{-3Ht}\int_0^t e^{3Hs}\int_{\mathbb{R}^3}\overline{\psi^T(x,s)}\gamma^0\cdot\psi\,(x,s)\,dx\,ds\,.$$

(iii) if $\psi(x,0)-z\gamma^2\overline{\psi(x,0)}=0$, then

$$\int_{\mathbb{R}^3}|\psi(x,t)-z\gamma^2\overline{\psi(x,t)}|^2\,dx \le 4|\Im(m)|e^{-3Ht}\int_0^t e^{3Hs}\int_{\mathbb{R}^3}\rho(x,s)\,dx\,ds\,.$$

Note, the statements (i),(iii) of Proposition I.1 [2] imply that for $z\in\mathbb{C}$, $|z|=1$, the condition $\psi-z\gamma^2\overline{\psi}=0$ is equivalent to $\rho^2(\psi)=0$.

3 Proof of Theorem 0.2

In order to prove the existence of the global solution to the Cauchy problem for the semilinear Dirac equation in the Minkowski space, Bachelot [2] used the successive approximations and appealed to the estimates, which are obtained by the replacing the generators of the Poincaré group with the Fermi operators. We use the energy estimate obtained in subsection 1.1 that allows us to expand the result from [2] to the Dirac equation in the de Sitter spacetime.

Let Ψ be the solution of

$$\begin{cases} \mathscr{D}_{dS}(x,t,\partial)\Psi(x,t) = 0\,, \\ \Psi(x,0) = \Psi_0(x)\,. \end{cases} \tag{3.1}$$

The finite propagation speed property for the equation in the de Sitter spacetime implies that the support of $\Psi = \Psi(x,t) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}_+; \mathbb{C}^4)$ is in the same compact subset of $\mathbb{R}^3$ for all $t > 0$. Then according to Lemma 1.1

$$e^{-\frac{1}{2}\delta_+ t}\|\Psi(t)\|_k \le c\|\Psi_0\|_k \quad \text{for all} \quad t > 0\,. \tag{3.2}$$

We look for the solution of (0.3) in the form

$$\psi = \Psi + \chi\,, \tag{3.3}$$

where Ψ solves (3.1).

Consider the nonlinear term. First of all, we note that, according to Corollary 2.4, (0.5), and (0.7), if $\Im(m) = 0$, then

$$F\left(\Psi^*(x,t)\gamma^0\Psi(x,t)\,,\, \Psi^*(x,t)\gamma^0\gamma^5\Psi(x,t)\right) = 0 \quad \text{for all} \quad t \ge 0, \quad x \in \mathbb{R}^3\,.$$

Further, (0.4) can be written as follows

$$F\left(\xi,\eta\right) = \alpha\left(\xi,\eta\right)\mathbb{I}_4 + i\beta\left(\xi,\eta\right)\gamma^5, \quad \xi = \psi^*\gamma^0\psi \in \mathbb{R}, \quad \eta = \psi^*\gamma^0\gamma^5\psi \in \mathbb{R}\,.$$

It is evident that with the solution $\Psi(x,t)$ we can write

$$\begin{aligned} |(\Psi(x,t)+\chi(x,t))^*\gamma^0(\Psi(x,t)+\chi(x,t)) - \Psi^*(x,t)\gamma^0\Psi(x,t)| &\le C\big(|\chi(x,t)| + |\Psi(x,t)|\big)^2, \\ |(\Psi(x,t)+\chi(x,t))^*\gamma^0\gamma^5(\Psi(x,t)+\chi(x,t)) - (\Psi^*(x,t)\gamma^0\gamma^5\Psi(x,t)| &\le C\big(|\chi(x,t)| + |\Psi(x,t)|\big)^2 \end{aligned}$$

and, consequently,

$$\begin{aligned} \alpha\left(\psi^*\gamma^0\psi\,,\, \psi^*\gamma^0\gamma^5\psi\right)\psi &= \alpha\left((\Psi+\chi)^*\gamma^0(\Psi+\chi)\,,\, (\Psi+\chi)^*\gamma^0\gamma^5(\Psi+\chi)\right)\psi \\ &= \alpha\left(\Psi^*\gamma^0\Psi\,,\, \Psi^*\gamma^0\gamma^5\Psi\right)\psi + \alpha_1\left(\chi,\Psi\right) \\ &= \alpha_1\left(\chi,\Psi\right)\,, \\ \beta\left(\psi^*\gamma^0\psi\,,\, \psi^*\gamma^0\gamma^5\psi\right)\psi &= \beta\left((\Psi+\chi)^*\gamma^0(\Psi+\chi)\,,\, (\Psi+\chi)^*\gamma^0\gamma^5(\Psi+\chi)\right)\psi \\ &= \beta\left(\Psi^*\gamma^0\Psi\,,\, \Psi^*\gamma^0\gamma^5\Psi\right)\psi + \beta_1\left(\chi,\Psi\right) \\ &= \beta_1\left(\chi,\Psi\right)\,, \end{aligned}$$

where $\alpha_1, \beta_1 \in C^\infty(\mathbb{C}^8; \mathbb{C}^4)$, and, as the functions of χ,

$$|\alpha_1\left(\chi\,,\, \Psi(x,t)\right)| = O\Big(|\chi|\big(|\chi| + |\Psi(x,t)|\big)^2\Big) \quad \text{as} \quad |\chi| \to 0\,, \tag{3.4}$$

$$|\beta_1\left(\chi\,,\, \Psi(x,t)\right)| = O\Big(|\chi|\big(|\chi| + |\Psi(x,t)|\big)^2\Big) \quad \text{as} \quad |\chi| \to 0\,. \tag{3.5}$$

Further, the Cauchy problem becomes

$$\begin{cases} i\gamma^0\mathscr{D}_{dS}(x,t,\partial)\chi = f_1\left(\chi\,,\, \Psi\right)\,, \\ \chi(x,0) = \varepsilon\chi_0(x)\,, \end{cases} \tag{3.6}$$

where, in view of (3.1), we have denoted

$$\begin{aligned} f_1\,(\chi\,,\,\Psi) \quad := \quad & F\left((\Psi^*+\chi^*)\gamma^0(\Psi+\chi)\,,\,(\Psi^*+\chi^*)\gamma^0\gamma^5(\Psi+\chi)\right)\Psi \\ & +F\left((\Psi^*+\chi^*)\gamma^0(\Psi+\chi)\,,\,(\Psi^*+\chi^*)\gamma^0\gamma^5(\Psi+\chi)\right)\chi \end{aligned}$$

that can be rewritten similar to (2.1) as $f_1\,(\chi\,,\,\Psi) = -A\chi$. Hence, $f_1 \in C^\infty(\mathbb{C}^8;\mathbb{C}^4)$, while (3.4) and (3.5) imply

$$|f_1\,(\chi\,,\,\Psi(x,t))\,| = O\Big(|\chi|\big(|\chi|+|\Psi(x,t)|\big)^2\Big) \quad \text{as} \quad |\chi|\to 0\,.$$

We look for the function χ as a limit of the sequence $\{\chi^{(k)}\}_1^\infty$ that is defined by

$$\begin{cases} i\gamma^0\mathscr{D}_{dS}(x,t,\partial)\chi^{(k)} = f_1\left(\chi^{(k-1)}\,,\,\Psi\right)\,, \\ \chi^{(k)}(x,0) = \varepsilon\chi_0(x), \quad k=1,2,\ldots\,, \end{cases}$$

and $\chi^{(0)}(x,t)\equiv 0$. The finite propagation speed property implies that the supports of the functions $\chi^{(k)} = \chi^{(k)}(x,t)$ are in the same compact subset of $\mathbb{R}^3$ for all $t>0$ and $k\geq 0$. Lemmas 1.1,2.1 and the estimate

$$\|\chi^{(1)}(t)\|_s \leq ce^{\frac12\delta_+ t}\|\chi^{(1)}(0)\|_s + ce^{\frac12\delta_+ t}\int_0^t e^{-\frac12\delta_+\tau}\|f_1\left(\chi^{(0)}(\tau)\,,\,\Psi(\tau)\right)\|_s\,d\tau \quad \text{for all} \quad t>0\,,$$

imply

$$\|\chi^{(1)}(t)\|_s = c\varepsilon e^{\frac12\delta_+ t}\|\chi_0\|_s \quad \text{for all} \quad t>0\,.$$

Corollary 6.4.5 [12] and Lemma 1.1 imply for $k=2,3,\ldots$ the estimate

$$\begin{aligned} & \|\chi^{(k)}(t)\|_s \\ \leq\ & e^{\frac12\delta_+ t}\left(c\varepsilon\|\chi_0(x)\|_s + 2c\int_0^t e^{-\frac12\delta_+\tau}\|f_1\left(\chi^{(k-1)}\,,\,\Psi\right)(\tau)\|_s\,d\tau\right) \\ \leq\ & e^{\frac12\delta_+ t}\left(c\varepsilon\|\chi_0(x)\|_s + 2c\int_0^t e^{-\frac12\delta_+\tau}\|\chi^{(k-1)}(\tau)\|_s\Big(|\chi^{(k-1)}(\tau)|_{\left[\frac{s}{2}\right]}+|\Psi(\tau)|_{\left[\frac{s}{2}\right]}\Big)^2 d\tau\right), \end{aligned} \tag{3.7}$$

where $\delta_+ = -3H+2|\Im(m)|<0$ and $\left[\frac{s}{2}\right]$ is the integer part of $\frac{s}{2}$, while

$$|\Psi(t)|_s := \sup_{|\alpha|\leq s}\|\partial_x^\alpha\Psi(x,t)\|_{L^\infty(\mathbb{R}^3)}.$$

Now we apply Sobolev embedding theorem and Lemma 1.1 to the function Ψ:

$$|\Psi(t)|_{\left[\frac{s}{2}\right]} \leq \|\Psi(t)\|_s \leq e^{\frac12\delta_+ t}\|\Psi_0\|_s \quad \text{for all} \quad t>0\,.$$

For the given s and n we define

$$a_n(t) := \sup_{0\leq\tau\leq t,\ 0\leq k\leq n} e^{-\frac12\delta_+\tau}\|\chi^{(k)}(\tau)\|_s\,, \qquad A_n := \sup_{t\in\mathbb{R}_+} a_n(t)\,.$$

Then for $s\geq 6$ by (3.7) we derive

$$\|\chi^{(k)}(t)\|_s \leq e^{\frac12\delta_+ t}\left(c\varepsilon\|\chi_0\|_s + 2c\int_0^t e^{\frac12\delta_+\tau}\|\chi^{(k-1)}(\tau)\|_s\Big(e^{-\frac12\delta_+\tau}\|\chi^{(k-1)}(\tau)\|_s + e^{-\frac12\delta_+\tau}\|\Psi(\tau)\|_s\Big)^2 d\tau\right)$$

and

$$\begin{aligned} a_n(t) \quad \leq\quad & c\varepsilon\|\chi_0\|_s + 2c\int_0^t e^{\delta_+\tau}\Big(e^{-\frac12\delta_+\tau}\|\chi^{(n-1)}(\tau)\|_s\Big)\Big(e^{-\frac12\delta_+\tau}\|\chi^{(n-1)}(\tau)\|_s + e^{-\frac12\delta_+\tau}\|\Psi(\tau)\|_s\Big)^2 d\tau \\ \leq\quad & c\varepsilon\|\chi_0\|_s + 2c\int_0^t e^{\delta_+\tau}\Big(e^{-\frac12\delta_+\tau}\|\chi^{(n-1)}(\tau)\|_s\Big)\Big(e^{-\frac12\delta_+\tau}\|\chi^{(n-1)}(\tau)\|_s + \|\Psi_0\|_s\Big)^2 d\tau \end{aligned}$$

that implies (with the new c depending on $\|\Psi_0\|_s$) for $n = 1, 2, \dots$

$$\begin{aligned} a_n(t) &\leq c\varepsilon\|\chi_0\|_s + 2c\big(1 + A_{n-1}\big)^2 \int_0^t e^{\delta_+\tau} a_{n-1}(\tau)\, d\tau \\ &\leq c\varepsilon\|\chi_0\|_s + 2c\big(1 + A_{n-1}\big)^2 \int_0^t e^{\delta_+\tau} a_n(\tau)\, d\tau\,. \end{aligned} \tag{3.8}$$

Denote

$$y(t) := \int_0^t e^{\delta_+\tau} a_n(\tau)\, d\tau \quad \text{then} \quad y'(t) = e^{\delta_+ t} a_n(t)$$

and according to (3.8) we obtain

$$y'(t) \leq e^{\delta_+ t} c\varepsilon\|\chi_0\|_s + 2c\big(1 + A_{n-1}\big)^2 e^{\delta_+ t} y(t)\,.$$

It follows

$$\frac{d}{dt}\left(e^{-2c\left(1+A_{n-1}\right)^2 \frac{1}{\delta_+}\left(e^{\delta_+ t}-1\right)} y(t) \right) \leq e^{-2c\left(1+A_{n-1}\right)^2 \frac{1}{\delta_+}\left(e^{\delta_+ t}-1\right)} e^{\delta_+ t} c\varepsilon\|\chi_0\|_s$$

and, consequently,

$$e^{-2c\left(1+A_{n-1}\right)^2 \frac{1}{\delta_+}\left(e^{\delta_+ t}-1\right)} y(t) \leq c\varepsilon\|\chi_0\|_s \int_0^t e^{-2c\left(1+A_{n-1}\right)^2 \frac{1}{\delta_+}\left(e^{\delta_+ \tau}-1\right)+\delta_+\tau}\, d\tau\,,$$

since $y(0) = 0$. Hence

$$\begin{aligned} y(t) &\leq e^{-2c\left(1+A_{n-1}\right)^2 \frac{1}{\delta_+}\left(1-e^{\delta_+ t}\right)} c\varepsilon\|\chi_0\|_s \int_0^t e^{-2c\left(1+A_{n-1}\right)^2 \frac{1}{\delta_+}\left(e^{\delta_+ \tau}-1\right)+\delta_+\tau}\, d\tau \\ &\leq e^{2c\left(1+A_{n-1}\right)^2 \frac{1}{\delta_+} e^{\delta_+ t}} c\varepsilon\|\chi_0\|_s \int_0^t e^{-2c\left(1+A_{n-1}\right)^2 \frac{1}{\delta_+} e^{\delta_+ \tau}+\delta_+\tau}\, d\tau\,. \end{aligned}$$

Further, in view of (3.8) we obtain

$$\begin{aligned} a_n(t) &\leq c\varepsilon\|\chi_0\|_s + 2c\big(1 + A_{n-1}\big)^2 \int_0^t e^{\delta_+\tau} a_n(\tau)\, d\tau \\ &\leq c\varepsilon\|\chi_0\|_s + 2c\big(1 + A_{n-1}\big)^2 y(t) \\ &\leq c\varepsilon\|\chi_0\|_s + 2c\big(1 + A_{n-1}\big)^2 e^{2c\left(1+A_{n-1}\right)^2 \frac{1}{\delta_+} e^{\delta_+ t}} c\varepsilon\|\chi_0\|_s \int_0^t e^{-2c\left(1+A_{n-1}\right)^2 \frac{1}{\delta_+} e^{\delta_+ \tau}+\delta_+\tau}\, d\tau \\ &\leq c\varepsilon\|\chi_0\|_s \left\{ 1 + 2c\big(1 + A_{n-1}\big)^2 e^{2c\left(1+A_{n-1}\right)^2 \frac{1}{\delta_+} e^{\delta_+ t}} \int_0^t e^{-2c\left(1+A_{n-1}\right)^2 \frac{1}{\delta_+} e^{\delta_+ \tau}+\delta_+\tau}\, d\tau \right\}\,. \end{aligned}$$

It follows

$$A_n \leq c\varepsilon\|\chi_0\|_s \left\{ 1 + 2c\big(1 + A_{n-1}\big)^2 e^{2c\left(1+A_{n-1}\right)^2 \frac{1}{\delta_+} e^{\delta_+ t}} \int_0^t e^{-2c\left(1+A_{n-1}\right)^2 \frac{1}{\delta_+} e^{\delta_+ \tau}+\delta_+\tau}\, d\tau \right\}\,.$$

On the other hand,

$$\int_0^t e^{-2c\left(1+A_{n-1}\right)^2 \frac{1}{\delta_+} e^{\delta_+ \tau}+\delta_+\tau}\, d\tau \leq \frac{1}{2c\big(1 + A_{n-1}\big)^2} e^{\frac{-2c(1+A_{n-1})^2}{\delta_+}}$$

leads to

$$A_n \quad \leq \quad c\varepsilon\|\chi_0\|_s \left\{ 1 + e^{2c\left(1+A_{n-1}\right)^2 \frac{1}{\delta_+} e^{\delta_+ t} - \frac{2c(1+A_{n-1})^2}{\delta_+}} \right\} \leq c\varepsilon\|\chi_0\|_s \left\{ 1 + e^{-\frac{2c(1+A_{n-1})^2}{\delta_+}} \right\}\,.$$

Finally,

$$
\begin{aligned}
A_n &\leq 2c\varepsilon\|\chi_0\|_s \exp\left\{-\frac{2c}{\delta_+}\right\} \exp\left\{-\frac{4cA_{n-1}(1+A_{n-1})}{\delta_+}\right\} \\
&\leq C\varepsilon \exp\left\{-\frac{4c}{\delta_+}A_{n-1}(1+A_{n-1})\right\}, \quad C := 2c\|\chi_0\|_s \exp\left\{-\frac{2c}{\delta_+}\right\}. \qquad (3.9)
\end{aligned}
$$

Let ε_0 be such that

$$
2\varepsilon_0 C < 1 \quad \text{and} \quad -\frac{16c}{\delta_+}2C\varepsilon < \ln 2\,.
$$

If

$$
A_{n-1} \leq 2C\varepsilon \quad \text{and} \quad \varepsilon \leq \varepsilon_0\,,
$$

then due to (3.9) we obtain

$$
\begin{aligned}
A_n &\leq C\varepsilon \exp\left\{-\frac{4c}{\delta_+}A_{n-1}(1+A_{n-1})\right\} \\
&\leq C\varepsilon \exp\left\{-\frac{4c}{\delta_+}2C\varepsilon(1+2C\varepsilon)\right\} \\
&\leq C\varepsilon \exp\left\{-\frac{16c}{\delta_+}2C\varepsilon\right\} \\
&\leq 2C\varepsilon\,.
\end{aligned}
$$

Thus, for given s and for all $n \geq 1$ we have proved the estimate

$$
\sup_{t\in\mathbb{R}_+}\ \sup_{0\leq\tau\leq t\,,\,0\leq k\leq n} e^{-\frac{1}{2}\delta_+\tau}\|\chi^{(k)}(\tau)\|_s \leq 4c\|\chi_0\|_s \exp\left\{-\frac{2c}{\delta_+}\right\}\varepsilon \quad \text{for all} \quad n \geq 1\,.
$$

The last estimate, (3.2), and Sobolev inequality imply

$$
\sup_{n=0,1,2\dots}\ \sup_{x\in\mathbb{R}^3,\ t\in\mathbb{R}_+} \left\{e^{-\frac{1}{2}\delta_+t}|\chi^{(n)}(x,t)|, e^{-\frac{1}{2}\delta_+t}|\Psi(x,t)|\right\} = r < \infty\,. \qquad (3.10)
$$

Hence, for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}_+$ we have

$$
\begin{aligned}
&\left|f_1(\chi^{(k-1)}(x,t);\Psi(x,t)) - f_1(\chi^{(k-2)}(x,t);\Psi(x,t))\right| \qquad (3.11) \\
\leq\ &\left|\chi^{(k-1)}(x,t) - \chi^{(k-2)}(x,t)\right| \sup_{\xi,\eta\in\mathbb{C}\ |\xi|,|\eta|\leq r} |\nabla_{\xi,\eta}f_1(\xi,\eta)|\,.
\end{aligned}
$$

Consider

$$
i\gamma^0\mathscr{D}_{dS}(x,t,\partial)\Big(\chi^{(k)} - \chi^{(k-1)}\Big) = f_1\left(\chi^{(k-1)}\,,\,\Psi\right) - f_1\left(\chi^{(k-2)}\,,\,\Psi\right)\,, \quad k = 1,2,\dots\,.
$$

By Lemma 1.1 for the solution of the last equation considering (3.11) and the initial values, one has

$$
\begin{aligned}
&e^{-\frac{1}{2}\delta_+t}\|\chi^{(k)}(t) - \chi^{(k-1)}(t)\|_{L^2(\mathbb{R}^3)} \\
\leq\ & c\int_0^t e^{-\frac{1}{2}\delta_+s}\|f_1\left(\chi^{(k-1)}(x,\tau)\,,\,\Psi(x,\tau)\right) - f_1\left(\chi^{(k-2)}(x,\tau)\,,\,\Psi(x,\tau)\right)\|_{L^2(\mathbb{R}^3)}\,ds \\
\leq\ & c\Bigg(\sup_{\xi,\eta\in\mathbb{C},\ \|\xi|,|\eta|\leq r}|\nabla_{\xi,\eta}f_1(\xi,\eta)|\Bigg)\int_0^t e^{-\frac{1}{2}\delta_+s}\|\chi^{(k-1)}(x,t) - \chi^{(k-2)}(x,t)\|_{L^2(\mathbb{R}^3)}\,ds\,.
\end{aligned}
$$

It follows

$$
\begin{aligned}
&e^{-\frac{1}{2}\delta_+t}\|\chi^{(k)}(t) - \chi^{(k-1)}(t)\|_{L^2(\mathbb{R}^3)} \\
\leq\ & c^2\Bigg(\sup_{\xi,\eta\in\mathbb{C},\ |\xi|,|\eta|\leq r}|\nabla_{\xi,\eta}f_1(\xi,\eta)|\Bigg)^2\int_0^t ds\int_0^s e^{-\frac{1}{2}\delta_+s_1}\|\chi^{(k-2)}(x,s_1) - \chi^{(k-3)}(x,s_1)\|_{L^2(\mathbb{R}^3)}\,ds_1
\end{aligned}
$$

and, consequently,

$$e^{-\frac{1}{2}\delta_+ t}\|\chi^{(k)}(t)-\chi^{(k-1)}(t)\|_{L^2(\mathbb{R}^3)} \le C\|\chi_0(x)\|_{L^2(\mathbb{R}^3)}\frac{(Ct)^k}{k!}, \quad \text{for all} \quad k=1,2,\dots .$$

Hence, the sequence $e^{-\frac{1}{2}\delta_+ t}\chi^{(k)}(t)$ converges to some $e^{-\frac{1}{2}\delta_+ t}\chi \in C^0([0,\infty);(L^2(\mathbb{R}^3)^4)$, that is,

$$\lim_{k\to\infty} e^{-\frac{1}{2}\delta_+ t}\chi^{(k)}(t) = e^{-\frac{1}{2}\delta_+ t}\chi(t)$$

uniformly on every compact subset of $\mathbb{R}^3$. By (3.10)

$$\lim_{k\to\infty} f_1\left(\chi^{(k)}\,,\,\Psi\right) = f_1\left(\chi\,,\,\Psi\right) \quad \text{in} \quad C^0([0,\infty);(L^2(\mathbb{R}^3)^4)\,.$$

Thus, χ solves (3.6), while ψ solves (0.3) and

$$\sup_{t\in\mathbb{R}_+} e^{-\frac{1}{2}\delta_+ t}\|\psi(t)\|_s < \infty \quad \text{for} \quad s\ge 6 \tag{3.12}$$

implies

$$\sup_{t\in\mathbb{R}_+} e^{-\frac{1}{2}\delta_+ t}|\psi(x,t)|_{s'} < \infty \quad \text{for} \quad s'\le s-1\,.$$

By Gagliardo-Nirenberg inequality for any integer $s\ge 6$ we have

$$\left\|F\left(\psi^*\gamma^0\psi\,,\,\psi^*\gamma^0\gamma^5\psi\right)\psi(t)\right\|_{(H_{(s)}(\mathbb{R}^3))^4} \le C_s\left\|\psi(t)\right\|_{(H_{(s)}(\mathbb{R}^3))^4}\,. \tag{3.13}$$

For every $s\ge 6$ the local Cauchy problem for (0.3) is well posed in $C^0([0,T_s);(H_{(s)}(\mathbb{R}^3))^4)$ for some $0<T_s$. According to (3.13) and

$$\|\psi(t)\|_s \le C_s e^{\frac{1}{2}\delta_+ t} + C_s e^{\frac{1}{2}\delta_+ t}\int_0^t e^{-\frac{1}{2}\delta_+ \tau}\|\psi(x,\tau)\|_s\,d\tau\,, \qquad t\in[0,T_s)\,,$$

from the last inequality we conclude that $T_s=\infty$ and $\psi(t)\in(C_0^\infty(\mathbb{R}^3))^4$. The equation (0.3) implies $\psi\in C^1([0,\infty);(C_0^\infty(\mathbb{R}^3))^4)$. Theorem is proved. $\square$

4 Large time asymptotics for large data solution

Theorem 4.1 *Let $\psi=\psi(x,t)$ be a solution of the problem*

$$\begin{cases} \left(i\gamma^0\partial_0 + ie^{-Ht}\displaystyle\sum_{\ell=1,2,3}\gamma^\ell\partial_\ell + i\frac{3}{2}H\gamma^0 - m\mathbb{I}_4 + \gamma^0 V(x,t)\right)\psi = F\left(\psi^*\gamma^0\psi\,,\,\psi^*\gamma^0\gamma^5\psi\right)\psi\,,\\ \psi(x,0)=\Psi_0(x)+\varepsilon\chi_0(x)\,, \end{cases}$$

given by Theorem 0.2. Assume that $F\left(\psi^*\gamma^0\psi\,,\,\psi^*\gamma^0\gamma^5\psi\right)\psi$ is the Lipschitz continuous function with exponent $\alpha>0$ in the space $H_{(6)}(\mathbb{R}^3)$.

Then the limit

$$\lim_{t\to\infty}\int_0^t S(0,\tau)\gamma^0 F\left(\psi^*(x,\tau)\gamma^0\psi(x,\tau)\,,\,\psi^*(x,\tau)\gamma^0\gamma^5\psi(x,\tau)\right)\psi(x,\tau)\,d\tau \tag{4.1}$$

exists in the space $H_{(6)}(\mathbb{R}^3)$. Furthermore, the solution $\psi^+(x,t)$ of the Cauchy problem for the free Dirac equation (1.6), where

$$\begin{aligned}\psi_0^+(x) &= \Psi_0(x)+\varepsilon\chi_0(x)\\ &\quad -i\lim_{t\to\infty}\int_0^t S(0,\tau)\gamma^0 F\left(\psi^*(x,\tau)\gamma^0\psi(x,\tau)\,,\,\psi^*(x,\tau)\gamma^0\gamma^5\psi(x,\tau)\right)\psi(x,\tau)\,d\tau\,,\end{aligned}$$

satisfies

$$\lim_{t\to+\infty} \left\|\psi(x,t)-\psi^+(x,t)\right\|_{(H_{(6)}(\mathbb{R}^3))^4} = 0$$

and $\mathscr{S}^+ : \psi_0 \longmapsto \psi_0^+$ *is the continuous operator.*

Moreover, if $V(x,t)=0$*, then the function* $\psi^+(x,t)$ *is given by (1.12).*

Proof. It is enough to prove the convergence in the space $(H_{(6)}(\mathbb{R}^3))^4$. The solution ψ can be written as in (3.3), $\psi=\Psi+\chi$, where the function $\Psi=\Psi(x,t)$ solves (3.1) while the function $\chi=\chi(x,t)$ solves (3.6), where

$$|f_1(\chi\,,\,\Psi)\,| = O\Big(|\chi|\big(|\chi|+|\Psi|\big)^2\Big)\,.$$

According to (3.12)

$$\begin{aligned}\left\|F\left(\psi^*\gamma^0\psi\,,\,\psi^*\gamma^0\gamma^5\psi\right)\psi(t)\right\|_{(H_{(6)}(\mathbb{R}^3))^4} &\le C_s\left\|\psi(t)\right\|^{1+\alpha}_{(H_{(6)}(\mathbb{R}^3))^4}\\ &\le C_s e^{\frac{1}{2}\delta_+(1+\alpha)t}\,.\end{aligned}$$

At the same time

$$\begin{aligned}\left\|S(0,\tau)\gamma^0F\left(\psi^*\gamma^0\psi\,,\,\psi^*\gamma^0\gamma^5\psi\right)\psi(\tau)\right\|_{(H_{(6)}(\mathbb{R}^3))^4} &\le C_s e^{-\frac{1}{2}\delta_-\tau}\left\|\psi(\tau)\right\|^{1+\alpha}_{(H_{(6)}(\mathbb{R}^3))^4}\\ &\le C_s e^{-\frac{1}{2}\delta_-\tau}e^{\frac{1}{2}\delta_+(1+\alpha)\tau}\end{aligned}$$

that implies the convergence of (4.1) in $(H_{(6)}(\mathbb{R}^3))^4$. □

5 Nonexistence of global in time solution

Consider the semilinear Dirac equation

$$\left(i\gamma^0\partial_0+ie^{-Ht}\sum_{\ell=1,2,3}\gamma^\ell\partial_\ell+i\frac{3}{2}H\gamma^0-m\mathbb{I}_4+i\gamma^0V(x,t)\right)\psi = iG(\psi)\gamma^0\psi\,,$$

where $H\in\mathbb{R}$, $m\in\mathbb{C}$, $V^*(x,t)=V(x,t)$, and the matrix-valued term $G(\psi)$ commutes with γ^0, $\gamma^0G(\psi)=G(\psi)\gamma^0$. The equation can also be written in the equivalent form of the following symmetric hyperbolic system

$$\mathscr{D}_{dS}(x,t,\partial)\psi = G(\psi)\psi\,.$$

Lemma 5.1 *Let* $H\in\mathbb{R}$*,* $m\in\mathbb{C}$*. Then for the derivative of the energy integral we have*

$$\frac{d}{dt}E(t)=\int_{\mathbb{R}^3}\Big(2\Re(G_{jk}(\psi)\psi_k(x,t)\overline{\psi_j(x,t)})-3H|\psi(x,t)|^2+2(\Im(m))\psi^*(x,t)\gamma^0\psi(x,t)\Big)\,dx.$$

Proof. The arguments have been used in the proof of Lemma 1.1 complete the proof of lemma. □

5.1 Nonexistence of global solution in the expanding universe

The next theorem gives blow up result for the solution with the large data.

Theorem 5.2 *Consider the Cauchy problem*

$$\begin{cases}\left(i\gamma^0\partial_0+ie^{-Ht}\displaystyle\sum_{\ell=1,2,3}\gamma^\ell\partial_\ell+i\frac{3}{2}H\gamma^0-m\mathbb{I}_4+i\gamma^0V(x,t)\right)\psi = iG(\psi)\gamma^0\psi\,,\\ \psi(x,0)=\psi_0(x)\end{cases}\tag{5.1}$$

with $\psi_0(x)$ *such that* $\operatorname{supp}\psi_0(x) \subseteq \{x \in \mathbb{R}^3 \,|\, |x| \le R\}$*. Assume that* $H>0$*,* $V^*(x,t)=V(x,t)$*, and*

$$\begin{aligned} & G(\zeta) = O(|\zeta|), \quad \gamma^0 G(\zeta) = G(\zeta)\gamma^0, \quad \Re(G(\zeta)\zeta,\overline{\zeta}) \ge c_0|\zeta|^{2+\alpha}, \quad \alpha>0, \\ & \int_{\mathbb{R}^3} |\psi_0(x)|^2\,dx > \left(\frac{3H+2|\Im(m)|}{c_0}\right)^{2/\alpha}\left(R+\frac{1}{H}\right)^3 . \end{aligned} \tag{5.2}$$

Then the solution ψ *of (5.1) that obeys the finite propagation speed property* $\operatorname{supp}\psi(x,t) \subseteq \big\{x \in \mathbb{R}^3 \,\big|\, |x| \le R+\phi(t)\big\}$ *blows up at finite time. More precisely, for the time* T *defined by*

$$T = -\frac{2}{\alpha(3H+2|\Im(m)|)}\ln\left(1-\frac{3H+2|\Im(m)|}{c_0}\left(\int_{\mathbb{R}^3}|\psi_0(x)|^2\,dx\right)^{-\alpha/2}\left(R+\frac{1}{H}\right)^{3\alpha/2}\right)$$

the following is true

$$\lim_{t\nearrow T}\int_{\mathbb{R}^3}|\psi(x,t)|^2\,dx = \infty\,.$$

Proof. According to Lemma 5.1 if $H\in\mathbb{R}$, $m\in\mathbb{C}$, then

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_{\mathbb{R}^3}\Big(2\Re(G_{jk}(\psi)\psi_k(x,t)\overline{\psi_j(x,t)}) - 3H|\psi(x,t)|^2 + 2\Im(m)\psi(x,t)\gamma^0\overline{\psi(x,t)}\Big)\,dx \\ &\ge \int_{\mathbb{R}^3}\Big(c_0|\psi(x,t)|^{2+\alpha} - 3H|\psi(x,t)|^2 + 2\Im(m)\psi(x,t)\gamma^0\overline{\psi(x,t)}\Big)\,dx \\ &\ge c_0\int_{\mathbb{R}^3}|\psi(x,t)|^{2+\alpha}\,dx - (3H+2|\Im(m)|)\int_{\mathbb{R}^3}|\psi(x,t)|^2\,dx\,. \end{aligned}$$

Further, if the solution obeys the finite speed propagation property, we obtain

$$\int_{\mathbb{R}^3}|\psi(x,t)|^2\,dx \le \left(\int_{\operatorname{supp}\psi}|\psi(x,t)|^{2+\alpha}\,dx\right)^{2/(2+\alpha)}(R+\phi(t))^{3\alpha/(2+\alpha)}\,.$$

It follows

$$(R+\phi(t))^{-3\alpha/2}\left(\int_{\mathbb{R}^3}|\psi(x,t)|^2\,dx\right)^{(2+\alpha)/2} \le \left(\int_{\operatorname{supp}\psi}|\psi(x,t)|^{2+\alpha}\,dx\right). \tag{5.3}$$

Then

$$\frac{d}{dt}E(t) \ge K(t)E(t)^{(2+\alpha)/2} - AE(t)\,,$$

where

$$K(t) := c_0\,(R+\phi(t))^{-3\alpha/2}\,, \quad A := (3H+2|\Im(m)|)\,.$$

Hence

$$\frac{d}{dt}\left(E(t)e^{At}\right) \ge K(t)e^{-A\alpha t/2}\left(E(t)e^{At}\right)^{(2+\alpha)/2}\,.$$

For the function

$$F(t) := E(t)e^{At}$$

the inequality leads to

$$\frac{d}{dt}F(t)^{-\frac{\alpha}{2}} \le -\frac{\alpha}{2}K(t)e^{-A\frac{\alpha}{2}t}\,.$$

After integration we obtain

$$F(t)^{-\frac{\alpha}{2}} \le F(0)^{-\frac{\alpha}{2}} - \frac{\alpha}{2}c_0\int_0^t (R+\phi(s))^{-3\frac{\alpha}{2}}\,e^{-As\frac{\alpha}{2}}\,ds\,.$$

Since the function $K(s)$ is monotonically decreasing, we obtain

$$F(t)^{-\frac{\alpha}{2}} \leq F(0)^{-\alpha/2} - \frac{1}{A}c_0\left(R+\frac{1}{H}\right)^{-3\frac{\alpha}{2}}(1-e^{-At\frac{\alpha}{2}})\,.$$

The condition (5.2) guarantees that the solution blows up no later than time T such that

$$F(0)^{-\alpha/2} = \frac{1}{A}c_0\left(R+\frac{1}{H}\right)^{-3\frac{\alpha}{2}}(1-e^{-AT\frac{\alpha}{2}})$$

provided that $F(0)$ is sufficiently large. Theorem is proved. □

5.2 Nonexistence of global solution in the contracting universe

In the next theorem $F\,(a,b;c;z)$ is the hypergeometric function (see, e.g., [3]).

Theorem 5.3 *Consider the Cauchy problem (5.1) with $\psi_0(x)$ such that* $\operatorname{supp}\psi_0(x) \subseteq \{x \in \mathbb{R}^3 \,|\, |x| \leq R\}$. *Assume that $H<0$, $V^*(x,t) = V(x,t)$, and for G, m, and H there is a constant $c_{G,H,m} > 0$ such that*

$$G(\zeta) = O(|\zeta|), \quad \gamma^0 G(\zeta) = G(\zeta)\gamma^0 \quad \text{for all} \quad \zeta \in \mathbb{C}^4, \tag{5.4}$$

$$2\Re(G(\zeta)\zeta,\overline{\zeta}) - 3H|\zeta|^2 + 2\Im(m)(\zeta,\gamma^0\overline{\zeta}) \geq c_{G,H,m}|\zeta|^3 \quad \text{for all} \quad \zeta \in \mathbb{C}^4\,. \tag{5.5}$$

Then the solution ψ of (5.1) that obeys the finite propagation speed property $\operatorname{supp}\psi(x,t) \subseteq \big\{x \in \mathbb{R}^3 \,\big|\, |x| \leq R+\phi(t)\big\}$ *blows up at finite time. More precisely, there is $T_{ls} < \infty$ defined by*

$$\int_{\mathbb{R}^3} |\psi_0(x)|^2\,dx = \left\{\frac{c_{G,H,m}\alpha}{2}\int_0^{T_{ls}} (R+\phi(s))^{-3\alpha/2}\,ds\right\}^{-2/\alpha},$$

depending on c_0, α, m, ψ_0, and H such that if

$$\int_{\mathbb{R}^3} |\psi_0(x)|^2\,dx > \left\{\frac{c_{G,H,m}R^{1-\frac{3}{2}}}{3}F\left(1,1;\frac{3\alpha}{2}+1;HR+1\right)\right\}^{2/\alpha},$$

then

$$\lim_{t\nearrow T_{ls}} \int_{\mathbb{R}^3} |\psi(x,t)|^2\,dx = \infty\,.$$

Proof. From Lemma 5.1

$$\frac{d}{dt}E(t) = \int_{\mathbb{R}^3}\Big(2\Re(G_{jk}(\psi)\psi_k(x,t)\overline{\psi_j(x,t)}) - 3H|\psi(x,t)|^2 + 2(\Im(m))\psi(x,t)\gamma^0\overline{\psi(x,t)}\Big)\,dx$$

and conditions (5.4), (5.5) we derive

$$\frac{d}{dt}E(t) \geq c\int_{\mathbb{R}^3} |\psi(x,t)|^{2+\alpha}\,dx\,.$$

Then, since the solution obeys the finite propagation speed property, we obtain (5.3) and, consequently,

$$E(t)^{-(2+\alpha)/2}\frac{d}{dt}E(t) \geq c_{G,H,m}\,(R+\phi(t))^{-3\alpha/2}\,.$$

We integrate the last inequality and obtain

$$-E(t)^{-\alpha/2} + E(0)^{-\alpha/2} \geq \frac{\alpha}{2}c_{G,H,m}\int_0^t (R+\phi(t))^{-3\alpha/2}\,ds$$

that leads to

$$E(t) \geq \left(\frac{1}{E(0)^{\alpha/2}} - \frac{c_{G,H,m}\alpha}{2} \int_0^t (R+\phi(t))^{-3\alpha/2}\, ds \right)^{-2/\alpha} .$$

Now for $H<0$ we calculate

$$\begin{aligned}
& \int_0^t (R+\phi(s))^{-3\alpha/2}\, ds \\
= \;& \frac{2}{3\alpha} \left\{ R^{1-\frac{3\alpha}{2}} F\left(1,1;\frac{3\alpha}{2}+1;HR+1\right) \right. \\
& \left. +|H|^{\frac{3\alpha}{2}-1} \left((HR+1)e^{Ht}-1\right) \left(|H|R+e^{-Ht}-1\right)^{-\frac{3\alpha}{2}} F\left(1,1;\frac{3\alpha}{2}+1;e^{Ht}(HR+1)\right) \right\} .
\end{aligned}$$

Here

$$\lim_{t\to\infty} \int_0^t (R+\phi(s))^{-3\alpha/2}\, ds = \frac{2}{3\alpha} R^{1-\frac{3\alpha}{2}} F\left(1,1;1+\frac{3\alpha}{2};HR+1\right), \quad H<0,$$

that gives a blowup of the solution with the large data such that

$$\frac{1}{E(0)^{\alpha/2}} < \frac{c_{G,H,m}\alpha}{2} \frac{2}{3\alpha} R^{1-\frac{3\alpha}{2}} F\left(1,1;1+\frac{3\alpha}{2};HR+1\right), \quad H<0\,,$$

and with the lifespan T_{ls} that can be obtained from

$$\frac{1}{E(0)^{\alpha/2}} = \frac{c_{G,H,m}\alpha}{2} \int_0^{T_{ls}} (R+\phi(s))^{-3\alpha/2}\, ds\,.$$

The theorem is proved. □

Remark 5.4 *We do not know whether for small data $E(0)$ such that*

$$\int_{\mathbb{R}^3} |\psi_0(x)|^2\, dx < \left\{ \frac{c_{G,H,m} R^{1-\frac{3}{2}}}{3} F\left(1,1;\frac{3\alpha}{2}+1;HR+1\right) \right\}^{2/\alpha} ,$$

the global solution exists. Finally, we note that if $\psi_0(x) \in C_0^\infty(\mathbb{R}^3)$, then the solution of the problem obeys the finite propagation speed property, which is proved in Section 6.

6 Finite speed propagation property

We are going to prove that the dependence domain for the classical solution $u(t,x)$ at the point (T,x_0) of the semilinear equation coincides with the dependence domain of the solution of the linear equation (0.1). More precisely, for $(T,x_0) \in [0,\infty)\times\mathbb{R}^3$ with $T>0$ let

$$\Sigma^-(T,x_0) := \left\{ (t,x) \in [0,T]\times\mathbb{R}^3 \,\middle|\, |x-x_0| = -(\phi(t)-\phi(T)) \right\}$$

be a part of the backward "curved light cone" (nullcone), where $\phi(t) := (1-e^{-Ht})/H$. Let also

$$D_-(T,x_0) = \left\{ (t,x) \in [0,T]\times\mathbb{R}^3 \,\middle|\, |x-x_0| \leq -\left(\phi(t)-\phi(T)\right) \right\}$$

be the region defined in (1.10), whose boundary contains $\Sigma^-(T,x_0)$. In the proof of the next theorem, we follow [13]. (See also [22, Ch.2, §6] and [24, Ch.16, §1].)

Theorem 6.1 *Let ψ be a C^1 solution of the equation*

$$\left(\partial_0 + e^{-Ht} \sum_{\ell=1,2,3} \alpha^\ell \partial_\ell + \frac{3}{2} H \mathbb{I}_4 + im\gamma^0 - iV(x,t) \right) \psi = F(\psi,\psi')\,, \tag{6.6}$$

in the backward curved light cone $D_-(T,x_0)$ through $(T,x_0) \in (0,\infty)\times\mathbb{R}^3$ with $T>0$. Assume that the potential $V \in C([0,\infty)\times\mathbb{R}^3)$ and the nonlinear term $F(\psi,\psi') \in C^1$ is such that

$$F(0,\psi') = 0 \quad \textit{for all} \quad \psi' = (\partial_t\psi, \nabla_x\psi)\,. \tag{6.7}$$

If

$$\psi(x,0) = 0 \quad \textit{for all} \quad x \in D_-(T,x_0)\cap\{t=0\}\,, \tag{6.8}$$

then ψ vanishes in $D_-(T,x_0)$.

Proof. The interior of the domain $D_-(T,x_0)$ can be filled up by one-parameter family of the smooth spacelike surfaces $\Sigma_s^-(T,x_0)$, where s is a parameter. In order to find the equation $t=\tau(s,x)$ of such surfaces we slightly modify the equation of the light cone $\phi(T)-\phi(t)=|x-x_0|,\ t\le T$, by equipping it with the parameter s in the following way

$$(\phi(T)-\phi(t))^2 = (s-\phi(T))^2 + (\phi(T))^{-2}\left(2s\phi(T)-s^2\right)|x-x_0|^2, \quad 0\le s\le\phi(T)\,.$$

Then for the values of the parameter $0\le s<\phi(T)$, by the implicit function theorem, the last equation can be solved for t since $\phi'(t)(\phi(T)-\phi(t))\neq 0$. The solution $t=\tau(s,x)$ is

$$\tau(s,x) = -\frac{1}{H}\ln\Big(1-H\phi(T)+H[(s-\phi(T))^2+(\phi(T))^{-2}\left(2s\phi(T)-s^2\right)|x-x_0|^2]^{1/2}\Big)\,.$$

Here

$$\tau(0,x)=0, \qquad \lim_{s\to\phi(T)}\tau(s,x) = -\frac{1}{H}\ln\Big(1-H(\phi(T)-|x-x_0|)\Big)\,.$$

For every given s, $0\le s\le\phi(T)$, we consider x such that

$$|x-x_0|^2 \le \frac{(\phi(T))^2[(\phi(T)-\phi(t))^2-(s-\phi(T))^2]}{(2s\phi(T)-s^2)}\,.$$

The region bounded by the surface $\Sigma_s^-(T,x_0)$ is

$$D_{-,s}(T,x_0) = \left\{(t,x)\,\middle|\, 0\le t\le\phi(s,x), |x-x_0|^2 \le \frac{(\phi(T))^2[(\phi(T)-\phi(t))^2-(s-\phi(T))^2]}{(2s\phi(T)-s^2)}\right\}\,.$$

Hence

$$D_-(T,x_0) = \bigcup_{0\le s\le\phi(T)} D_{-,s}(T,x_0)\,.$$

The surface $\Sigma_s^-(T,x_0)$ is space-like. Indeed, its outward unit normal at $(\tau(s,x),x)$ is

$$n(s,x) = \frac{(1,-\nabla_x\tau(s,x))}{\sqrt{1+|\nabla_x\tau(s,x)|^2}}\,,$$

where

$$|\nabla_x\tau(s,x)| = \frac{e^{H\tau(s,x)}(\phi(T))^{-2}\left(2s\phi(T)-s^2\right)|x-x_0|}{[(s-\phi(T))^2+(\phi(T))^{-2}\left(2s\phi(T)-s^2\right)|x-x_0|^2]^{1/2}}\,,$$

and, consequently,

$$n(s,x)^*g^{-1}(\tau(s,x))n(s,x) = (1-e^{-2H\tau(s,x)}|\nabla_x\tau(s,x)|^2)\frac{1}{1+|\nabla_x\tau(s,x)|^2} > 0\,.$$

Next, we apply the energy method to the equation (6.6). We write the identity

$$\begin{aligned}
&\partial_0|\psi|^2 + \sum_{\ell=1,2,3}\partial_\ell(e^{-Ht}\psi^*\alpha^\ell\psi) + 3H|\psi|^2 - 2\Im(m)\psi^*\gamma^0\psi + 2\psi^*\Im(V(x,t))\psi \\
=\;& \psi^*F(\psi,\psi') + F(\psi,\psi')^*\psi\,.
\end{aligned}$$

By the divergence theorem and the vanishing initial conditions (6.8) we obtain

$$
\begin{aligned}
& \int_{D_{-,s}(T,x_0)} \left(\partial_0 |\psi|^2 + \sum_{\ell=1,2,3} \partial_\ell (e^{-Ht}\psi^* \alpha^\ell \psi) \right) dt\, dx \\
= & \int_{\Sigma_s^-(T,x_0)} \left(|\psi|^2 + \sum_{\ell=1,2,3} e^{-Ht}\psi^* \alpha^\ell \psi (\partial_\ell \tau(s,x)) \right) \frac{1}{\sqrt{1+|\nabla_x \tau(s,x)|^2}}\, d\sigma \,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \int_{\Sigma_s^-(T,x_0)} \left(|\psi|^2 + \sum_{\ell=1,2,3} e^{-Ht}\psi^* \alpha^\ell \psi (\partial_\ell \tau(s,x)) \right) \frac{1}{\sqrt{1+|\nabla_x \tau(s,x)|^2}}\, d\sigma \\
& + \int_{D_{-,s}(T,x_0)} \left(3H|\psi|^2 - 2\Im(m)\psi^*\gamma^0\psi + 2\psi^*\Im(V(t))\psi \right) dt\, dx \\
= & \int_{D_{-,s}(T,x_0)} (\psi^* F(\psi,\psi') + F(\psi,\psi')^*\psi)\, dt\, dx \,. \qquad (6.9)
\end{aligned}
$$

We are going to estimate

$$
\sum_{\ell=1,2,3} e^{-H\tau(s,x)}\psi^* \alpha^\ell \psi (\partial_\ell \tau(s,x)) \frac{1}{\sqrt{1+|\nabla_x \tau(s,x)|^2}}
$$

on the surface $\Sigma_s^-(T,x_0)$. If we restrict the parameter s by $0 \le s \le s_0 < \phi(T)$, then

$$
\begin{aligned}
e^{-H\tau(s,x)}|\nabla_x \tau(s,x)| & = \frac{(\phi(T))^{-2}\left(2s\phi(T)-s^2\right)|x-x_0|}{[(s-\phi(T))^2 + (\phi(T))^{-2}\left(2s\phi(T)-s^2\right)|x-x_0|^2]^{1/2}} \\
& \le \vartheta(s_0) < 1 \,.
\end{aligned}
$$

Consider the hermitian matrix

$$
A = \mathbb{I}_4 - \sum_{\ell=1,2,3} \alpha^\ell a_\ell, \quad a_\ell := e^{-H\tau(s,x)} \frac{\partial_\ell \tau(s,x)}{\sqrt{1+|\nabla_x \tau(s,x)|^2}}, \quad \ell = 1,2,3.
$$

It has two double eigenvalues

$$
1 - \sqrt{a_1^2+a_2^2+a_3^2} > 0 \quad \text{and} \quad 1 + \sqrt{a_1^2+a_2^2+a_3^2} > 0 \quad \text{for all} \quad s \in [0,s_0].
$$

Hence, there is $\delta(s_0) < 1$ such that

$$
\left| \sum_{\ell=1,2,3} e^{-H\tau(s,x)}\psi^* \alpha^\ell \psi (\partial_\ell \tau(s,x)) \frac{1}{\sqrt{1+|\nabla_x \tau(s,x)|^2}} \right| \le \delta(s_0)|\psi|^2 \quad \text{for all} \quad s \in [0,s_0] \,.
$$

The equation (6.9) and condition (6.7) imply

$$
\begin{aligned}
& \int_{\Sigma_s^-(T,x_0)} |\psi|^2 \frac{1}{\sqrt{1+|\nabla_x \tau(s,x)|^2}} d\sigma \\
\le \; & C(s_0) \int_0^s \int_{\Sigma_\lambda^-(T,x_0)} |\psi|^2 \frac{\partial_\lambda \tau(\lambda,x)}{\sqrt{1+|\nabla_x \tau(\lambda,x)|^2}}\, d\sigma\, d\lambda \quad \text{for all} \quad s \le s_0 \,. \qquad (6.10)
\end{aligned}
$$

Here

$$
\partial_s \tau(s,x) = \frac{\phi(T) - s - s(\phi(T))^{-2}(\phi(T)-1)|x-x_0|^2}{\phi'(\tau(s,x))(\phi(T)-\phi(\tau(s,x)))}\,, \quad 0 \le s < \phi(T)\,.
$$

If we set

$$I(s) = \int_{\Sigma_s^-(T,x_0)} |\psi|^2 \frac{1}{\sqrt{1+|\nabla_x \tau|^2}}\, d\sigma\,,$$

then from (6.10) it follows

$$I(s) \leq C(s_0) \left(\max_{0\leq t\leq s_0} |\partial_\lambda \tau(t,x)| \right) \int_0^s I(\lambda)\, d\lambda \quad \text{for all} \quad s \leq s_0\,,$$

and Gronwall's inequality completes the proof of theorem. □

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