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General bright and dark soliton solutions to the massive Thirring model via KP hierarchy reductions

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Abstract

In the present paper, we are concerned with the tau function and its connection with the Kadomtsev-Petviashvili (KP) theory for the massive Thirring (MT) model. First, we bilinearize the massive Thirring model under both the vanishing and nonvanishing boundary conditions. Starting from a set of bilinear equations of two-component KP-Toda hierarchy, we derive the multi-bright solution to the MT model by the KP hierarchy reductions. Then, we show that the discrete KP equation can generate a set of bilinear equations of a deformed KP-Toda hierarchy through Miwa transformation. By imposing constraints on the parameters of the tau function, the general dark soliton solution to the MT model is constructed from the tau function of the discrete KP equation. Finally, the dynamics and properties of one- and two-soliton for both the bright and dark cases are analyzed in details.

1 Introduction

The massive Thirring (MT) model

$$iu_x + v + \sigma u|v|^2 = 0, \quad (1)$$

$$iv_t + u + \sigma v|u|^2 = 0, \quad (2)$$

with $\sigma = \pm 1$, was derived by Thirring in 1958 [1] in the context of general relativity. It represents a relativistically invariant nonlinear Dirac equation in the space of one dimension. It is one of the most remarkable solvable field theory models. Its complete integrability was firstly approved by Mikhailov [2] and Orfanidis [3] independently. The inverse scattering transform for the MT model was studied by Kuznetsov and Mikhailov [4] and many others [5, 6, 7, 8, 9]. The Darboux transformation, Bäcklund transformation of the MT model and its connection with other integrable systems have been investigated by Kaup and Newell [10], Lee [11, 12], Prikarpatskii [13, 14], Franca *et al.* [15] and Degasperis [16].

Since the pioneer work by Date [17], the soliton solutions to the MT model including the ones with nonvanishing background were constructed by many authors [18, 19, 20, 21, 22, 23, 24]. In addition, the algebro-geometric solutions to the MT model has attracted attention and has been studied from late 1970s [25, 26] to 1980s [27, 28] until recently [29, 30]. It should be pointed out that the rogue wave solutions to the MT model were recently investigated in [31, 32, 33] by the Darboux transformation.

The MT model admits the following Lax pair

$$\Phi_x = U\Phi, \quad \Phi_t = V\Phi, \quad (3)$$

where

$$U = -\frac{i}{2} \begin{pmatrix} \lambda^2 - \sigma v v^* & 2\lambda v \\ 2\lambda \sigma v^* & -\lambda^2 + \sigma v v^* \end{pmatrix}$$

$$V = -\frac{i}{2} \begin{pmatrix} \lambda^{-2} - \sigma u u^* & 2\lambda^{-1} u \\ 2\lambda^{-1} \sigma u^* & -\lambda^{-2} + \sigma u u^* \end{pmatrix},$$

in the sense that the compatibility condition $U_t - V_x + [U, V] = 0$ gives the MT model (1)–(2). It is noted that the MT model can also be written in the laboratory coordinates as follows

$$i(u_T + u_X) + v + \sigma u |v|^2 = 0, \quad (4)$$

$$i(v_T - v_X) + u + \sigma v |u|^2 = 0, \quad (5)$$

via the following transformation

$$X = x + t, \quad T = x - t.$$

The MT model in the laboratory coordinates has application in nonlinear optics to describe pulse propagation in Bragg gratings [34]. Alexeeva *et al.* studied the PT-symmetry extensions of the MT model [35]. Regarding the integrable discretization, Nijhoff *et al.* gave the integrable discretization

of the MT model in light cone coordinates [36, 37] in 1980s. Most recently, Pelinovsky *et al.* proposed a semi-discrete integrable MT model in laboratory coordinates [38] and studied its solution via Darboux transformation [39].

Surprisingly, as far as we are aware, the bilinear formulation is missing in the literature. As far as we are aware, no paper published either in finding solutions via Hirota's bilinear method [40] or revealing the connection of the MT model to the KP theory by Kyoto school [41]. By combining the Hirota's bilinear method and the KP hierarchy reduction method, we have constructed general soliton solutions to many soliton equations such as the vector nonlinear Schrödinger equation [42] and the complex short pulse equation [43, 44] for both the vanishing boundary condition (VBC) and non-vanishing boundary condition (NVBC). Therefore, the motivation and the goal of the present study is to bilinearize the MT model and find its soliton solutions under VBC and NVBC.

The remainder of the present paper is organized as follows. In section 2, we first bilinearize the MT model into a set of four equations corresponding to the bright soliton solution. Starting from a set of bilinear equations satisfied by the tau functions of two-component KP hierarchy, we arrive at above four bilinear equations satisfied by the bright soliton solutions by a series of reductions such as dimension and complex conjugate reductions. On the other hand, in section 3, we show that the discrete KP equation can generate a single-component KP-Toda hierarchy with four flows: one in positive flow and three in negative flows, along with its tau-function and a set of bilinear equations. By imposing one constraint on parameters, three reduction relations are achieved simultaneously, by which we are finally able to construct general N -dark soliton solution to the MT model. The dynamical behaviors and properties of one- and two-soliton solutions for both the bright and dark ones are analyzed in section 4. The paper is concluded in section 5 by some comments and further topics.

2 Bright solitons in the MT model

2.1 Bilinearization of the MT model under VBC

The bilinearization of the MT system is established by the following proposition.

Proposition 1. *By means of the dependent variable transformations*

$$u = \frac{g}{f^*}, \quad v = \frac{h}{f}, \quad (6)$$

the MT model (1)–(2) is transformed into the following bilinear equations

$$iD_x g \cdot f + hf^* = 0, \quad (7)$$

$$iD_x f \cdot f^* = -\sigma hh^*, \quad (8)$$

$$iD_t h \cdot f^* + gf = 0, \quad (9)$$

$$iD_t f \cdot f^* = \sigma gg^*. \quad (10)$$

where D is the Hirota D -operator defined by [40]

$$D_s^n D_y^m f \cdot g = \left(\frac{\partial}{\partial s} - \frac{\partial}{\partial s'} \right)^n \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^m f(y, s)g(y', s')|_{y=y', s=s'}.$$

Proof. By rewriting the dependent variable transformations (6)

$$u = \frac{g}{f} \frac{f}{f^*}, \quad v = \frac{h}{f^*} \frac{f^*}{f},$$

and substituting into Eq.(1), we have

$$\left[i \left(\frac{g}{f} \right)_x \frac{f}{f^*} + \frac{h}{f} \right] + \left[i \frac{g}{f} \left(\frac{f}{f^*} \right)_x + \sigma \frac{hh^*}{ff^*} \frac{g}{f^*} \right] = 0. \quad (11)$$

Bilinear equations (7) and (8) are deduced by taking zero for each group inside bracket. Similarly, we can drive bilinear equations (9) and (10) by substituting (6) into Eq.(2). \square

2.2 Reductions to the multi-bright solution from the two-component KP-Toda hierarchy

Define the following tau functions of two-component KP-Toda hierarchy,

$$f_{nm} = \begin{vmatrix} A_n & I \\ -I & B_m \end{vmatrix},$$

$$g_{nm} = \begin{vmatrix} A_n & I & \Phi_n^T \\ -I & B_m & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi}_m & 0 \end{vmatrix}, \quad \bar{g}_{nm} = \begin{vmatrix} A'_n & I & \mathbf{0}^T \\ -I & B_m & \Psi_m^T \\ -\bar{\Phi}_n & \mathbf{0} & 0 \end{vmatrix},$$

where A_n and B_m are $N \times N$ matrices whose elements are

$$a_{ij}(n) = \frac{\mu \bar{p}_j}{p_i + \bar{p}_j} \left(-\frac{p_i}{\bar{p}_j} \right)^n e^{\xi_i + \bar{\xi}_j}, \quad a'_{ij}(n) = -\frac{\mu p_i}{p_i + \bar{p}_j} \left(-\frac{p_i}{\bar{p}_j} \right)^n e^{\xi_i + \bar{\xi}_j},$$

$$b_{ij}(m) = \frac{\nu}{q_i + \bar{q}_j} \left(-\frac{q_i}{\bar{q}_j} \right)^m e^{\eta_i + \bar{\eta}_j},$$

with

$$\begin{aligned} \xi_i &= \frac{1}{p_i} x_{-1} + p_i x_1 + \xi_{i0}, & \bar{\xi}_j &= \frac{1}{\bar{p}_j} x_{-1} + \bar{p}_j x_1 + \bar{\xi}_{j0}, \\ \eta_i &= \frac{1}{q_i} y_{-1} + q_i y_1 + \eta_{i0}, & \bar{\eta}_j &= \frac{1}{\bar{q}_j} y_{-1} + \bar{q}_j y_1 + \bar{\eta}_{j0}, \end{aligned}$$

where Φ_n , Ψ_n , $\bar{\Phi}_n$ and $\bar{\Psi}_n$ are N -component row vectors

$$\begin{aligned} \Phi_n &= (p_1^n e^{\xi_1}, \dots, p_N^n e^{\xi_N}), & \bar{\Phi}_n &= \left((-\bar{p}_1)^{-n} e^{\bar{\xi}_1}, \dots, (-\bar{p}_N)^{-n} e^{\bar{\xi}_N} \right), \\ \Psi_m &= (q_1^m e^{\eta_1}, \dots, q_N^m e^{\eta_N}), & \bar{\Psi}_m &= \left((-\bar{q}_1)^{-m} e^{\bar{\eta}_1}, \dots, (-\bar{q}_N)^{-m} e^{\bar{\eta}_N} \right). \end{aligned}$$

Then we have the following lemma:

Lemma 1. *The above tau functions of two-component KP-Toda hierarchy satisfy the following bilinear equations*

$$D_{x_{-1}} g_{nm} \cdot f_{nm} = g_{n-1,m} f_{n+1,m}, \quad (12)$$

$$D_{x_1} g_{n,m+1} \cdot f_{n+1,m} = g_{n+1,m+1} f_{nm}, \quad (13)$$

$$D_{y_1} f_{n+1,m} \cdot f_{n,m} = \mu \nu g_{n,m} \bar{g}_{nm}, \quad (14)$$

$$D_{y_{-1}} f_{n+1,m} \cdot f_{n,m} = -\mu \nu g_{n,m+1} \bar{g}_{n,m-1}. \quad (15)$$

Proof. Let us take the following notations:

$$\begin{aligned} \phi_i(n) &= p_i^n e^{\xi_i}, & \bar{\phi}_i(n) &= (-\bar{p}_i)^{-n} e^{\bar{\xi}_i}, \\ \psi_i(m) &= q_i^m e^{\eta_i}, & \bar{\psi}_i(m) &= (-\bar{q}_i)^{-m} e^{\bar{\eta}_i}, \end{aligned}$$

then the above matrix elements possess the differential and difference rules:

$$\begin{aligned} \partial_{x_1} a_{ij}(n) &= -\mu \phi_i(n) \bar{\phi}_j(n-1), & \partial_{x_{-1}} a_{ij}(n) &= \mu \phi_i(n-1) \bar{\phi}_j(n), \\ a_{ij}(n+1) &= a_{ij}(n) - \mu \phi_i(n) \bar{\phi}_j(n), \\ \partial_{y_1} b_{ij}(m) &= \nu \psi_i(m) \bar{\psi}_j(m), & \partial_{y_{-1}} b_{ij}(m) &= -\nu \psi_i(m-1) \bar{\psi}_j(m+1), \\ b_{ij}(m+1) &= b_{ij}(m) + \nu \psi_i(m) \bar{\psi}_j(m+1), \\ \partial_{x_s} \phi_i(n) &= \phi_i(n+s), & \partial_{x_s} \bar{\phi}_i(n) &= -\bar{\phi}_i(n-s), \\ \partial_{y_s} \psi_i(m) &= \psi_i(m+s), & \partial_{y_s} \bar{\psi}_i(m) &= -\bar{\psi}_i(m-s), \quad (s = \pm 1). \end{aligned}$$

By using the differential formula of determinant, it can be checked that the derivatives and shifts of the tau functions are expressed by the bordered

determinants as follows:

$$\begin{aligned}
f_{n+1,m} &= \begin{vmatrix} A_n & I & \Phi_n^T \\ -I & B_m & \mathbf{0}^T \\ \mu\bar{\Phi}_n & \mathbf{0} & 1 \end{vmatrix}, \quad \partial_{x_{-1}}f_{n,m} = \begin{vmatrix} A_n & I & \Phi_{n-1}^T \\ -I & B_m & \mathbf{0}^T \\ -\mu\bar{\Phi}_n & \mathbf{0} & 0 \end{vmatrix}, \\
\partial_{x_1}f_{n+1,m} &= \begin{vmatrix} A_n & I & \Phi_{n+1}^T \\ -I & B_m & \mathbf{0}^T \\ \mu\bar{\Phi}_n & \mathbf{0} & 0 \end{vmatrix}, \quad \partial_{y_1}f_{n,m} = \begin{vmatrix} A_n & I & \mathbf{0}^T \\ -I & B_m & \Psi_m^T \\ \mathbf{0} & -\nu\bar{\Psi}_m & 0 \end{vmatrix}, \\
\partial_{y_{-1}}f_{n,m} &= \begin{vmatrix} A_n & I & \mathbf{0}^T \\ -I & B_m & \Psi_{m-1}^T \\ \mathbf{0} & \nu\bar{\Psi}_{m+1} & 0 \end{vmatrix}, \quad \partial_{y_1}f_{n+1,m} = \begin{vmatrix} A_n & I & \Phi_n^T & \mathbf{0}^T \\ -I & B_m & \mathbf{0}^T & \Psi_m^T \\ \mu\bar{\Phi}_n & \mathbf{0}^T & 1 & 0 \\ \mathbf{0} & -\nu\bar{\Psi}_m & 0 & 0 \end{vmatrix}, \\
\partial_{y_{-1}}f_{n+1,m} &= \begin{vmatrix} A_n & I & \Phi_n^T & \mathbf{0}^T \\ -I & B_m & \mathbf{0}^T & \Psi_{m-1}^T \\ \mu\bar{\Phi}_n & \mathbf{0}^T & 1 & 0 \\ \mathbf{0} & \nu\bar{\Psi}_{m+1} & 0 & 0 \end{vmatrix}, \quad g_{n-1,m} = \begin{vmatrix} A_n & I & \Phi_{n-1}^T \\ -I & B_m & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi}_m & 0 \end{vmatrix}, \\
g_{n,m+1} &= \begin{vmatrix} A_n & I & \Phi_n^T \\ -I & B_m & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi}_{m+1} & 0 \end{vmatrix}, \quad g_{n+1,m+1} = \begin{vmatrix} A_n & I & \Phi_{n+1}^T & \Phi_n^T \\ -I & B_m & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi}_{m+1} & 0 & 0 \\ \bar{\Phi}_n & \mathbf{0} & 0 & 1 \end{vmatrix}, \\
\partial_{x_{-1}}g_{n,m} &= - \begin{vmatrix} A_n & I & \Phi_n^T & \Phi_{n-1}^T \\ -I & B_m & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi}_m & 0 & 0 \\ \mu\bar{\Phi}_n & \mathbf{0} & 1 & 0 \end{vmatrix}, \quad \partial_{x_1}g_{n,m+1} = \begin{vmatrix} A_n & I & \Phi_{n+1}^T \\ -I & B_m & \mathbf{0}^T \\ \mathbf{0} & -\bar{\Psi}_{m+1} & 0 \end{vmatrix}, \\
\bar{g}_{n,m} &= \begin{vmatrix} A_{n+1} & I & \mathbf{0}^T \\ -I & B_m & \Psi_m^T \\ -\bar{\Phi}_n & \mathbf{0} & 0 \end{vmatrix} = \begin{vmatrix} A_n & I & \mathbf{0}^T \\ -I & B_m & \Psi_m^T \\ -\bar{\Phi}_n & \mathbf{0} & 0 \end{vmatrix}, \\
\bar{g}_{n,m-1} &= \begin{vmatrix} A_{n+1} & I & \mathbf{0}^T \\ -I & B_{m-1} & \Psi_{m-1}^T \\ -\bar{\Phi}_n & \mathbf{0} & 0 \end{vmatrix} = \begin{vmatrix} A_n & I & \mathbf{0}^T \\ -I & B_m & \Psi_{m-1}^T \\ -\bar{\Phi}_n & \mathbf{0} & 0 \end{vmatrix}.
\end{aligned}$$

By using the Jacobi identity of determinant, we can obtain the following relations

$$\begin{aligned}
f_{nm}\partial_{x_{-1}}g_{nm} &= g_{nm}\partial_{x_{-1}}f_{nm} + g_{n-1,m}f_{n+1,m}, \\
f_{nm}g_{n+1,m+1} &= f_{n+1,m}\partial_{x_1}g_{n,m+1} - g_{n,m+1}\partial_{x_1}f_{n+1,m}, \\
f_{n,m}\partial_{y_1}f_{n+1,m} &= f_{n+1,m}\partial_{y_1}f_{n,m} + \mu\nu g_{n,m}\bar{g}_{nm}, \\
f_{n,m}\partial_{y_{-1}}f_{n+1,m} &= f_{n+1,m}\partial_{y_{-1}}f_{n,m} - \mu\nu g_{n,m+1}\bar{g}_{n,m-1},
\end{aligned}$$

which completes the proof of bilinear equations (12)-(15). \square

In what follows, we proceed to the reduction procedure. To this end, we impose the reduction condition

$$q_i = \bar{p}_i, \quad \bar{q}_i = p_i, \quad (16)$$

then one can easily show

$$f_{n+1,m+1} = f_{n,m}, \quad g_{n+1,m+1} = -g_{n,m}, \quad (17)$$

which leads to

$$D_{x_{-1}} g_{nm} \cdot f_{nm} = -g_{n,m+1} f_{n+1,m}, \quad (18)$$

$$D_{x_1} g_{n,m+1} \cdot f_{n+1,m} = -g_{n,m} f_{nm} \quad (19)$$

from Eqs. (12)–Eqs. (13). On the other hand, by performing row and column operations, we rewrite the tau function f_{nm} as

$$f_{nm} = \left| \begin{array}{c|c} \tilde{A}_n & I \\ \hline -I & \tilde{B}_m \end{array} \right|,$$

with the elements

$$\tilde{a}_{ij}(n) = \frac{\mu \bar{p}_j}{p_i + \bar{p}_j} \left(-\frac{p_i}{\bar{p}_j} \right)^n, \quad \tilde{b}_{ij}(m) = \frac{\nu}{q_i + \bar{q}_j} \left(-\frac{q_i}{\bar{q}_j} \right)^m e^{n+\bar{q}_j+\xi_j+\bar{\xi}_i},$$

From the condition (16), one can deduce

$$\partial_{x_1} \tilde{b}_{ij}(m) = \partial_{y_1} \tilde{b}_{ij}(m), \quad \partial_{x_{-1}} \tilde{b}_{ij}(m) = \partial_{y_{-1}} \tilde{b}_{ij}(m), \quad (20)$$

which implies

$$\partial_{x_1} f_{nm} = \partial_{y_1} f_{nm}, \quad \partial_{x_{-1}} f_{nm} = \partial_{y_{-1}} f_{nm}. \quad (21)$$

Thus Eqs. (14)–(15) become

$$D_{x_1} f_{n+1,m} \cdot f_{n,m} = \mu \nu g_{n,m} \bar{g}_{nm}, \quad (22)$$

$$D_{x_{-1}} f_{n+1,m} \cdot f_{n,m} = -\mu \nu g_{n,m+1} \bar{g}_{n,m-1}. \quad (23)$$

Next, we consider complex conjugate condition by giving the following lemma:

Lemma 2. Assume $\mu = -i$, $\nu = \sigma$, $n = m = 0$,

$$\bar{p}_i = p_i^*, \quad \bar{\xi}_{i0} = \xi_{i0}^*, \quad e^{\eta_{i0}} = \alpha_i, \quad e^{\bar{\eta}_{i0}} = \alpha_i^*, \quad (24)$$

and drop the dummy variables y_1 and y_{-1} , then

$$a'_{ij}(0) = a_{ji}^*(0), \quad b_{ij}^*(0) = b_{ji}(0), \quad (25)$$

which implies

$$f_{00} = f_{10}^*, \quad \bar{g}_{00} = -g_{00}^*, \quad \bar{g}_{01} = g_{0,-1}^*. \quad (26)$$

Proof. Let $C = (\alpha_1, \alpha_2, \dots, \alpha_N)$ be a row vector. Direct calculations give

$$\begin{aligned}
f_{00} &= \begin{vmatrix} -\frac{ip_j^*}{p_i+p_j^*}e^{\xi_i+\xi_j^*} & I \\ -I & \frac{\sigma\alpha_i\alpha_j^*}{p_i^*+p_j} \end{vmatrix} = \begin{vmatrix} -\frac{ip_i^*}{p_j+p_i^*}e^{\xi_j+\xi_i^*} & -I \\ I & \frac{\sigma\alpha_j\alpha_i^*}{p_j^*+p_i} \end{vmatrix} \\
&= \begin{vmatrix} -\frac{ip_i^*}{p_j+p_i^*}e^{\xi_j+\xi_i^*} & I \\ -I & \frac{\sigma\alpha_j\alpha_i^*}{p_j^*+p_i} \end{vmatrix} = \begin{vmatrix} \frac{ip_i}{p_i+p_j^*}e^{\xi_i+\xi_j^*} & I \\ -I & \frac{\sigma\alpha_i\alpha_j^*}{p_i^*+p_j} \end{vmatrix}^* = f_{10}^*, \\
\bar{g}_{00} &= \begin{vmatrix} \frac{ip_i}{p_i+p_j^*}e^{\xi_i+\xi_j^*} & I & \mathbf{0}^T \\ -I & \frac{\sigma\alpha_i\alpha_j^*}{p_i^*+p_j} & C^T \\ -\Phi^* & \mathbf{0} & 0 \end{vmatrix} = \begin{vmatrix} \frac{ip_j}{p_j+p_i^*}e^{\xi_j+\xi_i^*} & -I & -\Phi^{*T} \\ I & \frac{\sigma\alpha_j\alpha_i^*}{p_j^*+p_i} & \mathbf{0}^T \\ \mathbf{0} & C & 0 \end{vmatrix} \\
&= \begin{vmatrix} \frac{ip_j}{p_j+p_i^*}e^{\xi_j+\xi_i^*} & I & -\Phi^{*T} \\ -I & \frac{\sigma\alpha_j\alpha_i^*}{p_j^*+p_i} & \mathbf{0}^T \\ \mathbf{0} & -C & 0 \end{vmatrix} = -g_{00}^*, \\
\bar{g}_{0,-1} &= \begin{vmatrix} \frac{ip_i}{p_i+p_j^*}e^{\xi_i+\xi_j^*} & I & \mathbf{0}^T \\ -I & \frac{\sigma\alpha_i\alpha_j^*}{p_i^*+p_j} \left(-\frac{p_j}{p_i^*}\right) & \frac{C^T}{p_i^*} \\ -\Phi^* & \mathbf{0} & 0 \end{vmatrix} = \begin{vmatrix} \frac{ip_j}{p_j+p_i^*}e^{\xi_j+\xi_i^*} & -I & -\Phi^{*T} \\ I & \frac{\sigma\alpha_j\alpha_i^*}{p_j^*+p_i} \left(-\frac{p_i}{p_j^*}\right) & \mathbf{0}^T \\ \mathbf{0} & \frac{C}{p_i^*} & 0 \end{vmatrix} \\
&= \begin{vmatrix} \frac{ip_j}{p_j+p_i^*}e^{\xi_j+\xi_i^*} & I & -\Phi^{*T} \\ -I & \frac{\sigma\alpha_j\alpha_i^*}{p_j^*+p_i} \left(-\frac{p_i}{p_j^*}\right) & \mathbf{0}^T \\ \mathbf{0} & -\frac{C}{p_i^*} & 0 \end{vmatrix} = g_{0,1}^*.
\end{aligned}$$

□

Finally, let $x_1 = x$ and $x_{-1} = -t$, we have

$$D_t g_{00} \cdot f_{00} = g_{01} f_{10}, \quad (27)$$

$$D_x g_{01} \cdot f_{10} = -g_{00} f_{00}, \quad (28)$$

$$D_x f_{10} \cdot f_{00} = -i\sigma g_{00} \bar{g}_{00}, \quad (29)$$

$$D_t f_{10} \cdot f_{00} = -i\sigma g_{01} \bar{g}_{0,-1}. \quad (30)$$

Therefore, we arrive at exactly the same set of bilinear equations (7)–(10) by setting

$$f_{00} = f^*, \quad f_{10} = f, \quad g_{00} = h, \quad \bar{g}_{00} = -h^*, \quad g_{01} = ig, \quad \bar{g}_{0,-1} = -ig^*. \quad (31)$$

In summary, we can give the N -bright soliton solution to the MT model by the following theorem:

Theorem 1. *The MT model (1)–(2) admits the multi-bright soliton solution $u = \frac{g}{f^*}$, $v = \frac{h}{f}$ where f , f^* , g and h are the following determinant solution*

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad f^* = \begin{vmatrix} A' & I \\ -I & B \end{vmatrix}, \quad (32)$$

$$g = -i \begin{vmatrix} A' & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & \frac{C^*}{p_j} & 0 \end{vmatrix}, \quad h = \begin{vmatrix} A' & I & \Phi^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & -C^* & 0 \end{vmatrix}, \quad (33)$$

where I is an $N \times N$ identity matrix, A , A' , and B are $N \times N$ matrices whose entries are

$$a_{ij} = \frac{ip_i}{p_i + p_j^*} e^{\xi_i + \xi_j^*}, \quad a'_{ij} = -\frac{ip_j^*}{p_i + p_j^*} e^{\xi_i + \xi_j^*}, \quad b_{ij} = \frac{\sigma \alpha_i \alpha_j^*}{p_i^* + p_j},$$

and $\mathbf{0}$ is a N -component zero-row vector, Φ and C are N -component row vectors given by

$$\Phi = (e^{\xi_1}, e^{\xi_2}, \dots, e^{\xi_N}), \quad C = (\alpha_1, \alpha_2, \dots, \alpha_N),$$

with $\xi_i = p_i x - \frac{1}{p_i} t + \xi_{i0}$. Here p_i , α_i and ξ_{i0} are arbitrary complex parameters for $i = 1, \dots, N$.

3 Dark solitons in the MT model

3.1 Bilinearization of the MT model under NVBC

The bilinearization of the MT model (1)–(2) under NVBC is established by the following proposition.

Proposition 2. *By means of the dependent variable transformations*

$$u = \rho_1 \frac{g}{f^*} e^{i(1+\sigma\rho_1\rho_2)\left(\frac{\rho_2}{\rho_1}x + \frac{\rho_1}{\rho_2}t\right)}, \quad v = \rho_2 \frac{h}{f} e^{i(1+\sigma\rho_1\rho_2)\left(\frac{\rho_2}{\rho_1}x + \frac{\rho_1}{\rho_2}t\right)}, \quad (34)$$

where ρ_1 and ρ_2 are real constants, the MT model (1)–(2) is transformed into the following bilinear equations

$$(iD_x - \frac{\rho_2}{\rho_1})g \cdot f = -\frac{\rho_2}{\rho_1}hf^*, \quad (35)$$

$$(iD_x - \sigma\rho_2^2)f \cdot f^* = -\sigma\rho_2^2hh^*, \quad (36)$$

$$(iD_t - \frac{\rho_1}{\rho_2})h \cdot f^* = -\frac{\rho_1}{\rho_2}gf, \quad (37)$$

$$(iD_t - \sigma\rho_1^2)f^* \cdot f = -\sigma\rho_1^2gg^*. \quad (38)$$

Proof. By rewriting the dependent variable transformations (34)

$$u = \rho_1 \frac{g}{f} \frac{f}{f^*} e^{i(1+\sigma\rho_1\rho_2)\left(\frac{\rho_2}{\rho_1}x + \frac{\rho_1}{\rho_2}t\right)}, \quad v = \rho_2 \frac{h}{f^*} \frac{f^*}{f} e^{i(1+\sigma\rho_1\rho_2)\left(\frac{\rho_2}{\rho_1}x + \frac{\rho_1}{\rho_2}t\right)}$$

and substituting into Eq.(1), one has

$$\begin{aligned} & \left[i \left(\frac{g}{f} \right)_x \frac{f}{f^*} - \frac{\rho_2}{\rho_1} \frac{g}{f^*} + \frac{\rho_2}{\rho_1} \frac{h}{f} \right] \\ & + \left[i \frac{g}{f} \left(\frac{f}{f^*} \right)_x - \sigma \rho_2^2 \frac{g}{f^*} + \sigma \rho_2^2 \frac{hh^*}{ff^*} * \frac{g}{f^*} \right] = 0. \end{aligned} \quad (39)$$

Bilinear equations (35) and (36) are deduced by taking zero for each group inside bracket. Similarly, we can drive bilinear equations (37) and (38) by substituting (34) into Eq.(2). \square

3.2 Discrete KP equation and bilinear equations for the KP-Toda hierarchy

Let us start with a concrete form of the Gram determinant expression of the tau functions for the extended KP hierarchy with negative flows

$$\tau_{nkl} = \left| m_{ij}^{nkl} \right|_{1 \leq i, j \leq N}, \quad (40)$$

where

$$m_{ij}^{nkl} = \delta_{ij} + \frac{ip_i}{p_i + \bar{p}_j} \varphi_i^{nkl} \psi_j^{nkl},$$

$$\varphi_i^{nkl} = p_i^n (p_i - a)^k (p_i - b)^l e^{\xi_i}, \quad \psi_j^{nkl} = \left(-\frac{1}{\bar{p}_j} \right)^n \left(-\frac{1}{\bar{p}_j + a} \right)^k \left(-\frac{1}{\bar{p}_j + b} \right)^l e^{\bar{\xi}_j},$$

with

$$\begin{aligned} \xi_i &= \frac{1}{p_i} x_{-1} + p_i x_1 + \frac{1}{p_i - a} t_a + \frac{1}{p_i - b} t_b + \xi_{i0}, \\ \bar{\xi}_j &= \frac{1}{\bar{p}_j} x_{-1} + \bar{p}_j x_1 + \frac{1}{\bar{p}_j + a} t_a + \frac{1}{\bar{p}_j + b} t_b + \bar{\xi}_{j0}. \end{aligned}$$

Here $p_i, \bar{p}_j, \xi_{i0}, \bar{\xi}_{j0}, a, b$ are constants. We have the following lemma regarding the bilinear equations satisfied by above tau function:

Lemma 3. *The discrete KP equation generates a set of bilinear equations*

$$(D_{x_1} + a)\tau_{n,k+1,l} \cdot \tau_{n+1,k,l} = a\tau_{n+1,k+1,l}\tau_{n,k,l}, \quad (41)$$

$$(bD_{x_{-1}} + 1)\tau_{n,k,l+1} \cdot \tau_{n,k,l} = \tau_{n-1,k,l+1}\tau_{n+1,k,l}, \quad (42)$$

$$(aD_{t_a} - 1)\tau_{n+1,k,l} \cdot \tau_{n,k,l} = -\tau_{n+1,k-1,l}\tau_{n,k+1,l}, \quad (43)$$

$$(bD_{t_b} - 1)\tau_{n+1,k,l} \cdot \tau_{n,k,l} = -\tau_{n+1,k,l-1}\tau_{n,k,l+1}. \quad (44)$$

satisfied by above tau function (40).

Proof. The discrete KP equation, or the so-called Hirota–Miwa equation [46, 47],

$$(a_1 - a_2)\tau_{12}\tau_3 + (a_2 - a_3)\tau_{23}\tau_1 + (a_3 - a_1)\tau_{13}\tau_2 = 0, \quad (45)$$

is a three-dimensional discrete integrable system where lattice parameters a_k are distinct constants, $k = 1, 2, 3$, and for $\tau = \tau(k_1, k_2, k_3)$ each subscript i denotes a forward shift in the corresponding discrete variable k_i . It is found by Ohta *et al.* that the discrete KP equation admits a general solution in terms of the following Gram-type determinant [48]

$$\tau(k_1, k_2, k_3) = \left| c_{ij} + \frac{d_{ij}}{p_i + q_j} \left(-\frac{p_i - a_1}{q_j + a_1} \right)^{k_1} \left(-\frac{p_i - a_2}{q_j + a_2} \right)^{k_2} \left(-\frac{p_i - a_3}{q_j + a_3} \right)^{k_3} \right|. \quad (46)$$

Notice that the element in (46) can be rewritten as

$$c_{ij} + \frac{d_{ij}}{p_i + q_j} \left(-\frac{\tilde{p}_i}{\tilde{q}_j} \right)^{k_1} \left(-\frac{\tilde{p}_i + a}{\tilde{q}_j - a} \right)^{k_3} \left(\frac{1 - b\tilde{p}_i}{1 + b\tilde{q}_j} \right)^{k_2} \left(\frac{1 - a_3\tilde{p}_i^{-1}}{1 + a_3\tilde{q}_j^{-1}} \right)^{k_3}$$

by reparametrizing $p_i - a_1 = \tilde{p}_i$, $q_i + a_1 = \tilde{q}_i$ and $a_2 - a_1 = b^{-1}$ and $a = a_1$. Set $a_3 = 0$ and redefine $k_3 = k$, $k_1 = n$, then the discrete KP equation (45) has the degenerate form

$$\begin{aligned} a\tau_{n,k}(k_2 + 1)\tau_{n+1,k+1}(k_2) + b^{-1}\tau_{n,k+1}(k_2)\tau_{n+1,k}(k_2 + 1) \\ (b^{-1} + a)\tau_{n,k+1}(k_2 + 1)\tau_{n+1,k+1}(k_2) = 0. \end{aligned} \quad (47)$$

Applying Miwa transformation by taking $b \rightarrow 0$ and $x_1 = -k_2b$, i.e., $\tau_{n,k}(k_2 + 1) \rightarrow \tau_{n,k} - b\partial_{x_1}\tau_{n,k}$ one obtains

$$(D_{x_1} + a)\tau_{n+1,k} \cdot \tau_{n,k+1} = a\tau_{n+1,k+1}\tau_{n,k}, \quad (48)$$

which is equivalent to (41) by taking $\tilde{p}_i \rightarrow p_i$, $\tilde{q}_i \rightarrow \bar{p}_i$, $c_{ij} = \delta_{ij}$, $d_{ij} = ip_i$, $a \rightarrow -a$ and adding l to each tau function.

In what follows, we further show that bilinear equation (41) can generate (42)–(44) by a dual relation between positive flow and negative flow. To be specific, we notice that

$$\begin{aligned} \delta_{ij} + \frac{ip_i}{p_i + \bar{p}_j} \left(-\frac{p_i}{\bar{p}_j} \right)^n \left(-\frac{p_i - a}{\bar{p}_j + a} \right)^k e^{\xi_i + \bar{\xi}_j} \\ \rightarrow \delta_{ij} + \frac{ip_i}{p_i + \bar{p}_j} \left(-\frac{p_i^{-1}}{\bar{p}_j^{-1}} \right)^{-(n+k)} \left(-\frac{p_i^{-1} - a^{-1}}{\bar{p}_j^{-1} + a^{-1}} \right)^k e^{\xi_i + \bar{\xi}_j}. \end{aligned}$$

Therefore, x_1 and x_{-1} is exchangeable by redefining $p_i^{-1} \rightarrow p_i$, $\bar{p}_j^{-1} \rightarrow \bar{p}_j$. Furthermore, by redefining index $n+k \rightarrow -n$, $k \rightarrow l$ and $a^{-1} = b$, (41) is converted into

$$(D_{x_{-1}} + b^{-1})\tau_{n,l+1} \cdot \tau_{n,l} = b^{-1}\tau_{n-1,l}\tau_{n+1,l}$$

which is nothing but Eq. (42).

On the other hand, by reparametrizing $p_i^{-1} - a^{-1} = \tilde{p}_i$, $\bar{p}_j^{-1} + a^{-1} = \tilde{\bar{p}}_j$

$$\begin{aligned} c_{ij} + \frac{d_{ij}}{p_i + \bar{p}_j} \left(-\frac{p_i}{\bar{p}_j} \right)^n \left(-\frac{p_i - a}{\bar{p}_j + a} \right)^k e^{\xi_i + \bar{\xi}_j} \\ \rightarrow c_{ij} + \frac{d_{ij}}{p_i + \tilde{\bar{p}}_j} \left(-\frac{\tilde{p}_i}{\tilde{\bar{q}}_j} \right)^k \left(-\frac{\tilde{p}_i + a^{-1}}{\tilde{\bar{q}}_j - a^{-1}} \right)^{-(n+k)} e^{\xi_i + \bar{\xi}_j} \end{aligned}$$

by redefining indices $k = n'$, $n+k = -k'$. Since

$$\begin{aligned} \xi_i = p_i x_1 + p_i^{-1} x_{-1} &\rightarrow \frac{1}{\tilde{p}_i + a^{-1}} x_1 + (\tilde{p}_i + a^{-1}) x_{-1}, \\ \bar{\xi}_j = \bar{p}_j x_1 + \bar{p}_j^{-1} x_{-1} &\rightarrow \frac{1}{\tilde{\bar{p}}_j - a^{-1}} x_1 + (\tilde{\bar{p}}_j - a^{-1}) x_{-1}, \end{aligned}$$

which makes positive flow (x_1) and negative flow (x_{-1}) exchangeable. Thus, by taking $x_1 \rightarrow t_a$, and $a^{-1} \rightarrow -a$, one obtains

$$\begin{aligned} (D_{x_1} + a)\tau_{n,k+1} \cdot \tau_{n+1,k} &= a\tau_{n,k}\tau_{n+1,k+1} \\ \rightarrow (D_{t_a} - a^{-1})\tau_{n'+1,k'} \cdot \tau_{n',k'} &= -a^{-1}\tau_{n'+1,k'-1}\tau_{n',k'+1} \\ \rightarrow (aD_{t_a} - 1)\tau_{n'+1,k'} \cdot \tau_{n',k'} &= -\tau_{n'+1,k'-1}\tau_{n',k'+1} \end{aligned}$$

which is exactly Eq. (43) by dropping the prime. Eq. (44) is just a parallel copy of Eq. (43) from $(a, t_a, k) \rightarrow (b, t_b, l)$. \square

Remark. It is very interesting to observe that the discrete KP equation can generate KP-Toda hierarchy with asymmetric positive flow and negative flow. Furthermore, the positive flow and negative flow are exchangeable by reparametrizing the wave numbers in the tau function.

Remark. It is noted that some of the bilinear equations derived above are also bilinear equations of the Fokas-Lenells equation [49], the complex short pulse equation [44] and the modified Camassa-Holm equation [50, 51]. In other words, the tau function behind these equations is the same before the reductions.

Remark. We can also give proof of above bilinear equations via determinant identities such as the Jacobi identity. However, the proof we give here is more systematic instead of technical. Moreover, in scrutinizing the connection between the discrete KP equation and the nonlinear PDEs, it makes possible for us to construct the integrable discrete analogues of the nonlinear PDEs.

3.3 Reduction to the dark soliton of the MT model

In what follows, we briefly show the reduction processes of reducing bilinear equations of extended KP hierarchy (41)–(44) to the bilinear equation (35)–(38). Firstly, we start with dimension reduction by noting that the determinant expression of τ_{nkl} ,

$$\tau_{nkl} = \left| \delta_{ij} + \frac{ip_i}{p_i + \bar{p}_j} \varphi_i^{nkl} \psi_j^{nkl} \right|_{1 \leq i, j \leq N},$$

can be alternatively expressed by

$$\tau_{nkl} = \prod_{i=1}^N \varphi_i^{nkl} \psi_i^{nkl} \left| \frac{\delta_{ij}}{\varphi_i^{nkl} \psi_i^{nkl}} + \frac{ip_i}{p_i + \bar{p}_j} \right|_{1 \leq i, j \leq N},$$

by dividing j -th column by ψ_j^{nkl} and i -th row by φ_i^{nkl} for $1 \leq i, j \leq N$.

By imposing the reduction condition

$$(p_i - b)(\bar{p}_i + b) = b(a - b), \quad (49)$$

or

$$|p_i|^2 = -\frac{b}{a-b}(p_i - a)(\bar{p}_i + a), \quad (50)$$

one can easily check the following relations hold

$$p_i + \bar{p}_i = b(a - b) \left(\frac{1}{p_i - b} + \frac{1}{\bar{p}_i + b} \right), \quad (51)$$

$$\frac{1}{p_i} + \frac{1}{\bar{p}_i} = -\frac{a-b}{b} \left(\frac{1}{p_i - a} + \frac{1}{\bar{p}_i + a} \right), \quad (52)$$

$$\left(-\frac{p_i}{\bar{p}_i} \right) \left(-\frac{p_i - a}{\bar{p}_i + a} \right) \left(-\frac{p_i - b}{\bar{p}_i + b} \right)^{-1} = 1, \quad (53)$$

which implies

$$\partial_{t_b} = \frac{1}{b(a-b)} \partial_{x_1}, \quad (54)$$

$$\partial_{t_a} = -\frac{b}{a-b} \partial_{x_{-1}}, \quad (55)$$

$$\tau_{n-1, k, l+1} = \tau_{n, k+1, l}, \quad \tau_{n, k, l+1} = \tau_{n+1, k+1, l}. \quad (56)$$

Therefore the bilinear equations (41)–(44) can be recast into

$$(D_{x_1} + a)\tau_{n,k+1,l} \cdot \tau_{n+1,k,l} = a\tau_{n,k,l+1}\tau_{n,k,l}, \quad (57)$$

$$(bD_{x_{-1}} + 1)\tau_{n,k,l+1} \cdot \tau_{n,k,l} = \tau_{n,k+1,l}\tau_{n+1,k,l}, \quad (58)$$

$$\left(-\frac{ab}{(a-b)}D_{x_{-1}} - 1\right)\tau_{n+1,k,l} \cdot \tau_{n,k,l} = -\tau_{n+1,k-1,l}\tau_{n,k+1,l}, \quad (59)$$

$$\left(\frac{1}{a-b}D_{x_1} - 1\right)\tau_{n+1,k,l} \cdot \tau_{n,k,l} = -\tau_{n+1,k,l-1}\tau_{n,k,l+1}. \quad (60)$$

By setting

$$x_1 = -\frac{\rho_2}{ia\rho_1}x, \quad x_{-1} = -\frac{\rho_1 b}{i\rho_2}t, \quad (61)$$

i.e.,

$$\partial_{x_1} = -\frac{ia\rho_1}{\rho_2}\partial_x, \quad \partial_{x_{-1}} = -\frac{i\rho_2}{\rho_1 b}\partial_t, \quad (62)$$

and assuming $b = a(1 + \sigma\rho_1\rho_2)$, we have the following bilinear equations

$$(iD_x - \frac{\rho_2}{\rho_1})\tau_{n,k+1,l} \cdot \tau_{n+1,k,l} = -\frac{\rho_2}{\rho_1}\tau_{n,k,l+1}\tau_{n,k,l}, \quad (63)$$

$$(iD_t - \frac{\rho_1}{\rho_2})\tau_{n,k,l+1} \cdot \tau_{n,k,l} = -\frac{\rho_1}{\rho_2}\tau_{n,k+1,l}\tau_{n+1,k,l}, \quad (64)$$

$$(-iD_t - \sigma\rho_1^2)\tau_{n+1,k,l} \cdot \tau_{n,k,l} = -\sigma\rho_1^2\tau_{n+1,k-1,l}\tau_{n,k+1,l}, \quad (65)$$

$$(iD_x - \sigma\rho_2^2)\tau_{n+1,k,l} \cdot \tau_{n,k,l} = -\sigma\rho_2^2\tau_{n+1,k,l-1}\tau_{n,k,l+1}, \quad (66)$$

Next, we proceed to the complex conjugate reduction. To this end, by taking a and b pure imaginary, and letting \bar{p}_i to be complex conjugate of p_i : $\bar{p}_i = p_i^*$ and $\bar{\xi}_{i0} = \xi_{i0}^*$ and $n = -1, k = 0, l = 0$, one can find that

$$\tau_{-1,0,0} = \tau_{0,0,0}^*, \quad \tau_{-1,1,0} = \tau_{0,-1,0}^*, \quad \tau_{-1,0,1} = \tau_{0,0,-1}^* \quad (67)$$

In summary, by defining

$$\tau_{0,0,0} = f, \quad \tau_{-1,0,0} = f^*, \quad \tau_{-1,1,0} = g, \quad \tau_{0,-1,0} = g^*, \quad \tau_{-1,0,1} = h, \quad \tau_{0,0,-1} = h^*$$

we arrive at exactly the set of bilinear equations (41)–(44). Therefore, the reduction process is complete. As a result, we can provide the determinant solution to the MT model by the following theorem. In summary, we can give the multi-dark soliton solution to the MT system by the following theorem by taking $a = i\alpha$ and $b = i\alpha(1 + \sigma\rho_1\rho_2)$.

Theorem 2. *The MT system (1)–(2) admits the multi-dark soliton solution*

$$u = \rho_1 \frac{g}{f^*} e^{i(1+\sigma\rho_1\rho_2)\left(\frac{\rho_2}{\rho_1}x + \frac{\rho_1}{\rho_2}t\right)}, \quad v = \rho_2 \frac{h}{f} e^{i(1+\sigma\rho_1\rho_2)\left(\frac{\rho_2}{\rho_1}x + \frac{\rho_1}{\rho_2}t\right)},$$

where f, f^*, g and h are the following $N \times N$ determinant

$$f = \left| \delta_{ij} + \frac{ip_i}{p_i + p_j^*} e^{\xi_i + \bar{\xi}_j} \right|, \quad f^* = \left| \delta_{ij} - \frac{ip_j^*}{p_i + p_j^*} e^{\xi_i + \xi_j^*} \right|, \quad (68)$$

$$g = \left| \delta_{ij} + \frac{-ip_j^*}{p_i + p_j^*} \left(-\frac{p_i - i\alpha}{p_j^* + i\alpha} \right) e^{\xi_i + \xi_j^*} \right|, \quad (69)$$

$$h = \left| \delta_{ij} + \frac{-ip_j^*}{p_i + p_j^*} \left[-\frac{p_i - i\alpha(1 + \sigma\rho_1\rho_2)}{p_j^* + i\alpha(1 + \sigma\rho_1\rho_2)} \right] e^{\xi_i + \xi_j^*} \right|, \quad (70)$$

with

$$\xi_i = \frac{\rho_2}{\alpha\rho_1} p_i x - \frac{\rho_1\alpha(1 + \sigma\rho_1\rho_2)}{\rho_2} \frac{t}{p_i} + \xi_{i0}.$$

Here p_i, ξ_{i0} are complex constants and α is a real constant which need to satisfy the constraint condition:

$$[p_i - i\alpha(1 + \sigma\rho_1\rho_2)][p_i^* + i\alpha(1 + \sigma\rho_1\rho_2)] = \alpha^2\rho_1\rho_2(\sigma + \rho_1\rho_2). \quad (71)$$

4 Dynamics of bright and dark soliton solutions

4.1 One- and two-bright solitons and their dynamics

Based on the N -bright soliton solution to the MT model, one- and two-soliton solutions are calculated as follows. For simplicity, we reparameterize $\xi'_i = \xi_i + \tilde{\xi}'_i$ with $\alpha_i = e^{\tilde{\xi}'_i}$, and redefine the complex constants p_i and ξ'_i as

$$p_i = a_i + ib_i, \quad \xi'_i = \eta_{i0} + i\theta_{i0}, \quad i = 1, 2, \quad (72)$$

where a_i, b_i, η_{i0} and θ_{i0} are real constants. Then the variables ξ'_1 and ξ'_2 are rewritten as

$$\xi'_i = \eta_i + i\theta_i, \quad \eta_i = a_i \left(x - \frac{1}{a_i^2 + b_i^2} t \right) + \eta_{i0}, \quad \theta_i = b_i \left(x + \frac{1}{a_i^2 + b_i^2} t \right) + \theta_{i0}, \quad i = 1, 2. \quad (73)$$

By taking $N = 1$ in (32)-(33), we obtain the tau functions for one-soliton solution

$$f = 1 + \frac{i\sigma p_1 |\alpha_1|^2}{(p_1 + p_1^*)^2} e^{\xi_1 + \xi_1^*}, \quad g = \frac{i\alpha_1^*}{p_1} e^{\xi_1}, \quad h = \alpha_1^* e^{\xi_1}, \quad (74)$$

or

$$f = 1 + \frac{i\sigma(a_1 + ib_1)}{4a_1^2}e^{2\eta_1}, \quad g = \frac{i}{a_1 + ib_1}e^{\eta_1 + i\theta_1}, \quad h = e^{\eta_1 + i\theta_1}. \quad (75)$$

This leads to the square of the modulus of one-soliton solution for the MT model (1) and (2)

$$|u|^2 = \frac{2a_1^2}{(a_1^2 + b_1^2)^{\frac{3}{2}}} \frac{1}{\cosh(2\eta_1 + 2\delta) - \frac{\sigma b_1}{\sqrt{a_1^2 + b_1^2}}}, \quad (76)$$

$$|v|^2 = \frac{2a_1^2}{(a_1^2 + b_1^2)^{\frac{1}{2}}} \frac{1}{\cosh(2\eta_1 + 2\delta) - \frac{\sigma b_1}{\sqrt{a_1^2 + b_1^2}}}, \quad (77)$$

with $e^{2\delta} = \frac{\sqrt{a_1^2 + b_1^2}}{4a_1^2}$. Then the amplitudes A_u and A_v are given by

$$A_u = \sqrt{\frac{2}{a_1^2 + b_1^2}(\sqrt{a_1^2 + b_1^2} + \sigma b_1)} = \sqrt{2(v_1^{\frac{1}{2}} + \sigma b_1 v_1)}, \quad (78)$$

$$A_v = \sqrt{2(\sqrt{a_1^2 + b_1^2} + \sigma b_1)} = \sqrt{2(v_1^{-\frac{1}{2}} + \sigma b_1)}, \quad (79)$$

which means the amplitude-velocity relations with the velocity $\kappa_1 = \frac{1}{a_1^2 + b_1^2}$. For a fixed value of the imaginary part of wave number b_1 , it can be seen that when $\sigma = 1$, A_u (A_v) is an increasing (decreasing) function of κ_1 , while when $\sigma = -1$, A_u increases in the interval $0 < \kappa_1 \leq \frac{1}{4b_1^2}$ and decreases in the interval $\frac{1}{4b_1^2} \leq \kappa_1 < \frac{1}{b_1^2}$, A_v is a decreasing function in the interval $0 < \kappa_1 < \frac{1}{b_1^2}$. Two cases of the amplitude-velocity relations are depicted in Fig. 1.

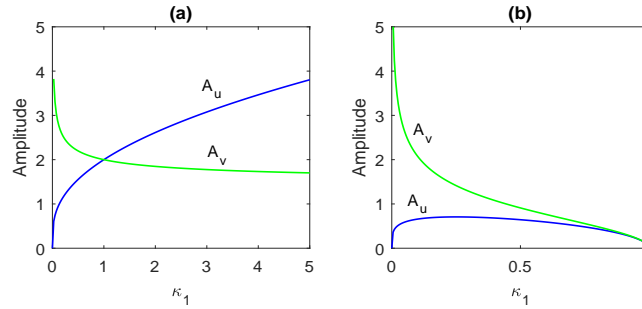


Figure 1: The amplitude-velocity relations with $b_1 = 1$: (a) $\sigma = 1$ and (b) $\sigma = -1$.

Besides, we need to point out that the amplitudes A_u and A_v remain finite in the limit of $a_1 \rightarrow 0$. In this limit case, the bright one-soliton solution (75) degenerates to the following algebraic soliton solution

$$u = \pm \frac{2\sqrt{\frac{\sigma}{b_1}} e^{i\left[b_1\left(x + \frac{1}{b_1^2}t\right) + \theta'_{10}\right]}}{b_1\left[2\left(x - \frac{1}{b_1^2}t + x_0\right) + \frac{i}{b_1}\right]}, \quad v = \pm \frac{2\sqrt{\frac{\sigma}{b_1}} e^{i\left[b_1\left(x + \frac{1}{b_1^2}t\right) + \theta'_{10}\right]}}{2\left(x - \frac{1}{b_1^2}t + x_0\right) + \frac{i}{b_1}}, \quad (80)$$

where we take $e^{m_{10} + i\theta_{10}} = \mp 2a_1 \sqrt{\frac{\sigma}{b_1}} e^{a_1 x_0 + i\theta'_{10}}$. The profiles of one-soliton are plotted in Fig. 2 with different values of wave number p_1 .

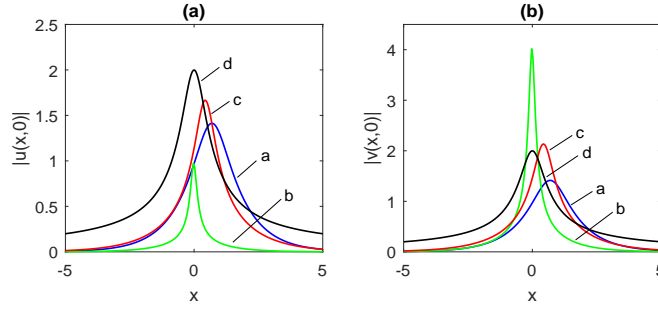


Figure 2: The bright one-soliton with the parameters $\sigma = 1$ and $\xi_{10} = x_0 = 0$: (a) $p_1 = 1$, (b) $p_1 = 1 + 4i$, (c) $p_1 = 0.8 + i$ and (d) the algebraic soliton with $p_1 = i$.

The tau functions corresponding to two-bright soliton solution can be obtained by taking $N = 2$ in (32)-(33)

$$f = 1 + c_{11^*} e^{\xi_1 + \xi_1^*} + c_{21^*} e^{\xi_2 + \xi_1^*} + c_{12^*} e^{\xi_1 + \xi_2^*} + c_{22^*} e^{\xi_2 + \xi_2^*} + c_{121^* 2^*} e^{\xi_1 + \xi_2 + \xi_1^* + \xi_2^*}, \quad (81)$$

$$g = \frac{i\alpha_1^*}{p_1} e^{\xi_1} + \frac{i\alpha_2^*}{p_2} e^{\xi_2} - \frac{ip_1^*}{p_1 p_2} c_{121^*} e^{\xi_1 + \xi_2 + \xi_1^*} - \frac{ip_2^*}{p_1 p_2} c_{122^*} e^{\xi_1 + \xi_2 + \xi_2^*}, \quad (82)$$

$$h = \alpha_1^* e^{\xi_1} + \alpha_2^* e^{\xi_2} + c_{121^*} e^{\xi_1 + \xi_2 + \xi_1^*} + c_{122^*} e^{\xi_1 + \xi_2 + \xi_2^*}, \quad (83)$$

where

$$c_{ij^*} = \frac{i\sigma p_i \alpha_i^* \alpha_j}{(p_i + p_j^*)^2}, \quad c_{12i^*} = (p_1 - p_2) p_i^* \left[\frac{\alpha_2^* c_{1i^*}}{p_1(p_2 + p_i^*)} - \frac{\alpha_1^* c_{2i^*}}{p_2(p_1 + p_i^*)} \right],$$

$$c_{121^* 2^*} = |p_1 - p_2|^2 \left[\frac{c_{11^*} c_{22^*}}{(p_1 + p_2^*)(p_2 + p_1^*)} - \frac{c_{12^*} c_{21^*}}{(p_1 + p_1^*)(p_2 + p_2^*)} \right].$$

The interaction of two-bright solitons are illustrated in Fig.3 for different parameters. As shown in Fig. 3(a) and (b), two-bright solitons undergo the regular collisions. The bound soliton state is exhibited in Fig. 3(c), in which two solitons move with the same velocity $\frac{1}{a_1^2 + b_1^2} = \frac{1}{a_2^2 + b_2^2}$.

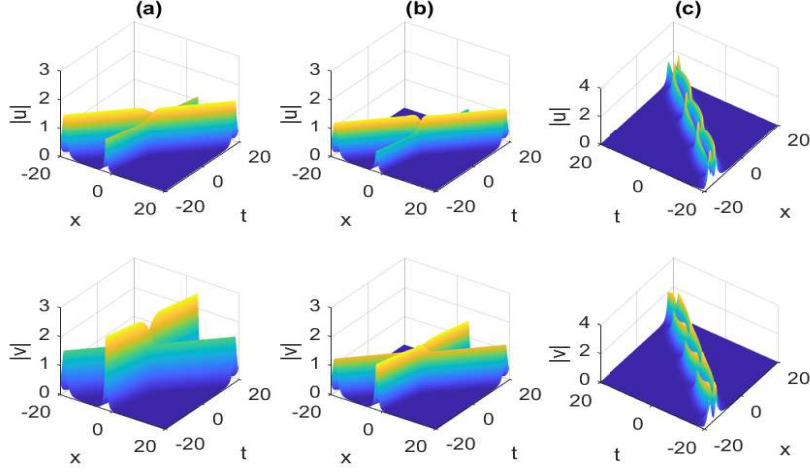


Figure 3: The bright two-soliton with the parameters $\alpha_1 = \alpha_2 = 1$, $\xi_{10} = \xi_{20} = 0$, $p_1 = 1 + \frac{1}{5}i$ and $p_2 = 2 + i$: (a) $\sigma = 1$ and (b) $\sigma_1 = -1$; (c) the bound state with $\sigma = 1$, $p_1 = 1 + \frac{2}{5}i$ and $p_2 = \frac{4}{5} + \frac{\sqrt{13}}{5}i$.

4.2 One- and two-dark solitons and their dynamics

In this subsection, we list one- and two-dark soliton solutions to the MT model. Same as the bright soliton case, we rewrite the complex constants p_i and ξ_i as

$$p_i = a_i + ib_i, \quad \xi_i = \eta_{i0} + i\theta_{i0}, \quad i = 1, 2, \quad (84)$$

where a_i , b_i , η_{i0} and θ_{i0} are real constants. Then the variables ξ_1 and ξ_2 are rewritten as

$$\begin{aligned} \xi_i &= \eta_i + i\theta_i, \quad \kappa_i = \frac{\alpha^2 \rho_1^2 (1 + \sigma \rho_1 \rho_2)}{\rho_2^2 (a_i^2 + b_i^2)}, \\ \eta_i &= \frac{\rho_2}{\alpha \rho_1} a_i (x - \kappa_i t) + \eta_{i0}, \quad \theta_i = \frac{\rho_2}{\alpha \rho_1} b_i (x + \kappa_i t) + \theta_{i0}, \quad i = 1, 2. \end{aligned} \quad (85)$$

By taking $N = 1$, we have tau functions for one-dark soliton solution

$$f = 1 + \frac{ip_1}{(p_1 + p_1^*)} e^{\xi_1 + \xi_1^*}, \quad g = 1 + \frac{ip_1^*}{(p_1 + p_1^*)} \left(\frac{p_1 - i\alpha}{p_1^* + i\alpha} \right) e^{\xi_1 + \xi_1^*}, \quad (86)$$

$$h = 1 + \frac{ip_1^*}{(p_1 + p_1^*)} \left[\frac{p_1 - i\alpha(1 + \sigma \rho_1 \rho_2)}{p_1^* + i\alpha(1 + \sigma \rho_1 \rho_2)} \right] e^{\xi_1 + \xi_1^*}, \quad (87)$$

which leads to the squares of the modulus of u and v :

$$|u|^2 = \rho_1^2 \left[1 + \frac{2\text{sgn}(a_1)\alpha a_1^2}{[a_1^2 + (b_1 - \alpha)^2]\sqrt{a_1^2 + b_1^2}} \frac{1}{\cosh(2\eta_1 + 2\delta') - \frac{\text{sgn}(a_1)b_1}{\sqrt{a_1^2 + b_1^2}}} \right], \quad (88)$$

$$|v|^2 = \rho_2^2 \left[1 + \frac{2\text{sgn}(a_1)(\alpha + \sigma\alpha\rho_1\rho_2)a_1^2}{[a_1^2 + (b_1 - \alpha - \sigma\alpha\rho_1\rho_2)^2]\sqrt{a_1^2 + b_1^2}} \frac{1}{\cosh(2\eta_1 + 2\delta') - \frac{\text{sgn}(a_1)b_1}{\sqrt{a_1^2 + b_1^2}}} \right], \quad (89)$$

with $e^{4\delta'} = \frac{a_1^2 + b_1^2}{4a_1^2}$. This implies that u exhibits a dark soliton when $\alpha a_1 < 0$ and an anti-dark soliton on the background ρ_1 when $\alpha a_1 > 0$, while v represents a dark soliton when $(\alpha + \sigma\alpha\rho_1\rho_2)a_1 < 0$ and an anti-dark soliton on the background ρ_1 when $(\alpha + \sigma\alpha\rho_1\rho_2)a_1 > 0$.

Besides, the dark one-soliton solution can be rewritten as

$$u = \frac{\rho_1}{2} e^{i(1+\sigma\rho_1\rho_2)\left(\frac{\rho_2}{\rho_1}x + \frac{\rho_1}{\rho_2}t\right)} \left[1 + e^{2i\phi_1} + (e^{2i\phi_1} - 1) \tanh(\eta_1 + \eta_0 + i\phi_0) \right],$$

$$v = \frac{\rho_2}{2} e^{i(1+\sigma\rho_1\rho_2)\left(\frac{\rho_2}{\rho_1}x + \frac{\rho_1}{\rho_2}t\right)} \left[1 + e^{4i\phi_0 + 2i\phi_2} + (e^{4i\phi_0 + 2i\phi_2} - 1) \tanh(\eta_1 + \eta_0 - i\phi_0) \right],$$

where

$$e^{2\eta_0 + 2i\phi_0} = \frac{-ip_1^*}{p_1 + p_1^*}, \quad e^{2i\phi_1} = -\frac{p_1 - i\alpha}{p_1^* + i\alpha}, \quad e^{2i\phi_2} = -\frac{p_1 - i\alpha(1 + \sigma\rho_1\rho_2)}{p_1^* + i\alpha(1 + \sigma\rho_1\rho_2)},$$

Therefore, the phase of u and v acquire shifts in the amount of $2\phi_1$ and $4\phi_0 + 2\phi_2$ when η_1 varies from $-\infty$ to $+\infty$, and the grayness of two components are $|\rho_1 \cos \phi_1|$ and $|\rho_2 \cos(2\phi_0 + \phi_2)|$, respectively.

Moreover, the constraint condition (71) becomes

$$a_1^2 + [b_1 - \alpha(1 + \sigma\rho_1\rho_2)]^2 = \sigma\rho_1\rho_2(1 + \sigma\rho_1\rho_2)\alpha^2, \quad (90)$$

which implies that the condition $\sigma\rho_1\rho_2(1 + \sigma\rho_1\rho_2) > 0$ needs to be hold. If the above constraint condition is expressed in terms of the velocity κ_1 , one can find that

$$b_1 = \frac{1}{2} \left(\alpha + \frac{\rho_1^2}{\kappa_1 \rho_2^2} \right), \quad a_1^2 = \frac{\alpha^2}{4\kappa_1^2} (\kappa_{1,max} - \kappa_1)(\kappa_1 - \kappa_{1,min}), \quad \kappa_{1,min} < \kappa_1 < \kappa_{1,max}, \quad (91)$$

where $\kappa_{1,min} = \hat{k}_1^-$, $\kappa_{1,max} = \hat{k}_1^+$ when $\alpha > 0$, and $\kappa_{1,min} = \hat{k}_1^+$, $\kappa_{1,max} = \hat{k}_1^-$ when $\alpha < 0$ with $\hat{k}_1^\pm = \frac{\rho_1^2}{\alpha\rho_2^2} [1 + 2\sigma\rho_1\rho_2 \pm 2\sqrt{\sigma\rho_1\rho_2(1 + \sigma\rho_1\rho_2)}]$. In the following, we discuss the maximum amplitude-velocity relations. Without

loss of generality, we consider $\alpha > 0$, $1 + \sigma\rho_1\rho_2 > 0$, $\rho_1, \rho_2 > 0$ which lead to $\kappa_1 > 0$ and $\sigma = 1$, then there are two cases corresponding to the sign of a_1 .

Case 1: $a_1 > 0$. Two components exhibit as anti-dark soliton and the amplitude-velocity relations are given by

$$A_u = \sqrt{\rho_1^2 + \frac{2\alpha\rho_1^2(\sqrt{a_1^2 + b_1^2} + b_1)}{a_1^2 + (b_1 - \alpha)^2}} - \rho_1 = \sqrt{\rho_1^2 + \Delta_1^+} - \rho_1, \quad (92)$$

$$A_v = \sqrt{\rho_2^2 + \frac{2\alpha(1 + \rho_1\rho_2)\rho_2^2(\sqrt{a_1^2 + b_1^2} + b_1)}{a_1^2 + (b_1 - \alpha - \alpha\rho_1\rho_2)^2}} - \rho_2 = \sqrt{\rho_2^2 + \frac{\Delta_1^+}{\alpha\kappa_1}} - \rho_2, \quad (93)$$

with $\Delta_1^+ = \frac{\rho_1^2 + \alpha\kappa_1\rho_2^2 + 2\sqrt{\alpha\kappa_1(1 + \rho_1\rho_2)}\rho_1^2\rho_2^2}{\rho_1\rho_2}$. A_u is an increasing function whereas A_v is a decreasing function in the interval $\hat{k}_1^- < \kappa_1 < \hat{k}_1^+$, which is shown in Fig.4(a).

Case 2: $a_1 < 0$. Two components behave as dark soliton and the amplitude-velocity relations read

$$A_u = \rho_1 - \sqrt{\rho_1^2 - \frac{2\alpha\rho_1^2(\sqrt{a_1^2 + b_1^2} - b_1)}{a_1^2 + (b_1 - \alpha)^2}} = \rho_1 - \sqrt{\rho_1^2 + \Delta_1^-}, \quad (94)$$

$$A_v = \rho_2 - \sqrt{\rho_2^2 - \frac{2\alpha(1 + \rho_1\rho_2)\rho_2^2(\sqrt{a_1^2 + b_1^2} - b_1)}{a_1^2 + (b_1 - \alpha - \alpha\rho_1\rho_2)^2}} = \rho_2 - \sqrt{\rho_2^2 + \frac{\Delta_1^-}{\alpha\kappa_1}}, \quad (95)$$

with $\Delta_1^- = \frac{\rho_1^2 + \alpha\kappa_1\rho_2^2 - 2\sqrt{\alpha\kappa_1(1 + \rho_1\rho_2)}\rho_1^2\rho_2^2}{\rho_1\rho_2}$. A_u increases in the interval $\hat{k}_1^- < \kappa_1 < \hat{k}_0^u$ with $\hat{k}_0^u = \frac{(1 + \rho_1\rho_2)\rho_1^2}{\alpha\rho_2^2}$ and decreases in the interval $\hat{k}_0^u < \kappa_1 < \hat{k}_1^+$, while A_v increases in the interval $\hat{k}_1^- < \kappa_1 < \hat{k}_0^v$ with $\hat{k}_0^v = \frac{\rho_1^2}{\alpha\rho_2^2(1 + \rho_1\rho_2)}$ and decreases in the interval $\hat{k}_0^v < \kappa_1 < \hat{k}_1^+$, which is displayed in Fig.4(b). It is noted that the black soliton occurs at two critical points \hat{k}_0^u and \hat{k}_0^v , which correspond to $p_1 = \frac{-\alpha\sqrt{(4 + 3\rho_1\rho_2)\rho_1\rho_2 + i(2 + \rho_1\rho_2)}\alpha}{2(1 + \rho_1\rho_2)}$ and $p_1 = \frac{\alpha}{2}[-\sqrt{(4 + 3\rho_1\rho_2)\rho_1\rho_2} + i(2 + \rho_1\rho_2)]$ respectively.

By taking the limit $a_1 \rightarrow 0$, the tau functions have the following expansion formulae

$$f = 1 - \text{sgn}(a_1)\text{sgn}(b_1) \left[1 + 2a_1 \left(\hat{\eta}_1 - \frac{i}{2b_1} \right) \right] + O(a_1^2), \quad (96)$$

$$g = 1 - \text{sgn}(a_1)\text{sgn}(b_1) \left[1 + 2a_1 \left(\hat{\eta}_1 + \frac{i}{2b_1} \frac{\alpha + b_1}{\alpha - b_1} \right) \right] + O(a_1^2), \quad (97)$$

$$h = 1 - \text{sgn}(a_1)\text{sgn}(b_1) \left[1 + 2a_1 \left(\hat{\eta}_1 + \frac{i}{2b_1} \frac{\alpha(1 + \sigma\rho_1\rho_2) + b_1}{\alpha(1 + \sigma\rho_1\rho_2) - b_1} \right) \right] + O(a_1^2), \quad (98)$$

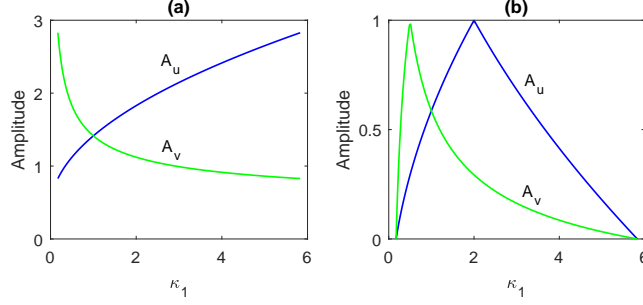


Figure 4: The amplitude-velocity relations with $\alpha = \rho_1 = \rho_2 = 1$: (a) anti-dark soliton ($a_1 > 0$) and (b) dark soliton ($a_1 < 0$).

with $\eta_{10} = x_0 - \delta'$ and $\hat{\eta}_1 = \frac{\rho_2}{\alpha\rho_1}x - \frac{(1+\sigma\rho_1\rho_2)\alpha\rho_1}{\rho_2 b_1^2}t + x_0$. This suggests that only $\text{sgn}(a_1)\text{sgn}(b_1) = 1$ gives rise to the algebraic soliton solution:

$$u = \rho_1 e^{i(1+\sigma\rho_1\rho_2)\left(\frac{\rho_2}{\rho_1}x + \frac{\rho_1}{\rho_2}t\right)} \frac{\hat{\eta}_1 + \frac{i}{2b_1} \frac{\alpha+b_1}{\alpha-b_1}}{\hat{\eta}_1 + \frac{i}{2b_1}}, \quad (99)$$

$$v = \rho_2 e^{i(1+\sigma\rho_1\rho_2)\left(\frac{\rho_2}{\rho_1}x + \frac{\rho_1}{\rho_2}t\right)} \frac{\hat{\eta}_1 + \frac{i}{2b_1} \frac{\alpha(1+\sigma\rho_1\rho_2)+b_1}{\alpha(1+\sigma\rho_1\rho_2)-b_1}}{\hat{\eta}_1 - \frac{i}{2b_1}}, \quad (100)$$

with $b_1 = \alpha[1 + \sigma\rho_1\rho_2 + \sqrt{\sigma\rho_1\rho_2(1 + \sigma\rho_1\rho_2)}]$. The usual dark soliton, black soliton and algebraic soliton are illustrated in 5 under the different parameters' values.

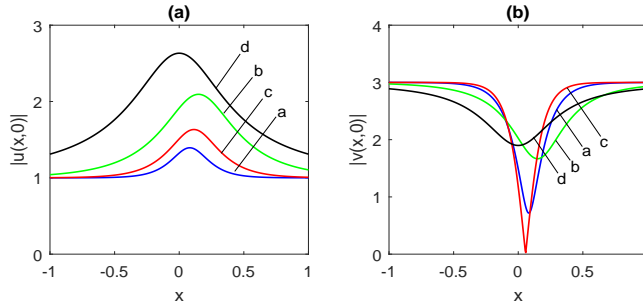


Figure 5: The dark one-soliton with the parameters $\rho_1 = \alpha = -\sigma = \rho_2/3 = 1$ and $\xi_{10} = x_0 = 0$: (a) $p_1 = \sqrt{2}$, (b) $p_1 = \frac{1+(\sqrt{23}-4)i}{2}$, (c) $p_1 = \frac{\sqrt{5}+i}{4}$ for u and $p_1 = \frac{\sqrt{15}-i}{2}$ for v (v is a black soliton) and (d) the algebraic soliton with $p_1 = (\sqrt{6} - 2)i$.

For the dark two-soliton with $N = 2$ in (68)-(70), we obtain the tau

functions

$$f = 1 + d_{11}^* e^{\xi_1 + \xi_1^*} + d_{22}^* e^{\xi_2 + \xi_2^*} + d_{11}^* d_{22}^* \Omega_{12} e^{\xi_1 + \xi_2 + \xi_1^* + \xi_2^*}, \quad (101)$$

$$g = 1 + d_{11}^* K_1 e^{\xi_1 + \xi_1^*} + d_{22}^* K_2 e^{\xi_2 + \xi_2^*} + d_{11}^* d_{22}^* K_1 K_2 \Omega_{12} e^{\xi_1 + \xi_2 + \xi_1^* + \xi_2^*}, \quad (102)$$

$$h = 1 + d_{11}^* H_1 e^{\xi_1 + \xi_1^*} + d_{22}^* H_2 e^{\xi_2 + \xi_2^*} + d_{11}^* d_{22}^* H_1 H_2 \Omega_{12} e^{\xi_1 + \xi_2 + \xi_1^* + \xi_2^*}, \quad (103)$$

with

$$d_{ii}^* = \frac{ip_i}{p_i + p_i^*}, \quad K_i = -\frac{p_i - i\alpha}{p_i^* + i\alpha}, \quad H_i = -\frac{p_i - i\alpha(1 + \sigma\rho_1\rho_2)}{p_i^* + i\alpha(1 + \sigma\rho_1\rho_2)}, \quad \Omega_{12} = \frac{|p_1 - p_2|^2}{|p_1 + p_2^*|^2}.$$

Since the dark one-soliton solution exhibits dark and anti-dark soliton, the dark two-soliton solution may possess three types of the interaction: dark-dark solitons, dark-anti-dark solitons, and anti-dark-anti-dark solitons, which are displayed in Fig.6. From the constraint conditions (71), if we assume the same velocity κ_i , one cannot get different value of p_i . Hence there does not exist bound state for the dark-soliton in the MT model.

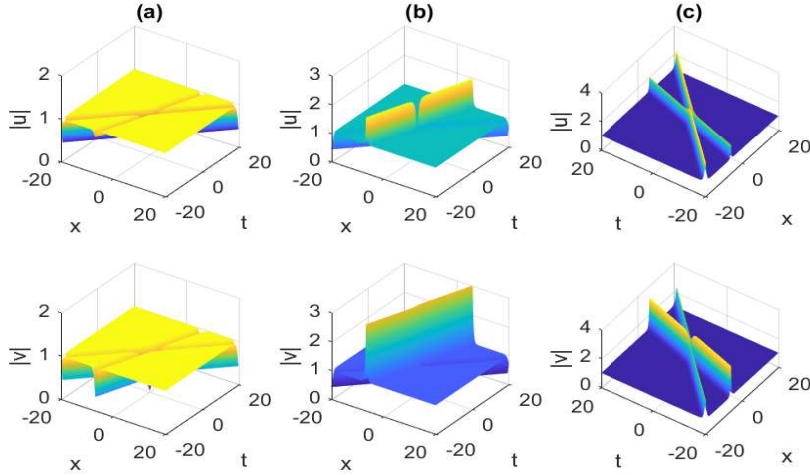


Figure 6: The dark two-soliton with the parameters $\rho_1 = \rho_2 = \alpha = \sigma = 1$ and $\xi_{10} = \xi_{20} = 0$: (a) dark-dark soliton $p_1 = 1 + i$, $p_2 = \sqrt{2} + 2i$; (b) dark-anti-dark soliton $p_1 = 1 + i$, $p_2 = -\sqrt{2} + 2i$; (c) anti-dark-anti-dark soliton $p_1 = -1 + i$, $p_2 = -\sqrt{2} + 2i$.

5 Concluding Remarks

In this paper, we give bilinear formulation of the massive Thirring model which is missing in the literature. Based on the bilinear equations, we constructed both the bright and dark soliton solutions to the massive Thirring

model under VBC and NVBC, respectively, via the KP hierarchy reduction technique. Especially, we have shown that the discrete KP equation can generate a set of four bilinear equations, which are reduced to bilinear equations corresponding to the dark soliton of the MT model. We have also listed one- and two- bright and dark soliton solutions and analyzed their properties in details.

We should point out that, even if we give the bright and dark soliton solution to the MT model in the light cone coordinates, one can easily obtain the corresponding soliton solutions in the laboratory coordinates. Moreover, based on the bilinear formulation established in the present work, we can also construct general breather and rogue wave solutions of the MT model (1)–(2) via the KP hierarchy reduction method. We will report our results elsewhere in the near future.

Finally, we comment that, since we have established a link between the discrete KP equation and the MT model in terms of tau functions and bilinear equations, similar to our recent work on modified Camassa-Holm (mCH) equation [51], it paves a way for constructing integrable semi-discrete and fully discrete MT model, which definitely deserves immediate investigation.

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