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Intractability of Integration and Derivative for Multivariate Polynomial and Trigonometric Function

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INTRACTABILITY OF INTEGRATION AND DERIVATIVE FOR MULTIVARIATE POLYNOMIAL AND TRIGONOMETRIC FUNCTION

A Thesis

by

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ABSTRACT

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We study the hardness of some basic linear operators which involve high dimension integration or derivative. For a multivariate polynomial $p(x_1, \cdots, x_d)$ which has format $\prod \sum S_1$, we show that there is no any factor polynomial time approximation for the integration of $p(x_1, \cdots, x_d)$ in the unit cube $[0,1]^d$ unless P = NP. In addition to polynomials, we extend the discussion to nonlinear function. For a trigonometric function $g(x_1, \cdots, x_d)$ of format $\prod \sum^* T_{sin}$, we show that it is #P-hard to compute derivative $\frac{\partial g^{(d)}(x_1,\cdots,x_d)}{\partial x_1 \cdots \partial x_d}$ at the origin point $(x_1, \cdots, x_d) = (0, \cdots, 0)$.

Consider the linear operator $L(f) = \int_{P^d} f(x_1, \cdots, x_d) e^{-i(x_1,\cdots,x_d)} dx_1 \cdots dx_d$, we show that it is NP-hard to compute $L(f)$ for a $\prod \sum T_{cos}$ trigonometric function with the range $P = [0, \pi]$. And there is no any factor approximation to compute $L(f)$ for the $\prod \sum T_{cos}$ trigonometric function with the range $P = [0, \pi]$.

DEDICATION

The completion of my master studies would not have been possible without the love and support of my family. My wife, Haiqi Wang, my father, Liming Ding and my mother Lili He, wholeheartedly inspired, motivated and supported me by all means to accomplish this degree. Thank you for your love and patience.

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TABLE OF CONTENTS

	Page
ABSTRACT…………………………………………………………………………………	iii
DEDICATION………………………………………………………….……………...	iv
ACKNOWLEDGEMENTS…………………………………………………………………	v
TABLE OF CONTENTS…………………………………………………………………	vi
CHAPTER I. INTRODUCTION…………………………………………………………	1
CHAPTER II. PRELIMINARIES………………………………..……………………..	4
CHAPTER III. OVERVIEW OF OUR METHODS………………………………..........	6
CHAPTER IV. INTRACTABILITY OF $\prod\sum S_1$ POLYNOMIAL INTEGRATION..........	9
CHAPTER V. THE COMPLEXITY OF DERIVATIVE FOR TRIGONOMETRIC FUNCTION..	20
CHAPTER VI. INTRACTABILITY OF LINEAR OPERATOR *L*..................................	22
CHAPTER VII. CONCLUSION…...……………………………………………………..	31
REFERENCES……………………………………………………………………….....	32
BIOGRAPHICAL SKETCH……………………………………………………….….....	34

CHAPTER I

INTRODUCTION

Integration is a basic concept in mathematics and, together with differentiation, is one of the two important operations in calculus. As the rigorous mathematical definition was given, they became the basic tools of science and engineering. There are many useful variations of integration. Integrations with a large number of variables have been found applications in many area such as finance, nuclear physics, and quantum system, etc.

The complexity and the algorithms for approximating multivariate integration have been studied for many years. In 1991, Brightwell and Winkler [2] proved that the exact integration of a multivariate function is under the conjectured hardness of ♯P, which implies that the exact computation of the volume of an n-dimensional polytope is ♯P-complete. In the same year, an amazed result was showed by Dyer, Frieze and Kannan [3]. They gave a fully polynomial randomized approximation scheme for the volume of n-dimensional convex bodies. Successively, Applegate and Kannan [4] extended this result to positive smooth and nearly log-concave functions. After that, Sloan and Wozniakowski [5] used the method of measuring the number of function evaluations to prove an exponential lower bounds 2^s in order to obtain an approximation with error less than the integration itself, which has s variables. As a complement of Applegate and Kannan's work, Koutis [6] proved in 2003 that for any fixed natural k, it is NP-hard to 2^{n^k}-approximate the integral of positive, polynomial time computable functions, that are smooth in the sense that $|f(x) - f(y)| \leq \alpha(\max_{i\in[1,n]} |x_i - y_i|)$ with $\alpha = poly(n)$.

Originally, our inspiration came from the recent techniques developed by Koutis [7] and Williams [8] about path and packing problem. They utilized the algebrization and randomization tools to product faster algorithms for the path and packing problems. The recently developed monomial testing theory [9, 10, 11] lay the foundation of our studies. In the first step of the study, Fu [1] has showed that for the multivarivate polynomial defined with format $f(x_1,x_2,\cdots,x_d) = p_1(x_1,x_2,\cdots,x_d)p_2(x_1,x_2,\cdots,x_d)\cdots p_k(x_1,x_2,\cdots,x_d)$, where each $p_i(x_1,\cdots,x_d) = \sum_{j=1}^{d} q_j(x_j)$ has degree 2, there is no any factor approximation for the integration of it in the region $[0,1]^n$ unless P=NP. Particularly if each p_i has $0,1$ coefficients, it was showed that there is no any factor polynomial time approximation to its derivative $\frac{f^{(d)}(x_1,\cdots,x_d)}{\partial x_1\cdots\partial x_d}$ at the origin point $(x_1,\cdots,x_d) = (0,\cdots,0)$ unless P=NP. In this paper, we prove that the same intractable result also holds even if each $q_j(x_j)$ has degree 1. Since the integration and the derivative has wide range of applications, this inspires us to consider other mathematic systems which involve high dimension integration or derivative. For the complexity of derivative for the multivariate trigonometric functions, we show that it is $\sharp$-hard to compute $\frac{\partial g(x_1,\cdots,x_d)}{\partial x_1\cdots\partial x_d}$ at the point $(0,\cdots,0)$ for a $\prod\sum^{*} T_{\sin}$ trigonometric function $g(x_1,\cdots,x_d)$.

Fourier series was first introduced by Joseph Fourier as a method to solve problems about the flow of heat through ordinary materials. Then it was named using Joseph Fourier's last name to memorize his important contribution to the study of trigonometric series. Fourier series has provided one of the principal methods of analysis for mathematical physics, engineering, and signal processing. Moveover, in the theory or application of mathematics, Fourier series is also a very useful tool. Inspired by the construction of Fourier series, for a multivariate function $f(x_1,\cdots,x_n)$ defined on P^d, consider a linear operator L with $L(f) = \int_{P^d} f(x_1,\cdots,x_d)e^{-i(x_1+\cdots+x_d)}\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\,\mathrm{d}x_d$. In our paper, we use the standard computation model to consider the complexity to compute $L(f)$ for the $\prod\sum^{*} T_{\cos}$ multivariate trigonometric functions with $P = [0,\pi]$. We show that it is NP-hard to compute $L(f)$ for a $\prod\sum T_{\cos}$ trigonometric function with the range $P = [0,\pi]$. And there is no any factor polynomial time approximation to compute $L(f)$ for the $\prod\sum T_{\cos}$ trigonometric functions with the range $P = [0,\pi]$.

The paper is organized as follows. We first define the $\prod\sum S_1$ polynomials and the $\prod\sum T$ trigonometric functions in section II. Then section III gives an overview of about our method. In section IV, the inapproximation result for integration of the $\prod\sum S_1$ polynomials is presented. The inapproximation results for high order derivative of the $\prod\sum^{*} T_{\sin}$ trigonometric functions are showed in section V. In section VI, the complexity of computing the linear operator $L(f)$ for the $\prod\sum T_{\cos}$ trigonometric functions is discussed. At last, in section VII, we conclude the paper.

CHAPTER II

PRELIMINARIES

Let $N = \{0,1,2,\cdots\}$ be the set of all natural numbers. Let $N^+ = \{1,2,3,\cdots\}$ be the set of all positive natural numbers.

Assume that function $r(n)$ is from N to N^+. For a function $F(\cdot)$, which converts a multivariate polynomial into a real number, an algorithm $A(\cdot)$ gives an *$r(n)$-factor approximation* to $F(f)$ if it satisfies the following conditions: if $F(f) \geq 0$, then $\frac{F(f)}{r(n)} \leq A(f) \leq r(n)F(f)$; and if $F(f) < 0$, then $r(n)F(f) \leq A(f) \leq \frac{F(f)}{r(n)}$, where n is the number of variables in f. Assume that functions $r(n)$ and $s(n)$ are from N to N^+. For a function $F(\cdot)$, an algorithm $A(\cdot)$ gives an *$(r(n),s(n))$-factor approximation* to $F(f)$ such that if $F(f) \geq 0$, then $\frac{F(f)}{r(n)} - s(n) \leq A(f) \leq r(n)F(f) + s(n)$; and if $F(f) < 0$, then $r(n)F(f) - s(n) \leq A(f) \leq \frac{F(f)}{r(n)} + s(n)$, where n is the number of variables in f.

For variables $x_1,\cdots,x_n$, let $\mathscr{P}[x_1,\cdots,x_n]$ denote the communicative ring of all the n-variate polynomials with coefficients from $\mathscr{P}$. For $1 \leq i_1 \leq \cdots \leq i_k \leq n$, $\pi = x_{i_1}^{j_1} \cdots x_{i_k}^{j_k}$ is called a *monomial*. The *degree of* π, denote by $\deg(\pi)$, is $\sum_{s=1}^{k} j_s$. π is *multilinear*, if $j_1 = \cdots = j_k = 1$, i.e., π is linear in all its variables $x_{i_1},\cdots,x_{i_k}$.

By definition, any polynomial $p(x_1,\cdots,x_n)$ can be expressed as a sum of a list of monomials, called the *sum-product expansion*. The *degree* of the polynomial is the largest degree of its monomials in the expansion. We use $\mathscr{R}$ to represent the set of real numbers.

Definition 1. *Let $f(x_1,\cdots,x_d) \in \mathscr{R}[x_1,\cdots,x_d]$ and $c \geq 1$ be an integer. $f(x_1,\cdots,x_d)$ is said to be a $\prod\sum S_c$ polynomial, if f has format $f(x_1,\cdots,x_d) = p_1(x_1,\cdots,x_d)p_2(x_1,\cdots,x_d)\cdots p_k(x_1,\cdots,x_d)$, where each $p_i(x_1,\cdots,x_d) = \sum_{j=1}^{d} q_j(x_j)$ with each single variable polynomial $q_j(x_j)$ of degree at most c.*

The second part of this paper extends the discussion of complexity for the integration from the multivariate polynomials to the multivariate trigonometric functions. For the multivariate trigonometric functions, we have

Definition 2. *Let m, n, w be three integers. Let b be a binary number which is either* 0 *or* 1.

- $f(x_1, \cdots, x_d)$ *is said to be a* $\prod\sum T$ *trigonometric function, if* f *has format* $f(x_1, \cdots, x_d) = p_1(x_1, \cdots, x_d) p_2(x_1, \cdots, x_d) \cdots p_k(x_1, \cdots, x_d)$, *where each* $p_i(x_1, \cdots, x_d) = \sum_{j=1}^{d} q_j(x_j)$ *with each single variable polynomial* $q_j(x_j)$ *of format* $w \sin(m x_j + n)$ *or* $w \cos(m x_j + n)$.

- $f(x_1, \cdots, x_d)$ *is said to be a* $\prod\sum T_{\cos}$ *trigonometric function, if* f *has format* $f(x_1, \cdots, x_d) = p_1(x_1, \cdots, x_d) p_2(x_1, \cdots, x_d) \cdots p_k(x_1, \cdots, x_d)$, *where each* $p_i(x_1, \cdots, x_d) = \sum_{j=1}^{d} q_j(x_j)$ *with each single variable polynomial* $q_j(x_j)$ *of format* $w \cos(m x_j + n)$.

- $f(x_1, \cdots, x_d)$ *is said to be a* $\prod\sum^* T_{\sin}$ *trigonometric function, if* f *has format* $f(x_1, \cdots, x_d) = p_1(x_1, \cdots, x_d) p_2(x_1, \cdots, x_d) \cdots p_k(x_1, \cdots, x_d)$, *where each* $p_i(x_1, \cdots, x_d) = \sum_{j=1}^{d} q_j(x_j)$ *with each single variable polynomial* $q_j(x_j)$ *of format* $\sin x_j$.

Our third result consider the effect of linear operator $L(f) = \int_{P^d} f(x_1, \cdots, x_d) e^{-i(x_1 + \cdots + x_d)} \, dx_1 \, dx_2 \cdots dx_d$ on the $\prod\sum T_{\cos}$ trigonometric function. The notations and definitions about the linear operator L will be introduced separately in chapter VI.

An algorithm is *subexponential time* if it runs in $2^{o(1)}$ time for all inputs of length n. Define *subE* to be the class of languages that have subexpoenential time algorithms.

In computational complexity theory, the complexity *class* $\sharp P$ is the set of the counting problems associated with the decision problems in the set *NP*. More formally, $\sharp P$ is the class of function problems of the form compute $f(x)$, where f is the number of accepting paths of a nondeterministic Turing machine running in polynomial time. Unlike the most well-known complexity classes, it is not a class of decision problems but a class of function problems.

CHAPTER III

OVERVIEW OF OUR METHODS

The key technique used in this paper is similar to the one used in the first step of the study [1]. We show the intractability of the integration through reducing the well-known 3SAT problem to it. 3SAT is an NP-complete problem proved by Cook [12]. It is still NP-hard to decide a conjunctive normal form that each variable appears at most three times with at most one negative time. We assume that each variable has its negation appears at most one time (Otherwise, we replace it by its negation).

Take the $\prod\sum S_1$ polynomial as an example, we show in Lemma 2 that there exist integer coefficients polynomial functions $g_1(x) = bx + c$, $g_2(x) = ux + v$, $f(x) = px + q$ satisfy that $\int_0^1 g_1(x)\,\mathrm{d}x$, $\int_0^1 g_2(x)\,\mathrm{d}x$, $\int_0^1 f(x)\,\mathrm{d}x$, and $\int_0^1 g_1(x)g_2(x)\,\mathrm{d}x$ are all positive numbers which are greater than 1, and $\int_0^1 g_1(x)f(x)\,\mathrm{d}x$, $\int_0^1 g_2(x)f(x)\,\mathrm{d}x$, $\int_0^1 g_1(x)g_2(x)f(x)\,\mathrm{d}x$ are all equal to 0.

In order to make the conversion from logical operation to algebraic operation, we represent conjunctive normal form with the following format. For example, the formula $(x_1 + x_2)(x_1 + \bar{x}_2)(\bar{x}_1 + x_2)$ is used to represent a conjunctive normal form $(x_1 \vee x_2) \wedge (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2)$ with two boolean variables x_1 and x_2, where $+$ represents the logical $\vee$, and $\cdot$ represent the logical $\wedge$.

Example 1. Consider the logical formula $F_1 = (x_1 + x_2)(x_1 + \bar{x}_2)x_2$, which has the sum of product expansion $x_1x_1x_2 + x_1\bar{x}_2x_2 + x_2x_1x_2 + x_2\bar{x}_2x_2$. The term $x_1x_1x_2$ can bring a truth assignment $x_1 = true$ and $x_2 = true$ to make F true. As each variable appears at most 3 times with at most one negative appearance, the first positive x_i is replaced by $g_1(y_i)$, the second positive x_i is replaced by

$g_2(y_i)$ and the negative $\bar{x}_i$ is replaced by $f(y_i)$. It is converted into the polynomial

$$\begin{aligned} p_1(y_1,y_2) &= (g_1(y_1)+g_1(y_2))(g_2(y_1)+f(y_2))g_2(y_2) \\ &= g_1(y_1)g_2(y_1)g_2(y_2)+g_1(y_1)f(y_2)g_2(y_2)+g_1(y_2)g_2(y_1)g_2(y_2)+g_1(y_2)f(y_2)g_2(y_2) \end{aligned}$$

Consider the integration $\int_{[0,1]^2} p_1(y_1,y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2$. The integration can be distributed into those product terms. $\int_{[0,1]^2} g_1(y_1)g_2(y_1)g_2(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2$ is one of them. We have

$$\int_{[0,1]^2} g_1(y_1)g_2(y_1)g_2(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 = (\int_0^1 g_1(y_1)g_2(y_1)\,\mathrm{d}y_1)(\int_0^1 g_2(y_2)\,\mathrm{d}y_2) > 1\cdot 1 = 1$$

Then integration for other terms are all non-negative. Thus the integration $\int_{[0,1]^2} p_1(y_1,y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2$ is a positive number greater than 1 due to the satisfiability of F_1.

Example 2. Consider the logical formula $F_2 = (x_1+x_2)\bar{x}_1\bar{x}_2 = x_1\bar{x}_1\bar{x}_2 + x_2\bar{x}_1\bar{x}_2$. Neither $x_1\bar{x}_1\bar{x}_2$ nor $x_2\bar{x}_1\bar{x}_2$ can be satisfied. As each variable appears at most 3 times with at most one negative appearance, the first positive x_i is replaced by $g_1(y_i)$, the second positive x_i is replaced by $g_2(y_i)$, and the negative $\bar{x}_i$ is replaced by $f(y_i)$. Thus F_2 is converted into the polynomial

$$p_2(y_1,y_2) = (g_1(y_1)+g_1(y_2))f(y_1)f(y_2) = g_1(y_1)f(y_1)f(y_2)+g_1(y_2)f(y_1)f(y_2)$$

Consider the integration $\int_{[0,1]^2} p_2(y_1,y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2$, which is identical to $\int_{[0,1]^2} g_1(y_1)f(y_1)f(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 + \int_{[0,1]^2} g_1(y_2)f(y_1)f(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2$. We have

$$\int_{[0,1]^2} g_1(y_1)f(y_1)f(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 = (\int_0^1 g_1(y_1)f(y_1)\,\mathrm{d}y_1)(\int_0^1 f(y_2)\,\mathrm{d}y_2) = 0\cdot\int_0^1 f(y_2)\,\mathrm{d}y_2 = 0$$

and

$$\int_{[0,1]^2} g_1(y_2)f(y_1)f(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 = (\int_0^1 f(y_1)\,\mathrm{d}y_1)(\int_0^1 g_1(y_2)f(y_2)\,\mathrm{d}y_2) = \int_0^1 f(y_1)\,\mathrm{d}y_1\cdot 0 = 0$$

Thus the integration $\int_{[0,1]^2} p_2(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 = 0$ due to the unsatisfiability of F_2. Therefore, for any factor $a(n) > 0$, a polynomial time factor $a(n)$-approximation to the integration of a $\prod\sum S_1$ polynomial implies a polynomial time decision for the satisfiability of the corresponding boolean formula.

Chapter IV

Intractability of the $\prod\sum S_1$ Polynomial Integration

In this section, we show that the integration for the $\prod\sum S_1$ polynomial in the cube $[0,1]^d$ does not have any factor approximation. We will prove the result by means of similar technique which was used by Fu [1]. The main technique is to convert a logical formula into a polynomial. We use a basic property of integration, which can be found in some standard text books of calculus. Assume function $f(x_1,\cdots,x_d) = f_1(x_{i_1},\cdots,x_{i_{d_1}})f_2(x_{j_1},\cdots,x_{j_{d_2}})$, where $\{x_1,\cdots,x_d\}$ is the disjoint union of $\{x_{i_1},\cdots,x_{i_{d_1}}\}$ and $\{x_{j_1},\cdots,x_{j_{d_2}}\}$. Then we have

$$
\begin{aligned}
& \int_{[0,1]^d} f(x_1,\cdots,x_d)\,\mathrm{d}x_1\cdots\,\mathrm{d}x_d \\
= \; & \left(\int_{[0,1]^{d_1}} f_1(x_{i_1},\cdots,x_{i_{d_1}})\,\mathrm{d}x_{i_1}\cdots\,\mathrm{d}x_{i_{d_1}}\right)\cdot\left(\int_{[0,1]^{d_2}} f_2(x_{j_1},\cdots,x_{j_{d_1}})\,\mathrm{d}x_{j_1}\cdots\,\mathrm{d}x_{j_{d_1}}\right)
\end{aligned}
$$

The core idea of our proof is that $(3,3)$-SAT problem can be polynomial-time reducible to the $\prod\sum S_1$ polynomial integration. We first give the definitions of 3SAT and $(3,3)$-SAT.

Definition 3. *A* 3SAT instance *is a conjunctive form $C_1 \cdot C_2 \cdots C_m$ such that each C_i is a disjunction of at most three literals.* 3SAT *is the language of those 3SAT instances that have satisfiable assignments.*

Definition 4. *A* $(3,3)$-SAT *instance is an instance G for 3SAT such that for each variable x, the total number of occurrences of x and $\bar{x}$ in G is at most 3, and the total number of occurrences of $\bar{x}$ in G is at most 1.* $(3,3)$-SAT *is the language of those $(3,3)$-SAT instances that have satisfiable assignments.*

For examples, $(\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2) \wedge (x_1 \vee \bar{x}_3)$ is a 3SAT instance but not a $(3,3)$-SAT instance since $\bar{x}_1$ appears twice in the formula. On the other hand, $(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee x_2 \vee x_3)$ is both 3SAT and $(3,3)$-SAT instance, and hence belongs to both 3SAT and $(3,3)$-SAT. It is well known that 3SAT is a NP-complete problem, the following lemma shows that $(3,3)$-SAT is polynomial-time reducible to 3SAT.

Lemma 1. *[1][13] There is a polynomial time reduction $f(.)$ from 3SAT to $(3,3)$-SAT.*

The following lemma is the main technique. It is used to convert a $(3,3)$-SAT instance into a $\prod\sum S_1$ polynomial.

Lemma 2. *There exist b,c,u,v,p and q such that the functions $g_1(x) = bx + c$, $g_2(x) = ux + v$, $f(x) = px + q$ satisfy that*

1. *$\int_0^1 g_1(x)\,\mathrm{d}x$, $\int_0^1 g_2(x)\,\mathrm{d}x$, $\int_0^1 f(x)\,\mathrm{d}x$, and $\int_0^1 g_1(x)g_2(x)\,\mathrm{d}x$ are all positive numbers which are greater than 1, and*
2. *$\int_0^1 g_1(x)f(x)\,\mathrm{d}x$, $\int_0^1 g_2(x)f(x)\,\mathrm{d}x$, $\int_0^1 g_1(x)g_2(x)f(x)\,\mathrm{d}x$ are all equal to 0.*

Proof. We give the details how to derive the functions $g_1(x)$, $g_2(x)$ and $f(x)$ to satisfy the conditions of the lemma.

$$\int_0^1 g_1(x)\,\mathrm{d}x = \int_0^1 (bx+c)\,\mathrm{d}x = \frac{b}{2}x^2 + cx \,|_0^1 = \frac{b}{2} + c \tag{IV.1}$$

$$\int_0^1 g_2(x)\,\mathrm{d}x = \int_0^1 (ux+v)\,\mathrm{d}x = \frac{u}{2}x^2 + vx \,|_0^1 = \frac{u}{2} + v \tag{IV.2}$$

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 (px+q)\,\mathrm{d}x = \frac{p}{2}x^2 + qx \,|_0^1 = \frac{p}{2} + q \tag{IV.3}$$

$$
\begin{aligned}
\int_0^1 g_1(x)g_2(x)\,\mathrm{d}x &= \int_0^1 (bx+c)(ux+v)\,\mathrm{d}x \\
&= \int_0^1 (bux^2+bvx+cux+cv)\,\mathrm{d}x \\
&= \frac{bu}{3}x^3+\frac{bv+cu}{2}x^2+cvx\,|_0^1 \\
&= \frac{bu}{3}+\frac{bv+cu}{2}+cv \qquad \text{(IV.4)}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 g_1(x)f(x)\,\mathrm{d}x &= \int_0^1 (bx+c)(px+q)\,\mathrm{d}x \\
&= \int_0^1 (bpx^2+bqx+cpx+cq)\,\mathrm{d}x \\
&= \frac{bp}{3}x^3+\frac{bq+cp}{2}x^2+cqx\,|_0^1 \\
&= \frac{bp}{3}+\frac{bq+cp}{2}+cq \qquad \text{(IV.5)}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 g_2(x)f(x)\,\mathrm{d}x &= \int_0^1 (ux+v)(px+q)\,\mathrm{d}x \\
&= \int_0^1 (upx^2+uqx+vpx+vq)\,\mathrm{d}x \\
&= \frac{up}{3}x^3+\frac{uq+vp}{2}x^2+vqx\,|_0^1 \\
&= \frac{up}{3}+\frac{uq+vp}{2}+vq \qquad \text{(IV.6)}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 g_1(x)g_2(x)f(x)\,\mathrm{d}x &= \int_0^1 (bx+c)(ux+v)(px+q)\,\mathrm{d}x \\
&= \int_0^1 [bupx^3+(buq+bvp+cup)x^2+(bvq+cuq+cvp)x+cvq]\,\mathrm{d}x \\
&= \frac{bup}{4}x^4+\frac{buq+bvp+cup}{3}x^3+\frac{bvq+cuq+cvp}{2}x^2+cvqx\,|_0^1 \\
&= \frac{bup}{4}+\frac{buq+bvp+cup}{3}+\frac{bvq+cuq+cvp}{2}+cvq \qquad \text{(IV.7)}
\end{aligned}
$$

By summarizing (IV.1), (IV.2), (IV.3), (IV.4), (IV.5), (IV.6), (IV.7) and conditions *i*, *ii*, we get the following equation set:

$$
\begin{aligned}
&\frac{b}{2}+c>0 && \text{(IV.8)} \\
&\frac{u}{2}+v>0 && \text{(IV.9)} \\
&\frac{p}{2}+q>0 && \text{(IV.10)} \\
&\frac{bu}{3}+\frac{bv+cu}{2}+cv>0 && \text{(IV.11)} \\
&\frac{bp}{3}+\frac{bq+cp}{2}+cq=0 && \text{(IV.12)} \\
&\frac{up}{3}+\frac{uq+vp}{2}+vq=0 && \text{(IV.13)} \\
&\frac{bup}{4}+\frac{(buq+bvp+cup)}{3}+\frac{(bvq+cuq+cvp)}{2}+cvq=0 && \text{(IV.14)}
\end{aligned}
$$

From (IV.14), we have

$$
u(\frac{bp}{4}+\frac{bq+cp}{3}+\frac{cq}{2})+v(\frac{bp}{3}+\frac{bq+cp}{2}+cq)=0 \qquad \text{(IV.15)}
$$

Due to (IV.12), (IV.15) becomes

$$u\left(\frac{bp}{4}+\frac{bq+cp}{3}+\frac{cq}{2}\right)=0 \tag{IV.16}$$

Here $u \neq 0$. Otherwise if we let $u = 0$, then $v > 0$ from (IV.9), and from (IV.13) we also have $v(\frac{p}{2}+q) = 0$, but equation (IV.10) tells us that $\frac{p}{2}+q > 0$, thus it results in a contradiction obviously. Since $u \neq 0$, we have the following two equations from (IV.16) and (IV.12):

$$\frac{bp}{4}+\frac{bq+cp}{3}+\frac{cq}{2}=0$$
$$\frac{bp}{3}+\frac{bq+cp}{2}+cq=0$$

Here we consider b, c as variables and p, q as coefficients. After simplifying, we have

$$(3p+4q)b+(4p+6q)c=0 \tag{IV.17}$$

$$(2p+3q)b+(3p+6q)c=0 \tag{IV.18}$$

In order to get nonzero solution, the determinant of coefficients for (IV.17) and (IV.18) must be zero, thus we have

$$\begin{vmatrix} 3p+4q & 4p+6q \\ 2p+3q & 3p+6q \end{vmatrix} = 0$$

Therefore we have

$$(3p+4q)(3p+6q)-(4p+6q)(2p+3q)=0 \tag{IV.19}$$

$$p^2+6pq+6q^2=0 \tag{IV.20}$$

$$p=\frac{-6q\pm\sqrt{36q^2-24q^2}}{2}=(-3\pm\sqrt{3})q \tag{IV.21}$$

So for equation set (IV.17) and (IV.18), only the coefficients p and q satisfy equation (IV.21), the variables b and c have nonzero solution. In order to get whole solution, we first fix the two variables p and q. Let $q=2\sqrt{3}$, then $p=6-6\sqrt{3}$, we can see that this assignment satisfies equation (IV.10). Equations (IV.9) and (IV.13) can be used to determine u and v, through (IV.13), we have

$$(1-\sqrt{3})u+\frac{\sqrt{3}}{2}u+\frac{3(1-\sqrt{3})}{2}v+\sqrt{3}v=0$$
$$u=-(\sqrt{3}+3)v$$

Because of (IV.9), we assign $u=-1$, then $v=\sqrt{3}+3$. Similarly, we can determine b, c using equations (IV.8) and (IV.12). Here we assign $b=2\sqrt{3}$, $c=1-\sqrt{3}$. Thus, $g_1(x)=bx+c=2\sqrt{3}x+1-\sqrt{3}$, $g_2(x)=ux+v=-x+\sqrt{3}+3$, $f(x)=px+q=(6-6\sqrt{3})x+2\sqrt{3}$. We have the following equations to satisfy the conditions in the lemma.

$$\begin{aligned}
\int_0^1 g_1(x)\,\mathrm{d}x &= 1 \\
\int_0^1 g_2(x)\,\mathrm{d}x &= \frac{5}{2}+\sqrt{3} \\
\int_0^1 f(x)\,\mathrm{d}x &= 3-\sqrt{3} \\
\int_0^1 g_1(x)g_2(x)\,\mathrm{d}x &= \frac{5\sqrt{3}+15}{6} \\
\int_0^1 g_1(x)f(x)\,\mathrm{d}x &= 0 \\
\int_0^1 g_2(x)f(x)\,\mathrm{d}x &= 0 \\
\int_0^1 g_1(x)g_2(x)f(x)\,\mathrm{d}x &= 0
\end{aligned}$$

□

Lemma 3. *There is a polynomial time algorithm h such that given a $(3,3)$-SAT instance $s(x_1,\cdots,x_n)$, it generates a $\prod\sum S_1$ polynomial $h(s(x_1,\cdots,x_n)) = p(y_1,\cdots,y_n)$ which satisfies the following two conditions:*

1. *if $s(x_1,\cdots,x_n)$ is satisfiable, then integration $\int_{[0,1]^n} p(y_1,\cdots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n$ is equal to 1; and*

2. *if $s(x_1,\cdots,x_n)$ is not satisfiable, then the integration $\int_{[0,1]^n} p(y_1,\cdots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n$ is 0.*

Proof. Let polynomial $g_1(y), g_2(y)$ and $f(y)$ be defined according to those in Lemma 2. For a (3,3)-SAT problem $s(x_1,\cdots,x_n)$, polynomial $p(x_1,\cdots,x_n)$ can be constructed as follows:

1. For the first positive literal x_i in $s(x_1,\cdots,x_n)$, replace it with $g_1(y_i)$.

2. For the second positive literal x_i in $s(x_1,\cdots,x_n)$, replace it with $g_2(y_i)$.

3. For the negative literal $\bar{x}_i$ in $s(x_1,\cdots,x_n)$, replace it with $f(y_i)$.

The formula $s(x_1,\cdots,x_n)$ has a sum of product form. It is satisfiable if and only if there exists one term which does not contain a positive and negative literals for the same variable. If a term contains both x_i and $\bar{x}_i$, the corresponding term in the sum of product for $p(\cdot)$ contains both $g_j(y_i)$ and $f(y_i)$ for some $j \in \{1,2\}$. This makes it zero after the integration in the range $[0,1]$ by Lemma 2. Furthermore, if $s(x_1,\cdots,x_n)$ is satisfiable, as we did in the example one in section III, the integration of p should be a positive number larger than 1 by Lemma 2. Therefore, $s(x_1,\cdots,x_n)$ is satisfiable if and only if $\int_{[0,1]^n} p(y_1,\cdots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n$ is not zero. The computational time of h is clearly polynomial since we convert s to $h(s)$ by replacing each literal by a single variable function of degree 1.

□

Theorem 1. *Let $a(n)$ be an arbitrary function from N to N^+. Then there is no polynomial time $a(n)$-factor approximation for the integration of a $\prod\sum S_1$ polynomial $p(x_1,\cdots,x_n)$ in the region $[0,1]^n$ unless $P = NP$.*

Proof. Assume that $A(\cdot)$ is a polynomial time $a(n)$-factor approximation for the integration $\int_{[0,1]^n} p(y_1,\cdots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n$ with $\prod\sum S_1$ polynomial $p(y_1,\cdots,y_n)$.

For a (3,3)-SAT instance $s(x_1,\cdots,x_n)$, let $p(y_1,\cdots,y_n) = h(s(x_1,\cdots,x_n))$ according to Lemma 3. By lemma 3, a (3,3)-SAT instance $s(x_1,\cdots,x_n)$ is satisfiable if and only if the integration $J = \int_{[0,1]^n} p(y_1,\cdots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n$ is not zero. Assume that s is not satisfiable, then we have $A(J) \in [J/a(n), J\cdot a(n)]$=[0,0], which implies $A(J) = 0$. On the other hand, if s is satisfiable, then we have $A(J) \in [J/a(n), J\cdot a(n)] \subseteq (1,+\infty)$, which implies $A(J) > 0$. Thus, $s(x_1,\cdots,x_n)$ is satisfiable if and only if $A(J) > 0$.

Therefore, there is a polynomial time algorithm for solving (3,3)-SAT, which is NP-complete by Lemma 1. So, P = NP.

□

Theorem 2. *Let $a(n)$ be an arbitrary function from N to N^+. Then there is no subexponential time $a(n)$-factor approximation for the integration of a $\prod\sum S_1$ polynomial $p(x_1,\cdots,x_n)$ in the region $[0,1]^n$ unless $NP \subseteq subE$.*

Proof. Assume that $A(\cdot)$ is a subexponential time $a(n)$-factor approximation for the integration $\int_{[0,1]^n} p(y_1,\cdots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n$ with $\prod\sum S_1$ polynomial $p(y_1,\cdots,y_n)$.

For a (3,3)-SAT instance $s(x_1,\cdots,x_n)$, let $p(y_1,\cdots,y_n) = h(s(x_1,\cdots,x_n))$ according to Lemma 3. By lemma 3, a (3,3)-SAT instance $s(x_1,\cdots,x_n)$ is satisfiable if and only if the integration $J = \int_{[0,1]^n} p(y_1,\cdots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n$ is not zero. Assume that s is not satisfiable, then we have $A(J) \in [J/a(n), J\cdot a(n)]$=[0,0], which implies $A(J) = 0$. On the other hand, if s is satisfiable, then we have $A(J) \in [J/a(n), J\cdot a(n)] \subseteq (1,+\infty)$, which implies $A(J) > 0$. Thus, $s(x_1,\cdots,x_n)$ is satisfiable if and only if $A(J) > 0$.

Therefore, there is a subexponential time algorithm for solving (3,3)-SAT, which is NP-complete by Lemma 1. So, NP $\subseteq$ subE.

□

Lemma 4. *Assume that $a(1^n)$ is a polynomial time computable function from N to N^+ with $a(1^n) > 0$ for n. There is polynomial time algorithm that given a $(3,3)$-SAT instance $s(x_1,\cdots,x_n)$, it generates a $\prod\sum S_1$ polynomial $p(y_1,\cdots,y_n)$ such that if $s(x_1,\cdots,x_n)$ is satisfiable, then $\int_{[0,1]^n} p(y_1,\cdots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n$ is a positive number at least $3a(1^n)^2$; and if $s(x_1,\cdots,x_n)$ is not satisfiable, $\int_{[0,1]^n} p(y_1,\cdots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n$ is zero.*

Proof. For a (3,3)-SAT problem $s(x_1,\cdots,x_n)$, let $q(y_1,\cdots,y_n) = h(s(x_1,\cdots,x_n))$ be constructed as Lemma 3. Since $a(1^n)$ is polynomial time computable, let $p(y_1,\cdots,y_n) = 3a(1^n)^2 q(y_1,\cdots,y_n)$, which can be computed in a polynomial time.

□

Theorem 3. *Let $a(1^n)$ be a polynomial time computable function from N to N^+. Then there is no polynomial time $(a(1^n), a(1^n))$-approximation for the integration problem $\int_{[0,1]^n} f(x_1,\cdots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n$ for a $\prod\sum S_1$ polynomial $f(\cdot)$ unless $P = NP$.*

Proof. Assume that there is a polynomial time $(a(1^n), a(1^n))$-approximation $App(\cdot)$ for the integration problem $\int_{[0,1]^n} f(x_1,\cdots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n$ for a $\prod\sum S_1$ polynomial $f(\cdot)$.

Let $s(x_1,\cdots,x_n)$ be an arbitrary (3,3)-SAT instance. Let $p(y_1,\cdots,y_n)$ be the polynomial according to Lemma 4. Let $J=\int_{[0,1]^n} p(y_1,\cdots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n$. If $s(x_1,\cdots,x_n)$ is satisfiable, then $J=0$. Otherwise, $J\geq 3a(1^n)^2$.

Assume that $s(x_1,\cdots,x_n)$ is not satisfiable. Since $App(J)$ is an $(a(1^n),a(1^n))$-approximation, we have $App(J)\leq J\cdot a(1^n)+a(1^n)=a(1^n)$ by the definition in section II. On the other hand, assume that $s(x_1,\cdots,x_n)$ is satisfiable. Since $App(J)$ is an $(a(1^n),a(1^n))$-approximation, we have $App(J)\geq J/a(1^n)-a(1^n)\geq 3a(1^n)^2/a(1^n)-a(1^n)=2a(1^n)$ by the definition in section II.

Therefore, $s(x_1,\cdots,x_n)$ is satisfiable if and only if $App(J)\geq 2a(1^n)$. Thus, if there is a polynomial time $(a(1^n),a(1^n))$-approximation, then there is a polynomial time algorithm for solving $(3,3)$-SAT. By Lemma 1, $\mathrm{P}=\mathrm{NP}$.

□

The well known exponential time hypothesis says $\mathrm{NP}\not\subseteq$ subE [16]. Basing on such a hypothesis, we have the following stronger result about the intractability of high dimension integration.

Theorem 4. *Let $a(1^n)$ be a polynomial time computable function from N to N^+. Then there is no subexponential time $(a(1^n),a(1^n))$-approximation for the integration problem $\int_{[0,1]^n} p(x_1,\cdots,x_n)\,\mathrm{d}x_1\cdots x_n$ for a $\prod\sum S_1$ polynomial $f(\cdot)$ unless* $\mathrm{NP}\subseteq$ subE.

Proof. Assume that there is a subexponential time $(a(1^n),a(1^n))$-approximation $App(\cdot)$ for the integration problem $\int_{[0,1]^n} f(x_1,\cdots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n$ for a $\prod\sum S_1$ polynomial $f(\cdot)$.

Let $s(x_1,\cdots,x_n)$ be an arbitrary (3,3)-SAT instance. Let $p(y_1,\cdots,y_n)$ be the polynomial according to Lemma 4. Let $J=\int_{[0,1]^n} p(y_1,\cdots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n$. If $s(x_1,\cdots,x_n)$ is satisfiable, then $J=0$. Otherwise, $J\geq 3a(1^n)^2$.

Assume that $s(x_1,\cdots,x_n)$ is not satisfiable. Since $App(J)$ is an $(a(1^n),a(1^n))$-approximation, we have $App(J)\leq J\cdot a(1^n)+a(1^n)=a(1^n)$ by the definition in section II. On the other hand, assume that $s(x_1,\cdots,x_n)$ is satisfiable. Since $App(J)$ is an $(a(1^n),a(1^n))$-approximation, we have $App(J)\geq J/a(1^n)-a(1^n)\geq 3a(1^n)^2/a(1^n)-a(1^n)=2a(1^n)$ by the definition in section II.

Therefore, $s(x_1, \cdots, x_n)$ is satisfiable if and only if $App(J) \geq 2a(1^n)$. Thus, if there is a subexponential time $(a(1^n), a(1^n))$-approximation, then there is a subexponential time algorithm for solving $(3,3)$-SAT. By Lemma 1, NP $\subseteq$ subE.

□

CHAPTER V

THE COMPLEXITY OF DERIVATIVE FOR TRIGONOMETRIC FUNCTION

In this chapter, we show that it is $\sharp$-hard to compute $\frac{\partial f(x_1,\cdots,x_d)}{\partial x_1\cdots\partial x_d}$ at the point $(0,\cdots,0)$ for a $\prod\sum^* T_{\sin}$ function $f(x_1,\cdots,x_d)$. We first give the following definitions which are consistent with the monomial testing theory [9, 10, 11].

Definition 5. *Let $p(x_1,\cdots,x_n)\in\mathscr{P}[x_1,\cdots,x_n]$ be any given polynomial. Let $m,s,t\geq 1$ be integers.*

- *$p(x_1,\cdots,x_n)$ is said to be a $\prod_m\sum_s\prod_t$ polynomial, if $p(x_1,\cdots,x_n)=\prod_{i=1}^m F_i$, $F_i=\sum_{j=1}^{r_i}X_{ij}$ and $1\leq r_i\leq s$, and $deg(X_{ij})\leq t$. We call each F_i a clause. Note that X_{ij} is not a monomial in the sum-product expansion of $p(x_1,\cdots,x_n)$ unless $m=1$. To differentiate this subtlety, we call X_{ij} term.*

- *In particular, we say $p(x_1,\cdots,x_n)$ is a $\prod_m\sum_s$ polynomial, if it is a $\prod_m\sum_s\prod_1$ polynomial. Here, each clause is a linear addition of single variables. In other word, each term has degree* 1.

- *When no confusing arises from the correct, we use $\prod\sum\prod$ and $\prod\sum$ to stand for $\prod_m\sum_s\prod_t$ and $\prod_m\sum_s$ respectively. Similarly, we use $\prod\sum_s\prod$ and $\prod\sum_s$ to stand for $\prod_m\sum_s\prod_t$ and $\prod_m\sum_s$ respectively, emphasizing that every clause in a polynomial has at most s terms or is a linear addition of at most s single variables.*

We show that the derivative for a $\prod\sum^* T_{\sin}$ trigonometric function has a polynomial time approximation scheme. Chen and Fu [9] derived the following results by a reduction from the number of perfect matchings in bipartite graph.

Lemma 5. *It is ♯P-hard to compute the coefficient of any given multilinear monomial in an d-variate $\prod_m \sum_s$ polynomial.*

Lemma 6. *There is a polynomial time randomized approximation schemes to compute the coefficient of a $\prod\sum^*$ polynomial.*

Lemma 5 and Lemma 6 imply the following theorem.

Theorem 5. *1. It is ♯P-hard to compute $\frac{\partial g(x_1,\cdots,x_d)^{(d)}}{\partial x_1\cdots\partial x_d}$ at the point $(0,\cdots,0)$ for an d-variate $\prod\sum^* T_{\sin}$ trigonometric function $g(x_1,\cdots,x_d)$.*

2. Given a $\prod\sum^* T_{\sin}$ trigonometric function $g(x_1,\cdots,x_d)$, there is a polynomial time randomized approximation schemes to compute $\frac{\partial g(x_1,\cdots,x_d)^{(n)}}{\partial x_1\cdots\partial x_d}$ at the point $(0,\cdots,0)$.

Proof. Since $(\sin x)'|_{x=0} = \cos x|_{x=0} = 1$, for a $\prod\sum^* T_{\sin}$ trigonometric function $g(x_1,\cdots,x_d)$, the derivative $\frac{\partial g(x_1,\cdots,x_d)^{(d)}}{\partial x_1\cdots\partial x_d}$ at the point $(0,\cdots,0)$ is identical to the coefficient of the term $\sin x_1 \sin x_2 \cdots \sin x_d$ in the expansion of $g(x_1,\cdots,x_d)$. Thus if we regard each $\sin x_i$ as an variable x_i in the $\prod\sum^*$ polynomial, the derivative $\frac{\partial g(x_1,\cdots,x_d)^{(d)}}{\partial x_1\cdots\partial x_d}$ at the point $(0,\cdots,0)$ for the $\prod\sum^* T_{\sin}$ trigonometric function can be reduced to the derivative $\frac{\partial f(x_1,\cdots,x_d)^{(d)}}{\partial x_1\cdots\partial x_d}$ at the point $(0,\cdots,0)$ for the $\prod_m \sum_s$ polynomial $f(x_1,\cdots,x_d)$. Therefore, by Lemma 5, it is ♯P-hard to compute $\frac{\partial g(x_1,\cdots,x_d)^{(d)}}{\partial x_1\cdots\partial x_d}$ at the point $(0,\cdots,0)$ for the $\prod\sum^* T_{\sin}$ trigonometric function. And the polynomial time randomized approximation schemes for computing the coefficient of a $\prod\sum^*$ polynomial also can be used to approximate the derivative $\frac{\partial g(x_1,\cdots,x_d)^{(n)}}{\partial x_1\cdots\partial x_d}$ at the point $(0,\cdots,0)$ of the $\prod\sum^* T_{\sin}$ trigonometric function.

□

CHAPTER VI

INTRACTABILITY OF LINEAR OPERATOR L

We first explain the motivation of the linear operator $L(f)$. Let P^d be the Cartesian product of d copies of P. A typical element of P^d is thus a d-tuple

$$\mathbf{x} = (x_1, x_2, \cdots, x_d)$$

where $x_k \in P$ for $k = 1, 2, \cdots, d$. Let $\mathbf{C}$ be the set of complex numbers, we study the functions defined on P^d which has format f: $P^d \to \mathbf{C}$. Each function f is a rule that to each $x \in P^d$ assigns a complex number $f(x) = f(x_1, x_2, \cdots, x_d)$. We also work with d-tuples of integers, which we write as

$$\mathbf{n} = (n_1, n_2, \cdots, n_d).$$

These are the elements of the set $\mathbf{Z}^d$. For $\mathbf{x} \in P^d$ and $\mathbf{n} \in \mathbf{Z}^d$, the "scalar product" is formed by

$$\mathbf{n} \cdot \mathbf{x} = n_1 x_1 + n_2 x_2 + \cdots + n_d x_d = \sum_{k=1}^{d} n_k x_d.$$

For this product, it is easy to verify that the following two rules are holds.

$$(\mathbf{m} + \mathbf{n}) \cdot \mathbf{x} = \mathbf{m} \cdot \mathbf{x} + \mathbf{n} \cdot \mathbf{x},$$

$$\mathbf{n} \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{n} \cdot \mathbf{x} + \mathbf{n} \cdot \mathbf{y}.$$

For a function f, the operator L is defined as

$$\begin{aligned} L(f) &= \int_{P^d} f(\mathbf{x})e^{-i\mathbf{n}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x} \\ &= \int_{P^d} f(x_1,\cdots,x_d)e^{-i(n_1x_1+n_2x_2+\cdots+n_dx_d)}\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\,\mathrm{d}x_d. \end{aligned}$$

It is easy to see that when $P=[c,c+2\pi]$, $(2\pi)^d\cdot L(f)$ is the Fourier coefficients $c_{\mathbf{n}}$ of f. For more analysis of multi-dimensional Fourier series, please refer to Vretblad's book [14].

In the following paragraphs, we show that it is NP-hard to compute $L(f)$ for a $\prod\sum T_{\cos}$ trigonometric function with the range $P=[0,\pi]$ and $n=(1,\cdots,1)$. And there is no any factor polynomial time approximation to compute $L(f)$ for the $\prod\sum T_{\cos}$ trigonometric functions with the range $P=[0,\pi]$ and $n=(1,\cdots,1)$. Similar to the technique used in section IV, we have the following lemma to convert a (3,3)-SAT instance into a $\prod\sum T_{\cos}$ trigonometric function.

Lemma 7. *There exist integers m_1,m_2, n, p, q and r such that the functions $g_1(x)=p\cos m_1x$, $g_2(x)=q\cos m_2x$, $f(x)=r\cos nx$ satisfy that*

1. *all the integrals $\int_0^\pi g_1(x)e^{-ix}\,\mathrm{d}x$, $\int_0^\pi g_2(x)e^{-ix}\,\mathrm{d}x$, $\int_0^\pi f(x)e^{-ix}\,\mathrm{d}x$, and $\int_0^\pi g_1(x)g_2(x)e^{-ix}\,\mathrm{d}x$ are not equal to* 0*, and*

2. *$\int_0^\pi g_1(x)f(x)e^{-ix}\,\mathrm{d}x$, $\int_0^\pi g_2(x)f(x)e^{-ix}\,\mathrm{d}x$, $\int_0^\pi g_1(x)g_2(x)f(x)e^{-ix}\,\mathrm{d}x$ are all equal to* 0.

Proof. First, it is easy to calculate the following indefinite integral

$$\int e^{ax}\cos bx\,\mathrm{d}x = \frac{b\sin bx + a\cos bx}{a^2+b^2}e^{ax}+C,$$

where C is an arbitrary constant. Obviously, the above equation doesn't hold for $b=1$ and $a=-i$, but this particular case is very useful in the construction of coefficients. According to the famous Euler Formula $e^{ix}=\cos x + i\sin x$, we have

$$\int e^{-ix}\cos x\,\mathrm{d}x = \int(\cos x - i\sin x)\cos x\,\mathrm{d}x$$
$$= \frac{1}{4}(\sin 2x + i\cos 2x) + \frac{1}{2}x + C.$$

When $x = \pi$, $e^{-i\pi} = \cos\pi - i\sin\pi = -1$. Thus we have the following definite integrals

$$\int_0^{\pi} e^{-ix}\cos bx\,\mathrm{d}x = \frac{i(\cos b\pi + 1) - b\sin b\pi}{b^2 - 1} \tag{VI.1}$$
$$\int_0^{\pi} e^{-ix}\cos x\,\mathrm{d}x = \frac{\pi}{2}. \tag{VI.2}$$

By (VI.1), the integrals of $g_1(x)$, $g_2(x)$ and $f(x)$ can be easily calculated:

$$\frac{1}{p}\int_0^{\pi} e^{-ix}g_1(x)\,\mathrm{d}x = \int_0^{\pi} e^{-ix}\cos m_1x\,\mathrm{d}x = \frac{i(\cos m_1\pi + 1) - m_1\sin m_1\pi}{m_1^2 - 1} \tag{VI.3}$$
$$\frac{1}{q}\int_0^{\pi} e^{-ix}g_2(x)\,\mathrm{d}x = \int_0^{\pi} e^{-ix}\cos m_2x\,\mathrm{d}x = \frac{i(\cos m_2\pi + 1) - m_2\sin m_2\pi}{m_2^2 - 1} \tag{VI.4}$$
$$\frac{1}{r}\int_0^{\pi} e^{-ix}f(x)\,\mathrm{d}x = \int_0^{\pi} e^{-ix}\cos nx\,\mathrm{d}x = \frac{i(\cos n\pi + 1) - n\sin n\pi}{n^2 - 1} \tag{VI.5}$$

To satisfy the condition *i*, we cannot make the results of above three integrals equal to 0. Thus the coefficients m_1, m_2 and n cannot be odd integers except 1. Next considering the condition *ii*, we have

$$
\begin{aligned}
\frac{1}{pr}\int_0^{\pi} g_1(x)f(x)e^{-ix}\,\mathrm{d}x &= \int_0^{\pi} e^{-ix}\cos m_1x\cos nx\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\pi} e^{-ix}\cos(m_1+n)\,\mathrm{d}x + \frac{1}{2}\int_0^{\pi} e^{-ix}\cos(m_1-n)\,\mathrm{d}x \\
&= \frac{1}{2}\frac{i[\cos(m_1+n)\pi+1]-(m_1+n)\sin(m_1+n)\pi}{(m_1+n)^2-1} \\
&\quad +\frac{1}{2}\frac{i[\cos(m_1-n)\pi+1]-(m_1-n)\sin(m_1-n)\pi}{(m_1-n)^2-1}
\end{aligned}
$$

Let $\mathbf{Z}$ be the set of integers. To make $\int_0^{\pi} g_1(x)f(x)\,\mathrm{d}x = 0$, a direct integer solution is $m_1+n = 1+2k_1$ and $m_1-n = 1+2k_2$ where $k_1,k_2 \in \mathbf{Z}\setminus 0$, which means m_1+n and m_1-n should be odd integers except 1 simultaneously. Similarly, to make $\int_0^{\pi} g_2(x)f(x)\,\mathrm{d}x = 0$, we have $m_2+n = 1+2k_3$ and $m_2-n = 1+2k_4$ where $k_3,k_4 \in \mathbf{Z}\setminus 0$, which means m_2+n and m_2-n should be odd integers except 1 simultaneously. In the end, we consider $\int_0^{\pi} g_1(x)g_2(x)f(x)\,\mathrm{d}x$:

$$
\begin{aligned}
\int_0^{\pi} g_1(x)g_2(x)f(x)e^{-ix}\,\mathrm{d}x &= \int_0^{\pi}\cos m_1x\cos m_2x\cos nxe^{-ix}\,\mathrm{d}x \\
&= \int_0^{\pi}\frac{1}{2}e^{-ix}[\cos(m_1+m_2)x+\cos(m_1-m_2)x]\cos nx\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\pi} e^{-ix}\cos(m_1+m_2)x\cos nx\,\mathrm{d}x + \frac{1}{2}\int_0^{\pi} e^{-ix}\cos(m_1-m_2)x\cos nx\,\mathrm{d}x \\
&= \frac{1}{4}\int_0^{\pi} e^{-ix}[\cos(m_1+m_2+n)x+\cos(m_1+m_2-n)x]\,\mathrm{d}x \\
&\quad +\frac{1}{4}\int_0^{\pi} e^{-ix}[\cos(m_1-m_2+n)x+\cos(m_1-m_2-n)x]\,\mathrm{d}x \\
&= \frac{1}{4}[\int_0^{\pi} e^{-ix}\cos(m_1+m_2+n)x\,\mathrm{d}x + \int_0^{\pi} e^{-ix}\cos(m_1+m_2-n)x\,\mathrm{d}x \\
&\quad +\int_0^{\pi} e^{-ix}\cos(m_1-m_2+n)x\,\mathrm{d}x + \int_0^{\pi} e^{-ix}\cos(m_1-m_2-n)x\,\mathrm{d}x]
\end{aligned}
$$

Using the similar analysis as above, if m_1+m_2+n, m_1+m_2-n, m_1-m_2+n and m_1-m_2-n satisfy that they are all odd integers except 1, then $\int_0^{\pi} g_1(x)g_2(x)f(x)\,\mathrm{d}x = 0$ holds. Through summarizing all the results, we let $m_1 = 8$, $m_2 = 4$, $n = 1$. Then $m_1+n = 9$, $m_1-n = 7$, $m_2+n = 5$,

$m_2 - n = 3$, thus $\int_0^\pi g_1(x)f(x)\,\mathrm{d}x = 0$ and $\int_0^\pi g_2(x)f(x)\,\mathrm{d}x = 0$ hold. Moreover, since $m_1 + m_2 + n = 13$, $m_1 + m_2 - n = 11$, $m_1 - m_2 + n = 5$ and $m_1 - m_2 - n = 3$, $\int_0^\pi g_1(x)g_2(x)f(x)\,\mathrm{d}x = 0$ also holds. Up to now we have made condition *ii* satisfy.

On the other hand, for condition *i*, by formula (VI.3), (VI.4)

$$\begin{aligned}
\frac{1}{p}\int_0^\pi e^{-ix} g_1(x)\,\mathrm{d}x &= \int_0^\pi e^{-ix}\cos 8x\,\mathrm{d}x = \frac{2i}{63} \\
\frac{1}{q}\int_0^\pi e^{-ix} g_2(x)\,\mathrm{d}x &= \int_0^\pi e^{-ix}\cos 4x\,\mathrm{d}x = \frac{2i}{15}
\end{aligned}$$

And by formula (VI.2),

$$\frac{1}{r}\int_0^\pi e^{-ix} f(x)\,\mathrm{d}x = \frac{\pi}{2}$$

The last term of condition *i* is

$$\begin{aligned}
\frac{1}{pr}\int_0^\pi g_1(x)g_2(x)e^{-ix}\,\mathrm{d}x &= \int_0^\pi e^{-ix}\cos m_1 x\cos m_2 x\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^\pi e^{-ix}\cos(m_1+m_2)\,\mathrm{d}x + \frac{1}{2}\int_0^\pi e^{-ix}\cos(m_1-m_2)\,\mathrm{d}x \\
&= \frac{1}{2}\cdot\frac{i[\cos(m_1+m_2)\pi+1]-(m_1+m_2)\sin(m_1+m_2)\pi}{(m_1+m_2)^2-1} \\
&\quad +\frac{1}{2}\cdot\frac{i[\cos(m_1-m_2)\pi+1]-(m_1-m_2)\sin(m_1-m_2)\pi}{(m_1-m_2)^2-1} \\
&= \frac{1}{2}\cdot\frac{2i}{143}+\frac{1}{2}\cdot\frac{2i}{15} = (\frac{1}{143}+\frac{1}{15})i
\end{aligned}$$

Therefore, let $p = 63$, $q = 15$ and $r = 2$, it is easily to verify that the condition *i* holds. Thus we have satisfied the two conditions and hence proven the lemma.

□

Lemma 8. *Let $x_1 = 2i$, $x_2 = \pi$, $x_3 = 69\frac{87}{143}i$ and m be a positive integer. Then for any positive integer k, there doesn't exist positive integer vectors $\{s_1^1, s_1^2, s_1^3\}$, $\{s_2^1, s_2^2, s_2^3\}$, $\cdots$, $\{s_k^1, s_k^2, s_k^3\}$ such that $x_1^{s_1^1}x_2^{s_1^2}x_3^{s_1^3} + x_1^{s_2^1}x_2^{s_2^2}x_3^{s_2^3} + \cdots + x_1^{s_k^1}x_2^{s_k^2}x_3^{s_k^3} = 0$ if $s_1^1 + s_1^2 + 2s_1^3 = s_2^1 + s_2^2 + 2s_2^3 = \cdots = s_k^1 + s_k^2 + 2s_k^3 = m$.*

Proof. We prove the lemma by induction on k. Consider $k = 1$, since m is a positive integer and $s_1^1 + s_1^2 + 2s_1^3 = m$, it is obvious that $x_1^{s_1^1}x_2^{s_1^2}x_3^{s_1^3}$ cannot be zero. Let

$$X_k = x_1^{s_1^1}x_2^{s_1^2}x_3^{s_1^3} + x_1^{s_2^1}x_2^{s_2^2}x_3^{s_2^3} + \cdots + x_1^{s_k^1}x_2^{s_k^2}x_3^{s_k^3}.$$

Assume that for $k \leq t$, the formula X_k cannot be zero. Consider $k = t+1$, suppose $X_{t+1} = 0$, we have two cases:

Case 1: $s_{t+1}^2 > 0$. Since $x_2 = \pi$ is an irrational number, there must exist s_i^2 where $i \in \{1, 2, \cdots, t\}$ such that $s_i^2 = s_{t+1}^2$. Otherwise, if all the s_i^2 are different from s_{t+1}^2, we have $X_t \neq -x_1^{s_{t+1}^1}x_2^{s_{t+1}^2}x_3^{s_{t+1}^3}$. It contradicts with $X_{t+1} = 0$. Therefore, we can select those terms with $s_j^2 \neq s_{t+1}^2$ and add them together. Since $X_{t+1} = 0$ and the number of terms with $s_j^2 \neq s_{t+1}^2$ is less than t, we have

$$\sum_{s_j^2 \neq s_{t+1}^2} x_1^{s_j^1}x_2^{s_j^2}x_3^{s_j^3} = 0$$

This violates our inductive assumption. If all the s_i^2 for $i \in \{1, 2, \cdots, t\}$ are equal to s_{t+1}^2, since $x_2^{s_{t+1}^2} \neq 0$, all the $x_2^{s_{t+1}^2}$ cancel. Then it is equivalent to case 2.

Case 2: $s_{t+1}^2 = 0$. If there exist some s_i^2 for $i \in \{1, 2, \cdots, t\}$ that $s_i^2 \neq 0$, then

$$\sum_{s_i^2 \neq 0} x_1^{s_i^1}x_2^{s_i^2}x_3^{s_i^3} = 0$$

This violates our inductive assumption. Otherwise, if all the s_i^2 for $i \in \{1, 2, \cdots, t\}$ equal to 0, the problem becomes $X_{t+1} = x_1^{s_1^1}x_3^{s_1^3} + x_1^{s_2^1}x_3^{s_2^3} + \cdots + x_1^{s_{t+1}^1}x_3^{s_{t+1}^3} = 0$ with $s_1^1 + 2s_1^3 = s_2^1 + 2s_2^3 = \cdots =$

$s_{t+1}^1 + 2s_{t+1}^3 = m$. Thus we have

$$\begin{aligned}
X_{t+1} &= (2i)^{s_1^1}(69\frac{87}{143}i)^{s_1^3} + (2i)^{s_2^1}(69\frac{87}{143}i)^{s_2^3} + \cdots + (2i)^{s_{t+1}^1}(69\frac{87}{143}i)^{s_{t+1}^3} \\
&= (2)^{s_1^1}(69\frac{87}{143})^{s_1^3} i^{s_1^1+s_1^3} + (2)^{s_2^1}(69\frac{87}{143})^{s_2^3} i^{s_2^1+s_2^3} + \cdots + (2)^{s_{t+1}^1}(69\frac{87}{143})^{s_{t+1}^3} i^{s_{t+1}^1+s_{t+1}^3} \\
&= (2)^{s_1^1}(69\frac{87}{143})^{s_1^3} i^{m-s_1^3} + (2)^{s_2^1}(69\frac{87}{143})^{s_2^3} i^{m-s_2^3} + \cdots + (2)^{s_{t+1}^1}(69\frac{87}{143})^{s_{t+1}^3} i^{m-s_{t+1}^3}
\end{aligned}$$

Since for any positive integers x and y, the formula $2^x = (69\frac{3\times 29}{11\times 13})^y$ doesn't hold. This illustrates that X_{t+1} cannot be 0 which contradicts with our supposition.

The analysis of both cases indicate that $X_{t+1} \neq 0$. Therefore we finish our induction.

□

Lemma 9. *There is a polynomial time algorithm h such that given a* (3,3)-SAT *instance $s(x_1,\cdots,x_n)$, it produces a $\prod\sum T_{\cos}$ trigonometric function $h(s(x_1,\cdots,x_n)) = g(y_1,\cdots,y_n)$ to satisfy the following two conditions:*

1. *if $s(x_1,\cdots,x_n)$ is satisfiable, then $L(g) = \int_{[0,\pi]^d} g(x_1,\cdots,x_n)e^{-i(x_1+\cdots+x_n)}\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n$ is not equal to zero; and*
2. *if $s(x_1,\cdots,x_n)$ is not satisfiable, then $L(g) = \int_{[0,\pi]^d} g(x_1,\cdots,x_n)e^{-i(x_1+\cdots+x_n)}\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n$ is zero.*

Proof. Let trigonometric function $g_1(y), g_2(y)$ and $f(y)$ be defined according to those in Lemma 7. For a (3,3)-SAT problem $s(x_1,\cdots,x_n)$, the function $g(x_1,\cdots,x_n)$ can be constructed as follows:

1. For the first positive literal x_i in $s(x_1,\cdots,x_n)$, replace it with $g_1(y_i)$.
2. For the second positive literal x_i in $s(x_1,\cdots,x_n)$, replace it with $g_2(y_i)$.
3. For the negative literal $\bar{x}_i$ in $s(x_1,\cdots,x_n)$, replace it with $f(y_i)$.

The formula $s(x_1,\cdots,x_n)$ has a sum of product form. It is satisfiable if and only if there exists one term which does not contain a positive and negative literals for the same variable. If a term contains both x_i and $\bar{x}_i$, the corresponding term in the sum of product for $p(\cdot)$ contains both $g_j(y_i)$ and $f(y_i)$ for some $j \in \{1,2\}$. This makes it zero after the effect of operator L by Lemma 2. Furthermore, if $s(x_1,\cdots,x_n)$ is satisfiable, the integration of g should be not identical to zero by Lemma 2 and Lemma 8. Therefore, $s(x_1,\cdots,x_n)$ is satisfiable if and only if $L(g) = \int_{[0,\pi]^d} g(x_1,\cdots,x_n)e^{-i(x_1+\cdots+x_n)}\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n$ is not zero. The computational time of h is clearly polynomial since we convert s to $h(s)$ by replacing each literal by a single variable function of degree 1.

□

Since the result of the operator L is a complex number, we need to redefine the $r(n)$-factor approximation $A(\cdot)$.

Let $N = \{0,1,2,\cdots\}$ be the set of all natural numbers. Let $N^+ = \{1,2,3,\cdots\}$ be the set of all positive natural numbers. Assume that function $r(n)$ is from N to N^+. For a function $F(\cdot)$, which converts a multivariate trigonometric function into a complex number $a_1 + b_1 i$, an algorithm $A(\cdot)$ gives an $r(n)$-*factor approximation* $A(f) = a_2 + b_2 i$ to $F(f)$ if it satisfies the following condition: $\frac{(|a_1|+|b_1|)}{r(n)} \leq |a_2| + |b_2| \leq r(n)\cdot(|a_1|+|b_1|)$, where n is the number of variables in f.

Theorem 6. *Let $a(n)$ be an arbitrary function from N to N^+. Then there is no polynomial time $a(n)$-factor approximation for computing $L(g)$ for a $\prod\sum T_{\cos}$ trigonometric function $g(x_1,\cdots,x_n)$ with the region $P = [0, pi]$ and* $\mathrm{n} = \{1,\cdots,1\}$ *unless* P = NP.

Proof. Assume that $A(\cdot)$ is a polynomial time $a(n)$-factor approximation for $L(g)$ with $\prod\sum T_{\cos}$ trigonometric function $g(x_1,\cdots,x_n)$ and $A(L(g)) = a_2 + b_2 i$.

For a (3,3)-SAT instance $s(x_1,\cdots,x_n)$, let $g(y_1,\cdots,y_n) = h(s(x_1,\cdots,x_n))$ according to Lemma 7. By lemma 9, a (3,3)-SAT instance $s(x_1,\cdots,x_n)$ is satisfiable if and only if $L(g) = \int_{[0,\pi]^d} g(x_1,\cdots,x_n)e^{-i(x_1+\cdots+x_n)}\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n = a_1 + b_1 i$ is not zero. Assume that s is not satisfiable, then we have $|a_2|+|b_2| \in [\frac{(|a_1|+|b_1|)}{r(n)}, r(n)\cdot(|a_1|+|b_1|)] = [0,0]$, which implies $A(J) = 0$. On the other hand, if s is satisfiable, then we have $|a_2|+|b_2| \in [\frac{(|a_1|+|b_1|)}{r(n)}, r(n)\cdot(|a_1|+|b_1|)] = (0,+\infty)$, which implies $A(J) \neq 0$. Thus, $s(x_1,\cdots,x_n)$ is satisfiable if and only if $A(J) \neq 0$.

Therefore, there is a polynomial time algorithm for solving (3,3)-SAT, which is NP-complete by Lemma 1. So, P = NP.

□

Theorem 7. *Let $a(n)$ be an arbitrary function from N to N^+. Then there is no subexponential time $a(n)$-factor approximation for computing $L(g)$ for a $\prod\sum T_{\cos}$ trigonometric function $g(x_1,\cdots,x_n)$ with the region $P=[0,pi]$ and* $\mathrm{n}=\{1,\cdots,1\}$ *unless* NP ⊆ subP.

Proof. Assume that $A(\cdot)$ is a subexponential time $a(n)$-factor approximation for $L(g)$ with $\prod\sum T_{\cos}$ trigonometric function $g(x_1,\cdots,x_n)$ and $A(L(g)) = a_2 + b_2 i$.

For a (3,3)-SAT instance $s(x_1,\cdots,x_n)$, let $g(y_1,\cdots,y_n) = h(s(x_1,\cdots,x_n))$ according to Lemma 7. By lemma 9, a (3,3)-SAT instance $s(x_1,\cdots,x_n)$ is satisfiable if and only if $L(g) = \int_{[0,\pi]^d} g(x_1,\cdots,x_n)e^{-i(x_1+\cdots+x_n)}\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\,\mathrm{d}x_n = a_1 + b_1 i$ is not zero. Assume that s is not satisfiable, then we have $|a_2|+|b_2| \in [\frac{(|a_1|+|b_1|)}{r(n)}, r(n)\cdot(|a_1|+|b_1|)] = [0,0]$, which implies $A(J)=0$. On the other hand, if s is satisfiable, then we have $|a_2|+|b_2| \in [\frac{(|a_1|+|b_1|)}{r(n)}, r(n)\cdot(|a_1|+|b_1|)] = (0,+\infty)$, which implies $A(J)\neq 0$. Thus, $s(x_1,\cdots,x_n)$ is satisfiable if and only if $A(J)\neq 0$.

Therefore, there is a subexponential time algorithm for solving (3,3)-SAT, which is NP-complete by Lemma 1. So, NP ⊆ subP.

□

CHAPTER VII

CONCLUSION

By means of the theory of NP-hardness and using the similar approach in our first step, we further show that the intractability of approximation for two fundamental mathematical operations: integration and derivative in high dimensional space. When dimension is high, integration and derivative are computational hard to approximate for some linear or nonlinear function. The derivative is not easier than integration in high dimension. We see in chapter VI that the operator L is close to the Fourier transform. We still cannot figure whether this approach is suitable for interpret the complexity of the Fourier transform. We leave it as future work.

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BIOGRAPHICAL SKETCH

Liang Ding received the Bachelor of Science degree in Mathematics from the Zhengzhou University of China in 2007 and he accomplished two years graduate study in Mathematics before moving to Texas, USA to pursue Master degree of Computer Science. He started his graduate studies in January, 2009. After two years studies, he decided to take up advanced studies for Ph.D. degree. His address is 1809 West Schunior Street, Apartment 208, Edinburg, Texas. In the coming fall, he will move to Athens, GA to start a new life.

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