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Zero Inflated Exponential Distribution and It's Variants

Sougata Dhar
University of Texas-Pan American

ZERO INFLATED EXPONENTIAL DISTRIBUTION AND IT'S VARIANTS

A Thesis

by

SOUGATA DHAR

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SOUGATA DHAR

COMMITTEE MEMBERS

Dr. Santanu Chakraborty
Chair of Committee

Dr. Paul Bracken
Committee Member

Dr. Arunava Mukherjea
Committee Member

Dr. George Yanev
Committee Member

August 2011

ABSTRACT

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There have been a lot of studies with respect to the popular Zero Inflated versions of some discrete distributions, like Zero Inflated Poisson and Zero Inflated Negative Binomial distributions. They arise naturally in the literature when one aims to model count data sets having more than usual number of zeros by well known probability distributions. But it can be argued and established that Zero Inflated versions of continuous distributions also make sense. In this thesis, we first provide some motivations for studying Zero Inflated versions of exponential distribution and its variants and then study some parametric and Bayesian aspects related to the Zero Inflated Exponential Distribution.

DEDICATION

The completion of my masters studies would not have been possible without the love and support of my family and friends. My mother Mrs. Kanika Dhar and my father Mr. Santi Ranjan Dhar wholeheartedly motivated and supported me throughout the period of my study. Without their kindness and inspiration, it was not possible for me to accomplish the degree.

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CHAPTER I

INTRODUCTION

The real life count data sets are expected to be modeled by well known discrete probability distributions like a Poisson or a Negative Binomial distribution. However, it is sometimes found that the number of zeros in such a dataset is more than what is allowed by a Poisson model.So in order to incorporate this feature, a zero inflated Poisson distribution was introduced in the literature. See for example, Cohen (1960), Singh (1963), Goraski (1977), Kemp (1986), Heilborn (1989), Lambert (1992), Al-Saleh and Al-Batainah (2003), one can come across such attempts. There had been studies on Bayesian analysis of zero-inflated Poisson regression models in the works of Ghosh et al (2006). One can, in fact, find Zero Inflated version of generalized Poisson distribution and corresponding Bayesian study in the works of Angers and Biswas (2003). For this, one needs to understand the generalized Poisson distribution that was introduces in the works of Consul and Jain (1973).

But so far there had been no relevant study regarding zero inflated versions of the well known continuous probability distributions, say, exponential or gamma or normal distributions. In fact, there could be real life situations where one can come across zero inflated version of an exponential or a normal distribution. For example, the life span of an electric bulb is generally assumed to be exponential. But if a cheap brand of bulb exists in the market, it may have a shorter life span as compared to an electric bulb of a standard brand. In that case, the probability at zero or on an interval around zero may be more than that for a usual exponential distribution. Similarly, one can think of a real life example where a zero inflated version of a normal distribution could be meaningful. Generally, the errors made by a measuring instrument are assumed to be normal with a mean of zero. But for a highly accurate measuring instrument, the probability at zero or on an interval around it could be higher than its corresponding normal counterpart. From all these considerations, it is worth looking at the Zero Inflated versions of these continuous probability distributions.

In this thesis, we look at the zero inflated version of the exponential distribution.The usual exponential distribution with parameter λ has the following distribution:

$$\begin{aligned} f(x) &= \lambda e^{-\lambda x} \quad \text{for } x \geq 0 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Now, for situations where one anticipates that the probability around zero is going to be more than that for the usual exponential and as a result, it is less in the other part of the positive real half compared to the usual exponential, it is imperative to inflate the probability distribution around zero and deflate it in the other part. If we decide to inflate a usual exponential with parameter λ in a small interval $(0, \varepsilon)$ where ε is a very small positive number close to zero and deflate it in (ε, ∞), then the probability density function looks like:

$$\begin{aligned} f(x) &= \frac{\phi}{\varepsilon} + (1-\phi)\lambda e^{-\lambda x} \quad \text{for } x \in (0, \varepsilon) \\ &= (1-\phi)\lambda e^{-\lambda x} \quad \text{for } x \in (\varepsilon, \infty) \\ &= 0 \quad \text{otherwise} \end{aligned}$$

As it appears from the above formula for the zero inflated exponential distribution is a mixture of a uniform $(0, \varepsilon)$ and exponential (λ) with weights ϕ and $(1-\phi)$ respectively. There have been a lot of studies in the theory of mixture distributions in the last 50 years. Quite a few of them deal with mixture of uniform distributions only, say E. Janvresse and T De La Rue (2004), A. Gning, L. Mihaylova and F abdallah (2010) or mixture of exponential distributions only, say, Paul R Rider (1961), Nicholas P Jwell (1982).But to the best of our knoledge, this is the very first attempt with mixture of uniform and exponential distributions. We do not go beyond the set up described here for such a mixture. But it could be of interest to future researchers for more generalized studies on such mixtures.

CHAPTER II

DEFINITION & MOMENTS

Exponential Distribution

In this thesis, we will be looking at Zero Inflated version of the exponential distribution and its variants. Apart from the example of lifetime of an electric bulb, there are other instances also where one comes across exponential distribution. For example, the waiting time to get service in a queue, the waiting time for a child to be born in a family etc. are common examples of exponential distributions. One can easily give meanings to the Zero Inflated versions of these distributions. For exponential distribution with parameter λ, $E(X)$, $V(X)$, $M_X(t)$ and $\Phi_X(t)$ are as follows:

$$E(X) = \frac{1}{\lambda}$$

$$V(X) = \frac{1}{\lambda^2}$$

$$M_X(t) = \frac{\lambda}{\lambda - t}$$

$$\Phi_X(t) = \frac{\lambda}{\lambda - it}$$

Zero Inflated Exponential Distribution

As mentioned in the introduction, the zero inflated exponential distribution is defined as follows

$$\begin{aligned} f(x) &= \frac{\phi}{\varepsilon} + (1-\phi)\lambda e^{-\lambda x} \quad \text{for } x \in (0,\varepsilon) \\ &= (1-\phi)\lambda e^{-\lambda x} \quad \text{for } x \in (\varepsilon,\infty) \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Using indicator function, this can be rewritten as follows

$$f(x) = \frac{\phi}{\varepsilon} I(0 < x < \varepsilon) + (1-\phi)\lambda e^{-\lambda x} \quad \text{for } x > 0$$
$$= 0 \quad \text{otherwise}$$

We will call this distribution as Zero Inflated Exponetial Distribution (ZIED). To emphasize its dependence on ε, ϕ and λ, we will denote it as ZIED(ε,ϕ,λ). In this thesis, for any study related to ZIED, we will first consider $\varepsilon = 1$ and then generalize to the case for any value of ε.

For $\varepsilon = 1$, the probability density function takes the following form:

$$f(x) = \phi + (1-\phi)\lambda e^{-\lambda x} \quad \text{for } 0 < x < 1$$
$$= (1-\phi)\lambda e^{-\lambda x} \quad \text{for } x > 1$$
$$= 0 \quad \text{otherwise}$$

where $0 < \phi < 1$ is a constant, and $\lambda > 0$. As before, using indicatior function, this can be rewritten as :

$$f(x) = \phi I(0 < x < 1) + (1-\phi)\lambda e^{-\lambda x} \quad \text{for } x > 0$$

The $E(X)$,$V(X)$,$M_X(t)$ and $\Phi_X(t)$ for this distribution function are as follows:

$$E(X) = \frac{\phi}{2} + \frac{1-\phi}{\lambda}$$

$$V(X) = \frac{1-\phi}{\lambda}\left\{\frac{1+\phi}{\lambda} - \phi\right\} + \frac{\phi}{12}(4-3\phi)$$

$$M_X(t) = \frac{\phi(e^t-1)}{t} + \frac{\lambda(1-\phi)}{\lambda - t}$$

$$\Phi_X(t) = \frac{\phi\left(e^{it}-1\right)}{it} + \frac{\lambda(1-\phi)}{\lambda - it}$$

For a general ZIED(ε,ϕ,λ), $E(X)$, $V(X)$, $M_X(t)$ and $\Phi_X(t)$ for this generalized version are:

$$E(X) = \frac{\phi\varepsilon}{2} + \frac{1-\phi}{\lambda}$$
$$V(X) = \frac{\varepsilon^2\phi}{12}(4-3\phi) + \frac{1-\phi^2}{\lambda^2} - \frac{\varepsilon\phi(1-\phi)}{\lambda}$$
$$M_X(t) = \frac{\phi}{\varepsilon t}(e^{\varepsilon t}-1) + \frac{\lambda(1-\phi)}{\lambda - t}$$
$$\Phi_X(t) = \frac{\phi}{i\varepsilon t}(e^{i\varepsilon t}-1) + \frac{\lambda(1-\phi)}{\lambda - it}$$

CHAPTER III

DISCUSSION ON n $i.i.d$ OBSERVATIONS

Joint Density of n i.i.d Observations

For n i.i.d. observations $X_1, \cdots, X_n$ from ZIED$(\varepsilon, \phi, \lambda)$ with $\varepsilon = 1$, the joint density is

$$f(x_1, \cdots, x_n|\lambda, \phi) = \prod_{i=0}^{n}[\phi I(0 < x_i < 1) + \lambda(1-\phi)e^{-\lambda x_i}]$$

$$= \sum_{l=0}^{n} \sum_{i_1, i_2, \cdots, i_l} \phi^{n-l}(1-\phi)^l \lambda^l e^{-\lambda \sum_{j=1}^{l} x_{ij}} I(0 < x_j < 1,\ j \neq i_1, i_2, \cdots, i_l)$$

For n i.i.d. observations $X_1, \cdots, X_n$ from ZIED$(\varepsilon, \phi, \lambda)$ for a general ε, the joint density is

$$f(x_1, \cdots, x_n|\lambda, \phi) = \prod_{i=0}^{n}[\frac{\phi}{\varepsilon} I(0 < x_i < \varepsilon) + \lambda(1-\phi)e^{-\lambda x_i}]$$

$$= \sum_{l=0}^{n} \sum_{i_1, i_2, \cdots, i_l} \left(\frac{\phi}{\varepsilon}\right)^{n-l} (1-\phi)^l \lambda^l e^{-\lambda \sum_{j=1}^{l} x_{ij}} I(0 < x_j < \varepsilon,\ j \neq i_1, i_2, \cdots, i_l)$$

Method of Moment Estimators

If $X_1, X_2, ...X_n$is a $i.i.d$ random sample from a ZIED with parameters (ϕ, λ) where both the parameters are unknown,then using the method of moments one can estimate ϕ and λ. The results are,

$$\lambda = \frac{(1-3m_2) \pm \sqrt{\left\{m_1(6+2\sqrt{3}) - 3m_2 + (2-\sqrt{3})\right\}\left\{m_1(6-2\sqrt{3}) - 3m_2 + (2+\sqrt{3})\right\}}}{2m_1 - 3m_2}$$

$$\phi = (6m_1 - 3m_2 - 1) \pm \sqrt{(6m_1 - 3m_2 - 1)^2 - 6\left(2m_1^2 - m_2\right)}$$

where $m_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

Unbiased Estimators

Let $X_1, X_2, ...X_\mathrm{n}$ be a *i.i.d* random sample from a ZIED with parameters (ϕ, λ). Then the unbiased estimators for $\frac{1}{\lambda}$ is $\frac{2X_i-\phi}{2-2\phi}$

and the unbiased estimator for ϕ is $\frac{2-2\lambda X_i}{2-\lambda}$ for $i = 1, 2, ..., n$.

PDF's Maximum & Minimum Order Statistics

Usually The k^{th} order statistics of any probability distribution is given by the following formulla

$$\frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)}[F(x_k)]^{k-1}[1-F(x_k)]^{n-k}f(x_k)$$

where $F(x_k)$ is the cumulative distribution function (c.d.f) of the k^{th} varriable and $f(x_k)$is the probability density function (p.d.f) of the k^{th} varriable. Now in order to get the density of the maximum order statictics we replace k by n. So the density of the maximum order statistics,denoted by $\xi(x_n)$ and is given by

$$\xi(x_n) = n[F(x_n)]^{n-1}f(x_n)$$

where

$$f(x_n) = \phi + \lambda(1-\phi)e^{-\lambda x_n} \quad 0 < x_n < 1$$
$$= \lambda(1-\phi)e^{-\lambda x_n} \quad x_n > 1$$

and

$$F(x_n) = \phi x_n + (1-\phi)(1-e^{-\lambda x_n}) \quad 0 < x_n < 1$$
$$= \phi + (1-\phi)(1-e^{-\lambda x}) \quad x > 1$$

Now putting the value of $F(x_n)$ and $f(x_n)$ we get the density of the maximum order statistics as

$$\xi(x_n) = n\{\phi x_n + (1-\phi)(1-e^{-\lambda x_n})\}^{n-1}\{\phi + \lambda(1-\phi)e^{-\lambda x_n}\} \quad 0 < x_n < 1$$

$$= n\lambda(1-\phi)e^{-\lambda x_n}\{\phi + (1-\phi)(1-e^{-\lambda x_n})\}^{n-1} \quad x_n > 1$$

Similarly putting $k = 1$ in the k^{th} order statistics we can find the density function of the minimum order statistics, denoted as $\eta(x_1)$ and is given by

$$\eta(x_1) = n[1-F(x_1)]^{n-1}f(x_1)$$

Again by putting the values of $F(x_n)$ and $f(x_n)$ we get the density of the minimum order statistics as

$$\eta(x_1) = n\{1-\phi x_1 - (1-\phi)(1-e^{-\lambda x_1})\}^{n-1}\{\phi + \lambda(1-\phi)e^{-\lambda x_1}\} \quad 0 < x_1 < 1$$

$$= n\lambda(1-\phi)^n e^{-n\lambda x_1} \quad x_1 > 1$$

CHAPTER IV

CONVOLUTION

Motivation

Our main aim in this thesis is to do some parametric inferential studies for the ZIED. For such a study, we start with independent identically distributed (i.i.d.) samples from Zero Inflated Exponetial distribution. For parametric inferential purposes, it is meaningful to obtain the sampling distributions of the sample mean and the sample variance constructed out of the i.i.d. sample. In other words, one needs to know the sampling distribution of the convolution of these i.i.d. observations. Let $X_1, X_2, \cdots, X_n$ be an i.i.d. sample of size n from a ZIED distribution. For simplicity, we first assume $\varepsilon = 1$ and obtain the distribution of $\sum_{i=1}^{n} X_i$ and then generalize it. For the obvious complicacy in deriving the expressions, we start with the case $n = 2$ and then generalize. From now on, we will always suppress the case of negative values while writing down the densities for the observations or their covolutions.

Convolution for two i.i.d ZIED random variables

Case 1 (ε=1)

First we consider $\varepsilon = 1$ and take only two i.i.d. observations from ZIED - X_1 and X_2. Then,

$$f(x_1) = \phi I(0 < x_1 < 1) + \lambda(1-\phi)e^{-\lambda x_1}$$

$$f(x_2) = \phi I(0 < x_2 < 1) + \lambda(1-\phi)e^{-\lambda x_2}$$

Our aim is to get the distribution of $X_1 + X_2$. We make the usual transformation: $(X_1, X_2) \longrightarrow (Y_1, Y_2)$ where $Y_1 = X_1$ and $Y_2 = X_1 + X_2$ to get the joint density of (Y_1, Y_2) as follows

$$g(y_1,y_2)=\phi^2 I(0<y_1,y_2-y_1<1)+\lambda^2(1-\phi)^2 e^{-\lambda y_2}+$$

$$\lambda\phi(1-\phi)\left[e^{-\lambda y_1}I(0<y_2-y_1<1)+e^{-\lambda(y_2-y_1)}I(0<y_1<1)\right]$$

Then, integrating out y_1, we get the density of Y_2 which is nothing but the density of X_1+X_2

$$h(y_2)=\phi^2 y_2+\lambda^2(1-\phi)^2 y_2 e^{-\lambda y_2}+2\phi(1-\phi)(1-e^{-\lambda y_2})\quad 0<y_2<1$$

$$=(2-y_2)\phi^2+\lambda^2(1-\phi)^2 y_2 e^{-\lambda y_2}+2\phi(1-\phi)(e^{\lambda}-1)e^{-\lambda y_2}\quad 1<y_2<2$$

$$=\lambda^2(1-\phi)^2 y_2 e^{-\lambda y_2}+2\phi(1-\phi)(e^{\lambda}-1)e^{-\lambda y_2}\qquad y_2>2$$

Case 2 (for a general ε)

Now considering a general ε, one can proceed similarly to obtain the density of the convolution of two i.i.d. observations as follows:

$$h(y_2)=\frac{\phi^2}{\varepsilon^2}y_2+\lambda^2(1-\phi)^2 y_2 e^{-\lambda y_2}+\frac{2\phi(1-\phi)}{\varepsilon}(1-e^{-\lambda y_2})\quad 0<y_2<\varepsilon$$

$$=\frac{\phi^2}{\varepsilon^2}(2\varepsilon-y_2)+\lambda^2(1-\phi)^2 y_2 e^{-\lambda y_2}+\frac{2\phi(1-\phi)}{\varepsilon}(e^{\lambda\varepsilon}-1)e^{-\lambda y_2}\quad \varepsilon<y_2<2\varepsilon$$

$$=\lambda^2(1-\phi)^2 y_2 e^{-\lambda y_2}+\frac{2\phi(1-\phi)}{\varepsilon}(e^{\lambda\varepsilon}-1)e^{-\lambda y_2}\quad y_2>2\varepsilon$$

Convolution for three i.i.d ZIED random variables

Case 1 (ε=1)

For doing it in the case of $n=3$, we consider three observations X_1, X_2 and X_3. One can now start with the joint density of $Y_2=X_1+X_2$ (obtained in the previous subsection) and X_3 which are independent and use a transformation to get thethe joint density of Y_2 and $Y_3=X_1+X_2+X_3$ and then integrate out Y_2 to get the marginal of Y_3. In the case $\varepsilon=1$, the density of Y_3 is as follows:

$$h(y_3)=\phi^3\Psi_{00}^{(3)}(y_3,\lambda)+3\phi^2(1-\phi)\Psi_{01}^{(3)}(y_3,\lambda)+3\phi(1-\phi)^2\Psi_{02}^{(3)}(y_3,\lambda)+(1-\phi)^3\Psi_{03}^{(3)}(y_3,\lambda),\quad 0<y_3<1$$

$$=\phi^3\Psi_{10}^{(3)}(y_3,\lambda)+3\phi^2(1-\phi)\Psi_{11}^{(3)}(y_3,\lambda)+3\phi(1-\phi)^2\Psi_{12}^{(3)}(y_3,\lambda)+(1-\phi)^3\Psi_{13}^{(3)}(y_3,\lambda),\quad 1<y_3<2$$

$$= \phi^3\Psi^{(3)}_{20}(y_3,\lambda)+3\phi^2(1-\phi)\Psi^{(3)}_{21}(y_3,\lambda)+3\phi(1-\phi)^2\Psi^{(3)}_{22}(y_3,\lambda)+(1-\phi)^3\Psi^{(3)}_{23}(y_3,\lambda), \quad 2<y_3<3$$

$$= \phi^3\Psi^{(3)}_{30}(y_3,\lambda)+3\phi^2(1-\phi)\Psi^{(3)}_{31}(y_3,\lambda)+3\phi(1-\phi)^2\Psi^{(3)}_{32}(y_3,\lambda)+(1-\phi)^3\Psi^{(3)}_{33}(y_3,\lambda), \quad y_3>3$$

where

$$\Psi^{(3)}_{00}(y_3,\lambda)=\frac{y_3^2}{2},\ \Psi^{(3)}_{10}(y_3,\lambda)=\frac{-2y_3^2+6y_3-3}{2},\ \Psi^{(3)}_{20}(y_3,\lambda)=\frac{(3-y_3)^2}{2},\ \Psi^{(3)}_{30}(y_3,\lambda)=0$$

$$\Psi^{(3)}_{01}(y_3,\lambda)=y_3-\frac{1}{\lambda}(1-e^{-\lambda y_3}),\ \Psi^{(3)}_{11}(y_3,\lambda)=(2-y_3)+\frac{1}{\lambda}(1+e^{-\lambda y_3}-2e^{-\lambda y_3+\lambda}),$$

$$\Psi^{(3)}_{21}(y_3,\lambda)=\Psi^{(3)}_{31}(y_3,\lambda)=\frac{1}{\lambda}e^{-\lambda y_3}(e^{\lambda}-1)^2$$

$$\Psi^{(3)}_{02}(y_3,\lambda)=1-e^{-\lambda y_3}(\lambda y_3+1),\ \Psi^{(3)}_{12}(y_3,\lambda)=\Psi^{(3)}_{22}(y_3,\lambda)=\Psi^{(3)}_{32}(y_3,\lambda)=e^{-\lambda y_3}[(\lambda y_3+1)(e^{\lambda}-1)-\lambda e^{\lambda}]$$

$$\Psi^{(3)}_{03}(y_3,\lambda)=\Psi^{(3)}_{13}(y_3,\lambda)=\Psi^{(3)}_{23}(y_3,\lambda)=\Psi^{(3)}_{33}(y_3,\lambda)=\frac{\lambda^3y_3^2e^{-\lambda y_3}}{2}$$

Case 2 (for a general ε)

For a general ε, the density of the convolution of three i.i.d. observations is as follows

$$h(y_3)=\left(\frac{\phi}{\varepsilon}\right)^3\Psi^{(3)}_{i0}(y_3,\lambda)+3\left(\frac{\phi}{\varepsilon}\right)^2(1-\phi)\Psi^{(3)}_{i1}(y_3,\lambda)+3\left(\frac{\phi}{\varepsilon}\right)\phi(1-\phi)^2\Psi^{(3)}_{i2}(y_3,\lambda)+(1-\phi)^3\Psi^{(3)}_{i3}(y_3,\lambda)$$

for $i<y_3<i+1$ with $i=0,1$ and 2 and

$$h(y_3)=\left(\frac{\phi}{\varepsilon}\right)^3\Psi^{(3)}_{30}(y_3,\lambda)+3\left(\frac{\phi}{\varepsilon}\right)^2(1-\phi)\Psi^{(3)}_{31}(y_3,\lambda)+3\left(\frac{\phi}{\varepsilon}\right)(1-\phi)^2\Psi^{(3)}_{32}(y_3,\lambda)+(1-\phi)^3\Psi^{(3)}_{33}(y_3,\lambda)$$

for $y_3>3$. The Ψs now involve ε s but for the sake of notational simplicity, we will supress the ε s. As before, we obtain the expressions for the Ψ s which are as follows

$$\Psi^{(3)}_{00}(y_3,\lambda)=\frac{y_3^2}{2},\ \Psi^{(3)}_{10}(y_3,\lambda)=\frac{-2y_3^2+6\varepsilon y_3-3\varepsilon^2}{2},\ \Psi^{(3)}_{20}(y_3,\lambda)=\frac{(3\varepsilon-y_3)^2}{2},\ \Psi^{(3)}_{30}(y_3,\lambda)=0$$

$$\Psi^{(3)}_{01}(y_3,\lambda)=y_3-\frac{1}{\lambda}(1-e^{-\lambda y_3}),\ \Psi^{(3)}_{11}(y_3,\lambda)=(2\varepsilon-y_3)+\frac{1}{\lambda}(1+e^{-\lambda y_3}-2e^{-\lambda y_3+\lambda}),$$

$$\Psi_{21}^{(3)}(y_3,\lambda)=\Psi_{31}^{(3)}(y_3,\lambda)=\frac{1}{\lambda}e^{-\lambda y_3}(e^{\lambda\varepsilon}-1)^2$$

$$\Psi_{02}^{(3)}(y_3,\lambda)=1-e^{-\lambda y_3}(\lambda y_3+1),\ \Psi_{12}^{(3)}(y_3,\lambda)=\Psi_{22}^{(3)}(y_3,\lambda)=\Psi_{32}^{(3)}(y_3,\lambda)=e^{-\lambda y_3}[(\lambda y_3+1)(e^{\lambda\varepsilon}-1)-\lambda\varepsilon e^{\lambda\varepsilon}]$$

$$\Psi_{03}^{(3)}(y_3,\lambda)=\Psi_{13}^{(3)}(y_3,\lambda)=\Psi_{23}^{(3)}(y_3,\lambda)=\Psi_{33}^{(3)}(y_3,\lambda)=\frac{\lambda^3 y_3^2 e^{-\lambda y_3}}{2}$$

Convolution for n i.i.d ZIED random variables

Case 1 (ε=1)

For the general case with $n>3$ observations $X_1,X_2,\cdots,X_n$, we follow the same technique. By virtue of induction, we have the marginals of $Y_1,\cdots,Y_n$ (say) which we can use to get the joint density of Y_n and $X_{n+1}$and then using a transformation, we obtain the joint density of Y_n and $Y_{n+1}=X_1+X_2+\cdots+X_{n+1}$. From there, we integrate out Y_n to get the marginal distribution of Y_{n+1} in the case $\varepsilon=1$. So, we first write down the joint distribution of Y_n and Y_{n+1}

$$g(y_n,y_{n+1})=\phi^{n+1}\Psi_{ik}^{(n)}(y_n,\lambda)+\sum_{k=1}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\phi^k(1-\phi)^{n-k+1}[\Psi_{ik-1}^{(n)}(y_n,\lambda)+\Psi_{ik}^{(n)}(y_n,\lambda)\Psi_{00}^{(1)}(y_{n+1}-y_n,\lambda)]$$

$$+(1-\phi)^{n+1}\Psi_{i0}^{(n)}(y_n,\lambda)\Psi_{00}^{(1)}(y_{n+1}-y_n,\lambda)$$

$$\text{for } i<y_n<i+1,\ \ 0<y_{n+1}-y_n<1 \quad i=0,1,2,...n$$

$$g(y_n,y_{n+1})=\sum_{k=1}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\phi^k(1-\phi)^{n-k+1}[\Psi_{nk-1}^{(n)}(y_n,\lambda)+\Psi_{nk}^{(n)}(y_n,\lambda)\Psi_{00}^{(1)}(y_{n+1}-y_n,\lambda)]$$

$$+(1-\phi)^{n+1}\Psi_{n0}^{(n)}(y_n,\lambda)\Psi_{00}^{(1)}(y_{n+1}-y_n,\lambda)$$

$$\text{for } y_n>n,\ \ 0<y_{n+1}-y_n<1$$

$$g(y_n,y_{n+1})=\sum_{k=1}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\phi^k(1-\phi)^{n-k+1}\Psi_{ik}^{(n)}(y_n,\lambda)\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda)$$

$$+(1-\phi)^{n+1}\Psi_{i0}^{(n)}(y_n,\lambda)\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda)$$

$$\text{for } i<y_n<i+1,\ \ y_{n+1}-y_n>1 \quad i=0,1,...n$$

$$g(y_n, y_{n+1}) = \sum_{k=1}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \phi^k (1-\phi)^{n-k+1} \Psi_{nk}^{(n)}(y_n, \lambda) \Psi_{10}^{(1)}(y_{n+1} - y_n, \lambda)$$

$$+ (1-\phi)^{n+1} \Psi_{n0}^{(n)}(y_n, \lambda) \Psi_{10}^{(1)}(y_{n+1} - y_n, \lambda)$$

$$\text{for } y_n > n, \;\; y_{n+1} - y_n > 1$$

As mentioned earlier, the density of y_{n+1} can now be obtained by integrating out y_n and is denoted by $h(y_{n+1})$:

$$h(y_{n+1}) = \sum_{k=0}^{n+1} \begin{pmatrix} n+1 \\ k \end{pmatrix} \phi^k (1-\phi)^{n-k+1} \Psi_{ik}^{(n+1)}(y_{n+1}, \lambda) \;\; i < y_{n+1} < i+1 \;\; i = 0, 1, ...n+1$$

$$= \sum_{k=0}^{n} \begin{pmatrix} n+1 \\ k \end{pmatrix} \phi^k (1-\phi)^{n-k+1} \Psi_{n+1\,k}^{(n)}(y_{n+1}, \lambda) \quad\; y_n > n+1$$

Definition of Ψ's

For $i = 0, \; k = n+1$

$$\Psi_{0n+1}^{(n+1)}(y_{n+1}, \lambda) = \int_0^{y_{n+1}} \Psi_{0n}^{(n)}(y_n, \lambda) dy_n$$

For $i = 0, \; k = 1, 2, \cdots, n$

$$\begin{pmatrix} n+1 \\ k \end{pmatrix} \Psi_{0k}^{(n+1)}(y_{n+1}, \lambda) = \begin{pmatrix} n \\ k \end{pmatrix} \int_0^{y_{n+1}} [\Psi_{0k-1}^{(n)}(y_n, \lambda) + \Psi_{0k}^{(n)}(y_n, \lambda) \Psi_{00}^{(1)}(y_{n+1} - y_n, \lambda)] dy_n$$

For $i = 0, \; k = 0$

$$\Psi_{00}^{(n+1)}(y_{n+1}, \lambda) = \int_0^{y_{n+1}} \Psi_{00}^{(n)}(y_n, \lambda) \Psi_{00}^{(1)}(y_{n+1} - y_n, \lambda) dy_n$$

For $i = 1, 2, \cdots, n, \; k = n+1$

$$\Psi_{in+1}^{(n+1)}(y_{n+1}, \lambda) = \int_{y_{n+1}-1}^{i} \Psi_{i-1\,n}^{(n)}(y_n, \lambda) dy_n \; + \int_i^{y_{n+1}} \Psi_{in}^{(n)}(y_n, \lambda) dy_n$$

For $i=1,2,\cdots,n,\ k=1,2,\cdots,n$

$$\Psi_{ik}^{(n+1)}(y_{n+1},\lambda)=\frac{\begin{pmatrix} n \\ k \end{pmatrix}}{\begin{pmatrix} n+1 \\ k \end{pmatrix}}G(\Psi_{jl}^{(n)}(y_n,\lambda),\,\Psi_{00}^{(0)}(y_{n+1}-y_n,\lambda),\,\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda);0\leq j\leq i,\,l=k-1,\,k)$$

where G equals

$$\sum_{j=0}^{i-2}\int_j^{j+1}\Psi_{jk}^{(n)}(y_n,\lambda)\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda)dy_n+\int_{i-1}^{y_{n+1}-1}\Psi_{i-1\,k}^{(n)}(y_n,\lambda)\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda)dy_n$$

$$+\int_{y_{n+1}-1}^{i}[\Psi_{i-1\,k-1}^{(n)}(y_n,\lambda)+\Psi_{i-1\,k}^{(n)}(y_n,\lambda)\Psi_{00}^{(1)}(y_{n+1}-y_n,\lambda)]dy_n$$

$$+\int_i^{y_{n+1}}[\Psi(y_n,\lambda)+\Psi_{ik}^{(n)}(y_n,\lambda)\Psi_{00}^{(1)}(y_{n+1}-y_n,\lambda)]dy_n]$$

Obviously, for $i=1,\ k=1,2,\cdots,n$

$$\sum\int_j^{j+1}\Psi_{jk}^{(n)}(y_n,\lambda)\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda)dy_n=0$$

For $i=1,2,\cdots,n,\ k=0$

$$\Psi_{i0}^{(n+1)}(y_{n+1},\lambda)=\sum_{j=0}^{i-2}\int_j^{j+1}\Psi_{j0}^{(n)}(y_n,\lambda)\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda)dy_n+\int_{i-1}^{y_{n+1}-1}\Psi_{i-1\,0}^{(n)}(y_n,\lambda)\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda)dy_n$$

$$+\int_{y_{n+1}-1}^{i}\Psi_{i-1\,0}^{(n)}(y_n,\lambda)\Psi_{00}^{(1)}(y_{n+1}-y_n,\lambda)dy_n+\int_i^{y_{n+1}}\Psi_{i0}^{(n)}(y_n,\lambda)\Psi_{00}^{(1)}(y_{n+1}-y_n,\lambda)dy_n$$

Once again, for $i=1,\ k=0$

$$\sum_{j=0}^{i-2}\int_j^{j+1}\Psi_{j0}^{(n)}(y_n,\lambda)\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda)dy_n=0$$

For $i = n+1,\ k = n+1$

$$\Psi^{(n+1)}_{n+1\,n+1}(y_{n+1},\lambda) = 0$$

For $i = n+1,\ k = 1,2,.........n$

$$\Psi^{(n+1)}_{n+1}(y_{n+1},\lambda) = \frac{\begin{pmatrix} n \\ k \end{pmatrix}}{\begin{pmatrix} n+1 \\ k \end{pmatrix}} H(\Psi^{(n)}_{jl}(y_n,\lambda),\, \Psi^{(0)}_{00}(y_{n+1}-y_n,\lambda),\, \Psi^{(1)}_{10}(y_{n+1}-y_n,\lambda); 0 \leq j \leq n,\, l = k-1,\, k)$$

where H equals

$$\sum_{j=0}^{n-1} \int_{j}^{j+1} \Psi^{(n)}_{jk}(y_n,\lambda)\Psi^{(1)}_{10}(y_{n+1}-y_n,\lambda)dy_n + \int_{n}^{y_{n+1}-1} \Psi^{(n)}_{nk}(y_n,\lambda)\Psi^{(1)}_{10}(y_{n+1}-y_n,\lambda)dy_n$$

$$+ \int_{y_{n+1}-1}^{y_n} [\Psi^{(n)}_{nk-1}(y_n,\lambda) + \Psi^{(n)}_{nk}(y_n,\lambda)\Psi^{(1)}_{00}(y_{n+1}-y_n,\lambda)]dy_n$$

Obviously, for $i = n+1,\ k = 0$

$$\Psi^{(n)}_{nk-1}(y_n,\lambda) = \Psi^{(n)}_{n-1}(y_n,\lambda) = 0$$

Case 2 (for a general ε)

It can be observed that for the general ε, the density of Y_{n+1} should have a similar formula:

$$h(y_{n+1}) = \sum_{k=0}^{n+1} \begin{pmatrix} n+1 \\ k \end{pmatrix} \left(\frac{\phi}{\varepsilon}\right)^k (1-\phi)^{n-k+1} \Psi^{(n+1)}_{ik}(y_{n+1},\lambda) \;\; i\varepsilon < y_{n+1} < (i+1)\varepsilon \;\; i = 0,...,n+1$$

$$= \sum_{k=0}^{n} \begin{pmatrix} n+1 \\ k \end{pmatrix} \left(\frac{\phi}{\varepsilon}\right)^k (1-\phi)^{n-k+1} \Psi^{(n+1)}_{n+1\,k}(y_{n+1},\lambda) \qquad y_n > (n+1)\varepsilon$$

Definition of Ψ's

In the definition the Ψ's will now involve ε's and will share similar recursive relations as in case of $\varepsilon = 1$:

For $i = 0,\ k = n+1$

$$\Psi_{0n+1}^{(n+1)}(y_{n+1},\lambda) = \int_0^{y_{n+1}\varepsilon} \Psi_{0n}^{(n)}(y_n,\lambda)dy_n$$

For $i = 0,\ k = 1,2,\cdots,n$

$$\binom{n+1}{k}\Psi_{0k}^{(n+1)}(y_{n+1},\lambda) = \binom{n}{k}\int_0^{y_{n+1}\varepsilon}[\Psi_{0k-1}^{(n)}(y_n,\lambda)+\Psi_{0k}^{(n)}(y_n,\lambda)\Psi_{00}^{(1)}(y_{n+1}-y_n,\lambda)]dy_n$$

For $i = 0,\ k = 0$

$$\Psi_{00}^{(n+1)}(y_{n+1},\lambda) = \int_0^{y_{n+1}\varepsilon}\Psi_{00}^{(n)}(y_n,\lambda)\Psi_{00}^{(1)}(y_{n+1}-y_n,\lambda)dy_n$$

For $i = 1,2,\cdots,n,\ k = n+1$

$$\Psi_{in+1}^{(n+1)}(y_{n+1},\lambda) = \int_{(y_{n+1}-1)\varepsilon}^{i\varepsilon}\Psi_{i-1n}^{(n)}(y_n,\lambda)dy_n + \int_{i\varepsilon}^{y_{n+1}\varepsilon}\Psi_{in}^{(n)}(y_n,\lambda)dy_n$$

For $i = 1,2,\cdots,n,\ k = 1,2,\cdots,n$

$$\Psi_{ik}^{(n+1)}(y_{n+1},\lambda) = \frac{\binom{n}{k}}{\binom{n+1}{k}}G(\Psi_{jl}^{(n)}(y_n,\lambda),\ \Psi_{00}^{(0)}(y_{n+1}-y_n,\lambda),\ \Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda);0\le j\le i,\ l=k-1,\ k)$$

where G equals

$$\sum_{j=0}^{i-2}\int_{j\varepsilon}^{(j+1)\varepsilon}\Psi_{jk}^{(n)}(y_n,\lambda)\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda)dy_n + \int_{(i-1)\varepsilon}^{(y_{n+1}-1)\varepsilon}\Psi_{i-1k}^{(n)}(y_n,\lambda)\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda)dy_n$$

$$+\int_{(y_{n+1}-1)\varepsilon}^{i\varepsilon}[\Psi_{i-1k-1}^{(n)}(y_n,\lambda)+\Psi_{i-1k}^{(n)}(y_n,\lambda)\Psi_{00}^{(1)}(y_{n+1}-y_n,\lambda)]dy_n$$

$$+\int_{i\varepsilon}^{y_{n+1}\varepsilon}[\Psi(y_n,\lambda)+\Psi_{ik}^{(n)}(y_n,\lambda)\Psi_{00}^{(1)}(y_{n+1}-y_n,\lambda)]dy_n]$$

Obviously, for $i=1,\ k=1,2,\cdots,n$

$$\sum\int_{j\varepsilon}^{(j+1)\varepsilon}\Psi_{jk}^{(n)}(y_n,\lambda)\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda)dy_n=0$$

For $i=1,2,\cdots,n,\ k=0$

$$\Psi_{i0}^{(n+1)}(y_{n+1},\lambda)=$$

$$=\sum_{j=0}^{i-2}\int_{j\varepsilon}^{(j+1)\varepsilon}\Psi_{j0}^{(n)}(y_n,\lambda)\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda)dy_n+\int_{(i-1)\varepsilon}^{(y_{n+1}-1)\varepsilon}\Psi_{i-10}^{(n)}(y_n,\lambda)\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda)dy_n$$

$$+\int_{(y_{n+1}-1)\varepsilon}^{i\varepsilon}\Psi_{i-10}^{(n)}(y_n,\lambda)\Psi_{00}^{(1)}(y_{n+1}-y_n,\lambda)dy_n+\int_{i\varepsilon}^{y_{n+1}\varepsilon}\Psi_{i0}^{(n)}(y_n,\lambda)\Psi_{00}^{(1)}(y_{n+1}-y_n,\lambda)dy_n$$

Once again, for $i=1,\ k=0$

$$\sum_{j=0}^{i-2}\int_{j\varepsilon}^{(j+1)\varepsilon}\Psi_{j0}^{(n)}(y_n,\lambda)\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda)dy_n=0$$

For $i=n+1,\ k=n+1$

$$\Psi_{n+1\,n+1}^{(n+1)}(y_{n+1},\lambda)=0$$

For $i=n+1,\ k=1,2,.........n$

$$\Psi_{n+1}^{(n+1)}(y_{n+1},\lambda)=\frac{\begin{pmatrix}n\\k\end{pmatrix}}{\begin{pmatrix}n+1\\k\end{pmatrix}}H(\Psi_{jl}^{(n)}(y_n,\lambda),\,\Psi_{00}^{(0)}(y_{n+1}-y_n,\lambda),\,\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda);0\le j\le n,\,l=k-1,\,k)$$

where H equals

$$\sum_{j=0}^{n-1}\int_{j\varepsilon}^{(j+1)\varepsilon}\Psi_{jk}^{(n)}(y_n,\lambda)\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda)dy_n+\int_{n\varepsilon}^{(y_{n+1}-1)\varepsilon}\Psi_{nk}^{(n)}(y_n,\lambda)\Psi_{10}^{(1)}(y_{n+1}-y_n,\lambda)dy_n$$

$$+\int_{(y_{n+1}-1)\varepsilon}^{y_n\varepsilon}[\Psi_{nk-1}^{(n)}(y_n,\lambda)+\Psi_{nk}^{(n)}(y_n,\lambda)\Psi_{00}^{(1)}(y_{n+1}-y_n,\lambda)]dy_n$$

Obviously, for $i=n+1,\ k=0$

$$\Psi_{nk-1}^{(n)}(y_n,\lambda)=\Psi_{n-1}^{(n)}(y_n,\lambda)=0$$

CHAPTER V

BAYESIAN PERSPECTIVE

Introduction

For a ZIED($\varepsilon, \phi, \lambda$) distribution, we consider ϕ and λ as the parameters of interest. The range for ϕ is the open interval $(0,1)$ and the range for λ is the open interval $(0,\infty)$. Therefore, from Bayesian perspective, it is appropriate to assume a beta prior, say, $\beta(a,b)$ a for ϕ and gamma prior, say, $\Gamma(\alpha,\theta)$ for λ. In order to find posteriors for these parameters, we will first consider one observation X and then consider the general situation of n i.i.d. observations. Like every other situation, we will assume $\varepsilon = 1$ to start with and then do for general ε.

Case 1 (ε=1)

First we consider the simplest case where $\varepsilon = 1$.

Posterior for One Observation

As mentioned above, we will first consider one observation and then generalize to n observations.

The Posterior density of λ

To start with, we assume ϕ to be a constant and λ having the $\Gamma(\alpha,\theta)$ density:

We define $\lambda \sim \Gamma(\alpha,\theta)$ as

$$g_1(\lambda) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}}\lambda^{\alpha}e^{-\frac{\lambda}{\theta}} \text{ for } \lambda > 0$$

As we know, the density for the observation X from ZIED$(1,\phi,\lambda)$ is

$$f\left(x|\lambda\right)=\phi I\left(0<x<1\right)+\left(1-\phi\right)\lambda e^{-\lambda x}\text{ for }x>0$$

Hence, the joint density of X and λ is

$$f(x,\lambda)=\frac{\phi}{\Gamma(\alpha)\theta^{\alpha}}\lambda^{\alpha-1}e^{-\frac{\lambda}{\theta}}I(0<x<1)+\frac{\lambda^{\alpha}}{\Gamma(\alpha)\theta^{\alpha}}(1-\phi)e^{-\lambda(x+\frac{1}{\theta})}$$

Then the marginal density of X is

$$f_0(x)=\int_0^{\infty}f(x,\lambda)d\lambda=\phi I(0<x<1)+(1-\phi)\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}$$

And the posterior of λ is given by

$$g_1(\lambda|x)=\frac{f(x,\lambda)}{f_0(x)}=\frac{\frac{\phi}{\Gamma(\alpha)\theta^{\alpha}}\lambda^{\alpha-1}e^{-\frac{\lambda}{\theta}}I(0<x<1)+\frac{\lambda^{\alpha}}{\Gamma(\alpha)\theta^{\alpha}}(1-\phi)e^{-\lambda(x+\frac{1}{\theta})}}{\phi I(0<x<1)+(1-\phi)\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}}\text{ for }\lambda>0$$

The Posterior density of ϕ

Now we consider λ to be a constant and ϕ having a $\beta(a,b)$ density, say,

$$g_2(\phi)=\frac{1}{\beta(a,b)}\phi^{a-1}(1-\phi)^{b-1}\text{ for }\phi\in(a,b)$$

Then the density for one observation X is

$$f\left(x|\phi\right)=\phi I\left(0<x<1\right)+\left(1-\phi\right)\lambda e^{-\lambda x}\text{ for }x>0$$

So, the joint density of X and ϕ is given by

$$f(x,\phi)=\frac{1}{\beta(a,b)}\phi^{a}(1-\phi)^{b-1}I(0<x<1)+\frac{1}{\beta(a,b)}\phi^{a-1}(1-\phi)^{b}\lambda e^{-\lambda x}$$

Then the marginal of X is

$$f_0(x) = \int_0^1 f(x,\phi)d\phi = \frac{a}{a+b}I(0<x<1) + \frac{b}{a+b}\lambda e^{-\lambda x}$$

And the posterior density of ϕ is as follows

$$g_2(\phi|x) = \frac{f(x,\phi)}{f_0(\phi)} = \frac{\frac{1}{\beta(a,b)}\phi^{a-1}(1-\phi)^{b-1}[\phi I(0<x<1)+\lambda(1-\phi)e^{-\lambda x}]}{\frac{a}{a+b}I(0<x<1)+\frac{b}{a+b}\lambda e^{-\lambda x}} \text{ for } \phi \in (0,1)$$

The Posterior density of λ and ϕ

Finally, we assume that $\lambda \sim \Gamma(\alpha,\theta)$ and $\phi \sim \beta(a,b)$ and they are independent. So, (λ,ϕ) has the joint prior

$$g(\lambda,\phi) = \left[\frac{1}{\Gamma(\alpha)\theta^{\alpha}}\lambda^{\alpha-1}e^{-\frac{\lambda}{\theta}}\right]\left[\frac{1}{\beta(a,b)}\phi^{a-1}(1-\phi)^{b-1}\right] \text{ for } \lambda>0,\, \phi\in(0,1)$$

The joint density of x, λ, and ϕ is as follows

$$f(x,\lambda,\theta) = [\phi I(0<x<1)+\lambda(1-\phi)e^{-\lambda x}]\left[\frac{1}{\Gamma(\alpha)\theta^{\alpha}}\lambda^{\alpha-1}e^{-\frac{\lambda}{\theta}}\right]\left[\frac{1}{\beta(a,b)}\phi^{a-1}(1-\phi)^{b-1}\right]$$

From here, the marginal density of X is obtained as follows:

$$f_0(x) = \int_0^1\int_0^{\infty} f(x,\lambda,\theta)d\lambda d\theta = \frac{a}{a+b}I(0<x<1) + \frac{b}{a+b}\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}$$

So the posterior joint density of λ and ϕ for $\lambda>0$, $\phi\in(0,1)$ is given by

$$f(\lambda,\phi|x) = \frac{f(x,\lambda,\phi)}{f_0(x)}$$

$$= \frac{[\phi I(0<x<1)+\lambda(1-\phi)e^{-\lambda x}]\left[\frac{1}{\Gamma(\alpha)\theta^{\alpha}}\lambda^{\alpha-1}e^{-\frac{\lambda}{\theta}}\right]\left[\frac{1}{\beta(a,b)}\phi^{a-1}(1-\phi)^{b-1}\right]}{\frac{a}{a+b}I(0<x<1)+\frac{b}{a+b}\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}}$$

From this, we can obtain the marginals $\lambda|x$ and $\phi|x$ as follows

$$f(\lambda|x) = \int_0^1 f(\lambda,\phi|x)d\phi$$

$$= \frac{\frac{1}{\Gamma(\alpha)\theta^\alpha}\lambda^{\alpha-1}e^{-\frac{\lambda}{\theta}}}{\frac{a}{a+b}I(0<x<1)+\frac{b}{a+b}\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}}\left[\frac{a}{a+b}I(0<x<1)+\frac{b}{a+b}\lambda e^{-\lambda x}\right]$$

$$f(\phi|x) = \int_0^\infty f(\lambda,\phi|x)d\lambda$$

$$= \frac{\frac{1}{\beta(a,b)}\phi^{a-1}(1-\phi)^{b-1}}{\frac{a}{a+b}I(0<x<1)+\frac{b}{a+b}\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}}\left[\phi I(0<x<1)+(1-\phi)\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}\right]$$

The Posterior mean and variance of λ and ϕ

The posterior mean of λ is

$$E(\lambda|X=x) = \frac{\alpha\theta}{a(1+\theta x)^{\alpha+1}I(0<x<1)+b\alpha\theta}\left[a(1+\theta x)^{\alpha+1}I(0<x<1)+b\frac{(\alpha+1)\theta}{1+\theta x}\right]$$

In order to get the posterior variance $V(\lambda|X=x)$, we first calculate $E(\lambda^2|x)$ which is

$$E(\lambda^2|x) = \frac{(\alpha+1)\alpha\theta^2}{a(1+\theta x)^{\alpha+1}I(0<x<1)+b\alpha\theta}\left[a(1+\theta x)^{\alpha+1}I(0<x<1)+b\frac{(\alpha+2)\theta}{(1+\theta x)^2}\right]$$

Then the posterior variance of λ is obtained as follows

$$V(\lambda|x) = E(\lambda^2|x) - \{E(\lambda|x)\}^2$$

The posterior mean of ϕ is

$$E(\phi|X=x) = \frac{\frac{a}{a+b+1}}{aI(0<x<1)+b\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}}\left[(a+1)I(0<x<1)+b\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}\right]$$

In order to obtain the variance, $V(\phi|X=x)$, we first obtain $E(\phi^2|x)$ which is

$$E(\phi^2|x) = \frac{\frac{a(a+1)}{(a+b+1)(a+b+2)}}{aI(0<x<1)+b\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}}\left[(a+2)I(0<x<1)+b\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}\right]$$

Then the variance is obtained as follows

$$V(\phi|x) = E(\phi^2|x) - \{E(\phi|x)\}^2$$

Posterior for n Observation

For n i.i.d. observations $X_1, \cdots, X_n$ from ZIED$(\varepsilon, \phi, \lambda)$ with $\varepsilon = 1$, the joint density is

$$f(x_1, \cdots, x_n|\lambda, \phi) = \prod_{i=0}^{n} [\phi I(0 < x_i < 1) + \lambda(1-\phi)e^{-\lambda x_i}]$$

$$= \sum_{l=0}^{n} \sum_{i_1, i_2, \cdots, i_l} \phi^{n-l}(1-\phi)^l \lambda^l e^{-\lambda \sum_{j=1}^{l} x_{ij}} I(0 < x_j < 1,\ j \neq i_1, i_2, \cdots, i_l)$$

The Posterior density of λ

As in the case of one observation, we first consider the case when ϕ is a constant and $\lambda \sim g_1(\lambda) \sim \Gamma(\alpha, \theta)$ so that the joint density of $X_1, X_2, ...X_n$ and λ is

$$f(x_1, \cdots, x_n, \lambda) = f(x_1, \cdots, x_n|\lambda) g_1(\lambda)$$

$$= \frac{1}{\Gamma(\alpha)\theta^\alpha} \sum_{l=0}^{n} \sum_{i_1, i_2, \cdots, i_l} \phi^{n-l} \lambda^{\alpha+l-1} (1-\phi)^l e^{-\lambda(\frac{1}{\theta} + \sum_{j=1}^{l} x_{ij})} I(0 < x_j < 1,\ j \neq i_1, i_2, \cdots, i_l)$$

Hence, the marginal of $X_1, \cdots, X_n$ is

$$f_0(x_1, \cdots, x_n) = \int_0^\infty f(x_1, \cdots, x_n, \lambda) d\lambda$$

$$= \sum_{l=0}^{n} \frac{\Gamma(\alpha+l)}{\Gamma(\alpha)} \sum_{i_1, i_2, \cdots, i_l} \phi^{n-l}(1-\phi)^l \frac{1}{\theta^\alpha (\frac{1}{\theta} + \sum_{j=1}^{l} x_{ij})^{\alpha+l}} I(0 < x_j < 1,\ j \neq i_1, i_2, \cdots, i_l)$$

So, the posterior density of λ is given as follows

$$f(\lambda|x_1, \cdots, x_n) = \frac{f(x_1, \cdots, x_n, \lambda)}{f_0(x_1, \cdots, x_n)}$$

$$= \frac{\sum_{l=0}^{n} \sum_{i_1, i_2, \cdots, i_l} \phi^{n-l} \lambda^{\alpha+l-1} (1-\phi)^l e^{-\lambda(\frac{1}{\theta} + \sum_{j=1}^{l} x_{ij})} I(0 < x_j < 1,\ j \neq i_1, i_2, \cdots, i_l)}{\sum_{l=0}^{n} \Gamma(\alpha+l) \sum_{i_1, i_2, \cdots, i_l} \phi^{n-l}(1-\phi)^l \frac{1}{(\frac{1}{\theta} + \sum_{j=1}^{l} x_{ij})^{\alpha+l}} I(0 < x_j < 1,\ j \neq i_1, i_2, \cdots, .i_l)}$$

The Posterior density of ϕ

Then we consider λ to be a constant and $\phi \sim g_2(\phi) \sim \beta(a,b)$ so that the joint density of so that the joint density of $X_1, \cdots, X_n$ and ϕ is

$$f(x_1, \cdots, x_n, \phi) = f(x_1, \cdots, x_n|\phi)g_2(\phi)$$

$$= \frac{1}{\beta(a,b)} \sum_{l=0}^{n} \sum_{i_1,i_2,\cdots,i_l} \phi^{n+a-l-1} \lambda^l (1-\phi)^{l+b-1} e^{-\lambda \Sigma_{j=1}^{l} x_{ij}} I(0 < x_j < 1,\ j \neq i_1, i_2, \cdots, i_l)$$

So, the marginal of $X_1, \cdots, X_n$ is

$$f_0(x_1, \cdots, x_n) = \int_0^1 f(x_1, \cdots, x_n, \phi) d\phi$$

$$= \sum_{l=0}^{n} \frac{\beta(n-l+a, l+b)}{\beta(a,b)} \sum_{i_1,i_2,\cdots,i_l} \lambda^l e^{-\lambda \Sigma_{j=1}^{l} x_{ij}} I(0 < x_j < 1,\ j \neq i_1, i_2, \cdots, i_l)$$

So, the posterior density of ϕ is as follows

$$f(\phi|x_1, \cdots, x_n) = \frac{f(x_1, \cdots, x_n, \phi)}{f_0(x_1, \cdots, x_n)}$$

$$= \frac{\sum_{l=0}^{n} \sum_{i_1,i_2,\cdots,i_l} \phi^{n+a-l-1} \lambda^l (1-\phi)^{l+b-1} e^{-\lambda \Sigma_{j=1}^{l} x_{ij}} I(0 < x_j < 1,\ j \neq i_1, i_2, \cdots, i_l)}{\sum_{l=0}^{n} \beta(n-l+a, l+b) \sum_{i_1,i_2,\cdots,i_l} \lambda^l e^{-\lambda \Sigma_{j=1}^{l} x_{ij}} I(0 < x_j < 1,\ j \neq i_1, i_2, \cdots, i_l)}$$

The Posterior density of λ and ϕ

Then we assume that $\lambda \sim \Gamma(\alpha, \theta)$ and $\phi \sim \beta(a,b)$ and they are independent with joint density being $g(\lambda, \phi)$. Then the joint density of $X_1, \cdots, X_n, \lambda$ and ϕ is

$$f(x_1, \cdots, x_n, \lambda, \phi)$$

$$= \frac{1}{\Gamma(\alpha)\theta^{\alpha}\beta(a,b)} \sum_{l=0}^{n} \sum_{i_1,i_2,\cdots,i_l} \phi^{n+a-l-1} \lambda^{l+\alpha-1} (1-\phi)^{l+b-1} e^{-\lambda(\frac{1}{\theta} + \Sigma_{j=1}^{l} x_{ij})} I(0 < x_j < 1,\ j \neq i_1, i_2, \cdots, i_l)$$

Then we integrate this with respect to ϕ and λ to get the marginal of $X_1, \cdots, X_n$

$$f_0(x_1,\cdots,x_n) = \int_0^\infty \int_0^1 f(x_1,\cdots,x_n,\lambda,\phi)d\phi d\lambda$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\theta^\alpha} \sum_{l=0}^{n} \sum_{i_1,i_2,\cdots,i_l} \frac{\beta(n-l+a,l+b)}{\beta(a,b)} \lambda^{\alpha+l-1} e^{-\lambda(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})} I(0<x_j<1,\ j\neq i_1,i_2,\cdots,i_l)d\lambda$$

$$= \sum_{l=0}^{n} \sum_{i_1,i_2,\cdots,i_l} \frac{\beta(n-l+a,l+b)}{\beta(a,b)} \frac{\Gamma(\alpha+l)}{\Gamma(\alpha)} \frac{1}{\theta^\alpha(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})^{\alpha+l}} I(0<x_j<1,\ j\neq i_1,i_2,\cdots,i_l)$$

So the posterior joint density of λ and ϕ is given as follows

$$f(\lambda,\phi|x_1,\cdots,x_n) = \frac{f(x_1,\cdots,x_n,\lambda,\phi)}{f_0(x_1,\cdots,x_n)}$$

$$= \frac{\sum_{l=0}^{n}\sum_{i_1,i_2\cdots,i_l} \phi^{n+a-l-1}\lambda^{l+\alpha-1}(1-\phi)^{l+b-1}e^{-\lambda(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})} I(0<x_j<1,\ j\neq i_1,i_2,\cdots,i_l)}{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l} \beta(n+a-l,l+b)\Gamma(\alpha+l)\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})^{\alpha+l}} I(0<x_j<1,\ j\neq i_1,i_2,\cdots,i_l)}$$

From this we obtain the marginals $\lambda\,|\,x_1,\cdots,x_n$ and $\phi\,|\,x_1,\cdots,x_n$ as follows

$$f(\lambda|x_1,\cdots,x_n) = \int_0^1 f(\lambda,\phi|x_1,\cdots,x_n)d\phi$$

$$= \frac{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l} \beta(n+a-l,l+b)\lambda^{\alpha+l-1}e^{-\lambda(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})} I(0<x_j<1,\ j\neq i_1,i_2,\cdots,i_l)}{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l} \beta(n+a-l,l+b)\Gamma(\alpha+l)\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})^{\alpha+l}} I(0<x_j<1,\ j\neq i_1,i_2,\cdots,i_l)}$$

and

$$f(\phi|x_1,\cdots,x_n) = \int_0^\infty f(\lambda,\phi|x_1,\cdots,x_n)d\lambda$$

$$= \frac{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l} \phi^{n+a-l-1}(1-\phi)^{l+b-1}\Gamma(\alpha+l)\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})^{\alpha+l}} I(0<x_j<1,\ j\neq i_1,\cdots,i_l)}{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l} \beta(n+a-l,l+b)\Gamma(\alpha+l)\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})^{\alpha+l}} I(0<x_j<1,\ j\neq i_1,\cdots,i_l)}$$

The Posterior mean and variance of λ and ϕ

Then the posterior mean of λ, which is $E(\lambda|x_1,\cdots,x_n)$ is given by

$$\frac{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\beta(n+a-l,l+b)\Gamma(\alpha+l+1)\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l}x_{ij})^{\alpha+l+1}}I(0<x_j<1,\ j\neq i_1,\cdots,i_l)}{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\beta(n+a-l,l+b)\Gamma(\alpha+l)\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l}x_{ij})^{\alpha+l}}I(0<x_j<1,\ j\neq i_1,\cdots,i_l)}$$

Also, $E(\lambda^2|x_1.\cdots,x_n)$ is obtained as

$$\frac{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\beta(n+a-l,l+b)\Gamma(\alpha+l+2)\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l}x_{ij})^{\alpha+l+2}}I(0<x_j<1,\ j\neq i_1,\cdots,i_l)}{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\beta(n+a-l,l+b)\Gamma(\alpha+l)\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l}x_{ij})^{\alpha+l}}I(0<x_j<1,\ j\neq i_1,\cdots,i_l)}$$

from which one can obtain the posterior variance of λ as,

$$V(\lambda|x)=E(\lambda^2|x)-\{E(\lambda|x)\}^2$$

Similarly, the posterior mean of ϕ, $E(\phi|x_1,\cdots,x_n)$ is as follows:

$$\frac{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\beta(n+a-l+1,l+b)\Gamma(\alpha+l)\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l}x_{ij})^{\alpha+l}}I(0<x_j<1,\ j\neq i_1,i_2,\cdots,i_l)}{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\beta(n+a-l,l+b)\Gamma(\alpha+l)\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l}x_{ij})^{\alpha+l}}I(0<x_j<1,\ j\neq i_1,i_2,\cdots,i_l)}$$

And $E(\phi^2|x_1,\cdots,x_n)$ is obtained as

$$\frac{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\beta(n+a-l+2,l+b)\Gamma(\alpha+l)\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l}x_{ij})^{\alpha+l}}I(0<x_j<1,\ j\neq i_1,i_2,\cdots,i_l)}{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\beta(n+a-l,l+b)\Gamma(\alpha+l)\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l}x_{ij})^{\alpha+l}}I(0<x_j<1,\ j\neq i_1,i_2,\cdots,i_l)}$$

from which one can obtain the posterior variance of ϕ as,

$$V(\phi|x)=E(\phi^2|x)-\{E(\phi|x)\}^2$$

Case 2 (For a general ε)

We are now going to see the case for a general ε.

Posterior for One Observation

Like before, we first consider one observation X and then consider n i.i.d. observations $X_1, \cdots, X_n$.

The Posterior density of λ

To start with, we assume ϕ to be a constant and λ having the $\Gamma(\alpha,\theta)$ density:

$$g_1(\lambda) = \frac{1}{\Gamma(\alpha)\theta^\alpha}\lambda^\alpha e^{-\frac{\lambda}{\theta}} \text{ for } \lambda > 0$$

As we know, the density for the observation X from ZIED$(\varepsilon, \phi, \lambda)$ is

$$f(x|\lambda) = \frac{\phi}{\varepsilon} I(0 < x < \varepsilon) + (1-\phi)\lambda e^{-\lambda x} \text{ for } x > 0$$

Hence, the joint density of X and λ is

$$f(x,\lambda) = \frac{\phi}{\varepsilon\Gamma(\alpha)\theta^\alpha}\lambda^{\alpha-1}e^{-\frac{\lambda}{\theta}}I(0<x<\varepsilon) + \frac{\lambda^\alpha}{\Gamma(\alpha)\theta^\alpha}(1-\phi)e^{-\lambda(x+\frac{1}{\theta})}$$

Then the marginal density of X is

$$f_0(x) = \int_0^\infty f(x,\lambda)d\lambda = \frac{\phi}{\varepsilon}I(0<x<\varepsilon) + (1-\phi)\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}$$

And the posterior of λ is given by

$$g_1(\lambda|x) = \frac{f(x,\lambda)}{f_0(x)} = \frac{\frac{\phi}{\varepsilon\Gamma(\alpha)\theta^\alpha}\lambda^{\alpha-1}e^{-\frac{\lambda}{\theta}}I(0<x<\varepsilon) + \frac{\lambda^\alpha}{\Gamma(\alpha)\theta^\alpha}(1-\phi)e^{-\lambda(x+\frac{1}{\theta})}}{\frac{\phi}{\varepsilon}I(0<x<\varepsilon) + (1-\phi)\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}} \text{ for } \lambda > 0$$

The Posterior density of ϕ

Next we consider λ to be a constant and ϕ having a $\beta(a,b)$ density, say,

$$g_2(\phi) = \frac{1}{\beta(a,b)}\phi^{a-1}(1-\phi)^{b-1} \text{ for } \phi \in (a,b)$$

Then the density for one observation X is

$$f(x|\phi) = \frac{\phi}{\varepsilon} I(0 < x < \varepsilon) + (1-\phi)\lambda e^{-\lambda x} \text{ for } x > 0$$

So, the joint density of X and ϕ is given by

$$f(x,\phi) = \frac{1}{\varepsilon\beta(a,b)}\phi^{a}(1-\phi)^{b-1}I(0<x<\varepsilon) + \frac{1}{\beta(a,b)}\phi^{a-1}(1-\phi)^{b}\lambda e^{-\lambda x}$$

Then the marginal of X is

$$f_0(x) = \int_0^1 f(x,\phi)d\phi = \frac{a}{\varepsilon(a+b)}I(0<x<\varepsilon) + \frac{b}{a+b}\lambda e^{-\lambda x}$$

And the posterior density of ϕ is as follows

$$g_2(\phi|x) = \frac{f(x,\phi)}{f_0(\phi)} = \frac{\frac{1}{\beta(a,b)}\phi^{a-1}(1-\phi)^{b-1}[\frac{\phi}{\varepsilon}I(0<x<\varepsilon)+\lambda(1-\phi)e^{-\lambda x}]}{\frac{a}{\varepsilon(a+b)}I(0<x<\varepsilon)+\frac{b}{a+b}\lambda e^{-\lambda x}} \text{ for } \phi \in (0,1)$$

The Posterior density of λ and ϕ

Finally, we assume that $\lambda \sim \Gamma(\alpha,\theta)$ and $\phi \sim \beta(a,b)$ and they are independent. So, (λ,ϕ) has the joint prior

$$g(\lambda,\phi) = \left[\frac{1}{\Gamma(\alpha)\theta^{\alpha}}\lambda^{\alpha-1}e^{-\frac{\lambda}{\theta}}\right]\left[\frac{1}{\beta(a,b)}\phi^{a-1}(1-\phi)^{b-1}\right] \text{ for } \lambda > 0,\ \phi \in (0,1)$$

The joint density of x, λ, and ϕ is as follows

$$f(x,\lambda,\theta) = [\frac{\phi}{\varepsilon}I(0<x<\varepsilon)+\lambda(1-\phi)e^{-\lambda x}]\left[\frac{1}{\Gamma(\alpha)\theta^{\alpha}}\lambda^{\alpha-1}e^{-\frac{\lambda}{\theta}}\right]\left[\frac{1}{\beta(a,b)}\phi^{a-1}(1-\phi)^{b-1}\right]$$

From here, the marginal density of X is obtained as follows:

$$f_0(x) = \int_0^1\int_0^{\infty} f(x,\lambda,\theta)d\lambda d\theta = \frac{a}{\varepsilon(a+b)}I(0<x<\varepsilon) + \frac{b}{a+b}\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}$$

So the posterior joint density of λ and ϕ for $\lambda > 0$, $\phi \in (0,1)$ is given by

$$f(\lambda,\phi|x) = \frac{f(x,\lambda,\phi)}{f_0(x)}$$

$$= \frac{[\frac{\phi}{\varepsilon}I(0<x<\varepsilon)+\lambda(1-\phi)e^{-\lambda x}]\left[\frac{1}{\Gamma(\alpha)\theta^{\alpha}}\lambda^{\alpha-1}e^{-\frac{\lambda}{\theta}}\right]\left[\frac{1}{\beta(a,b)}\phi^{a-1}(1-\phi)^{b-1}\right]}{\frac{a}{\varepsilon(a+b)}I(0<x<\varepsilon)+\frac{b}{a+b}\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}}$$

From this, we can obtain the marginals $\lambda\,|\,x$ and $\phi\,|\,x$ as follows

$$f(\lambda|x) = \int_0^1 f(\lambda,\phi|x)d\phi$$

$$= \frac{\frac{1}{\Gamma(\alpha)\theta^{\alpha}}\lambda^{\alpha-1}e^{-\frac{\lambda}{\theta}}}{\frac{a}{\varepsilon(a+b)}I(0<x<\varepsilon)+\frac{b}{a+b}\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}}\left[\frac{a}{\varepsilon(a+b)}I(0<x<\varepsilon)+\frac{b}{a+b}\lambda e^{-\lambda x}\right]$$

$$f(\phi|x) = \int_0^{\infty} f(\lambda,\phi|x)d\lambda$$

$$= \frac{\frac{1}{\beta(a,b)}\phi^{a-1}(1-\phi)^{b-1}}{\frac{a}{\varepsilon(a+b)}I(0<x<\varepsilon)+\frac{b}{a+b}\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}}\left[\frac{\phi}{\varepsilon}I(0<x<\varepsilon)+(1-\phi)\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}\right]$$

The Posterior mean and variance of λ and ϕ

Then the posterior mean of λ is

$$E(\lambda|X=x) = \frac{\alpha\theta}{\frac{a}{\varepsilon}(1+\theta x)^{\alpha+1}I(0<x<\varepsilon)+b\alpha\theta}\left[\frac{a}{\varepsilon}(1+\theta x)^{\alpha+1}I(0<x<\varepsilon)+b\frac{(\alpha+1)\theta}{1+\theta x}\right]$$

In order to get the posterior variance $V(\lambda|X=x)$, we first calculate $E(\lambda^2|x)$ which is

$$E(\lambda^2|x) = \frac{(\alpha+1)\alpha\theta^2}{\frac{a}{\varepsilon}(1+\theta x)^{\alpha+1}I(0<x<\varepsilon)+b\alpha\theta}\left[\frac{a}{\varepsilon}(1+\theta x)^{\alpha+1}I(0<x<\varepsilon)+b\frac{(\alpha+2)\theta}{(1+\theta x)^2}\right]$$

Then the posterior variance of λ is obtained as follows

$$V(\lambda|x) = E(\lambda^2|x) - \{E(\lambda|x)\}^2$$

The posterior mean of ϕ is

$$E(\phi|X=x) = \frac{\frac{a}{a+b+1}}{\frac{a}{\varepsilon}I(0<x<\varepsilon)+b\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}}\left[\frac{a+1}{\varepsilon}I(0<x<\varepsilon)+b\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}\right]$$

In order to obtain the variance, $V(\phi|X=x)$, we obtain $E(\phi^2|x)$ as

$$E(\phi^2|x) = \frac{\frac{a(a+1)}{(a+b+1)(a+b+2)}}{\frac{a}{\varepsilon}I(0<x<\varepsilon)+b\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}}\left[\frac{a+2}{\varepsilon}I(0<x<\varepsilon)+b\frac{\alpha\theta}{(1+\theta x)^{\alpha+1}}\right]$$

Then the variance is obtained as follows

$$V(\phi|x) = E(\phi^2|x) - \{E(\phi|x)\}^2$$

Posterior for n Observation

For n i.i.d. observations $X_1,\cdots,X_n$ from ZIED$(\varepsilon,\phi,\lambda)$ for a general ε, the joint density is

$$f(x_1,\cdots,x_n|\lambda,\phi) = \prod_{i=0}^{n}[\frac{\phi}{\varepsilon}I(0<x_i<\varepsilon)+\lambda(1-\phi)e^{-\lambda x_i}]$$

$$=\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\left(\frac{\phi}{\varepsilon}\right)^{n-l}(1-\phi)^l\lambda^l e^{-\lambda\sum_{j=1}^{l}x_{ij}}I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)$$

The Posterior density of λ

As in the case of one observation, we first consider the case when ϕ is a constant and $\lambda\sim g_1(\lambda)\sim\Gamma(\alpha,\theta)$ so that the joint density of $X_1,X_2,.......X_n$ and λ is

$$f(x_1,\cdots,x_n,\lambda) = f(x_1,\cdots,x_n|\lambda)g_1(\lambda)$$

$$=\frac{1}{\Gamma(\alpha)\theta^\alpha}\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\left(\frac{\phi}{\varepsilon}\right)^{n-l}\lambda^{\alpha+l-1}(1-\phi)^l e^{-\lambda(\frac{1}{\theta}+\sum_{j=1}^{l}x_{ij})}I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)$$

Hence, the marginal of $X_1,\cdots,X_n$ is

$$f_0(x_1,\cdots,x_n) = \int_0^\infty f(x_1,\cdots,x_n,\lambda)d\lambda$$

$$=\sum_{l=0}^{n}\frac{\Gamma(\alpha+l)}{\Gamma(\alpha)}\sum_{i_1,i_2,\cdots,i_l}\left(\frac{\phi}{\varepsilon}\right)^{n-l}(1-\phi)^l\frac{1}{\theta^\alpha(\frac{1}{\theta}+\sum_{j=1}^{l}x_{ij})^{\alpha+l}}I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)$$

So, the posterior density of λ is given as follows

$$f(\lambda|x_1,\cdots,x_n) = \frac{f(x_1,\cdots,x_n,\lambda)}{f_0(x_1,\cdots,x_n)}$$

$$=\frac{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\left(\frac{\phi}{\varepsilon}\right)^{n-l}\lambda^{\alpha+l-1}(1-\phi)^l e^{-\lambda(\frac{1}{\theta}+\sum_{j=1}^{l}x_{ij})}I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}{\sum_{l=0}^{n}\Gamma(\alpha+l)\sum_{i_1,i_2,\cdots,i_l}\left(\frac{\phi}{\varepsilon}\right)^{n-l}(1-\phi)^l\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l}x_{ij})^{\alpha+l}}I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,.i_l)}$$

The Posterior density of ϕ

Then we consider λ to be a constant and $\phi \sim g_2(\phi) \sim \beta(a,b)$ so that the joint density of so that the joint density of $X_1,\cdots,X_n$ and $\phi$is

$$f(x_1,\cdots,x_n,\phi) = f(x_1,\cdots,x_n|\phi)g_2(\phi)$$

$$= \frac{1}{\beta(a,b)}\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\frac{\phi^{n+a-l-1}}{\varepsilon^{n-l}}\lambda^l(1-\phi)^{l+b-1}e^{-\lambda\Sigma_{j=1}^{l}x_{ij}}I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)$$

So, the marginal of $X_1,\cdots,X_n$ is

$$f_0(x_1,\cdots,x_n) = \int_0^1 f(x_1,\cdots,x_n,\phi)d\phi$$

$$= \sum_{l=0}^{n}\frac{\beta(n-l+a,l+b)}{\varepsilon^{n-l}\beta(a,b)}\sum_{i_1,i_2,\cdots,i_l}\lambda^l e^{-\lambda\Sigma_{j=1}^{l}x_{ij}}I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)$$

So, the posterior density of ϕ is as follows

$$f(\phi|x_1,\cdots,x_n) = \frac{f(x_1,\cdots,x_n,\phi)}{f_0(x_1,\cdots,x_n)}$$

$$= \frac{\Sigma_{l=0}^{n}\Sigma_{i_1,i_2,\cdots,i_l}\frac{\phi^{n+a-l-1}}{\varepsilon^{n-l}}\lambda^l(1-\phi)^{l+b-1}e^{-\lambda\Sigma_{j=1}^{l}x_{ij}}I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}{\Sigma_{l=0}^{n}\frac{\beta(n-l+a,l+b)}{\varepsilon^{n-l}}\Sigma_{i_1,i_2,\cdots,i_l}\lambda^l e^{-\lambda\Sigma_{j=1}^{l}x_{ij}}I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}$$

The Posterior density of λ and ϕ

Then we assume that $\lambda \sim \Gamma(\alpha,\theta)$ and $\phi \sim \beta(a,b)$ and they are independent with joint density being $g(\lambda,\phi)$. Then the joint density of $X_1,\cdots,X_n,\lambda$ and ϕ is

$$f(x_1,\cdots,x_n,\lambda,\phi)$$

$$= \frac{1}{\Gamma(\alpha)\theta^{\alpha}\beta(a,b)}\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\frac{\phi^{n+a-l-1}}{\varepsilon^{n-l}}\lambda^{l+\alpha-1}(1-\phi)^{l+b-1}e^{-\lambda(\frac{1}{\theta}+\Sigma_{j=1}^{l}x_{ij})}I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)$$

Then we integrate this with respect to ϕ and λ to get the marginal of $X_1,\cdots,X_n$

$$f_0(x_1,\cdots,x_n) = \int_0^{\infty}\int_0^1 f(x_1,\cdots,x_n,\lambda,\phi)d\phi d\lambda$$

$$= \int_0^{\infty}\frac{1}{\Gamma(\alpha)\theta^{\alpha}}\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\frac{\beta(n-l+a,l+b)}{\varepsilon^{n-l}\beta(a,b)}\lambda^{\alpha+l-1}e^{-\lambda(\frac{1}{\theta}+\Sigma_{j=1}^{l}x_{ij})}I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)d\lambda$$

$$= \sum_{l=0}^{n} \sum_{i_1,i_2,\cdots,i_l} \frac{\beta(n-l+a,l+b)}{\varepsilon^{n-l}\beta(a,b)} \frac{\Gamma(\alpha+l)}{\Gamma(\alpha)} \frac{1}{\theta^{\alpha}(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})^{\alpha+l}} I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)$$

So the posterior joint density of $\lambda$and ϕ is given as follows

$$f(\lambda,\phi|x_1,\cdots,x_n) = \frac{f(x_1,\cdots,x_n,\lambda,\phi)}{f_0(x_1,\cdots,x_n)}$$

$$= \frac{\sum_{l=0}^{n}\sum_{i_1,i_2\cdots,i_l} \frac{\phi^{n+a-l-1}}{\varepsilon^{n-l}} \lambda^{l+\alpha-1}(1-\phi)^{l+b-1} e^{-\lambda(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})} I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l} \frac{\beta(n+a-l,l+b)}{\varepsilon^{n-l}} \Gamma(\alpha+l) \frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})^{\alpha+l}} I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}$$

From this we obtain the marginals $\lambda\,|\,x_1,\cdots,x_n$ and $\phi\,|\,x_1,\cdots,x_n$ as follows

$$f(\lambda|x_1,\cdots,x_n) = \int_0^1 f(\lambda,\phi|x_1,\cdots,x_n)d\phi$$

$$= \frac{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l} \frac{\beta(n+a-l,l+b)}{\varepsilon^{n-l}} \lambda^{\alpha+l-1} e^{-\lambda(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})} I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l} \frac{\beta(n+a-l,l+b)}{\varepsilon^{n-l}} \Gamma(\alpha+l) \frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})^{\alpha+l}} I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}$$

$$f(\phi|x_1,\cdots,x_n) = \int_0^{\infty} f(\lambda,\phi|x_1,\cdots,x_n)d\lambda$$

$$= \frac{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l} \frac{\phi^{n+a-l-1}}{\varepsilon^{n-l}} (1-\phi)^{l+b-1} \Gamma(\alpha+l) \frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})^{\alpha+l}} I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l} \frac{\beta(n+a-l,l+b)}{\varepsilon^{n-l}} \Gamma(\alpha+l) \frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})^{\alpha+l}} I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}$$

The Posterior mean and variance of λ and ϕ

Then the posterior mean of λ, which is $E(\lambda|x_1,\cdots,x_n)$ is given by

$$\frac{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l} \frac{\beta(n+a-l,l+b)}{\varepsilon^{n-l}} \Gamma(\alpha+l+1) \frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})^{\alpha+l+1}} I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l} \frac{\beta(n+a-l,l+b)}{\varepsilon^{n-l}} \Gamma(\alpha+l) \frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})^{\alpha+l}} I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}$$

Also, $E(\lambda^2|x_1.\cdots,x_n)$ is obtained as

$$\frac{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l} \frac{\beta(n+a-l,l+b)}{\varepsilon^{n-l}} \Gamma(\alpha+l+2) \frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})^{\alpha+l+2}} I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l} \frac{\beta(n+a-l,l+b)}{\varepsilon^{n-l}} \Gamma(\alpha+l) \frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l} x_{ij})^{\alpha+l}} I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}$$

from which one can obtain the posterior variance of λ as,

$$V(\lambda|x) = E(\lambda^2|x) - \{E(\lambda|x)\}^2$$

Similarly, the posterior mean of ϕ, $E(\phi|x_1,\cdots,x_n)$ is as follows

$$\frac{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\frac{\beta(n+a-l+1,l+b)}{\varepsilon^{n-l}}\Gamma(\alpha+l)\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l}x_{ij})^{\alpha+l}}I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\frac{\beta(n+a-l,l+b)}{\varepsilon^{n-l}}\Gamma(\alpha+l)\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l}x_{ij})^{\alpha+l}}I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}$$

And to get the posterior variance of ϕ,$E(\phi^2|x_1,\cdots,x_n)$ is obtained as

$$\frac{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\frac{\beta(n+a-l+2,l+b)}{\varepsilon^{n-l}}\Gamma(\alpha+l)\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l}x_{ij})^{\alpha+l}}I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}{\sum_{l=0}^{n}\sum_{i_1,i_2,\cdots,i_l}\frac{\beta(n+a-l,l+b)}{\varepsilon^{n-l}}\Gamma(\alpha+l)\frac{1}{(\frac{1}{\theta}+\sum_{j=1}^{l}x_{ij})^{\alpha+l}}I(0<x_j<\varepsilon,\ j\neq i_1,i_2,\cdots,i_l)}$$

Then the posterior variance of ϕ is,

$$V(\phi|x)=E(\phi^2|x)-\{E(\phi|x)\}^2$$

CHAPTER VI

CONCLUDING REMARKS

In most part of this thesis, we worked with i.i.d. observations. But it is interesting to study the situation when our observations are independent but not identically distributed, For example, consider two fast serving counters. Let the serving times for the two counters be indepedent random variables with distributions ZIED$(\varepsilon_1,\lambda,\phi)$ and ZIED$(\varepsilon_2,\lambda,\phi)$ respectively. Here we assume $\varepsilon_1 < \varepsilon_2$. Then their respective are as follows:

$$f(x_1) = \frac{\phi}{\varepsilon_1} I(0 < x_1 < \varepsilon_1) + \lambda(1-\phi)e^{-\lambda x_1}$$

$$f(x_2) = \frac{\phi}{\varepsilon_2} I(0 < x_2 < \varepsilon_2) + \lambda(1-\phi)e^{-\lambda x_2}$$

Now if one has to go through both the counters, one may be interested to know what should be total time taken in the two serving counters. In other words, we are interested to find the distribution of the convolution of these two random variables. Let Y_2 be the convolution sum of them. Then, we find out which is as follows

$$h(y_2) = \frac{\phi^2}{\varepsilon_1\varepsilon_2} y_2 + \lambda^2(1-\phi)y_2 e^{-\lambda y_2} + \phi(1-\phi)(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2})(1-e^{-\lambda y_2}) \quad 0 < y_2 < \varepsilon_1$$

$$= \frac{\phi^2}{\varepsilon_2} + \lambda^2(1-\phi)y_2 e^{-\lambda y_2} + \frac{\phi(1-\phi)}{\varepsilon_1}\left(e^{-\lambda(y_2-\varepsilon_1)} - e^{-\lambda y_2}\right) + \frac{\phi(1-\phi)}{\varepsilon_2}\left(1-e^{-\lambda y_2}\right) \quad \varepsilon_1 < y_2 < \varepsilon_2$$

$$= \frac{\phi^2}{\varepsilon_1\varepsilon_2}(\varepsilon_1+\varepsilon_2-y_2) + \lambda^2(1-\phi)y_2 e^{-\lambda y_2} + \sum_{i=1}^{2}\frac{\phi(1-\phi)}{\varepsilon_i}\left(e^{-\lambda(y_2-\varepsilon_i)} - e^{-\lambda y_2}\right) \quad \varepsilon_2 < y_2 < \varepsilon_1+\varepsilon_2$$

$$= \lambda^2(1-\phi)y_2 e^{-\lambda y_2} + \sum_{i=1}^{2}\frac{\phi(1-\phi)}{\varepsilon_i}\left(e^{-\lambda(y_2-\varepsilon_i)} - e^{-\lambda y_2}\right) \quad y_2 > \varepsilon_1+\varepsilon_2$$

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BIOGRAPHICAL SKETCH

Sougata Dhar was born in 1986 in a small town called Berhampore. His permanent address is 105/1, Nilmani Bhattachariya Lane, Kadai, Berhampore, Murshidabad, Pin-742101, West Bengal, India. He went to a local High School. He studied Electronics & Communication Engineering in his undergraduate program at Techno India, Saltlake in 2008. He did his masters degree in mathematics from University of Texas Pan American. He graduated in August 2011. After that He went to the department of mathematics at Northern Illinois University to join their PhD program in the fall 2011.