8-2011

Subdifferentials of Distance Functions and Applications to Facility Location Problems

Juan Salinas Jr.
University of Texas-Pan American

SUBDIFFERENTIALS OF DISTANCE FUNCTIONS

AND APPLICATIONS TO FACILITY

LOCATION PROBLEMS

A Thesis

by

JUAN SALINAS, JR.

Submitted to the Graduate School of the
University of Texas Pan American
In partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

August 2011

Major Subject: Mathematics

SUBDIFFERENTIALS OF DISTANCE FUNCTIONS AND APPLICATIONS TO FACILITY LOCATION PROBLEMS

A Thesis

by

JUAN SALINAS, JR.

COMMITTEE MEMBERS

Dr. Nguyen, Mau Nam
Chair of Committee

Dr. Bede, Barnabas
Committee Member

Dr. Bracken, Paul
Committee Member

Dr. Villalobos, Cristina
Committee Member

August 2011

ABSTRACT

Salinas Jr., Juan, Subdifferentials of Distance Functions and Applications to Facility Location Problems. Master of Science (MS), August, 2011, 77 pages, 9 figures, 21 titles.

The classical Heron problem states: *on a given straight line in the plane, find a point C such that the sum of the distances from C to the given points A and B is minimal*. This problem can be solved using standard geometry or differential calculus. In the light of modern convex analysis, we are able to investigate more general versions of this problem. In this thesis we propose and solve the following problem: on a given nonempty closed convex subset of $\mathbb{R}^s$, find a point such that the sum of the distances from that point to n given nonempty closed convex subsets of $\mathbb{R}^s$ is minimal.

DEDICATION

I would like to dedicate this thesis to my parents, Nimia and Juan Salinas, Sr., to my loving wife, Griselda Salinas, and to my two beautiful daughters, Nataly and Arely D. Salinas.

I want to thank my parents for teaching me to appreciate and be grateful for the things I have and for always being there for their children. They worked hard to teach each and everyone of us how important it is to care and help each other and to strongly believe in moral and family values. My two daughters are my inspiration and I want to set a good example for them. My greatest desire is for them to understand the importance of higher education and to always strive to do their best and achieve their goals in life.

Lastly, but not least, I want to thank my wife who always believes and supports my personal and professional goals. We agree in many things and always share and strive for our goals together. I want to thank her for her sacrifices and support during the time I was a full time graduate student. Most importantly, I want to thank her for always being by my side during our fifteen years of marriage and wish for many more.

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to Dr. Mau Nam Nguyen for his guidance and support. I thank him also for providing me with the opportunity to grow as a graduate student and researcher in mathematics. I am grateful for being able to participate under his supervision and mentorship on the UTPA's Summer Research Initiative for Graduate Students during the Summer of 2010, which allowed me to continue to do research in the applications of mathematics to facility location problems. I enjoyed all of his lecture presentations in and outside of class. In just one year, I have learned so much from him and only wish I could do it all over again. His dedication and excellent work ethic has inspired me to take my job as an educator to a higher level.

Great appreciation to my good friend and mentor Mr. Esteban Salinas for many years of guidance and motivation. He has taught me to appreciate teaching mathematics to students. I have known Mr. Salinas for twenty years and he is still enthusiastic and energetic teaching mathematics and other subjects to students who seek his help. He has been my role model and I respect him for the work he has done as an educator. I wish that some day I can come near to what he has accomplished in helping students succeed in life.

I also want to thank my thesis committee, Dr. Bede, Dr. Bracken, and Dr. Villalobos for taking time from their busy schedules to help me on my thesis work. Their advice and help is greatly appreciated.

TABLE OF CONTENTS

Page

ABSTRACT . . . iii

DEDICATION . . . iv

ACKNOWLEDGEMENTS . . . v

TABLE OF CONTENTS . . . vi

LIST OF FIGURES . . . viii

CHAPTER I. DISTANCE FUNCTIONS AND A GENERALIZED HERON PROBLEM . 1
- 1.1 Introduction and Problem Formulation . . . 1
- 1.2 Basic Properties of Distance Functions . . . 4

CHAPTER II. ELEMENTS OF CONVEX ANALYSIS AND OPTIMIZATION . . . 9
- 2.1 Basic Definitions and Properties . . . 9
- 2.2 Subgradient Method in Convex Optimization . . . 12
- 2.3 Projected Subgradient Method in Convex Optimization . . . 18

CHAPTER III. SUBDIFFERENTIALS OF DISTANCE FUNCTIONS . . . 20
- 3.1 Subdifferential Formulas for Distance Functions: Convex Case . . . 20
- 3.2 Subdifferential of Distance Functions with the Euclidean Norm . . . 28
- 3.3 Subdifferential of Distance Functions with the Sum Norm . . . 31
- 3.4 Subdifferential of Distance Functions with the Max Norm . . . 34

CHAPTER IV. APPLICATIONS TO THE GENERALIZED HERON PROBLEM . . . 38
- 4.1 Optimal Solutions to the Generalized Heron Problem . . . 38
- 4.2 Numerical Algorithm and Implementation . . . 50
- 4.3 Generalized Heron Problem with Euclidean Norm . . . 51

4.4 Generalized Heron Problem with Sum Norm 58
4.5 Generalized Heron Problem with Max Norm 61

REFERENCES 65

APPENDIX A 67

BIOGRAPHICAL SKETCH 77

LIST OF FIGURES

Page

Figure 4.1: A Generalized Heron Problem for Balls with Ball Constraint. 54

Figure 4.2: A Generalized Heron Problem for Balls with Square Constraint. 55

Figure 4.3: A Generalized Heron Problem for Squares with Line Constraint. 57

Figure 4.4: A Generalized Heron Problem for Squares with Ball Constraint. 57

Figure 4.5: A Generalized Heron Problem for Cubes with Ball Constraint in Three Dimensions. . . . 58

Figure 4.6: A Generalized Heron Problem for Squares with Ball Constraint and "Sum" Distances. . . 61

Figure 4.7: A Generalized Heron Problem for Squares with Square Constraint and "Sum" Distances. 61

Figure 4.8: A Generalized Heron Problem for Squares with Square Constraint and "Max" Distances. 63

Figure 4.9: A Generalized Heron Problem for Squares with Ball Constraint and "Max" Distances. . 63

CHAPTER I

DISTANCE FUNCTIONS AND A GENERALIZED HERON PROBLEM

The classical Heron problem states: *on a given straight line in the plane, find a point C such that the sum of the distances from C to the given points A and B is minimal*. This problem can be solved using standard geometry or differential calculus. In the light of modern convex analysis, we are able to investigate more general versions of this problem. In this thesis we propose and solve the following problem: on a given nonempty closed convex subset of $I\!R^s$, find a point such that the sum of the distances from that point to n given nonempty closed convex subsets of $I\!R^s$ is minimal.

1.1 Introduction and Problem Formulation

In this thesis we propose and largely investigate various extensions of the classical Heron problem, which seem to be mathematically interesting and important for applications. In particular, the one of this type is to replace two given points in the classical Heron problem by finitely many nonempty closed convex subsets of $I\!R^s$ with a given norm and to replace the straight line therein by another nonempty closed convex subset of this space.

Recall that the classical Heron problem was posted by Heron from Alexandria (10–75 AS) in his *Catroptica* as follows: find a point on a straight line in the plane such that the sum of the distances from it to two given points is minimal; see [4, 7] for more discussions. We formulate the *distance function version* of the *generalized Heron problem* as follows:

$$\text{minimize } \ D(x) := \sum_{i=1}^{n} d(x;\Omega_i) \ \text{ subject to } \ x \in \Omega, \tag{1.1.1}$$

where Ω and Ω_i, $i = 1, \ldots, n, n \geq 2$, are given nonempty closed convex subsets of $I\!R^s$ endowed with the norm $\|\cdot\|$, and where

$$d(x;Q) := \inf\big\{\|x-y\| \;\big|\; y \in Q\big\} \tag{1.1.2}$$

is the usual distance from $x \in I\!R^s$ to a set Q. Observe that in this new formulation the generalized Heron problem (1.1.1) is an extension of the *generalized Fermat-Torricelli* problem proposed and studied in [13]. The difference is that the latter problem in unconstrained, i.e., $\Omega = I\!R^s$ in (1.1.1), while the presence of the *geometric constraint* in the generalized Heron version (1.1.1) makes it more mathematically complicated and more realistic for applications. The Fermat-Torricelli and Heron problems as well as their generalized versions in this thesis are examples of facility location problems. In contrast to existing facility location models where the locations are of negligible sizes, represented by points, our approach in this thesis allows us to deal with facility location problems where the locations are of non-negligible sizes, now represented by sets. Among the most natural areas of applications we mention constrained problems arising in location science, optimal networks, wireless communications, etc. We refer the reader to the corresponding discussions and results in [13] and the bibliographies therein concerning unconstrained Fermat-Torricelli-Steiner-Weber versions. Needless to say that the presence of geometric constraints in (1.1.1) essentially changes these versions while referring us to the original Heron geometric problem.

A characteristic feature of the generalized Heron problem (1.1.1) is that it is *intrinsically nonsmooth* and *convex*, since the function (1.1.2) is nondifferentiable in general, while the convexity of (1.1.1) follows from the convexity of the sets Ω and Ω_i. This makes it natural to apply advanced methods and tools of convex analysis and generalized differentiation to study these problems. To proceed in this direction, we largely employ the recent results on generalized differentiation of the distance function (1.1.2) in convex settings as well as comprehensive rules of generalized differential calculus. As can be seen from the results in the thesis, the constraint nature of the Heron problem and its extensions leads to new structural phenomena in comparison with the corresponding Fermat-Torricelli counterparts. Note that a number of the results obtained in this thesis are new even for the unconstrained setting of the generalized Fermat-Torricelli problem.

The rest of the thesis is organized as follows. In Chapter 2, we present basic constructions and properties from convex analysis and optimization. We provide detailed proofs for important known results that will be used in the following chapters. Chapter 3 is devoted to the study of generalized differentiation properties of distance functions. Although most of the theorems presented in this chapter are known (see [12] and the references therein), our examples of distance functions generated by different norms for different convex sets are new. In Chapter 4 we study the generalized Heron problem theoretically and develop numerical algorithms of subgradient type to solve the problem. We derive necessary optimality conditions for solutions to the generalized Heron problem in the case of arbitrary closed convex sets Ω and Ω_i, $i = 1, \ldots, n$. We pay a special attention to the Euclidean space setting, which allows us to establish necessary and sufffficent optimality conditions in the most efficient forms. The links between the generalized Heron problem and the generalized Fermat-Torricelli problem are also established. Many examples are given to illustrate applications of general results in particular situations. New results in this chapter and further study of facility location problems involving sets are presented in our recent papers [14, 15, 16].

The notations throughout the entire thesis are basically standard in the area of convex analysis and generalized differentiation; see [1, 2, 20, 21]. Let $X = \mathbb{R}^s$ be a normed space where $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the norm and the scalar product, repectively. Recall that X^* is the dual space of the normed space X, that means that $X^* = X$ and the norm in X^* is defined by

$$\|x^*\| = \sup_{\|x\| \le 1} \langle x^*, x \rangle \text{ for all } x^* \in X^*.$$

As usual, the closed unit ball in the space X is given by $\mathbb{B} = \{v \in X \mid \|v\| \le 1\}$ and the closed unit ball in X^* is given by

$$\mathbb{B}^* = \{x^* \in X^* = \mathbb{R}^s \mid ||x^*|| = \sup_{\|x\| \le 1} \langle x^*, x \rangle \le 1\}.$$

We also denote S and S^* to be the unit spheres in X and X^*, respectively. For a subset A of X, co A is convex hull of A or the smallest convex sets that contains A.

1.2 Basic Properties of Distance Functions

In this section we are going to present important well known properties of distance functions. These properties play an important role in developing generalized differentiation rules for distance functions and applications to facility location problems involving sets. The setting in this section and throughout the thesis is finite dimensional although many results presented below hold in infinite dimensional.

Proposition 1.2.1. *Let Ω be a nonempty subset of X. Then the following hold:*

(a) $d(x;\Omega) < \infty$ *for all* $x \in X$*,*

(b) $d(x;\Omega) = 0$ *if and only if* $x \in \overline{\Omega}$*,*

(c) $|d(x;\Omega) - d(y;\Omega)| \le ||x - y||$ *for all* $x, y \in X$*.*

Proof: (a) Fix any $\omega \in \Omega$. It follows from the definition that

$$d(x;\Omega) \le ||x - \omega|| < \infty.$$

(b) Suppose that $d(x;\Omega) = 0$. For each $n \in I\!N$, there exists $\omega_n \in \Omega$ such that

$$0 = d(x;\Omega) \le ||x - \omega_n|| < d(x;\Omega) + \frac{1}{n} = \frac{1}{n}.$$

It follows that (ω_n) converges to x, and hence $x \in \overline{\Omega}$.

Conversely, suppose that $x \in \overline{\Omega}$. Then there exists a sequence $(\omega_n) \subset \Omega$ that converges to x. It follows that

$$0 \le d(x;\Omega) \le ||x - \omega_n|| \text{ for all } n \in I\!N.$$

This implies $d(x;\Omega) = 0$ because $||x - \omega_n|| \to 0$.

(c) For any $\omega \in \Omega$, one has

$$d(x;\Omega) \le ||x - \omega|| \le ||x - y|| + ||y - \omega||.$$

It follows that

$$d(x;\Omega) \leq ||x-y|| + \inf\{||y-\omega|| \bigm| \omega \in \Omega\} = ||x-y|| + d(y;\Omega).$$

Similarly,

$$d(y;\Omega) \leq ||y-x|| + d(x;\Omega).$$

Therefore,

$$|d(x;\Omega) - d(y;\Omega)| \leq ||x-y||.$$

The proof is now complete. □

For each $x \in X$, the projection from x to Ω is defined by

$$\Pi(x;\Omega) := \{\omega \in \Omega \bigm| ||x-\omega|| = d(x;\Omega)\}. \tag{1.2.1}$$

Proposition 1.2.2. *For any $x \in X$, the projection $\Pi(x;\Omega)$ is nonempty if Ω is closed.*

Proof: For each $n \in \mathbb{N}$, there exists $\omega_n \in \Omega$ such that

$$d(x;\Omega) \leq ||x-\omega_n|| < d(x;\Omega) + \frac{1}{n}.$$

It is clear that (ω_n) is a bounded sequence. Thus it has a subsequence (ω_{n_k}) that converges to ω. Since Ω is closed, $\omega \in \Omega$ and

$$d(x;\Omega) = ||x-\omega||.$$

This implies $\omega \in \Pi(x;\Omega)$. Therefore, $\Pi(x;\Omega)$ is nonempty. □

Corollary 1.2.3. *If Ω is a nonempty closed convex subset of X endowed with the Euclidean norm, then $\Pi(x;\Omega)$ is singleton.*

Proof: The fact that $\Pi(x;\Omega)$ is nonempty follows directly from Proposition 1.2.2. Let us now consider the case where X is equipped with the Euclidean norm. Suppose that $\omega_1, \omega_2 \in \Pi(x;\Omega)$ and $\omega_1 \neq \omega_2$. Then

$$||x-\omega_1|| = ||x-\omega_2|| = d(x;\Omega).$$

By the parallelogram equality one has

$$2||x-\omega_1||^2 = ||x-\omega_1||^2 + ||x-\omega_2||^2 = 2||x-\frac{\omega_1+\omega_2}{2}||^2 + \frac{||\omega_1-\omega_2||^2}{2}.$$

This implies

$$||x-\frac{\omega_1+\omega_2}{2}||^2 = ||x-\omega_1||^2 - \frac{||\omega_1-\omega_2||^2}{4} < ||x-\omega_1||^2 = [d(x;\Omega)]^2.$$

This is a contradiction because $\dfrac{\omega_1+\omega_2}{2} \in \Omega$. The proof is now complete. □

Proposition 1.2.4. *Suppose that Ω is a nonempty closed subset of X. Then the distance function $d(\cdot;\Omega)$ is convex if and only if Ω is convex.*

Proof: Let us first prove the converse. Suppose that Ω is convex. Fix any $x_1, x_2 \in X$ and $t \in (0,1)$. Then for any $\epsilon > 0$, there exists $\omega_i \in \Omega$ such that

$$||x_i - \omega_i|| < d(x_i;\Omega_i) + \epsilon \text{ for } i = 1, 2.$$

Since Ω is convex, $t\omega_1 + (1-t)\omega_2 \in \Omega$. Thus

$$\begin{aligned} d(tx_1 + (1-t)x_2;\Omega) &\leq ||tx_1 + (1-t)x_2 - [t\omega_1 + (1-t)\omega_2]|| \\ &\leq t||x_1-\omega_1|| + (1-t)||x_2-\omega_2|| \\ &\leq td(x_1;\Omega_1) + (1-t)d(x_2;\Omega_2) + \epsilon. \end{aligned}$$

It follows that

$$d(tx_1 + (1-t)x_2;\Omega) \leq td(x_1;\Omega) + (1-t)d(x_2;\Omega)$$

because ϵ is arbitrary.

Let us now prove the implication. Suppose that $d(\cdot;\Omega)$ is convex. Fix $\omega_i \in \Omega$ for $i = 1, 2$ and $t \in (0,1)$. Then by the convexity of $d(\cdot;\Omega)$, one has

$$d(t\omega_1 + (1-t)\omega_2;\Omega) \leq td(\omega_1;\Omega) + (1-t)d(\omega_2;\Omega) = 0.$$

Since Ω is closed, this implies $t\omega_1 + (1-t)\omega_2 \in \Omega$. Therefore, Ω is convex. □

Proposition 1.2.5. *Suppose that Ω is a nonempty closed convex subset of X with the Euclidean norm. If $\bar{\omega} \in \Pi(\bar{x};\Omega)$, then*

$$\langle \bar{x} - \bar{\omega}, \omega - \bar{\omega} \rangle \leq 0 \textit{ for all } \omega \in \Omega.$$

Proof: Fix any $\omega \in \Omega$ and $t \in (0,1)$, one has $\bar{\omega} + t(\omega - \bar{\omega}) \in \Omega$. Thus

$$\begin{aligned} ||\bar{x} - \bar{\omega}||^2 = [d(\bar{x};\Omega)]^2 &\leq ||\bar{x} - [\bar{\omega} + t(\omega - \bar{\omega})]||^2 \\ &= ||\bar{x} - \bar{\omega}||^2 - 2t\langle \bar{x} - \bar{\omega}, \omega - \bar{\omega} \rangle + t^2||\omega - \bar{\omega}||^2. \end{aligned}$$

It follows that

$$2\langle \bar{x} - \bar{\omega}, \omega - \bar{\omega} \rangle \leq t||\omega - \bar{\omega}||^2.$$

By letting $t \to 0$, we obtain the result. □

As we have proved, for a closed convex set Ω of X with the Euclidean norm, the projection $\Pi(x;\Omega)$ is singleton. In what follows we will show that the projection mapping is nonexpansive.

Proposition 1.2.6. *Let Ω be a nonempty closed convex set of X with the Euclidean norm. For any $x_1, x_2 \in X$, one has*

$$||\Pi(x_1;\Omega) - \Pi(x_2;\Omega)||^2 \leq \langle \Pi(x_1;\Omega) - \Pi(x_2;\Omega), x_1 - x_2 \rangle.$$

In particular,

$$||\Pi(x_1;\Omega)-\Pi(x_2;\Omega)||\leq||x_1-x_2||.$$

Proof: By the previous proposition, one has

$$\langle\Pi(x_2;\Omega)-\Pi(x_1;\Omega),x_1-\Pi(x_1;\Omega)\rangle\leq 0.$$

Similarly,

$$\langle\Pi(x_1;\Omega)-\Pi(x_2;\Omega),x_2-\Pi(x_2;\Omega)\rangle\leq 0.$$

Adding these inequalities, we obtain

$$\langle\Pi(x_1;\Omega)-\Pi(x_2;\Omega),x_2-x_1+\Pi(x_1;\Omega)-\Pi(x_2;\Omega)\rangle\leq 0.$$

Therefore,

$$||\Pi(x_1;\Omega)-\Pi(x_2;\Omega)||^2\leq\langle\Pi(x_1;\Omega)-\Pi(x_2;\Omega),x_1-x_2\rangle.$$

Finally, the second inequality follows from the fact that

$$||\Pi(x_1;\Omega)-\Pi(x_2;\Omega)||^2\leq\langle\Pi(x_1;\Omega)-\Pi(x_2;\Omega),x_1-x_2\rangle\leq||\Pi(x_1;\Omega)-\Pi(x_2;\Omega)||\cdot||x_1-x_2||.$$

The proof is now complete. □

CHAPTER II

ELEMENTS OF CONVEX ANALYSIS AND OPTIMIZATION

In this chapter we review some basic concepts of convex analysis used in subsequent chapters. This material and much more can be found, e.g., in the books [1, 2, 20, 21].

2.1 Basic Definitions and Properties

Let $f\colon X \to \overline{I\!R} := (-\infty, \infty]$ be an extended-real-valued function, which may be infinite at some points, and let

$$\text{dom } f := \big\{x \in X \;\big|\; f(x) < \infty\big\}$$

be its *effective domain*. The *epigraph* of f is a subset of $X \times I\!R$ defined by

$$\text{epi } f := \big\{(x, \alpha) \in X \times I\!R \;\big|\; x \in \text{dom } f \;\text{ and }\; \alpha \geq f(x)\big\}.$$

The function f is *closed* if its epigraph is closed, and it is *convex* is its epigraph is a convex subset of $X \times I\!R$. It is easy to check that f is convex if and only if

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) \;\text{ for all }\; x, y \in \text{dom } f \;\text{ and }\; \lambda \in [0,1].$$

Furthermore, as we have seen from the previous chapter, a nonempty closed subset Ω of X is convex if and only if the corresponding distance function $f(x) = d(x;\Omega)$ is a convex function. Moreover, the distance function $f(x) = d(x;\Omega)$ is Lipschitz continuous on X with modulus one, i.e.,

$$|f(x) - f(y)| \leq ||x - y|| \;\text{ for all }\; x, y \in X.$$

A typical example of an extended-real-valued function is the set *indicator function*

$$\delta(x;\Omega) := \begin{cases} 0 & \text{if } x \in \Omega, \\ \infty & \text{otherwise.} \end{cases} \tag{2.1.1}$$

It follows immediately from the definitions that the set $\Omega \subset X$ is convex if and only if the indicator function (2.1.1) is convex.

An element $x^* \in X^*$ is called a *subgradient* of a convex function $f\colon X \to \overline{I\!R}$ at $\bar{x} \in \text{dom} f$ if it satisfies the inequality

$$\langle x^*, x - \bar{x} \rangle \le f(x) - f(\bar{x}) \text{ for all } x \in X, \tag{2.1.2}$$

where $\langle \cdot, \cdot \rangle$ stands for the usual scalar product in X. The set of all the subgradients x^* in (2.1.2) is called the *subdifferential* of f at $\bar{x}$ and is denoted by $\partial f(\bar{x})$. If f is convex and differentiable at $\bar{x}$, then $\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$.

A well-recognized technique in optimization is to reduce a constrained optimization problem to an unconstrained one using the indicator function of the constraint. Indeed, $\bar{x} \in \Omega$ is a minimizer of the constrained optimization problem:

$$\text{minimize } f(x) \text{ subject to } x \in \Omega \tag{2.1.3}$$

if and only if it solves the unconstrained problem

$$\text{minimize } f(x) + \delta(x;\Omega), \quad x \in X. \tag{2.1.4}$$

By the definitions, for any convex function $\varphi\colon X \to \overline{I\!R}$,

$$\bar{x} \text{ is a minimizer of } \varphi \text{ if and only if } 0 \in \partial\varphi(\bar{x}), \tag{2.1.5}$$

which is *nonsmooth convex* counterpart of the classical *Fermat stationary rule*. Applying (2.1.5) to the constrained optimization problem (2.1.3) via its unconstrained description (2.1.4) requires the usage of *subdifferential calculus*.

The most fundamental calculus result of convex analysis is the following Moreau-Rockafellar theorem for representing the subdifferential of sums.

Theorem 2.1.1. *Let $\varphi_i\colon X \to \overline{I\!R}$, $i = 1, \ldots, m$, be closed convex functions. Assume that there is a point $\bar{x} \in \cap_{i=1}^m dom\,\varphi_i$ at which all but (except possibly one) of the functions $\varphi_1, \ldots, \varphi_m$ are continuous. Then we have the equality*

$$\partial\Big(\sum_{i=1}^m \varphi_i\Big)(x) = \sum_{i=1}^m \partial\varphi_i(x) = \Big\{\sum_{i=1}^m v_i \;\Big|\; v_i \in \partial\varphi_i(x)\; for\; i = 1, \ldots, m\Big\}$$

for all $x \in \cap_{i=1}^m dom\,\varphi_i$.

The following theorem presents the well known subdifferential rule that involves "max" functions.

Theorem 2.1.2. *Let $f_i : X \to I\!R$, $i = 1, \ldots, n$, be continuous convex functions. Define*

$$f(x) := \max\{f_i(x) \;\big|\; i = 1, \ldots, n\}.$$

Then

$$\partial f(\bar{x}) = co\,\{\partial f_i(\bar{x}) \;\big|\; i \in I(\bar{x})\},$$

where $I(\bar{x}) := \{i = 1, \ldots, n \;\big|\; f(\bar{x}) = f_i(\bar{x})\}$.

Given a convex set $\Omega \subset X$ and a point $\bar{x} \in \Omega$, the corresponding geometric counterpart of (2.1.2) is the *normal cone* to Ω at $\bar{x}$ defined by

$$N(\bar{x};\Omega) := \Big\{v \in X \;\Big|\; \langle v, x - \bar{x}\rangle \le 0 \;\text{ for all }\; x \in \Omega\Big\}. \tag{2.1.6}$$

It easily follows from the definitions that

$$\partial\delta(\bar{x};\Omega) = N(\bar{x};\Omega) \text{ for every } \bar{x} \in \Omega, \tag{2.1.7}$$

which allows us, in particular, to characterize minimizers of the constrained problem (2.1.3) in terms of the subdifferential (2.1.2) of f and the normal cone (2.1.6) to Ω by applying Theorem 2.1.1 to the function $\varphi(x) = f(x) + \delta(x;\Omega)$ in (2.1.5).

2.2 Subgradient Method in Convex Optimization

In this section we are going to prove detailed proofs of the convergence of iterative sequences generated by subgradient method for solving nonsmooth convex minimization problems. This method was originally developed by Naum Z. Shor and others in the 1960s and 1970s. The reader are referred to [1, 5] for more discussion related to the method.

Proposition 2.2.1. *Suppose that $f : X \to I\!R$ is a convex function which satisfies a Lipschitz condition with constant $\ell < \infty$. Then for any $\bar{x} \in X$ one has*

$$\|x^*\| \leq \ell \text{ for all } x^* \in \partial f(\bar{x}).$$

Proof: Take any $x^* \in \partial f(\bar{x})$. It follows from the definition of subdifferential that

$$\langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) \text{ for any } x \in X.$$

Since f satisfies a Lipschitz condition with constant ℓ, one has

$$\langle x^*, x - \bar{x} \rangle \leq \ell \|x - \bar{x}\| \text{ for all } x \in X.$$

Set $x = u + \bar{x}$. Then

$$\langle x^*, u + \bar{x} - \bar{x} \rangle \leq \ell \|u + \bar{x} - \bar{x}\| \text{ for all } u \in X,$$

which is equivalent to

$$\langle x^*, u \rangle \leq \ell \|u\| \text{ for any } u \in X.$$

This implies

$$\|x^*\| \leq \ell.$$

The proof in now complete. □

In what follows we will focus on the subgradient algorithm and its convergence. Let $f : X \to I\!R$ be a convex function. Given a sequence $\{\alpha_k\}$ with $\alpha_k \geq 0$ for all k, define the sequence $\{x_k\}$ as $k \in I\!N$ with an arbitrary starting point $x_1 \in X$ as follows

$$x_{k+1} = x_k - \alpha_k x_k^*, \tag{2.2.1}$$

where x_k^* is *any subgradient* of $\partial f(x_k)$. Consider the following unconstrained optimization problem

$$\text{minimize } f(x) \text{ subject to } x \in X. \tag{2.2.2}$$

Proposition 2.2.2. *Suppose that $f : X \to I\!R$ is a convex function that satisfies a Lipschitz condition with constant ℓ. Then for the sequence $\{x_k\}$ generated by algorithm* (2.2.1)*, we have that*

$$\|x_{k+1} - x\|^2 \leq \|x_k - x\|^2 - 2\alpha_k(f(x_k) - f(x)) + {\alpha_k}^2 \ell^2 \textit{ for all } x \in X \textit{ and all } k \in I\!N.$$

when $\|\cdot\|$ is the Euclidean norm on X.

Proof: From (2.2.1) we have that

$$\begin{aligned} \|x_{k+1} - x\|^2 &= \|x_k - \alpha_k x_k^* - x\|^2 \\ &= \|x_k - x - \alpha_k x_k^*\|^2 \\ &= \|x_k - x\|^2 - 2\alpha_k \langle x_k^*, x_k - x\rangle + {\alpha_k}^2 \|x_k^*\|^2 \end{aligned}$$

Since $x_k^* \in \partial f(x_k)$, one has $\|x_k^*\| \leq \ell$ and

$$\langle x_k^*, x_k - x\rangle \geq f(x) - f(x_k).$$

Thus

$$\|x_{k+1} - x\|^2 \leq \|x_k - x\|^2 - 2\alpha_k(f(x_k) - f(x)) + {\alpha_k}^2 \ell^2 \text{ for all } x \in X \text{ and all } k \in I\!N.$$

The proof in now complete. □

Let

$$V_k = \min\{f(x_1), \ldots, f(x_k)\}. \tag{2.2.3}$$

Proposition 2.2.3. *Suppose that problem* (2.2.2) *has an optimal solution, with the optimal value* V_*. *Under the same assumptions as in proposition* (2.2.2) *we have*

$$V_k - V_* \leq \frac{d(x_1; A^*)^2 + \ell^2 \sum_{i=1}^k \alpha_i^2}{2\sum_{i=1}^k \alpha_i}, \text{ for all } k \in I\!N,$$

where A *is the solution set of* (2.2.2) *and* $d(x_1; A)$ *is generated by the Euclidean distance.*

Proof: From Proposition 2.2.2 we have that

$$\|x_{k+1} - x\|^2 \leq \|x_k - x\|^2 - 2\alpha_k(f(x_k) - f(x)) + \alpha_k^2\ell^2 \text{ for all } x \in X \text{ and all } k \in I\!N.$$

Let $x = x_* \in A$. Then $V_* = f(x_*)$ and the preceding relation can be expressed as

$$\|x_{k+1} - x_*\|^2 \leq \|x_k - x_*\|^2 - 2\alpha_k(f(x_k) - V_*) + \alpha_k^2\ell^2 \text{ for all } k \in I\!N.$$

Applying the previous relation recursively, we have

$$\|x_{k+1} - x_*\|^2 \leq \|x_1 - x_*\|^2 - 2\sum_{i=1}^k \alpha_i(f(x_i) - V_*) + \ell^2 \sum_{i=1}^k \alpha_i^2 \text{ for all } k \in I\!N. \tag{2.2.4}$$

Using $\|x_{k+1} - x_*\|^2 \geq 0$, equation (2.2.4) yields to

$$2\sum_{i=1}^k \alpha_i(f(x_i) - V_*) \leq \|x_1 - x_*\|^2 + \ell^2 \sum_{i=1}^k {\alpha_i}^2 \text{ for all } k \in I\!N.$$

Moreover,

$$\sum_{i=1}^k \alpha_i(f(x_i) - V_*) \geq (\sum_{i=1}^k \alpha_i)(V_k - V_*),$$

and combined with the previous inequality we have

$$2(\sum_{i=1}^k \alpha_i)(V_k - V_*) \leq \|x_1 - x_*\|^2 + \ell^2 \sum_{i=1}^k \alpha_i^2 \text{ for all } k \in I\!N.$$

Since x_* is an arbitrary optimal solution, one has

$$V_k - V_* \leq \frac{d(x_1; A)^2 + \ell^2 \sum_{i=1}^k \alpha_i^2}{2\sum_{i=1}^k \alpha_i}, \text{ for all } k \in I\!N, \tag{2.2.5}$$

where A denotes the optimal set and $d(x_1; A)$ is the Euclidean distance of x_1 to the optimal set. The proof in now complete. □

This proposition yields the following direct consequence.

Corollary 2.2.4. *Under the same setting of Proposition 2.2.2, suppose that*

$$\sum_{k=1}^{\infty} \alpha_k = \infty \textit{ and } \sum_{k=1}^{\infty} \alpha_k^2 < \infty.$$

Then the sequence $\{V_k\}$ *defined in* (2.2.3) *converges to the optimal* V_*.

We are now going to prove a convergence criterion for the sequence $\{x_k\}$ generated by the subgradient method.

Lemma 2.2.5. *Suppose that problem* (2.2.2) *has the solution set* $\emptyset \neq A \subset X$ *with optimal value* V_*. *Under the same assumptions as in proposition* (2.2.2)*, and with*

$$\lim_{k\to\infty} \alpha_k = 0 \textit{ and } \sum_{k=1}^{\infty} \alpha_k = \infty,$$

the sequence $\{x_k\}$ *generated by algorithm* (2.2.1) *satisfies*

$$\liminf_{k\to\infty} f(x_k) = V_*.$$

Proof: Since V_* is the optimal value it is clear that $f(x_k) \geq V_*$ for any k. This implies

$$\liminf_{k\to\infty} f(x_k) \geq V_*.$$

Suppose by contradiction that

$$\liminf_{k\to\infty} f(x_k) > V_*.$$

Then there exists $\epsilon > 0$ such that

$$\liminf_{k\to\infty} f(x_k) - 2\epsilon > V_* = \inf_{x\in X} f(x).$$

Hence there exists $\bar{x} \in X$ such that

$$\liminf_{k\to\infty} f(x_k) - 2\epsilon > f(\bar{x}),$$

which implies

$$\liminf_{k\to\infty} f(x_k) - \epsilon > f(\bar{x}) + \epsilon.$$

There exists $k_0 \in I\!N$ such that for every $k \geq k_0$ we have that

$$f(x_k) > \liminf_{k\to\infty} f(x_k) - \epsilon.$$

Thus $f(x_k) - f(\bar{x}) > \epsilon$ for every $k \geq k_0$. From Proposition 2.2.2 and letting $x = \bar{x}$ we have

$$\|x_{k+1} - \bar{x}\|^2 \leq \|x_k - \bar{x}\|^2 - 2\alpha_k(f(x_k) - f(\bar{x})) + {\alpha_k}^2\ell^2 \text{ for every } k \geq k_0. \tag{2.2.6}$$

Let k_1 be such that $\alpha_k\ell^2 < \epsilon$ for all $k \geq k_1$, and let $\bar{k}_1 = \max\{k_1, k_0\}$. Since $\bar{k}_1 \geq k_0$, we can derive from (2.2.6) that

$$\|x_{k+1} - \bar{x}\|^2 \leq \|x_k - \bar{x}\|^2 - \alpha_k\epsilon \leq \|x_{k-1} - \bar{x}\|^2 - \epsilon(\alpha_{k-1} + \alpha_k) \leq \cdots \leq \left\|x_{\bar{k}_1} - \bar{x}\right\|^2 - \epsilon\sum_{j=\bar{k}_1}^{k} \alpha_j.$$

This is a contradiction because $\sum_{k=1}^{\infty} \alpha_k = \infty$. The proof in now complete. □

Lemma 2.2.6. *Suppose that problem* (2.2.2) *has the solution solution set* $\emptyset \neq A \subset X$ *with the optimal value* V_*. *Under the same assumptions as in proposition* (2.2.2)*, and with*

$$\sum_{k=1}^{\infty} \alpha_k = \infty; \textit{ and } \sum_{k=1}^{\infty} \alpha_k^2 < \infty,$$

the sequence $\{x_k\}$ *generated by algorithm* (2.2.1) *converges to some optimal solution.*

Proof: From Proposition 2.2.2 and letting $x = x_* \in A$, we have that

$$\|x_{k+1} - x_*\|^2 \le \|x_k - x_*\|^2 - 2\alpha_k(f(x_k) - V_*) + {\alpha_k}^2\ell^2 \text{ for all } x_* \in A \text{ and all } k \in I\!N. \quad (2.2.7)$$

Since $f(x_k) - V_* \ge 0$ for all k we can rewrite (2.2.7) as

$$\|x_{k+1} - x_*\|^2 \le \|x_k - x_*\|^2 + {\alpha_k}^2\ell^2 \text{ for all } x_* \in A \text{ and all } k \in I\!N.$$

By combining the previous two relations and repeating the same procedure down to $k = 1$, we get

$$\|x_{k+1} - x_*\|^2 \le \|x_1 - x_*\|^2 + \ell^2 \sum_{i=1}^{k} \alpha_i^2 \text{ for all } x_* \in A \text{ and all } k.$$

By the assumption that $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$, the sequence $\{x_k\}$ is bounded. Moreover, by Lemma 2.2.5, we have that

$$\liminf_{k\to\infty} f(x_k) = V_*.$$

Let $\{x_{k_j}\}$ be a subsequence of $\{x_k\}$ along which the above $\liminf$ is attained. Then

$$\lim_{j\to\infty} f(x_{k_j}) = V_*. \quad (2.2.8)$$

Since the sequence $\{x_{k_j}\}$ is also bounded, it contains a convergent subsequence. Without loss of generality we can assume that $x_{k_j} \to \bar{x}$. Since f is continuous and from (2.2.8), we have that $\bar{x} \in A$. Thus we can rewrite (2.2.7) with $x_* = \bar{x}$ for any j and any $k \ge k_j$ as

$$\|x_{k+1} - \bar{x}\|^2 \le \|x_k - \bar{x}\|^2 + {\alpha_k}^2\ell^2 \le \cdots \le \left\|x_{k_j} - \bar{x}\right\|^2 + \ell^2 \sum_{i=k_j}^{k} \alpha_i^2.$$

Taking first the limit as $k \to \infty$ and then the limit as $j \to \infty$, from the preceding relation, we obtain

$$\limsup_{k\to\infty} \|x_{k+1} - \bar{x}\|^2 \le \lim_{j\to\infty} \left\|x_{k_j} - \bar{x}\right\|^2 + \ell^2 \lim_{j\to\infty} \sum_{i=k_j}^{\infty} \alpha_i^2.$$

Since $x_{k_j} \to \bar{x}$ and $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$, one has

$$\limsup_{k\to\infty} \|x_{k+1} - \bar{x}\|^2 = 0.$$

Consequently, $x_k \to \bar{x}$ with $\bar{x} \in A$. The proof in now complete. $\square$

2.3 Projected Subgradient Method in Convex Optimization

In this section we will focus on subgradient method for constrained optimization problem called the *projected subgradient method*, which will be important for the study of the generalized Heron problem (1.1.1). We will show that the convergence results for the projected subgradient method follow from the standard subgradient method results discussed in the previous section.

Let $g : X \to I\!R$ be a convex function which satisfies a Lipschitz condition with constant ℓ and let $\Omega \subset X$ be a nonempty closed convex set. Consider the following constrained optimization problem

$$\text{minimize } g(x) \text{ subject to } x \in \Omega. \tag{2.3.1}$$

Given a sequence of nonnegative real numbers $\{\alpha_k\}$, the sequence $\{x_k\}$ as $k \in I\!N$ with a chosen starting point $x_1 \in \Omega$ is defined by

$$x_{k+1} = P(x_k - \alpha_k x_k^*; \Omega), \tag{2.3.2}$$

where x_k^* is any subgradient of $\partial g(x_k)$ and P denotes the Euclidean projection onto Ω.

Proposition 2.3.1. *Suppose that $g : X \to I\!R$ satisfy a Lipschitz condition with constant ℓ. Then the sequence $\{x_k\}$ generated by algorithm* (2.3.2) *satisfies the following*

$$\|x_{k+1} - x\|^2 \le \|x_k - x_*\|^2 - 2\alpha_k(g(x_k) - g(x)) + \alpha_k{}^2 l^2 \textit{ for all } k \in I\!N \textit{ and for all } x \in \Omega,$$

where $\|\cdot\|$ denotes the Euclidean norm.

Proof: Let $z_{k+1} = x_k - \alpha_k x_k^*$ and fix any $x \in \Omega$. Then by Proposition 2.2.2 we have that

$$\|z_{k+1} - x\|^2 \le \|x_k - x\|^2 - 2\alpha_k(g(x_k) - g(x)) + {\alpha_k}^2\ell^2 \text{ and all } k \in I\!N. \tag{2.3.3}$$

Now by projecting z_{k+1} onto Ω, by the convexity of Ω, we have

$$\|x_{k+1} - x\| = \|P(z_{k+1};\Omega) - P(x;\Omega)\| \le \|z_{k+1} - x\|.$$

By combining these two relations we conclude that

$$\|x_{k+1} - x\|^2 \le \|x_k - x\|^2 - 2\alpha_k(g(x_k) - g(x)) + {\alpha_k}^2\ell^2 \text{ for all } k \in I\!N$$

The proof in now complete. □

Based on this proposition we can easily obtain the same convergence results for the constrained problem (2.3.1) by exactly the same way as in the previous section.

CHAPTER III

SUBDIFFERENTIALS OF DISTANCE FUNCTIONS

In this chapter we study generalized differentiation properties of distance functions and present new examples of distance functions generated by different norms for different convex sets.

3.1 Subdifferential Formulas for Distance Functions: Convex Case

In this section we study subdifferential formulas for distance functions to convex sets which will play an important role in solving generalized facility location problems. This material and many other extensions can be found in [12] and the references therein. Throughout this section we assume that Ω is a closed convex subset of X.

Proposition 3.1.1. *Suppose that $\bar{x} \in \Omega$. Then*

$$\partial d(\bar{x};\Omega) = N(\bar{x};\Omega) \cap I\!B^*, \tag{3.1.1}$$

where $I\!B^*$ is the closed unit ball in X^*.

Proof: Fix any $x^* \in \partial d(\bar{x};\Omega)$. Then

$$\langle x^*, x - \bar{x}\rangle \le d(x;\Omega) - d(\bar{x};\Omega) = d(x;\Omega) \text{ for all } x \in X. \tag{3.1.2}$$

Since the distance function $d(\cdot;\Omega)$ satisfies a Lipschitz condition with the Lipschitz constant $\ell = 1$, one has

$$\langle x^*, x - \bar{x}\rangle \le ||x - \bar{x}|| \text{ for all } x \in X.$$

This implies $||x^*|| \leq 1$ or $x^* \in I\!\!B^*$. It also follows from (3.1.2) that

$$\langle x^*, x - \bar{x} \rangle \leq 0 \text{ for all } x \in \Omega.$$

Thus $x^* \in N(\bar{x}; \Omega)$. Therefore, $x^* \in N(\bar{x}; \Omega) \cap I\!\!B^*$.

Let us now prove the opposite inclusion. Fix any $x^* \in N(\bar{x}; \Omega) \cap I\!\!B^*$. Then $||x^*|| \leq 1$ and

$$\langle x^*, w - \bar{x} \rangle \leq 0 \text{ for all } w \in \Omega.$$

Thus for any $x \in X$ and for any $w \in \Omega$, one has

$$\begin{aligned}
\langle x^*, x - \bar{x} \rangle &= \langle x^*, x - w + w - \bar{x} \rangle \\
&= \langle x^*, x - w \rangle + \langle x^*, w - \bar{x} \rangle \\
&\leq \langle x^*, x - w \rangle \\
&\leq ||x^*|| ||x - w|| \leq ||x - w||.
\end{aligned}$$

This implies

$$\langle x^*, x - \bar{x} \rangle \leq d(x; \Omega) = d(x; \Omega) - d(\bar{x}; \Omega),$$

and hence $x^* \in \partial d(\bar{x}; \Omega)$. □

For any $r > 0$, define

$$\Omega_r := \{x \in X \mid d(x; \Omega) \leq r\}.$$

Recall that

$$S^* := \{x^* \in X^* \mid ||x^*|| = 1\}.$$

The following two lemmas play an important role in representing subdifferential of distance functions for the out-of-set case.

Lemma 3.1.2. *Let $r > 0$ and let $\bar{x} \notin \Omega_r$. Then*

$$d(x;\Omega) = d(x;\Omega_r) + r.$$

Proof: Since $x \notin \Omega_r$, one has $d(x;\Omega) > r$. For any $\varepsilon > 0$, there exists $w_r \in \Omega_r$ such that

$$||x - w_r|| < d(x;\Omega_r) + \varepsilon.$$

As $d(w_r, \Omega) \leq r$, there exists $w \in \Omega$ with

$$||w - w_r|| < r + \varepsilon.$$

Thus

$$d(x;\Omega) \leq ||x - w|| \leq ||x - w_r|| + ||w_r - w|| < r + d(x;\Omega_r) + 2\varepsilon.$$

This implies

$$d(x;\Omega) \leq r + d(x;\Omega_r).$$

Let us now prove the opposite inequality. For any $\varepsilon > 0$, there exists $w \in \Omega$ such that

$$||x - w|| < d(x;\Omega) + \varepsilon.$$

Consider the function

$$f(t) = d(x + t(w - x);\Omega) - r.$$

This function is continuous and $f(0) = d(x;\Omega) - r > 0$, $f(1) = -r < 0$. Thus there exists $t \in (0,1)$ such that

$$d(w_t;\Omega) = r, \text{ where } w_t := x + t(w - x).$$

Then $w_t \in \Omega_r$ and

$$r + d(x;\Omega_r) = d(w_t;\Omega) + d(x;\Omega_r) \leq ||w_t - w|| + ||x - w_t|| = ||x - w|| < d(x;\Omega) + \varepsilon.$$

Notice that the triangle inequality becomes equality for three points x, w, w_t because they are on the same line (we can also see this by using the specific form of w_t). It follows that

$$r + d(x; \Omega_r) \leq d(x; \Omega).$$

The proof is now complete. □

Lemma 3.1.3. *Let $\bar{x} \notin \Omega$ and let $x^* \in \partial d(\bar{x}; \Omega)$. Then $x^* \in S^*$.*

Proof: Fix any $x^* \in \partial d(\bar{x}; \Omega)$. Then

$$\langle x^*, x - \bar{x} \rangle \leq d(x; \Omega) - d(\bar{x}; \Omega) \leq ||x - \bar{x}|| \text{ for all } x \in X. \tag{3.1.3}$$

This implies $||x^*|| \leq 1$. For any $t > 0$, there exists $\bar{w} \in \Omega$ such that

$$d(\bar{x}; \Omega) \leq ||\bar{x} - \bar{w}|| < d(\bar{x}; \Omega) + t^2.$$

Consider $x_t = \bar{x} - t(\bar{x} - \bar{w})$. Then the following holds by (3.1.5),

$$\begin{aligned} -t\langle x^*, \bar{x} - \bar{w} \rangle &\leq d(\bar{x} - t(\bar{x} - \bar{w}; \Omega) - d(\bar{x}; \Omega) \\ &\leq ||\bar{x} - t(\bar{x} - \bar{w}) - \bar{w}|| - ||\bar{x} - \bar{w}|| + t^2 \\ &= (1 - t)||\bar{x} - \bar{w}|| - ||\bar{x} - \bar{w}|| + t^2 \\ &= -t||\bar{x} - \bar{w}|| + t^2. \end{aligned}$$

This implies

$$||\bar{x} - \bar{w}|| \leq ||x^*||||\bar{x} - \bar{w}|| + t,$$

and hence

$$1 \leq ||x^*|| + \frac{t}{||\bar{x} - \bar{w}||} \leq ||x^*|| + \frac{t}{d(\bar{x}; \Omega)}.$$

Letting $t \to 0$, one has $1 \leq ||x^*||$. The proof is complete. □

Theorem 3.1.4. *Let* $\bar{x} \notin \Omega$ *and let* $r = d(\bar{x}; \Omega)$. *Then*

$$\partial d(\bar{x}; \Omega) = N(\bar{x}; \Omega_r) \cap S^*.$$

Proof: Fix any $x^* \in \partial d(\bar{x}; \Omega)$. Then

$$\langle x^*, x - \bar{x} \rangle \leq d(x; \Omega) - d(\bar{x}; \Omega) \text{ for all } x \in X.$$

Thus for any $x \in \Omega_r$, one has $d(x; \Omega) \leq r = d(\bar{x}; \Omega)$, and hence

$$\langle x^*, x - \bar{x} \rangle \leq d(x; \Omega) - d(\bar{x}; \Omega) \leq 0.$$

This implies $x \in N(\bar{x}; \Omega_r)$. The fact that $x^* \in S^*$ follows from the previous lemma.

Let us now prove the converse. Fix any $x^* \in N(\bar{x}; \Omega_r) \cap S^*$. Then

$$\langle x^*, x - \bar{x} \rangle \leq 0 \text{ for all } x \in \Omega_r. \tag{3.1.4}$$

Moreover, by the in-set case

$$x^* \in N(\bar{x}; \Omega_r) \cap I\!B^* = \partial d(\bar{x}; \Omega_r).$$

Thus

$$\langle x^*, x - \bar{x} \rangle \leq d(x; \Omega_r) \text{ for all } x \in X.$$

If $x \notin \Omega_r$, then $d(x; \Omega) = d(x; \Omega_r) + r$. Thus

$$\langle x^*, x - \bar{x} \rangle \leq d(x; \Omega_r) = d(x; \Omega) - r = d(x; \Omega) - d(\bar{x}; \Omega).$$

Now suppose that $x \in \Omega_r$. Then $d(x; \Omega) \leq r$. Since $||x^*|| = 1$, for any $\varepsilon > 0$, there exists $q \in B^*$ such that

$$\langle x^*, q \rangle > 1 - \varepsilon.$$

Let $t = r - d(x;\Omega) \geq 0$ and let $w_t = x + tq$. Then

$$d(w_t;\Omega) \leq d(x;\Omega) + t||q|| \leq d(x;\Omega) + t = d(x;\Omega) + r - d(x;\Omega) = r.$$

It follows that $w_t \in \Omega_r$, and hence by (3.1.4), one has

$$\langle x^*, x + tq - \bar{x}\rangle \leq 0.$$

This implies

$$\langle x^*, x - \bar{x}\rangle \leq -t\langle x^*, q\rangle \leq -t(1-\varepsilon) = [d(x;\Omega) - d(\bar{x};\Omega)](1-\varepsilon).$$

Notice that ε does not depend on x. Thus in this case we also have

$$\langle x^*, x - \bar{x}\rangle \leq d(x;\Omega) - d(\bar{x};\Omega).$$

Therefore, $x^* \in \partial d(\bar{x};\Omega)$. The proof is complete. □

Let us now consider the case where $\bar{x} \notin \Omega$. Define

$$\Pi(\bar{x};\Omega) = \{\bar{w} \in \Omega \mid d(\bar{x};\Omega) = ||\bar{x} - \bar{w}||\}.$$

Theorem 3.1.5. *Suppose that $\bar{x} \notin \Omega$ and $\Pi(\bar{x};\Omega) \neq \emptyset$. Then*

$$\partial d(\bar{x};\Omega) = \partial p(\bar{x} - \bar{w}) \cap N(w;\Omega)$$

for any $\bar{w} \in \Pi(\bar{x};\Omega)$ (that means the representation does not depend on the choice of $\bar{w}$), where $p(x) = ||x||$.

Proof: Fix any $x^* \in \partial d(\bar{x};\Omega)$. Then

$$\langle x^*, x - \bar{x}\rangle \leq d(x;\Omega) - d(\bar{x};\Omega) \text{ for all } x \in X. \tag{3.1.5}$$

Fix any $\bar{w} \in \Pi(\bar{x}; \Omega)$. Then the following holds for any $w \in \Omega$.

$$\begin{aligned}\langle x^*, w - \bar{w}\rangle &= \langle x^*, (w - \bar{w} + \bar{x}) - \bar{x}\rangle \\ &\le d(w - \bar{w} + \bar{x}; \Omega) - d(\bar{x}; \Omega) \\ &\le d(w - \bar{w} + \bar{x}; \Omega) - ||\bar{x} - \bar{w}|| \\ &\le ||(w - \bar{w} + \bar{x}) - w|| - ||\bar{x} - \bar{w}|| = 0.\end{aligned}$$

Thus $x^* \in N(\bar{w}; \Omega)$. For any $x \in X$ and $t \in (0, 1)$. Considering $\tilde{x} = \bar{x} - \bar{w}$ and applying (3.1.5) for $\bar{x} + t(x - \tilde{x})$, where $x \in X$ is arbitrary, one has

$$\begin{aligned}t\langle x^*, x - \tilde{x}\rangle &= \langle x^*, \bar{x} + t(x - \tilde{x}) - \bar{x}\rangle \\ &\le d(\bar{x} + t(x - \tilde{x}); \Omega) - d(\bar{x}; \Omega) \\ &\le ||\bar{x} + t(x - \tilde{x}) - \bar{w}|| - ||\bar{x} - \bar{w}|| \\ &= ||tx - t\tilde{x} + \tilde{x}|| - ||\tilde{x}|| \\ &= ||tx + (1 - t)\tilde{x}|| - ||\tilde{x}|| \\ &\le t||x|| + (1 - t)||\tilde{x}|| - ||\tilde{x}|| \\ &= t(||x|| - ||\tilde{x}||) \\ &= t(p(x) - p(\tilde{x})).\end{aligned}$$

This implies $x^* \in \partial p(\tilde{x})$.

Let us now prove the opposite inclusion. Take any $x^* \in \partial p(\bar{x} - \bar{w}) \cap N(\bar{w}; \Omega)$. Since $\bar{x} - \bar{w} \neq 0$. By the property of norm function $||x^*|| = 1$ or $x^* \in S^*$. By Theorem (3.1.4). we only need to prove that $x^* \in N(\bar{x}; \Omega_r)$. Fix any $x \in \Omega_r$. Then for any $\epsilon > 0$, one has $d(x; \Omega) \le r < r + \epsilon$. Thus there exist $w \in \Omega$ such that $||x - w|| \leq r + \varepsilon$. Then

$$\begin{aligned}
\langle x^*, x - \bar{x} \rangle &= \langle x^*, x - w + w - \bar{x} \rangle \\
&= \langle x^*, x - w \rangle + \langle x^*, w - \bar{x} \rangle \\
&\leq ||x^*||||x - w|| + \langle x^*, w - \bar{w} + \bar{w} - \bar{x} \rangle \\
&\leq r + \varepsilon + \langle x^*, w - \bar{w} \rangle + \langle x^*, \bar{w} - \bar{x} \rangle \\
&= d(\bar{x}; \Omega) + \varepsilon + \langle x^*, 0 - (\bar{x} - \bar{w}) \rangle \\
&= d(\bar{x}; \Omega) + \varepsilon + p(0) - p(\bar{x} - \bar{w}) = \varepsilon
\end{aligned}$$

Since ε does not depend on x, one has

$$\langle x^*, x - \bar{x} \rangle \leq 0 \text{ for all } x \in \Omega_r.$$

It follows that $x^* \in N(\bar{x}; \Omega_r) \cap S^* = \partial d(\bar{x}; \Omega)$. The proof is complete. □

Corollary 3.1.6. *Let X be endowed with the Euclidean norm. The following holds*

$$\partial d(\bar{x}; \Omega) = \begin{cases} N(\bar{x}; \Omega) \cap I\!\!B, & \text{if } \bar{x} \in \Omega \\ \{\dfrac{\bar{x} - \Pi(\bar{x}; \Omega)}{d(\bar{x}; \Omega)}\}, & \text{if } \bar{x} \notin \Omega. \end{cases}$$

Proof: The formula for the case $\bar{x} \in \Omega$ follows from Proposition 3.1.1. Let $\bar{x} \notin \Omega$. Since X is endowed with the Euclidean norm, $\bar{w} := \Pi(\bar{x}; \Omega)$ is singleton. By Theorem 3.1.5, one has

$$\begin{aligned}
\partial d(\bar{x}; \Omega) &= \partial p(\bar{x} - \bar{w}) \cap N(\bar{w}; \Omega) \\
&= \{\frac{\bar{x} - \bar{w}}{||\bar{x} - \bar{w}||}\} \cap N(\bar{w}; \Omega) \\
&= \{\frac{\bar{x} - \bar{w}}{||\bar{x} - \bar{w}||}\} \\
&= \{\frac{\bar{x} - \Pi(\bar{x}; \Omega)}{d(\bar{x}; \Omega)}\}.
\end{aligned}$$

Notice that the last equality holds since $\dfrac{\bar{x} - \bar{w}}{||\bar{x} - \bar{w}||} \in N(\bar{w}; \Omega)$ as always. □

In the remaining sections of this chapter we are going to use the subdifferential formulas from this section to provide various examples of the subdifferential of the distance function in different settings and with different norms.

3.2 Subdifferential of Distance Functions with the Euclidean Norm

In this section let the distance function generated by a set Ω be defined as

$$d(x;\Omega) = \inf\{||x-w||_2 \;\big|\; w \in \Omega\},$$

where $||\cdot||_2$ is the Euclidean norm in X.

Proposition 3.2.1. *Let $X = I\!R^s$ with the Euclidean norm. The subdifferential $\partial p(\bar{x})$ of the function $p(x) = ||x||_2$ has the following explicit representation*

$$\partial p(\bar{x}) = \begin{cases} I\!B & \text{if } \bar{x} = 0, \\ \{\dfrac{\bar{x}}{||\bar{x}||_2}\} & otherwise. \end{cases}$$

Proof: Consider the case where $\bar{x} = 0$. We will prove that $\partial p(0) = I\!B$. Take any $x^* \in \partial p(0)$. Then

$$\langle x^*, x - \bar{x}\rangle \leq p(x) - p(\bar{x}) \text{ for all } x \in X.$$

This implies

$$\langle x^*, x\rangle \leq p(x) = ||x||_2 \text{ for all } x \in X.$$

Thus $||x^*||_2 \leq 1$ and

$$\partial p(0) \subset I\!B.$$

Now take any $x^* \in I\!B$. By the Cauchy Schwarz Inequality

$$\langle x^*, x\rangle \leq |\langle x^*, x\rangle| \leq ||x^*||_2||x||_2 \text{ for all } x \in X.$$

Since $||x^*||_2 \leq 1$, one has

$$\langle x^*, x\rangle \leq ||x||_2 \text{ for } x \in X,$$

which is equivalent to

$$\langle x^*, x - \bar{x}\rangle \leq ||x||_2 - ||0||_2 = p(x) - p(0) \text{ for all } x \in X.$$

Thus $I\!B \subset \partial p(0)$ and the equality is true for the case $\bar{x} = 0$. Now we consider the case where $\bar{x} \neq 0$. Let $x^* \in \partial p(\bar{x})$. Since $p(x) = ||x||_2$ is convex and differentiable at $\bar{x}$, one has

$$\partial p(\bar{x}) = \{\nabla p(\bar{x})\} = \big\{\frac{\bar{x}}{||\bar{x}||_2}\big\}.$$

The proof is now complete. □

Using Proposition 3.2.1 and Theorem 3.1.5 we are able to derive the subdifferential formula for the distance function $d(x;\Omega)$ where Ω is a set of particular shapes.

Example 3.2.2.

In this example we consider $\Omega = I\!B(a;r) \subset X$ endowed with the Euclidean norm.

Consider the case where $||\bar{x} - a|| > r$. In this case $\bar{x} \notin \Omega$ and

$$\begin{aligned}\partial d(\bar{x};\Omega) &= \partial p(\bar{x} - \bar{w}) \cap N(\bar{w};\Omega)\\ &= \big\{\frac{\bar{x} - \bar{w}}{d(\bar{x};\Omega)}\big\} \cap \big\{t\frac{\bar{w} - a}{||\bar{w} - a||_2} \bigm| t \geq 0\big\}\\ &= \big\{\frac{\bar{x} - a}{||\bar{x} - a||_2}\big\},\end{aligned}$$

where $\bar{w} \in \Pi(\bar{x};\Omega)$.

Consider the case where $||\bar{x} - a|| = r$. In this case $\bar{w} \in \Pi(\bar{x};\Omega) = \{\bar{x}\}$ and

$$\begin{aligned}\partial d(\bar{x};\Omega) &= \partial p(0) \cap N(\bar{x};\Omega)\\ &= I\!B \cap \big\{t\frac{\bar{x} - a}{||\bar{x} - a||_2} \bigm| t \geq 0\big\}\\ &= \big\{t\frac{\bar{x} - a}{||\bar{x} - a||_2} \bigm| 0 \leq t \leq 1\big\}.\end{aligned}$$

Consider the case where $||\bar{x} - a|| < r$. In this case $\bar{w} \in \Pi(\bar{x}; \Omega) = \{\bar{x}\}$ and

$$\partial d(\bar{x}; \Omega) = \partial p(0) \cap N(\bar{x}; \Omega) = I\!B \cap \{0\} = \{0\}.$$

Example 3.2.3.

In this example we consider $\Omega = [a_1 - r, a_1 + r] \times [a_2 - r, a_2 + r] \subset I\!R^2$. Let $a = (a_1, a_2)$ and $\bar{x} = (\bar{x}_1, \bar{x}_2)$. We are going to compute $\partial d(\bar{x}; \Omega)$ for all $\bar{x} \in I\!R^2$. Let the vertices of the square Ω be denoted by $w_1 = (a_1 + r, a_2 + r), w_2 = (a_2 - r, a_2 + r), w_3 = (a_1 - r, a_2 - r), w_4 = (a_1 + r, a_2 - r)$. Consider the case where $\bar{x}_1 < a_1 - r$ and $\bar{x}_2 > a_2 + r$. In this case $\bar{w} \in \Pi(\bar{x}; \Omega) = \{w_2\}$. Then

$$\begin{aligned}\partial d(\bar{x}; \Omega) &= \partial p(\bar{x} - w_2) \cap N(w_2; \Omega)\\ &= \{\frac{\bar{x} - w_2}{d(\bar{x}; \Omega)}\} \cap \{(v_1, v_2) \mid v_1 \le 0, v_2 \ge 0\}\\ &= \{\frac{\bar{x} - w_2}{||\bar{x} - w_2||_2}\}.\end{aligned}$$

Consider the case where $|\bar{x}_1 - a_1| \le r$ and $\bar{x}_2 > a_2 + r$. In this case $\bar{w} \in \Pi(\bar{x}; \Omega) = \{(\bar{x}_1, a_2 + r)\}$ and

$$\begin{aligned}\partial d(\bar{x}; \Omega) &= \partial p(\bar{x} - \bar{w}) \cap N(\bar{w}; \Omega)\\ &= \{\frac{\bar{x} - \bar{w}}{d(\bar{x}; \Omega)}\} \cap \{(0, t) \mid t \ge 0\}\\ &= \{(0, 1)\}.\end{aligned}$$

Consider the case where $\bar{x} = w_1$. In this case $\bar{w} \in \Pi(\bar{x}; \Omega) = \{\bar{x}\}$ and

$$\begin{aligned}\partial d(\bar{x}; \Omega) &= \partial p(0) \cap N(\bar{x}; \Omega)\\ &= \{(v_1, v_2) \mid v_1^2 + v_2^2 \le 1\} \cap \{(v_1, v_2) \mid v_1 \ge 0, v_2 \ge 0\}\\ &= \{(v_1, v_2) \mid v_1^2 + v_2^2 \le 1, v_1 \ge 0, v_2 \ge 0\}.\end{aligned}$$

Consider the case where $\bar{x} \in \text{int}\,\Omega$. In this case $\bar{w} \in \Pi(\bar{x}; \Omega) = \{\bar{x}\}$ and

$$\begin{aligned}\partial d(\bar{x}; \Omega) &= \partial p(0) \cap N(\bar{x}; \Omega)\\ &= \{(v_1, v_2) \mid v_1^2 + v_2^2 \le 1\} \cap \{0\} = \{0\}.\end{aligned}$$

Similarly, we can derive the formula for $\partial d(\bar{x}; \Omega)$ for all of the other cases.

3.3 Subdifferential of Distance Functions with the Sum Norm

In this section let the distance function generated by a set $\Omega \subset I\!R^2$ be defined as

$$d(x;\Omega) = \inf\{||x-w||_1 \bigm| w \in \Omega\},$$

where $||\cdot||_1$ is the "sum" norm in $I\!R^2$ defined by

$$p(u) = ||u||_1 = ||(u_1,u_2)||_1 = |u_1| + |u_2|$$

for $u = (u_1,u_2) \in I\!R^2$.

Proposition 3.3.1. *Let $X = I\!R^2$ with the "sum" norm. The subdifferential $\partial p(\bar{x}), \bar{x} = (\bar{x}_1,\bar{x}_2) \in X$ in Theorem 3.1.5 has the following explicit representation*

$$\partial p(\bar{x}_1,\bar{x}_2) = \begin{cases} [-1,1]\times[-1,1] & \textit{if } (\bar{x}_1,\bar{x}_2) = (0,0), \\ [-1,1]\times\{1\}, & \textit{if } \bar{x}_1 = 0, \bar{x}_2 > 0, \\ [-1,1]\times\{-1\}, & \textit{if } \bar{x}_1 = 0, \bar{x}_2 < 0, \\ \{1\}\times[-1,1], & \textit{if } \bar{x}_1 > 0, \bar{x}_2 = 0, \\ \{-1\}\times[-1,1], & \textit{if } \bar{x}_1 < 0, \bar{x}_2 = 0, \\ \{1\}\times\{1\}, & \textit{if } \bar{x}_1 > 0, \bar{x}_2 > 0, \\ \{1\}\times\{-1\}, & \textit{if } \bar{x}_1 > 0, \bar{x}_2 < 0, \\ \{-1\}\times\{1\}, & \textit{if } \bar{x}_1 < 0, \bar{x}_2 > 0, \\ \{-1\}\times\{-1\}, & \textit{if } \bar{x}_1 < 0, \bar{x}_2 < 0. \end{cases}$$

Proof: Let $f_1(x_1,x_2) = |x_1|$ and $f_2(x_1,x_2) = |x_2|$. Then

$$p(x) = f_1(x) + f_2(x).$$

By Theorem 2.1.1

$$\partial p(\bar{x}) = \partial f_1(\bar{x}) + \partial f_2(\bar{x}).$$

We have

$$\partial f_1(\bar{x}_1,\bar{x}_2) = \begin{cases} \{1\} & \text{if } \bar{x}_1 > 0, \\ [-1,1] & \text{if } \bar{x}_1 = 0, \\ \{-1\} & \text{if } x_1 < 0, \end{cases}$$

and

$$\partial f_2(\bar{x}_1, \bar{x}_2) = \begin{cases} \{1\} & \text{if } \bar{x}_2 > 0, \\ [-1, 1] & \text{if } \bar{x}_2 = 0, \\ \{-1\} & \text{if } x_2 < 0, \end{cases}$$

Let us show that $\partial p(\bar{x})$ is valid. For instance, if $\bar{x}_2 > 0$ and $\bar{x}_1 = 0$, then

$$\partial p(\bar{x}) = \partial f_1(\bar{x}) + \partial f_2(\bar{x}) = [-1, 1] \times \{1\}.$$

In the case where $\bar{x}_1 > 0$ and $\bar{x}_2 > 0$, one has

$$\partial p(\bar{x}) = \partial f_1(\bar{x}) + \partial f_2(\bar{x}) = \{1\} \times \{1\}.$$

For the case where $\bar{x}_1 = \bar{x}_2 = 0$, one has

$$\partial p(\bar{x}) = \partial f_1(\bar{x}) + \partial f_2(\bar{x}) = [-1, 1] \times [-1, 1].$$

The rest of the cases can be calculated in the same manner. $\square$

We are going to use Proposition 3.3.1 and Theorem 3.1.5 to derive the subdifferential formula for the distance function $d(x; \Omega)$ where Ω is a set of particular shapes.

Example 3.3.2.

In this example we consider $\Omega = [a - r, a + r] \times [b - r, b + r] \subset \mathbb{R}^2$. We are going to compute $\partial d(\bar{x}; \Omega)$ for all $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2$. Let the vertices of the square Ω be denoted by $w_1 = (a + r, b + r), w_2 = (a - r, b + r), w_3 = (a - r, b - r), w_4 = (a + r, b - r)$.

Consider the case where $\bar{x}_1 > a + r$ and $|\bar{x}_2 - b| < r$. In this case $\bar{w} \in \Pi(\bar{x}; \Omega) = (a + r, \bar{x}_2)$. Then

$$\begin{aligned} \partial d(\bar{x}; \Omega) &= \partial p(\bar{x} - \bar{w}) \cap N(\bar{w}; \Omega) \\ &= \{1\} \times [-1, 1] \cap \{(t, 0) \mid t \geq 0\} \\ &= \{(1, 0)\}. \end{aligned}$$

Consider the case where $\bar{x}_1 > a + r$ and $\bar{x}_2 > b + r$. In this case $\bar{w} \in \Pi(\bar{x}; \Omega) = \{w_1\}$. Then

$$\begin{aligned} \partial d(\bar{x}; \Omega) &= \partial p(\bar{x} - w_1) \cap N(w_1; \Omega) \\ &= \{(1,1)\} \cap \{(v_1, v_2) \mid v_1 \geq 0, v_2 \geq 0\} \\ &= \{(1,1)\}. \end{aligned}$$

Consider the case where $\bar{x} = w_1$. In this case $\bar{w} \in \Pi(\bar{x}; \Omega) = \{\bar{x}\}$ and

$$\begin{aligned} \partial d(\bar{x}; \Omega) &= \partial p(0) \cap N(\bar{x}; \Omega) \\ &= \{(v_1, v_2) \mid \max\{v_1, v_2\} \leq 1, v_1 \geq 0, v_2 \geq 0\}. \end{aligned}$$

Consider the case where $\bar{x} \in \text{int}[w_1, w_2]$. In this case $\bar{w} \in \Pi(\bar{x}; \Omega) = \{\bar{x}\}$ and

$$\partial d(\bar{x}; \Omega) = \partial p(0) \cap N(\bar{x}; \Omega) = \{(0, t) \mid 0 \leq t \leq 1\}.$$

Consider the case where $\bar{x} \in \text{int}\,\Omega$. In this case $\bar{w} \in \Pi(\bar{x}; \Omega) = \{\bar{x}\}$ and

$$\partial d(\bar{x}; \Omega) = \partial p(0) \cap N(\bar{x}; \Omega) = \{0\}.$$

Similarly, we can derive the formula for $\partial d(\bar{x}; \Omega)$ for all of the other cases.

Example 3.3.3.

In this example we consider $\Omega = \{(x_1, x_2) \mid |x_1 - a| + |x_2 - b| \leq r, r > 0\} \subset I\!R^2$. We are going to compute $\partial d(\bar{x}; \Omega)$ for all $\bar{x} = (\bar{x}_1, \bar{x}_2) \in I\!R^2$. Let the vertices of the diamond Ω be denoted by $w_1 = (a + r, b), w_2 = (a, b + r), w_3 = (a - r, b), w_4 = (a, b - r)$.

Consider the case where $\bar{x}_1 > a + r$ and $\bar{x}_2 = b$. In this case $\bar{w} \in \Pi(\bar{x}; \Omega) = \{w_1\}$. Then

$$\begin{aligned} \partial d(\bar{x}; \Omega) &= \partial p(\bar{x} - w_1) \cap N(w_1; \Omega) \\ &= \{1\} \times [-1, 1] \cap \{(v_1, v_2) \mid v_1 \geq |v_2|\} \\ &= \{(1, t) \mid -1 \leq t \leq 1\}. \end{aligned}$$

Consider the case where $|\bar{x}_1 - a| + |\bar{x}_2 - b| > r$, $\bar{x}_1 > a$, and $\bar{x}_2 > b$. In this case we choose $\bar{w} \in \Pi(\bar{x};\Omega)$ such that $\bar{w} \in \text{int}[w_1, w_2]$. Then

$$\begin{aligned}\partial d(\bar{x};\Omega) &= \partial p(\bar{x} - \bar{w}) \cap N(\bar{w};\Omega) \\ &= \{1\} \times \{1\} \cap \{(t,t) \mid 0 \le t\} \\ &= \{(1,1)\}.\end{aligned}$$

Consider the case where $\bar{x} = w_1$. In this case $\bar{w} \in \Pi(\bar{x};\Omega) = \{\bar{x}\}$ and

$$\begin{aligned}\partial d(\bar{x};\Omega) &= \partial p(0) \cap N(\bar{x};\Omega) \\ &= \{(v_1, v_2) \mid v_1 \ge |v_2|, 0 \le v_1 \le 1\}.\end{aligned}$$

Consider the case where $\bar{x} \in \text{int}[w_1, w_2]$. In this case $\bar{w} \in \Pi(\bar{x};\Omega) = \{\bar{x}\}$ and

$$\partial d(\bar{x};\Omega) = \partial p(0) \cap N(\bar{x};\Omega) = \{(t,t) \mid 0 \le t \le 1\}.$$

Consider the case where $\bar{x} \in \text{int}\,\Omega$. In this case $\bar{w} \in \Pi(\bar{x},\Omega) = \{\bar{x}\}$ and

$$\partial d(\bar{x};\Omega) = \partial p(0) \cap N(\bar{x};\Omega) = \{0\}.$$

Similarly, we can derive the formula for $\partial d(\bar{x};\Omega)$ for all of the other cases.

3.4 Subdifferential of Distance Functions with the Max Norm

In this section let the distance function generated by a set $\Omega \subset I\!\!R^2$ be defined as

$$d(x;\Omega) = \inf\{||x - w||_\infty \mid w \in \Omega\},$$

where $|| \cdot ||_1$ is the “max” norm in $I\!\!R^2$ defined by

$$p(u) = ||u||_\infty = ||(u_1, u_2)||_\infty = \max\{|u_1|, |u_2|\}$$

for $u = (u_1, u_2) \in I\!\!R^2$.

Proposition 3.4.1. *Let* $X = I\!R^2$ *with the "max" norm. The subdifferential* $\partial p(\bar{x}), \bar{x} = (\bar{x}_1, \bar{x}_2) \in X$ *in Theorem 3.1.5 has the following explicit representation*

$$\partial p(\bar{x}_1, \bar{x}_2) = \begin{cases} \{(v_1, v_2) \in I\!R^2 \mid |v_1| + |v_2| \le 1\} & \textit{if } (\bar{x}_1, \bar{x}_2) = (0,0), \\ \{(0,1)\} & \textit{if } |\bar{x}_1| < \bar{x}_2, \\ \{(0,-1)\} & \textit{if } \bar{x}_2 < -|\bar{x}_1|, \\ \{(1,0)\} & \textit{if } x_1 > |\bar{x}_2|, \\ \{(-1,0)\} & \textit{if } \bar{x}_1 < -|\bar{x}_2|, \\ \{(v_1, v_2) \in I\!R^2 \mid |v_1| + |v_2| = 1,\ v_1 \ge 0,\ v_2 \ge 0\} & \textit{if } \bar{x}_1 = \bar{x}_2 > 0, \\ \{(v_1, v_2) \in I\!R^2 \mid |v_1| + |v_2| = 1,\ v_1 \ge 0,\ v_2 \le 0\} & \textit{if } \bar{x}_1 = -\bar{x}_2 > 0, \\ \{(v_1, v_2) \in I\!R^2 \mid |v_1| + |v_2| = 1,\ v_1 \le 0,\ v_2 \le 0\} & \textit{if } \bar{x}_1 = \bar{x}_2 < 0, \\ \{(v_1, v_2) \in I\!R^2 \mid |v_1| + |v_2| = 1,\ v_1 \le 0,\ v_2 \ge 0\} & \textit{if } \bar{x}_1 = -\bar{x}_2 < 0. \end{cases}$$

Proof: Let $f_1(x_1, x_2) = |x_1|$ and $f_2(x_1, x_2) = |x_2|$ then

$$p(x) = \max\{f_1(x), f_2(x)\}.$$

By Theorem 2.1.2

$$\partial p(\bar{x}) = \text{co}\,\{\partial f_i(\bar{x}) \mid f_i(\bar{x}) = f(\bar{x})\}.$$

We have

$$\partial f_1(\bar{x}_1, \bar{x}_2) = \begin{cases} \{(1,0)\} & \text{if } \bar{x}_1 > 0, \\ [-1,1] \times \{0\} & \text{if } \bar{x}_1 = 0, \\ \{(-1,0)\} & \text{if } x_1 < 0, \end{cases}$$

and

$$\partial f_2(\bar{x}_1, \bar{x}_2) = \begin{cases} \{(0,1)\} & \text{if } \bar{x}_2 > 0, \\ \{0\} \times [-1,1] & \text{if } \bar{x}_2 = 0, \\ \{(0,-1)\} & \text{if } x_2 < 0, \end{cases}$$

Let us show that $\partial p(\bar{x})$ is valid. For instance, if $\bar{x}_2 > |\bar{x}_1|$, then $f_2(\bar{x}_1, \bar{x}_2) > f_1(\bar{x}_1, \bar{x}_2)$ and $f(\bar{x}) = f_2(\bar{x})$. Thus

$$\partial p(\bar{x}) = \text{co}\,\{\partial f_2(\bar{x})\} = \text{co}\,\{(0,1)\} = \{(0,1)\}.$$

In the case where $\bar{x}_1 = \bar{x}_2 > 0$, then $f_1(\bar{x}) = f_2(\bar{x}) = f(\bar{x})$. Thus

$$\begin{aligned}\partial p(\bar{x}) &= \text{co}\,\{\partial f_1(\bar{x}), \partial f_2(\bar{x})\}\\ &= \text{co}\,\{(1,0),(0,1)\}\\ &= \{(v_1,v_2) \in I\!R^2 \mid |v_1| + |v_2| = 1, v_1 > 0, v_2 > 0\}.\end{aligned}$$

Consider the case where $\bar{x}_1 = \bar{x}_2 = 0$. In this case $f_1(\bar{x}) = f_2(\bar{x}) = f(\bar{x})$. Thus

$$\begin{aligned}\partial p(\bar{x}) &= \text{co}\,\{\partial f_1(\bar{x}), \partial f_2(\bar{x})\}\\ &= \text{co}\,\{[-1,1] \times \{0\}, \{0\} \times [-1,1]\}\\ &= \{(v_1,v_2) \in I\!R^2 \mid |v_1| + |v_2| \le 1\}.\end{aligned}$$

The rest of the cases can be calculated in the same manner. The proof is now complete. □

Using Proposition 3.4.1 and Theorem 3.1.5 we are able to derive the subdifferential formula for the distance function $d(x;\Omega)$ where Ω is a set of particular shapes.

Example 3.4.2.

In this example we consider $\Omega = [a-r, a+r] \times [b-r, b+r] \subset I\!R^2$. We are going to compute $\partial d(\bar{x};\Omega)$ for all $\bar{x} = (\bar{x}_1, \bar{x}_2) \in I\!R^2$. Let the vertices of the square Ω be denoted by $w_1 = (a+r, b+r), w_2 = (a-r, b+r), w_3 = (a-r, b-r), w_4 = (a+r, b-r)$.

Consider the case where $\bar{x}_2 - b > |\bar{x}_1 - a|$ and $\bar{x}_2 > b + r$. In this case choose $\bar{w} \in \Pi(\bar{x};\Omega)$ such that $\bar{w} \in \text{int}[w_1, w_2]$. Then

$$\begin{aligned}\partial d(\bar{x};\Omega) &= \partial p(\bar{x} - \bar{w}) \cap N(\bar{w};\Omega)\\ &= \{(0,1)\} \cap \{(0,t) \mid t \ge 0\}\\ &= \{(0,1)\}.\end{aligned}$$

Consider the case where $\bar{x} = w_1$. In this case $\bar{w} \in \Pi(\bar{x};\Omega) = \{\bar{x}\}$ and

$$\begin{aligned}\partial d(\bar{x};\Omega) &= \partial p(0) \cap N(\bar{x};\Omega)\\ &= \{(v_1,v_2) \mid |v_1| + |v_2| \le 1, v_1 \ge 0, v_2 \ge 0\}.\end{aligned}$$

Consider the case where $\bar{x} \in \text{int}[w_1, w_2]$. In this case $\bar{w} \in \Pi(\bar{x}; \Omega) = \{\bar{x}\}$ and

$$\partial d(\bar{x}; \Omega) = \partial p(0) \cap N(\bar{x}; \Omega) = \{(0, t) \mid 0 \leq t \leq 1\}.$$

Consider the case where $\bar{x} \in \text{int}\,\Omega$. In this case $\bar{w} \in \Pi(\bar{x}; \Omega) = \{\bar{x}\}$ and

$$\partial d(\bar{x}; \Omega) = \partial p(0) \cap N(\bar{x}; \Omega) = \{0\}.$$

Similarly, we can derive the formula for $\partial d(\bar{x}; \Omega)$ for all of the other cases.

Example 3.4.3.

In this example we consider $\Omega = \{(x_1, x_2) \mid |x_1 - a| + |x_2 - b| \leq r\} \subset I\!R^2$. We are going to compute $\partial d(\bar{x}; \Omega)$ for all $\bar{x} = (\bar{x}_1, \bar{x}_2) \in I\!R^2$. Let the vertices of the diamond Ω be denoted by $w_1 = (a + r, b), w_2 = (a, b + r), w_3 = (a - r, b), w_4 = (a, b - r)$.

Consider the case where $\bar{x}_1 - (a + r) > |\bar{x}_2 - b|$. In this case $\bar{w} \in \Pi(\bar{x}; \Omega) = \{w_1\}$ and

$$\begin{aligned} \partial d(\bar{x}; \Omega) &= \partial p(\bar{x} - w_1) \cap N(\bar{x}; \Omega) \\ &= \{(1, 0)\}. \end{aligned}$$

Consider the case where $\bar{x}_2 - b \geq |\bar{x}_1 - (a + r)|$ and $\bar{x}_2 - (b + r) \leq |\bar{x}_1 - a|$. In this case we choose $\bar{w} \in \Pi(\bar{x}; \Omega)$ such that $\bar{w} \in [w_1, w_2]$. Then

$$\begin{aligned} \partial d(\bar{x}; \Omega) &= \partial p(\bar{x} - \bar{w}) \cap N(\bar{w}; \Omega) \\ &= \{(\frac{1}{2}, \frac{1}{2})\}. \end{aligned}$$

Consider the case where $\bar{x} = w_1$. In this case $\bar{w} \in \Pi(\bar{x}; \Omega) = \{\bar{x}\}$ and

$$\begin{aligned} \partial d(\bar{x}; \Omega) &= \partial p(0) \cap N(\bar{x}; \Omega) \\ &= \{(v_1, v_2) \mid |v_1| + |v_2| \leq 1, v_1 \geq |v_2|\}. \end{aligned}$$

Consider the case where $\bar{x} \in \text{int}\,\Omega$. In this case $\bar{w} \in \Pi(\bar{x}; \Omega) = \{\bar{x}\}$ and

$$\partial d(\bar{x}; \Omega) = \partial p(0) \cap N(\bar{x}; \Omega) = \{0\}.$$

Similarly, we can derive the formula for $\partial d(\bar{x}; \Omega)$ for all of the other cases.

CHAPTER IV

APPLICATIONS TO THE GENERALIZED HERON PROBLEM

In this chapter we study the generalized Heron Problem theoretically and develop numerical algorithms of subgradient type to solve the problem. Many examples are presented to illustrate applications in particular situations. The new results in this chapter and more on the study of facility location problems involving sets are presented in our recent papers [14, 15, 16].

4.1 Optimal Solutions to the Generalized Heron Problem

In this section we derive efficient characterizations of optimal solutions to the generalized Heron problem (1.1.1), which allow us to completely solve this problem in some important particular settings.

First let us present general conditions that ensure the *existence* of optimal solutions to (1.1.1).

Proposition 4.1.1. *Assume that one of the sets* Ω *and* Ω_i*,* $i = 1, \ldots, n$*, is bounded. Then the generalized Heron problem* (1.1.1) *admits an optimal solution.*

Proof: Consider the optimal value

$$\gamma := \inf_{x \in \Omega} D(x)$$

in (1.1.1) and take a minimizing sequence $\{x_k\} \subset \Omega$ with $D(x_k) \to \gamma$ as $k \to \infty$. If the constraint set Ω is bounded, then by the classical Bolzano-Weierstrass theorem the sequence $\{x_k\}$ contains a subsequence converging to some point $\bar{x}$, which belongs to the set Ω due to its closedness. Since the function $D(x)$ in (1.1.1) is continuous, we have $D(\bar{x}) = \gamma$, and thus $\bar{x}$ is an optimal solution to (1.1.1).

It remains to consider the case when one of the sets Ω_i, say Ω_1, is bounded. In this case we have for the above sequence $\{x_k\}$ when k is sufficiently large that

$$d(x_k; \Omega_1) \leq D(x_k) < \gamma + 1,$$

and thus there exists $w_k \in \Omega_1$ with $||x_k - w_k|| < \gamma + 1$ for such indexes k. Then

$$||x_k|| < ||w_k|| + \gamma + 1,$$

which shows that the sequence $\{x_k\}$ is bounded. The existence of optimal solutions follows in this case from the arguments above. $\square$

To characterize in what follows optimal solutions to the generalized Heron problem (1.1.1), for any nonzero vectors $u, v \in X$ define the quantity

$$\cos(v, u) := \frac{\langle v, u \rangle}{||v|| \cdot ||u||}. \tag{4.1.1}$$

We say that Ω has a *tangent space* at $\bar{x}$ if there exists a subspace $L = L(\bar{x}) \neq \{0\}$ such that

$$N(\bar{x}; \Omega) = L^{\perp} := \big\{ v \in X \big| \ \langle v, u \rangle = 0 \ \text{ whenever } \ u \in L \big\}. \tag{4.1.2}$$

The following theorems give necessary and sufficient conditions for optimal solutions to (1.1.1) in terms of projections (1.2.1) on Ω_i incorporated into quantities (4.1.1). This theorem and its consequences are also important in verifying the validity of numerical results in Section 4.2.

Define the index sets

$$I(x) := \big\{ i \in \{1, \ldots, n\} \ \big| \ x \in \Omega_i \big\} \ \text{ and } \ J(x) = \big\{ i \in \{1, \ldots, n\} \ \big| \ x \notin \Omega_i \big\}, \quad x \in X. \tag{4.1.3}$$

We have $I(x) \cup J(x) = \{1, \ldots, n\}$ and $I(x) \cap J(x) = \emptyset$ for all $x \in X = I\!R^s$.

Theorem 4.1.2. *Consider the generalized Heron problem* (1.1.1) *with a Euclidean space* $X = \mathbb{R}^s$ *the sets* Ω, $\Omega_i \subset \mathbb{R}^s$ *as* $i = 1, \ldots, n$ *are nonempty closed and convex, not necessarily disjoint. Then the following assertions hold:*

(i) Let $\bar{x} \in \Omega$ *be a solution to* (1.1.1). *Then for* $a_i(\bar{x}) \equiv A_i(\bar{x})$ *as* $i \in J(\bar{x})$ *we have*

$$-\sum_{i \in J(\bar{x})} a_i(\bar{x}) \in \sum_{i \in I(\bar{x})} A_i(\bar{x}) + N(\bar{x}; \Omega), \tag{4.1.4}$$

where each set $A_i(\bar{x})$ *is computed by*

$$A_i(\bar{x}) = \begin{cases} \dfrac{\bar{x} - \Pi(\bar{x}; \Omega_i)}{d(\bar{x}; \Omega_i)} & \textit{for } \bar{x} \notin \Omega_i, \\ \\ N(\bar{x}; \Omega_i) \cap \mathbb{B} & \textit{for } \bar{x} \in \Omega_i \end{cases} \tag{4.1.5}$$

for $i = 1, \ldots, n$.

(ii) Conversely, if $\bar{x}$ *satisfies* (4.1.4), *then* $\bar{x}$ *is a solution to* (1.1.1).

Proof: (i) Fix an optimal solution $\bar{x}$ to problem (1.1.1) and equivalently describe it as an optimal solution to the following unconstrained optimization problem:

$$\text{minimize } \ D(x) + \delta(x; \Omega), \quad x \in \mathbb{R}^s. \tag{4.1.6}$$

Applying the generalized Fermat rule to (4.1.6), we characterize $\bar{x}$ by

$$0 \in \partial\Big(\sum_{i=1}^{n} d(\cdot; \Omega_i) + \delta(\cdot; \Omega)\Big)(\bar{x}). \tag{4.1.7}$$

Since all of the functions $d(\cdot; \Omega_i)$, $i = 1, \ldots, n$, are convex and continuous, we employ the subdifferential sum rule of Theorem 2.1.1 to (4.1.7) and arrive at

$$\begin{aligned} 0 &\in \sum_{i \in J(\bar{x})} \partial d(\bar{x}; \Omega_i) + \sum_{i \in I(\bar{x})} d(\bar{x}; \Omega_i) + N(\bar{x}; \Omega) \\ &= \sum_{i \in J(\bar{x})} A_i(\bar{x}) + \sum_{i \in I(\bar{x})} A_i(\bar{x}) + N(\bar{x}; \Omega) \\ &= \sum_{i \in J(\bar{x})} a_i(\bar{x}) + \sum_{i \in I(\bar{x})} A_i(\bar{x}) + N(\bar{x}; \Omega). \end{aligned}$$

The latter implies that

$$-\sum_{i\in J(\bar{x})} a_i(\bar{x}) \in \sum_{i\in I(\bar{x})} A_i(\bar{x}) + N(\bar{x};\Omega).$$

(ii) To prove the converse, we simply take advantage of the "if and only if" property of the proof for (i). The proof is now complete. □

Next we will examine the generalize Heron problem (1.1.1) where the intersection of the constraint set Ω with the target sets Ω_i is empty.

Corollary 4.1.3. *Consider the generalized Heron problem* (1.1.1) *with the Euclidean space* $X = \mathbb{R}^s$ *where* $\Omega \subset X$ *and* $\Omega_i \subset X$ *as* $i = 1, \ldots, n$ *are nonempty closed and convex in which* $\Omega \cap \Omega_i = \emptyset$ *for* $i = 1, \ldots, n$. *Then the following holds:*

Given $\bar{x} \in \Omega$, *define the vectors*

$$a_i(\bar{x}) := \frac{\bar{x} - \Pi(\bar{x};\Omega_i)}{d(\bar{x};\Omega_i)} \neq 0, \quad i = 1, \ldots, n. \tag{4.1.8}$$

Then $\bar{x} \in \Omega$ *is an optimal solution to the generalized Heron problem* (1.1.1) *if and only if*

$$-\sum_{i=1}^{n} a_i(\bar{x}) \in N(\bar{x};\Omega). \tag{4.1.9}$$

Proof: In this case we have that $\bar{x} \notin \Omega_i$ for $i = 1, ..., n$, then $I(\bar{x}) = \emptyset$ and $J(\bar{x}) = \{1, ..., n\}$. Therefore, this is a direct consequence of Theorem 4.1.2. □

In the following we further provide a detailed proof for [13, Proposition 4.9]. We consider problem (1.1.1) with $\Omega = X$ and the three sets Ω_1, Ω_2, and Ω_3 are nonempty closed and convex. This is clearly an extension of the classical Fermat-Torricelli problem where the three points are replaced by three sets. First we consider the following lemmas, which will help in the proof of the succeeding theorem. For simplicity, we denote $a_i \equiv a_i(\bar{x})$.

Lemma 4.1.4. *Let a_1 and a_2 by any two unit vectors in X. We have $-a_1 - a_2 \in \mathbb{B}$ if and only if $\langle a_1, a_2 \rangle \leq -\frac{1}{2}$.*

Proof: Let $-a_1 - a_2 \in \mathbb{B}$. Then $a_1 + a_2 \in \mathbb{B}$, which implies

$$\|a_1 + a_2\| \leq 1.$$

By squaring both sides, we get

$$\langle a_1 + a_2, a_1 + a_2 \rangle \leq 1,$$

Thus

$$\begin{aligned}
\langle a_1 + a_2, a_1 + a_2 \rangle &= \langle a_1, a_1 + a_2 \rangle + \langle a_2, a_1 + a_2 \rangle \\
&= \langle a_1, a_1 \rangle + \langle a_1, a_2 \rangle + \langle a_2, a_1 \rangle + \langle a_2, a_2 \rangle \\
&= \|a_1\|^2 + \langle a_1, a_2 \rangle + \langle a_1, a_2 \rangle + \|a_2\|^2 \\
&= 2 + 2\langle a_1, a_2 \rangle \\
&\leq 1.
\end{aligned}$$

The previous relation implies $\langle a_1, a_2 \rangle \leq -\frac{1}{2}$. The converse proof follows from the forward direction. The proof is now complete. □

Lemma 4.1.5. *Let a_1, a_2, and a_3 by any three unit vectors in $\mathbb{R}^s$. We have $a_1 + a_2 + a_3 = 0$ if and only if $\langle a_1, a_2 \rangle = \langle a_2, a_3 \rangle = \langle a_3, a_1 \rangle = -\frac{1}{2}$.*

Proof: Let $a_1 + a_2 + a_3 = 0$. Then

$$\|a_1\|^2 + \langle a_1, a_2 \rangle + \langle a_1, a_3 \rangle = 0, \tag{4.1.10}$$

$$\langle a_2, a_1 \rangle + \|a_2\|^2 + \langle a_2, a_3 \rangle = 0, \tag{4.1.11}$$

$$\langle a_3, a_1\rangle + \langle a_3, a_2\rangle + \|a_3\|^2 = 0. \tag{4.1.12}$$

By subtracting (4.1.12) from (4.1.10), we get

$$\langle a_1, a_2\rangle = \langle a_2, a_3\rangle.$$

Similarly, $\langle a_2, a_3\rangle = \langle a_3, a_1\rangle$. Substituting these equivalences to (4.1.10), (4.1.11), and (4.1.12), we get $\langle a_1, a_2\rangle = \langle a_2, a_3\rangle = \langle a_3, a_1\rangle = -\frac{1}{2}$. Conversely, let $\langle a_1, a_2\rangle = \langle a_2, a_3\rangle = \langle a_3, a_1\rangle = -\frac{1}{2}$. We have that

$$\begin{aligned}
\|a_1 + a_2 + a_3\|^2 &= \|(a_1 + a_2) + a_3\|^2 \\
&= \|a_1 + a_2\|^2 + 2\langle a_1 + a_2, a_3\rangle + \|a_3\|^2 \\
&= \|a_1\|^2 + 2\langle a_1, a_2\rangle + \|a_2\|^2 + 2\langle a_1, a_3\rangle + 2\langle a_2, a_3\rangle + \|a_3\|^2 \\
&= 1 + 2(-\frac{1}{2}) + 1 + 2(-\frac{1}{2}) + 2(-\frac{1}{2}) + 1 \\
&= 0.
\end{aligned}$$

The previous relation implies $a_1 + a_2 + a_3 = 0$. The proof is now complete. □

Theorem 4.1.6. *Let $n = 3$ in the framework of Theorem 4.1.2 where Ω_1, Ω_2, and Ω_3 are pairwise disjoint closed convex subsets of X and $\Omega = \mathbb{R}^s$. The following alternatives hold for an optimal solution $\bar{x} \in \mathbb{R}^s$ with the sets $A_i(\bar{x})$ defined by* (4.1.5)*:*

(i) *If the point $\bar{x}$ belongs to one of the sets Ω_i, say Ω_1, then we have the relationships*

$$\langle a_2, a_3\rangle \le -1/2 \;\; \textit{and} \;\; -a_2 - a_3 \in N(\bar{x}; \Omega_1).$$

(ii) *If the point $\bar{x}$ does not belong to any of the three sets Ω_1, Ω_2, and Ω_3, then*

$$\langle a_i, a_j\rangle = -1/2 \;\; \textit{for} \;\; i \neq j \;\; \textit{as} \;\; i, j \in \{1, 2, 3\}.$$

Conversely, suppose that the sets Ω_i, $i = 1,2,3$, are convex and that $\bar{x}$ satisfies either (i) or (ii). Then it is an optimal solution to the problem under consideration.

Proof: (i)We have $\bar{x} \in \Omega_1$ is a solution if and only if

$$
\begin{aligned}
0 &\in \partial d(\bar{x};\Omega_1) + \partial d(\bar{x};\Omega_2) + \partial d(\bar{x};\Omega_3) \\
&= \partial d(\bar{x};\Omega_1) + a_2 + a_3 \\
&= N(\bar{x};\Omega_1) \cap \mathbb{B} + a_2 + a_3.
\end{aligned}
$$

Thus $-a_2 - a_3 \in \mathbb{N}(\bar{x};\Omega_1) \cap \mathbb{B}$, which is equivalent to $-a_2 - a_3 \in \mathbb{B}$ and $-a_2 - a_3 \in N(\bar{x};\Omega_1)$. Therefore, by Lemma 4.1.4 the latter is equivalent to

$$\langle a_2, a_3 \rangle \leq -1/2 \text{ and } -a_2 - a_3 \in N(\bar{x};\Omega_1).$$

(ii) Similarly, $\bar{x} \notin \Omega_i$ for $i = 1,2,3$ is a solution if and only if

$$
\begin{aligned}
0 &\in \partial d(\bar{x};\Omega_1) + \partial d(\bar{x};\Omega_2) + \partial d(\bar{x};\Omega_3) \\
&= a_1 + a_2 + a_3.
\end{aligned}
$$

Therefore, by Lemma 4.1.5 the latter is equivalent to

$$\langle a_i, a_j \rangle = -1/2 \text{ for } i \neq j \text{ as } i,j \in \{1,2,3\}.$$

Conversely, if $\Omega_1, \Omega_2, \Omega_3$ are convex and either (i) or (ii) are satisfied then

$$
\begin{aligned}
0 &\in \partial d(\bar{x};\Omega_1) + \partial d(\bar{x};\Omega_2) + \partial d(\bar{x};\Omega_3) \\
&= \partial(D(\bar{x})).
\end{aligned}
$$

Therefore $\bar{x}$ is an optimal solution to the problem under consideration. The proof is now complete.

□

Theorem 4.1.7. *Consider problem* (1.1.1) *in the Euclidean space* $X = I\!R^s$ *with*

$$\Omega_i \cap \Omega = \emptyset \ \textit{for all} \ \ i = 1, \ldots, n. \tag{4.1.13}$$

Suppose that the constraint set Ω *has a tangent space* L *at* $\bar{x}$*. Then (4.1.9) is equivalent to*

$$\sum_{i=1}^{n} \cos\big(a_i(\bar{x}), u\big) = 0 \ \ \textit{whenever} \ \ u \in L \setminus \{0\}. \tag{4.1.14}$$

Proof: Suppose that the constraint set Ω has a tangent space L at $\bar{x}$. Then the inclusion (4.1.9) is equivalent to

$$0 \in \sum_{i=1}^{n} a_i(\bar{x}) + L^{\perp},$$

which in turn can be written in the form

$$\Big\langle \sum_{i=1}^{n} a_i(\bar{x}), u \Big\rangle = 0 \ \ \text{for all} \ \ u \in L.$$

Taking into account that $||a_i(\bar{x})|| = 1$ for all $i = 1, \ldots, n$ by (4.1.8) and assumption (4.1.13), the latter equality is equivalent to

$$\sum_{i=1}^{n} \frac{\langle a_i(\bar{x}), v\rangle}{||a_i(\bar{x})|| \cdot ||u||} = 0 \ \ \text{for all} \ \ u \in L \setminus \{0\},$$

which gives (4.1.14) due to the notation (4.1.1) and thus completes the proof of the theorem. $\square$

To further specify the characterization of Theorem 4.1.7, recall that a set A of $I\!R^s$ is an *affine subspace* if there is a vector $a \in A$ and a (linear) subspace L such that $A = a + L$. In this case we say that A is parallel to L. Note that the subspace L parallel to A is uniquely defined by $L = A - A = \{x - y \mid x \in A, \ y \in A\}$ and that $A = b + L$ for any vector $b \in A$.

Corollary 4.1.8. *Let* Ω *be an affine subspace parallel to a subspace* L*, and let assumption* (4.1.13) *of Theorem 4.1.7 be satisfied. Then* $\bar{x} \in \Omega$ *is a solution to the generalized Heron problem* (1.1.1) *if and only if condition* (4.1.14) *holds.*

Proof: To apply Theorem 4.1.7, it remains to check that L is a tangent space of Ω at $\bar{x}$ in the setting of this corollary. Indeed, we have $\Omega = \bar{x} + L$, since Ω is an affine subspace parallel to L. Fix any $v \in N(\bar{x}; \Omega)$ and get by (2.1.6) that $\langle v, x - \bar{x} \rangle \leq 0$ whenever $x \in \Omega$ and hence $\langle v, u \rangle \leq 0$ for all $u \in L$. Since L is a subspace, the latter implies that $\langle v, u \rangle = 0$ for all $u \in L$, and thus $N(\bar{x}; \Omega) \subset L^{\perp}$. The opposite inclusion is trivial, which gives (4.1.2) and completes the proof of the corollary. □

The underlying characterization (4.1.14) can be easily checked when the subspace L in Theorem 4.1.7 is given as a span of fixed generating vectors.

Corollary 4.1.9. *Let $L = span\,\{u_1, \ldots, u_m\}$ with $u_j \neq 0$, $i = 1, \ldots, m$, in the setting of Theorem 4.1.7. Then $\bar{x} \in \Omega$ is an optimal solution to the generalized Heron problem* (1.1.1) *if and only if*

$$\sum_{i=1}^{n} \cos\big(a_i(\bar{x}), u_j\big) = 0 \;\textit{ for all }\; j = 1, \ldots, m. \tag{4.1.15}$$

Proof: We show that (4.1.14) is equivalent to (4.1.15) in the setting under consideration. Since (4.1.14) obviously implies (4.1.15), it remains to justify the opposite implication. Denote

$$a := \sum_{i=1}^{n} a_i(\bar{x})$$

and observe that (4.1.15) yields the condition

$$\langle a, u_j \rangle = 0 \;\text{ for all }\; j = 1, \ldots m, \tag{4.1.16}$$

since $u_j \neq 0$ for all $j = 1, \ldots, m$ and $||a_i|| = 1$ for all $i = 1, \ldots, n$. Taking now any vector $u \in L \setminus \{0\}$, we represent it in the form

$$u = \sum_{j=1}^{m} \lambda_j u_j \;\text{ with some }\; \lambda_j \in I\!R^n$$

and get from (4.1.16) the equalities

$$\langle a, u\rangle = \sum_{j=1}^{n} \lambda_j \langle a, u_j\rangle = 0.$$

This justifies (4.1.14) and completes the proof of the corollary. □

Let us further examine in more detail the case of two sets Ω_1 and Ω_2 in (1.1.1) with the normal cone to the constraint set Ω being a straight line generated by a given vector. This is a direct extension of the classical Heron problem to the setting when two points are replaced by closed and convex sets, and the constraint line is replaced by a closed convex set Ω with the property above. The next theorem gives a complete and verifiable solution to the new problem.

Theorem 4.1.10. *Let Ω_1 and Ω_2 be subsets of $I\!R^s$ as $s \geq 1$ with $\Omega \cap \Omega_i = \emptyset$ for $i = 1, 2$ in* (1.1.1)*. Suppose also that there is a vector $a \neq 0$ such that $N(\bar{x}; \Omega) = span\,\{a\}$. The following assertions hold, where $a_i := a_i(\bar{x})$ are defined in* (4.1.8)*:*

(i) *If $\bar{x} \in \Omega$ is an optimal solution to* (1.1.1)*, then*

$$\textit{either}\ \ a_1 + a_2 = 0\ \ \textit{or}\ \ \cos(a_1, a) = \cos\big(a_2, a\big). \tag{4.1.17}$$

(ii) *Conversely, if $s = 2$ and*

$$\textit{either}\ \ a_1 + a_2 = 0\ \ \textit{or}\ \ \big[a_1 \neq a_2\ \ \textit{and}\ \ \cos(a_1, a) = \cos(a_2, a)\big], \tag{4.1.18}$$

then $\bar{x} \in \Omega$ is an optimal solution to the generalized Heron problem (1.1.1).

Proof: It follows from the above (see the proof of Theorem 4.1.7) that $\bar{x} \in \Omega$ is an optimal solution to (1.1.1) if and only if $-a_1 - a_2 \in N(\bar{x}; \Omega)$. By the assumed structure of the normal cone to Ω the latter is equivalent to the alternative:

$$\text{either}\ \ a_1 + a_2 = 0\ \ \text{or}\ \ a_1 + a_2 = \lambda a\ \ \text{with some}\ \ \lambda \neq 0. \tag{4.1.19}$$

To justify (i), let us show that the second equality in (4.1.19) implies the corresponding one in (4.1.17). Indeed, we have $||a_1|| = ||a_1|| = 1$, and thus (4.1.19) implies that

$$\lambda^2||a||^2 = ||a_1 + a_2||^2 = ||a_1||^2 + ||a_2||^2 + 2\langle a_1, a_2\rangle = 2 + 2\langle a_1, a_2\rangle.$$

The latter yields in turn that

$$\begin{aligned}\langle a_1, \lambda a\rangle &= \langle \lambda a - a_2, \lambda a\rangle \\ &= \lambda^2||a||^2 - \lambda\langle a_2, a\rangle \\ &= 2 + 2\langle a_1, a_2\rangle - \lambda\langle a_2, a\rangle \\ &= 2\langle a_2, a_2\rangle + 2\langle a_1, a_2\rangle - \lambda\langle a_2, a\rangle \\ &= 2\langle a_2 + a_1, a_2\rangle - \lambda\langle a_2, a\rangle \\ &= 2\langle \lambda a, a_2\rangle - \lambda\langle a_2, a\rangle = \langle a_2, \lambda a\rangle,\end{aligned}$$

which ensures that $\langle a_1, a\rangle = \langle a_2, a\rangle$ as $\lambda \neq 0$. This gives us the equality $\cos(a_1, a) = \cos(a_2, a)$ due to $||a_1|| = ||a_2|| = 1$ and $a \neq 0$. Hence we arrive at (4.1.17).

To justify (ii), we need to prove that the relationships in (4.1.18) imply the fulfillment of

$$-a_1 - a_2 \in N(\bar{x}; \Omega) = \text{span}\{a\}. \tag{4.1.20}$$

If $-a_1 - a_2 = 0$, then (4.1.20) is obviously satisfied. Consider the alternative in(4.1.18) when $a_1 \neq a_2$ and $\cos(a_1, a) = \cos(a_2, a)$. Since we are in $I\!R^2$, represent $a_1 = (x_1, y_1)$, $a_2 = (x_2, y_2)$, and $a = (x, y)$ with two real coordinates. Then by (4.1.1) the equality $\cos(a_1, a) = \cos(a_2, a)$ can be written as

$$x_1x + y_1y = x_2x + y_2y, \text{ i.e., } (x_1 - x_2)x = (y_2 - y_1)y. \tag{4.1.21}$$

Since $a \neq 0$, assume without loss of generality that $y \neq 0$. By

$$||a_1||^2 = ||a_2||^2 \Longleftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$$

we have the equality $(x_1 - x_2)(x_1 + x_2) = (y_2 - y_1)(y_2 + y_1)$, which implies by (4.1.21) that

$$y(x_1 - x_2)(x_1 + x_2) = x(x_1 - x_2)(y_2 + y_1). \tag{4.1.22}$$

Note that $x_1 \neq x_2$, since otherwise we have from (4.1.21) that $y_1 = y_2$, which contradicts the condition $a_1 \neq a_2$ in (4.1.18). Dividing both sides of (4.1.22) by $x_1 - x_2$, we get

$$y(x_1 + x_2) = x(y_2 + y_1),$$

which implies in turn that

$$y(a_1 + a_2) = y(x_1 + x_2, y_1 + y_2) = \big(x(y_1 + y_2), y(y_1 + y_2)\big) = (y_1 + y_2)a.$$

In this way we arrive at the representation

$$a_1 + a_2 = \frac{y_1 + y_2}{y} a$$

showing that inclusion (4.1.20) is satisfied. This ensures the optimality of $\bar{x}$ in (1.1.1) and thus completes the proof of the theorem. □

Finally in this section, we present two examples illustrating the application of Theorem 4.1.7 and Corollary 4.1.9, respectively, to solving the corresponding generalized and classical Heron problems.

Example 4.1.11.

Consider problem (1.1.1) where $n = 2$, the sets Ω_1 and Ω_2 are two points A and B in the plane, and the constraint Ω is a disk that does not contain A and B. Condition (4.1.9) from Theorem 4.1.7 characterizes a solution $M \in \Omega$ to this generalized Heron problem as follows. In the first case the line segment AB intersects the disk; then the intersection is an optimal solution. In this case the problem may actually have infinitely many solutions. Otherwise, there is a unique point M on the circle such that a *normal vector* $\overrightarrow{n}$ to Ω at M is the angle bisector of angle AMB, and that is the only optimal solution to the generalized Heron problem under consideration.

Example 4.1.12.

Consider problem (1.1.1), where $\Omega_i = \{A_i\}$, $i = 1, \ldots, n$, are n points in the plane, and where $\Omega = \mathcal{L} \subset I\!R^2$ is a straight line that does not contain these points. Then, by Corollary 4.1.9 of Theorem 4.1.7, a point $M \in \mathcal{L}$ is a solution to this generalized Heron problem if and only if

$$\cos(\overrightarrow{MA_1}, \overrightarrow{a}) + \cdots + \cos(\overrightarrow{MA_n}, \overrightarrow{a}) = 0,$$

where $\overrightarrow{a}$ is a direction vector of $\mathcal{L}$. Note that the latter equation completely characterizes the solution of the classical Heron problem in the plane in both cases when A_1 and A_2 are on the same side and different sides of $\mathcal{L}$.

4.2 Numerical Algorithm and Implementation

Throughout this section we denote the *Euclidean projection* of a point $x \in I\!R^s$ to a closed subset Ω of $I\!R^s$ by $P(x; \Omega)$ to avoid confusion.

In this section we present and justify an iterative algorithm to solve the generalized Heron problem (1.1.1) numerically and illustrate its implementations by using MATLAB in several important settings and constraints. Here is the main algorithm.

Theorem 4.2.1. *Let Ω and Ω_i, $i = 1, \ldots, n$, be nonempty closed convex subsets of $I\!R^s$ such that at least one of them is bounded. Picking a sequence $\{\alpha_k\}$ of positive numbers and a starting point $x_1 \in \Omega$, consider the iterative algorithm:*

$$x_{k+1} = P\Big(x_k - \alpha_k \sum_{i=1}^{n} v_{ik}; \Omega\Big), \quad k = 1, 2, \ldots, \tag{4.2.1}$$

where the vectors v_{ik} *in* (4.2.1) *are constructed by*

$$v_{ik} := \frac{x_k - \omega_{ik}}{d(x_k; \Omega_i)} \;\text{ with }\; \omega_{ik} := \Pi(x_k; \Omega_i) \;\text{ if }\; x_k \notin \Omega_i \tag{4.2.2}$$

and $v_{ik} := 0$ *otherwise. Assume that the given sequence* $\{\alpha_k\}$ *in* (4.2.1) *satisfies the conditions*

$$\sum_{k=1}^{\infty} \alpha_k = \infty \;\text{ and }\; \sum_{k=1}^{\infty} \alpha_k^2 < \infty. \tag{4.2.3}$$

Then the iterative sequence $\{x_k\}$ *in* (4.2.2) *converges to an optimal solution of the generalized Heron problem* (1.1.1) *and the value sequence*

$$V_k := \min\left\{D(x_j)\Big|\; j = 1, \ldots, k\right\} \tag{4.2.4}$$

converges to the optimal value $\widehat{V}$ *in this problem.*

Proof: Observe first of all that algorithm (4.2.1) is well posed, since the projection to a convex set used in (4.2.2) is uniquely defined. Furthermore, all the iterates $\{x_k\}$ in (4.2.1) are feasible; see the proof of Proposition 4.1.1. This algorithm and its convergence under conditions (4.2.3) are based on the subgradient method for convex functions in the so-called “square summable but not summable developed in Section 2.3, the subdifferential sum rule of Theorem 2.1.1, and the subdifferential formula for the distance function given in Proposition 3.1.1. The reader can compare this algorithm and its justifications with the related developments in [13] for the numerical solution of the (unconstrained) generalized Fermat-Torricelli problem. □

4.3 Generalized Heron Problem with Euclidean Norm

Proposition 4.3.1. *In the setting of Theorem 4.2.1 the conclusions of Theorem 4.2.1 hold true, where the element* q_{ik} *has the following representation*

$$q_{ik} = \begin{cases} 0 & \text{if } x_k \in \Omega_i, \\ \dfrac{x_k - P(x_k; \Omega_i)}{d(x_k; \Omega_i)} & \text{if } x_k \notin \Omega_i. \end{cases}$$

Proof: In this case, because the sets Ω_i, $i = 1, \ldots, n$, are convex, we have the following subdifferential representation of the distance function

$$\partial d(u; \Omega_i) = \frac{u - P(u; \Omega_i)}{d(u; \Omega_i)}.$$

if $u \notin \Omega_i$ and $0 \in \partial d(u; \Omega_i) = N(u; \Omega_i) \cap I\!B^*$ otherwise. Notice that in this case $P(u; \Omega_i)$ is singleton. Therefore, the explicit representation of q_{ik} can be derived directly from Theorem 4.2.1. □

Let us continue with several examples to demonstrate the subgradient algorithm. We are going to start with the case where the Generalized Heron problem under consideration is generated by closed balls and the constraint Ω is a nonempty closed convex set in $I\!R^2$.

Corollary 4.3.2. *Consider the problem 1.1.1 in $I\!R^2$ with the Euclidean norm, were $\Omega_i = B(c_i, r_i), i = 1, \ldots, n$. Then the quantities q_{ik} in Theorem 4.2.1 are given by*

$$q_{ik} = \begin{cases} 0 & \text{if } \|x_k - c_i\| \le r_i, \\ \dfrac{x_k - c_i}{\|x_k - c_i\|} & \text{if } \|x_k - c_i\| > r_i. \end{cases}$$

The corresponding quantities V_k are evaluated by formula (4.2.4) with

$$D(x_j) = \sum_{i=1,\, x_j \notin \Omega_i}^{n} \big(\|x_j - c_i\| - r_i\big).$$

Proof: The representation of q_{ik} follows directly from the corresponding representation in Proposition 4.3.1. In this case the projection of x_k to Ω_i is

$$P(x_k; \Omega_i) = c_i + r_i \frac{x_k - c_i}{||x_k - c_i||},$$

if $x_k \notin \Omega_i$. If $x_k \in \Omega_i$ the projection and $P(x_k, \Omega_i) = x_k$. The formula for V_k can be derived directly from Theorem 4.2.1 using the special structures of balls in $I\!R^2$. $\square$

Consider the generalized Heron problem for balls in $I\!R^2$ with the Euclidean norm subject to a given ball constraint. Let $c_i = (a_i, b_i)$ and r_i, $i = 1, \ldots, n$, be the centers and the radii of the balls Ω_i under consideration, and let $c = (x_o, y_o)$ and r be the center and radius for the given ball constraint Ω. The subgradient algorithm is given by (4.2.1), where the projection $P(x, y) := \Pi((x, y); \Omega)$ is computed by

$$P(x,y) = (v_x + x_0, v_y + y_0) \text{ with } v_x = \frac{r(x - x_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}},\ v_y = \frac{r(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}},$$

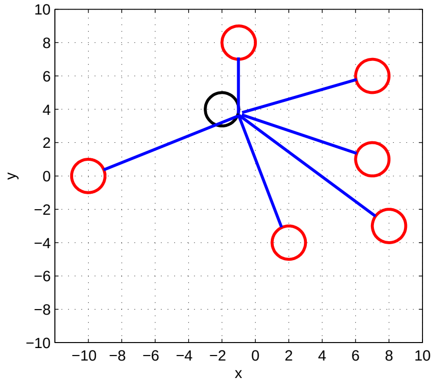
if $(x, y) \notin \Omega$. If $(x, y) \in \Omega$ the projection $P(x, y) = (x, y)$. The quantities q_{ik} and V_k are calculated in Corollary 4.3.2.

Example 4.3.3.

Consider the ball constraint Ω with center $(-2, 4)$ and radius 1. The sets $\Omega_i, i = 1, \ldots, 6$, are the balls with centers $(-10, 0)$, $(-1, 8)$, $(2, -4)$, $(7, 6)$, $(7, 1)$, and $(8, -3)$ and radius $r = 1$. A MATLAB program is performed for the sequence $\alpha_k = 1/k$ satisfying (4.2.3) and the starting point x_1 = (-1,4); see Figure 4.1. Observe that the numerical results are points on the ball constraint where the optimal solution $\bar{x} \approx (-1.07779, 3.61331)$ and the optimal value $\widehat{V} \approx 44.36969$.

Example 4.3.4.

Consider the implementation of the algorithm using a MATLAB program with the square constraint Ω of center (0,-4) and short radius $r = 1$ and the balls Ω_i, $i = 1, \ldots, 6$, of centers (-7,-3), (0,5), (-4,0), (2,-4), (6,0), and (6,7) with the same radius 0.5. Notice that the projection $P(x, y) = P((x, y); \Omega)$ is calculated by



MATLAB RESULT		
k	x_k	V_k
1	(-1,4)	44.58483
10	(-1.07737,3.61433)	44.36969
100	(-1.07779,3.61332)	44.36969
1000	(-1.07779,3.61331)	44.36969
10,000	(-1.07779,3.61331)	44.36969
100,000	(-1.07779,3.61331)	44.36969

Figure 4.1: A Generalized Heron Problem for Balls with Ball Constraint.

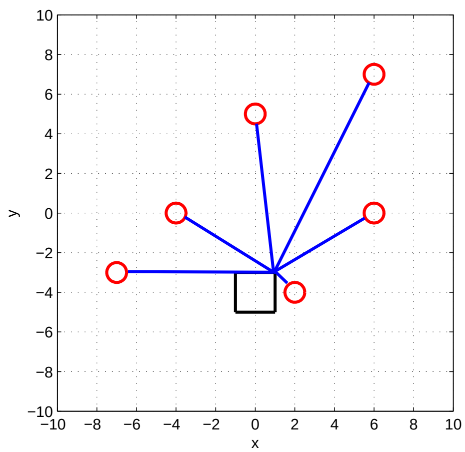
$$P(x,y) = \begin{cases} (a+r, b+r), & \text{if } \; y-b > r, \; x-a > r, \\ (x, b+r), & \text{if } \; |x-a| \le r, \; y-b > r, \\ (a-r, b+r), & \text{if } \; x-a < -r, \; y-b > r, \\ (a-r, y), & \text{if } \; |y-b| \le r, \; x-a < -r, \\ (a-r, b-r), & \text{if } \; y-b < -r, x-a < -r, \\ (x, b-r), & \text{if } \; |x-a| \le r, \; y-b < -r, \\ (a+r, b-r), & \text{if } \; x-a > r, \; y-b < -r, \\ (a+r, y), & \text{if } \; |y-b| \le r, \; x-a > r, \\ (x,y), & \text{if } \; (x,y) \in \Omega. \end{cases}$$

The quantities q_{ik} and V_k are given by Corollary 4.3.2.

The presented calculations are performed for the sequence $\alpha_k = 1/k$ satisfying (4.2.3) and the starting point $x_1 =$(-1,-4); see Figure 4.2. Observe that an optimal solution is $\bar{x} \approx (1.00000, -3.00000)$ and the optimal value is $\widehat{V} \approx 37.31872$.

Next we consider the generalized Heron problem (1.1.1) in $I\!R^2$ with the Euclidean norm generated by squares in *right positions*, which means the sides of the squares are parallel to the $x-$axis and the $y-$axis, and the constraint Ω is a nonempty closed convex set.

Corollary 4.3.5. *Consider the problem (1.1.1) in $I\!R^2$ with the Euclidean norm, where Ω_i is a square*



MATLAB RESULT		
k	x_k	V_k
1	(-1,-4)	41.23881
10	(0.64356,-3)	37.36652
50	(0.89884,-3)	37.32496
100	(0.95169,-3)	37.32091
150	(0.97352,-3)	37.31974
200	(0.98595,-3)	37.31920
250	(0.99413,-3)	37.31890
300	(1.00000,-3)	37.31872
350	(1.00000,-3)	37.31872

Figure 4.2: A Generalized Heron Problem for Balls with Square Constraint.

in right position with center $c_i = (a_i, b_i)$ *and short radius* r_i*,* $i = 1, \ldots, n$*. Let the vertices of the* i*th square be denoted by* $v_{1i} = (a_i + r_i, b_i + r_i), v_{2i} = (a_i - r_i, b_i + r_i), v_{3i} = (a_i - r_i, b_i - r_i), v_{4i} = (a_i + r_i, b_i - r_i)$ *and let* $x_k = (x_{1k}, x_{2k})$*. Then quantities* q_{ik} *in Theorem 4.2.1 are given by*

$$q_{ik} = \begin{cases} 0 & \textit{if } |x_{1k} - a_i| \le r_i \textit{ and } |x_{2k} - b_i| \le r_i, \\ \dfrac{x_k - v_{1i}}{\|x_k - v_{1i}\|} & \textit{if } x_{1k} - a_i > r_i \textit{ and } x_{2k} - b_i > r_i, \\ \dfrac{x_k - v_{2i}}{\|x_k - v_{2i}\|} & \textit{if } x_{1k} - a_i < -r_i \textit{ and } x_{2k} - b_i > r_i, \\ \dfrac{x_k - v_{3i}}{\|x_k - v_{3i}\|} & \textit{if } x_{1k} - a_i < -r_i \textit{ and } x_{2k} - b_i < -r_i, \\ \dfrac{x_k - v_{4i}}{\|x_k - v_{4i}\|} & \textit{if } x_{1k} - a_i > r_i \textit{ and } x_{2k} - b_i < -r_i, \\ (0,1) & \textit{if } |x_{1k} - a_i| \le r_i \textit{ and } x_{2k} - b_i > r_i, \\ (0,-1) & \textit{if } |x_{1k} - a_i| \le r_i \textit{ and } x_{2k} - b_i < -r_i, \\ (1,0) & \textit{if } |x_{1k} - a_i| > r_i \textit{ and } |x_{2k} - b_i| \le r_i, \\ (-1,0) & \textit{if } |x_{1k} - a_i| < -r_i \textit{ and } |x_{2k} - b_i| \le r_i. \end{cases}$$

for all $i = 1, \ldots, n$ *and* $k \in N$ *with the corresponding quantities* V_k *being defined by* (4.2.4).

Proof: The corollary is another direct consequence of Proposition 4.3.1. It follows from the property of the projection from a point to the square Ω_i. $\square$

Example 4.3.6.

Consider the generalized Heron problem (1.1.1) for squares in right positions in $I\!R^2$ with the Euclidean norm subject to a given straight line. Let $c_i = (a_i, b_i)$ and r_i, $i = 1, \ldots, n$, be the centers and the short radius of the squares Ω_i under consideration. The vertices of the ith square are denoted by $v_1 i = (a_i{+}r_i, b_i{+}r_i), v_2 i = (a_i{-}r_i, b_i{+}r_i), v_3 i = (a_i{-}r_i, b_i{-}r_i), v_4 i = (a_i{+}r_i, b_i{-}r_i)$. Let $v = [s, h]$ and $p = (x_o, y_o)$, be the direction vector and point for the given line Ω. Then the subgradient algorithm is written in this case as

$$x_{k+1} = P(x_k - \alpha_k \sum_{i=1}^{n} q_{ik}; \Omega),$$

where the projection $P(x, y) = P((x, y); \Omega)$ is calculated by

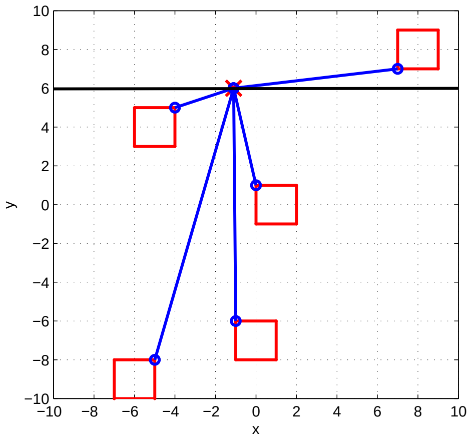
$$P(x, y) = (x_o + st, y_o + ht) \text{ and } t = \frac{s(x - x_o) + h(y - y_o)}{s^2 + h^2},$$

if $(x, y) \notin \Omega$. If $(x, y) \in \Omega$ the projection $P(x, y) = (x, y)$. The quantities q_{ik} and V_k for all $i = 1, \ldots, n$ and $k \in N$ are given by the Corollary 4.3.5.

Consider the implementation of this algorithm using a MATLAB program with line constraint described by direction vector $v = [1, 0]$ and point $(1, 6)$, and the convex sets $\Omega_i, i = 1, \ldots 5$, are squares of centers $(-6, -9)$, $(-5, 4)$, $(0, -7)$, $(1, 0)$, and $(8, 8)$ with the same short radius r=1. The presented calculations are performed for the sequence $\alpha_k = 1/k$ satisfying (4.2.3) and the starting point $x_1 = (-1, 6)$; see Figure 4.3. Observe that an optimal solution is $\bar{x} \approx (-1.0946, 6)$ and the optimal value is $\widehat{V} \approx 42.8821$.

Example 4.3.7.

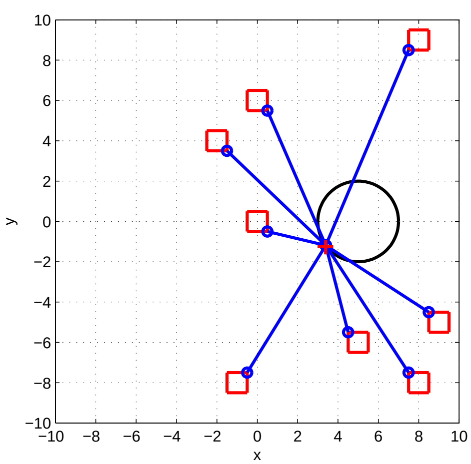
Similarly, we can also provide the subgradient method for the generalized Heron problems generated by squares in right positions with ball constraint.



MATLAB RESULT		
k	x_k	V_k
1	(-1,6)	42.8838
100	(-1.0826,6)	42.8821
1000	(-1.0896,6)	42.8821
100,000	(-1.0938,6)	42.8821
1,000,000	(-1.0944,6)	42.8821
5,000,000	(-1.0946,6)	42.8821
10,000,000	(-1.0946,6)	42.8821

Figure 4.3: A Generalized Heron Problem for Squares with Line Constraint.

Consider the implementation of this algorithm using a MATLAB program with the ball constraint Ω of center $(5,0)$ and radius 2 and the convex sets $\Omega_i, i = 1,\ldots,8$, are squares of centers $(-2,4)$, $(-1,-8)$, $(0,0)$, $(0,6)$, $(5,-6)$, $(8,-8)$, $(8,9)$, and $(9,-5)$ with the same short radius $r = 0.5$. The presented calculations are performed for the sequence $\alpha_k = 1/k$ satisfying (4.2.3) and the starting point x_1 = (5,-2); see Figure 4.4. Observe that the numerical results yield the optimal solution $\bar{x} \approx (3.39270, -1.19021)$ and an optimal value $\widehat{V} \approx 53.04363$.



MATLAB RESULT		
k	x_k	V_k
1	(5,-2)	54.41891
10	(3.51379,-1.33835)	53.05740
100	(3.41230,-1.21623)	53.04403
1000	(3.39607,-1.19475)	53.04364
100,000	(3.39279,-1.19033)	53.04363
600,000	(3.39271,-1.19022)	53.04363
1,000,000	(3.39271,-1.19021)	53.04363
1,200,000	(3.39270,-1.19021)	53.04363
1,400,000	(3.39270,-1.19021)	53.04363

Figure 4.4: A Generalized Heron Problem for Squares with Ball Constraint.

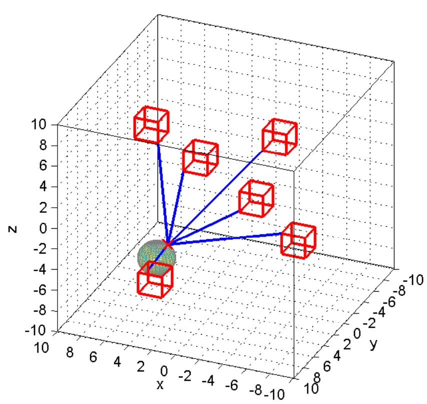
Example 4.3.8.

We finally consider a generalized Heron problem (1.1.1) for cubes of *right positions* in $\mathbb{R}^3$ subject to a ball constraint. In this case the subgradient algorithm (4.2.1) is

$$x_{k+1} = P\Big(x_k - \alpha_k \sum_{i=1}^{n} q_{ik}; \Omega\Big),$$

where the projection $P((x, y, z); \Omega)$ and quantities q_{ik} are computed similarly to Example 4.4.

Consider the implementation of this algorithm using a MATLAB program. Figure 4.5 and corresponding table present the calculation results for the ball constraint Ω with center $(5, 2, -6)$ and radius 1.5, the cubes Ω_i with centers $(8, -4, 3)$, $(-2, -6, 3)$, $(3, -2, 2)$, $(-4, -5, -6)$, $(-3, 1, 1)$, and $(3, 7, -5)$ with the same short radius $r = 1$, the starting point $x_1 = (5, .5, -6)$, and the sequence of $\alpha_k = 1/k$ in (4.2.1) satisfying (4.2.3); see Figure 4.5. The optimal solution and optimal value computed up to five significant digits are $\bar{x} \approx (4.23948, 1.53024, -4.79546)$ and $\widehat{V} \approx 47.19026$.



MATLAB RESULT		
k	x_k	V_k
1	(5,0.5,-6)	51.58786
10	(4.23949,1.52680,-4.79680)	47.19028
100	(4.23948,1.53023,-4.79546)	47.19026
1,000	(4.23948,1.53024,-4.79546)	47.19026
10,000	(4.23948,1.53024,-4.79546)	47.19026
100,000	(4.23948,1.53024,-4.79546)	47.19026
1,000,000	(4.23948,1.53024,-4.79546)	47.19026

Figure 4.5: A Generalized Heron Problem for Cubes with Ball Constraint in Three Dimensions.

4.4 Generalized Heron Problem with Sum Norm

Let us illustrate applications of the subgradient algorithm of Theorem 4.2.1 to solving the generalized Heron problem (1.1.1) formulated via the distance function with other norms. Consider the unit ball given by the *closed unit diamond* on the plane

$$I\!B = \{(x_1, x_2) \mid |x_1| + |x_2| \le 1\}.$$

In this case the norm in $I\!R^2$ is given by the formula

$$p(x) = \|x\|_1 = \|(x_1, x_2)\|_1 = |x_1| + |x_2|. \tag{4.4.1}$$

In the following proposition we compute a subgradient of the distance function (1.1.1) generated by the Sum norm and a square target in $I\!R^2$, which will be used to construct a subgradient algorithm to solve the corresponding generalized Heron problem.

Proposition 4.4.1. *Consider the problem (1.1.1) in $I\!R^2$ with the Sum norm, and let Ω be a square of right position in $I\!R^2$ centered at $c = (a, b)$ with short radius $r > 0$. Then a subgradient $v(\bar{x}_1, \bar{x}_2) \in \partial d(\bar{x}_1, \bar{x}_2)$(not necessarily uniquely defined) of the distance function d at $(\bar{x}_1, \bar{x}_2)$ is computed by*

$$v(\bar{x}_1, \bar{x}_2) = \begin{cases} (1,0), & \text{if } |\bar{x}_2 - b| \le r,\ \bar{x}_1 > a + r, \\ (-1,0), & \text{if } |\bar{x}_2 - b| \le r,\ \bar{x}_1 < a - r. \\ (0,1), & \text{if } |\bar{x}_1 - a| \le r,\ \bar{x}_2 > b + r, \\ (0,-1), & \text{if } |\bar{x}_1 - a| \le r,\ \bar{x}_2 < b - r, \\ (1,1), & \text{if } \bar{x}_1 > a + r, \bar{x}_2 > b + r, \\ (-1,1), & \text{if } \bar{x}_1 < a - r, \bar{x}_2 > b + r, \\ (-1,-1), & \text{if } \bar{x}_1 < a - r, \bar{x}_2 < b - r, \\ (1,-1), & \text{if } \bar{x}_1 > a + r, \bar{x}_2 < b - r, \\ 0, & \text{if } (\bar{x}_1, \bar{x}_2) \in \Omega. \end{cases} \tag{4.4.2}$$

Proof: Let us apply [12, Theorem 6.2]. For a vector $(\bar{x}_1, \bar{x}_2) \in I\!R^2$, it is easy to find the non-Euclidean projection $\Pi(\bar{x}_1, \bar{x}_2)$ to the square Ω. Moreover, using Proposition (3.3.1), which is a direct result of the sum rule from convex analysis, we only need to verify that $v(\bar{x}_1, \bar{x}_2) \in \partial d(\bar{x}_1, \bar{x}_2)$. The proof in now complete. □

Corollary 4.4.2. *Consider problem (1.1.1) generated by squares Ω_i, $i = 1, \ldots, n$, of* right position, *and Sum norm in $I\!R^2$. Let $c_i = (a_i, b_i)$ and r_i, $i = 1, \ldots, n$, be the centers and the short radii* *of the squares under consideration. Then the vertices of the ith square are denoted by* $v_{1i} = (a_i + r_i, b_i + r_i)$, $v_{2i} = (a_i - r_i, b_i + r_i)$, $v_{3i} = (a_i - r_i, b_i - r_i)$, *and* $v_{4i} = (a_i + r_i, b_i - r_i)$ *and let* $x_k = (x_{1k}, x_{2k})$. *Then the quantities* q_{ik} *from Theorem 4.2.1 are computed as follows:*

$$q_{ik} = \begin{cases} 0, & \text{if } |x_{1k} - a_i| \le r_i \text{ and } |x_{2k} - b_i| \le r_i, \\ (1,1), & \text{if } x_{1k} - a_i > r_i \text{ and } x_{2k} - b_i > r_i, \\ (-1,1), & \text{if } x_{1k} - a_i < -r_i \text{ and } x_{2k} - b_i > r_i, \\ (-1,-1), & \text{if } x_{1k} - a_i < -r_i \text{ and } x_{2k} - b_i < -r_i, \\ (1,-1), & \text{if } x_{1k} - a_i > r_i \text{ and } x_{2k} - b_i < -r_i, \\ (0,1), & \text{if } |x_{1k} - a_i| \le r_i \text{ and } x_{2k} - b_i > r_i, \\ (0,-1), & \text{if } |x_{1k} - a_i| \le r_i \text{ and } x_{2k} - b_i < -r_i, \\ (1,0), & \text{if } x_{1k} - a_i > r_i \text{ and } |x_{2k} - b_i| \le r_i, \\ (-1,0), & \text{if } x_{1k} - a_i < -r_i \text{ and } |x_{2k} - b_i| \le r_i. \end{cases}$$

for all $i \in \{1, \ldots, n\}$ *and* $k \in \mathbb{N}$.

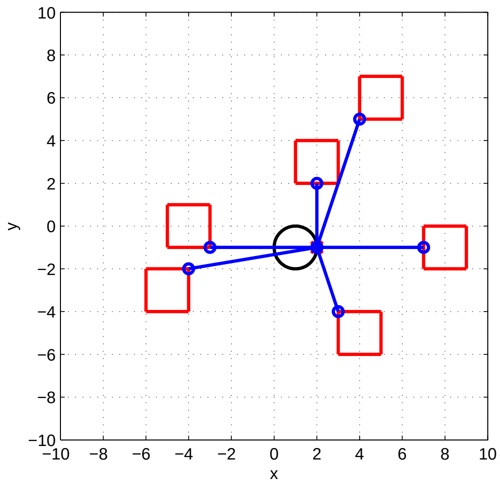
Proof: It follows from Proposition 4.4.1, comparison between the right-hand side of (4.2.2) and formula (3.1.1), and the square calculations of Corollary 4.3.5. □

Example 4.4.3.

Consider the implementation of this algorithm using a MATLAB program with the Euclidean ball constraint constraint Ω of center (1,-1) and radius 1 and the sets $\Omega_i, i = 1, \ldots, 6$, of centers (-5,-3), (-4,0), (2,3), (4,-5), (5,6), and (8,-1) with the same short radius r=1. The presented calculations are performed for the sequence $\alpha_k = 1/k$ satisfying (4.2.3) and the starting point $x_1 = (1, -2)$; see Figure 4.6. Observe that the numerical results are points on the ball constraint where the optimal solution $\bar{x} \approx (2.00000, -1.00000)$ and the optimal value $\widehat{V} \approx 32.00000$.

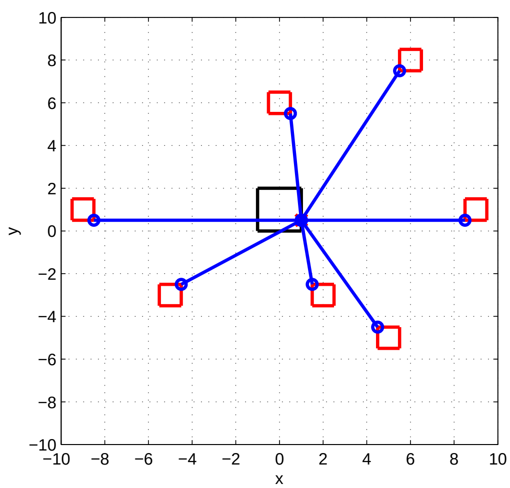
Example 4.4.4.

Consider the implementation of this algorithm using a MATLAB program with the square constraint Ω with center (0,1) and short radius 1 and the convex sets $\Omega_i, i = 1, \ldots 7$, are squares of centers $(-5, -3)$, $(-9, 1)$, $(0, 6)$, $(2, -3)$, $(6, 8)$, $(5, -5)$, and $(9, 1)$ with the same short radius r=0.5. The presented calculations are performed for the sequence $\alpha_k = 1/k$ satisfying (4.2.3) and the starting point $x_1 = (-1, 2)$; see Figure 4.7. Observe that the numerical results are points on the square constraint where the optimal solution $\bar{x} \approx (1, 0.50000)$ and the optimal value $\widehat{V} \approx 54.5$.



MATLAB RESULT		
k	x_k	V_k
1	(1,-2)	34
10	(1.98703,-0.83947)	32.01297
100	(1.99987,-0.98385)	32.00013
1,000	(2.00000,-0.99838)	32.00000
10,000	(2.00000,-0.99984)	32.00000
100,000	(2.00000,-0.99998)	32.00000
200,000	(2.00000,-0.99999)	32.00000
400,000	(2.00000,-1.00000)	32.00000
800,000	(2.00000,-1.00000)	32.00000

Figure 4.6: A Generalized Heron Problem for Squares with Ball Constraint and "Sum" Distances.



MATLAB RESULT		
k	x_k	V_k
1	(-1,2)	61
10	(1,0.57897)	54.5
100	(1,0.48990)	54.5
1000	(1,0.50100)	54.5
1500	(1,0.49933)	54.5
2000	(1,0.49950)	54.5
2500	(1,0.49960)	54.5
3000	(1,0.50000)	54.5
3500	(1,0.50000)	54.5

Figure 4.7: A Generalized Heron Problem for Squares with Square Constraint and "Sum" Distances.

4.5 Generalized Heron Problem with Max Norm

Let us illustrate applications of the subgradient algorithm of Theorem 4.2.1 to solving the generalized Heron problem 1.1.1, where the unit ball is given by the square $[-1, 1] \times [-1, 1]$ on the plane. In this case the Max norm in $I\!\!R^2$ is given by the formula

$$p(x) = \|x\|_\infty = \|(x_1, x_2)\|_\infty = \max\big\{|x_1|, |x_2|\big\}. \tag{4.5.1}$$

Similar method can be used to obtain the results below; see [13] for more detail.

Proposition 4.5.1. *Consider the problem (1.1.1) in $I\!\!R^2$ with the Max norm, and let Ω be a square of right position in $I\!\!R^2$ centered at $c = (a, b)$ with short radius $r > 0$. Then a subgradient $v(\bar{x}_1, \bar{x}_2) \in \partial d(\bar{x}_1, \bar{x}_2)$(not necessarily uniquely defined) of the distance function d at $(\bar{x}_1, \bar{x}_2)$ is computed by*

$$v(\bar{x}_1, \bar{x}_2) = \begin{cases} (1,0), & \text{if } |\bar{x}_2 - b| \le \bar{x}_1 - a,\ \bar{x}_1 > a + r, \\ (-1,0), & \text{if } |\bar{x}_2 - b| \le a - \bar{x}_1,\ \bar{x}_1 < a - r. \\ (0,1), & \text{if } |\bar{x}_1 - a| \le \bar{x}_2 - b,\ \bar{x}_2 > b + r, \\ (0,-1), & \text{if } |\bar{x}_1 - a| \le b - \bar{x}_2,\ \bar{x}_2 < b - r, \\ 0, & \text{if } (\bar{x}_1, \bar{x}_2) \in \Omega. \end{cases} \tag{4.5.2}$$

Proof: It is given in [13, Proposition 5.1]. □

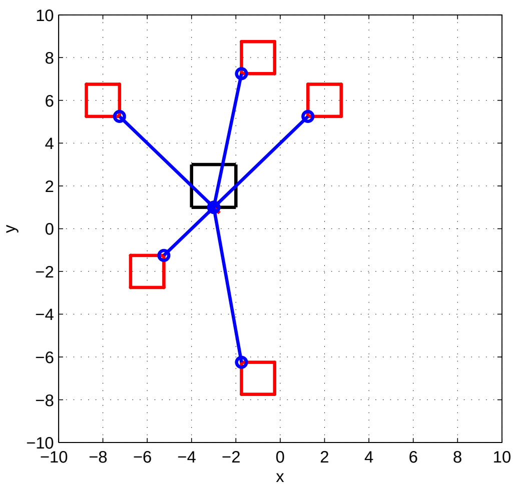
As a consequence of the proposition above, we calculate the quantities q_{ik} in algorithm (4.2.1) for the corresponding version of the generalized Heron problem.

Corollary 4.5.2. *Consider problem (1.1.1) generated by squares Ω_i, $i = 1, \ldots, n$, of* right position *and the Max norm. Let $c_i = (a_i, b_i)$ and r_i, $i = 1, \ldots, n$, be the centers and the short radii of the squares under consideration. Let the vertices of the ith square be denoted by $v_{1i} = (a_i + r_i, b_i + r_i)$, $v_{2i} = (a_i - r_i, b_i + r_i)$, $v_{3i} = (a_i - r_i, b_i - r_i)$, and $v_{4i} = (a_i + r_i, b_i - r_i)$ and let $x_k = (x_{1k}, x_{2k})$. Then the quantities q_{ik} from Theorem 4.2.1 are computed as follows:*

$$q_{ik} = \begin{cases} (1,0), & \text{if } |x_{2k} - b_i| \le x_{1k} - a_i \text{ and } x_{1k} > a_i + r_i, \\ (-1,0), & \text{if } |x_{2k} - b_i| \le a_i - x_{1k} \text{ and } x_{1k} < a_i - r_i, \\ (0,1), & \text{if } |x_{1k} - a_i| \le x_{2k} - b_i \text{ and } x_{2k} > b_i + r_i, \\ (0,-1), & \text{if } |x_{1k} - a_i| \le b_i - x_{2k} \text{ and } x_{2k} < b_i - r_i, \\ (0,0), & \text{otherwise.} \end{cases}$$

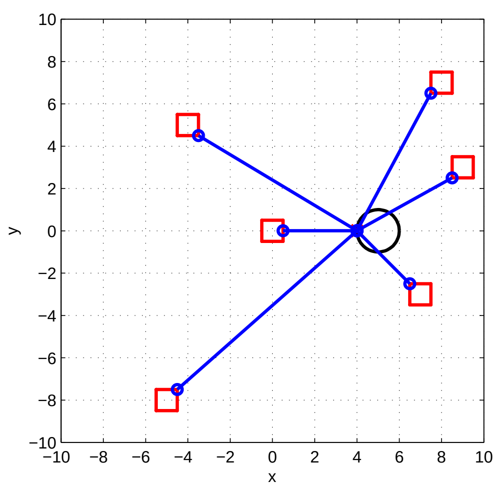
for all $i \in \{1, \ldots, n\}$ and $k \in I\!\!N$.

Proof: It follows from Proposition 4.5.1, comparison between the right-hand side of (4.2.2) and formula (3.1.1), and the square calculations of Corollary 4.3.5. □



MATLAB RESULT		
k	x_k	V_k
1	(-4,3)	26.25000
100	(-2.99136,1.00070)	24.25068
1,000	(-3.00133,1.00133)	24.25000
10,000	(-2.99996,1.00001)	24.25000
100,000	(-3.00000,1.00000)	24.25000
200,000	(-3.00001,1.00000)	24.25000
400,000	(-3.00000,1.00000)	24.25000
800,000	(-3.00000,1.00000)	24.25000

Figure 4.8: A Generalized Heron Problem for Squares with Square Constraint and "Max" Distances.



MATLAB RESULT		
k	x_k	V_k
1	(5,0)	35
10	(4.00062,0.03519)	33
100	(4.00000,0.00038)	33
200	(4.00000,0.00010)	33
400	(4.00000,0.00002)	33
800	(4.00000,0.00001)	33
1,000	(4.00000,0.00000)	33
1,200	(4.00000,0.00000)	33

Figure 4.9: A Generalized Heron Problem for Squares with Ball Constraint and "Max" Distances.

Example 4.5.3.

Consider the implementation of this algorithm using a MATLAB program with the square constraint Ω of center (-3,2) and short radius 1 and the sets $\Omega_i, i = 1, \ldots, 5$, of centers (-8,6), (-6,-2), (-1,8), (-1,-7), and (2,6) with the same short radius r=0.75. The presented calculations are performed for the sequence $\alpha_k = 1/k$ satisfying (4.2.3) and the starting point $x_1 = (-4, 3)$; see Figure 4.8. Observe that the numerical results are points on the square constraint where the optimal solution $\bar{x} \approx (-3.00000, 1.00000)$ and the optimal value $\widehat{V} \approx 24.25000$.

Example 4.5.4.

Consider the implementation of this algorithm using a MATLAB program with the Euclidean ball constraint Ω of center (5,0) and radius 1 and the sets $\Omega_i, i = 1, \ldots, 6$, of centers (-5,-8), (-4,5), (0,0), (8,7), (9,3),and (7,-3) with the same short radius r=0.5. The presented calculations are performed for the sequence $\alpha_k = 1/k$ satisfying (4.2.3) and the starting point $x_1 = (5, 0)$; see Figure 4.9. Observe that the numerical results are points on the ball constraint where the optimal solution $\bar{x} \approx (4.00000, 0.00000)$ and the optimal value $\widehat{V} \approx 33$.

REFERENCES

[1] Bertsekas, D., Nedic, A., and Ozdaglar, A. (2003), *Convex analysis and optimization*, Athena Scientific.

[2] Borwein, J.M., Lewis, A.S.: Convex Analysis and Nonlinear Optimization: Theory and Examples, 2nd edition. Springer, New York (2006)

[3] Borwein, J.M., Zhu, Q.J.: Techniques of Variational Analysis. Springer, CMS Books in Mathematics **20**, Springer, New York (2005)

[4] Courant, R., Robbins, R.: What Is Mathematics? An Elementary Approach to Ideas and Methods, Oxford University Press, New York (1941)

[5] Boyd, S., Xiao, L., and Mutapcic, A. (2003), *Subgradient methods*, Lecture Notes.

[6] Burke, J., Ferris, M.C., Quian, M.: On the Clarke subdifferential of the distance function of a closed set. J. Math. Anal. Appl. **166**, 199–213 (1992)

[7] Heath, T.L.: A History of Greek Mathematics, Vols. I and II, Oxford University Press, London (1921)

[8] B. S. Mordukhovich, *Maximum principle in problems of time optimal control with nonsmooth constgraints*, Appl. Math. Mech. 40 (1976), pp. 960–969.

[9] Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, I: Basic Theory, II: Applications, Grundlehren Series (Fundamental Principles of Mathematical Sciences), Vols. 330 and 331, Springer, Berlin (2006)

[10] Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, II: Applications. Grundlehren Series (Fundamental Principles of Mathematical Sciences) **331**, Springer, Berlin (2006)

[11] Mordukhovich, B.S., Nam, N.M.: Limiting subgradients of minimal time functions in Banach spaces. J. Global Optim. **46**, 615–633 (2010)

[12] Mordukhovich, B.S, Nam, N.M.: Subgradients of minimal time functions under minimal assumptions. Arxiv: 1009.1585 (2010), to appear in J. Convex Anal.

[13] Mordukhovich, B.S., Nam, N.M.: Applications of variational analysis to a generalized Fermat-Torricelli problem. J. Optim. Theory Appl. **148**, 431–454 (2011)

[14] Mordukhovich, B.S., Nam, N.M., Salinas, J.: Solving a generalized Heron problem by means of convex analysis, to appear in American Mathematical Monthly.

[15] Mordukhovich, B.S., Nam, N.M., Salinas, J.: Applications of variational analysis to a generalized Heron problem, to appear in Applicable Analysis.

[16] Nam, N.M., An. N.T, Salinas, J.: Applications of convex analysis to the smallest intersecting ball problem, to appear in Journal of Convex Analysis.

[17] Nickel, S., Puerto, J., Rodriguez-Chia, A.M.: An approach to location models involving sets as existing facilities. Math. Oper. Res. 28, 693–715 (2003)

[18] Rockafellar, R.T., Wets, R.J-B.: Variational Analysis, Grundlehren Series (Fundamental Principles of Mathematical Sciences), Vols. 317, Springer, Berlin (1998)

[19] Schirotzek, W.: Nonsmooth Analysis. Universitext, Springer, Berlin (2007)

[20] J.-B. Hiriart-Urruty and C. Lemaréchal: Convex Analysis and Minimization Algorithms I. Fundamentals, Springer-Verlag, Berlin (1993)

[21] Rockafellar, R.T.: Convex Analysis. Princeton University Press, Princeton, NJ (1970)

APPENDIX A

APPENDIX A

MATLAB NUMERICAL IMPLEMENTATION

This is the Matlab Source Code for the numerical implementation for the Generalized Heron Problem described in this thesis where the constraint is a disk and the target sets are squares.

```
*********************Filename 1: draw_disk.m*******************

function draw_disk (cx,cy,r)
hold all;
format compact                    % tightens loose format
format long e                     % makes numerical output in
                                  % double precision
theta = linspace(0,2*pi,100);     % create vector theta
x = cx + r * cos(theta);          % generate x-coordinate
y = cy + r * sin(theta);          % generate y-coordinate
plot(x,y);                        % plot circle
axis('equal');                    % set equal scale on axes
                                  % per pixel
end

*******************Filename 2: disk2squares.m******************

function disk2squares
format long
%the constraint set is a disk center (a,b) and radius r and the
%distance
%to squares
cont = 'y';
```

```
while cont == 'y' || cont == 'Y'

    clc
    num_squares = 0;

    a = input(['Enter x for the center of the constraint disk:
     ']);
    b = input(['Enter y for the center of the constraint disk:
     ']);
    r = input(['Enter the radius of the constraint circle: ']);

    for i = 1:intmax
      temp = input(['Enter x for square ', num2str(i),':
      [ENTER to break]']);
      if isempty(temp) == 1
        break
      else
        c(i,1) = temp;
        c(i,2) = input(['Enter y for square ', num2str(i),'
        : ']);
        rr(i) = input(['Enter the radius for square ',
        num2str(i),': ']);
        num_squares = num_squares + 1;
      end
    end

if num_squares == 0
    disp('Program Stopped.  You did not enter any points.');
else
    x1 = input(['Enter x for initial point:']);
    y1 = input(['Enter y for initial point:']);
    iterations = input(['Enter the number of iterations:']);
```

```
%draw constraint
draw_disk(a,b,r);
%draw squares
for i = 1:num_squares
    line([c(i,1)+rr(i) c(i,1)-rr(i)],[c(i,2)+rr(i) c(i,2)
    +rr(i)],'LineStyle','-','Color','r');
    line([c(i,1)-rr(i) c(i,1)-rr(i)],[c(i,2)+rr(i) c(i,2)
    -rr(i)],'LineStyle','-','Color','r');
    line([c(i,1)-rr(i) c(i,1)+rr(i)],[c(i,2)-rr(i) c(i,2)
    -rr(i)],'LineStyle','-','Color','r');
    line([c(i,1)+rr(i) c(i,1)+rr(i)],[c(i,2)-rr(i) c(i,2)
    +rr(i)],'LineStyle','-','Color','r');
end

f1 = 0;

px = x1;
py = y1;

for i=1:num_squares

    switch logical(true)
      %Euclidean Norm distance to squares
      case {px>c(i,1)+rr(i) && py>=c(i,2)-rr(i)&& py<=
      c(i,2)+rr(i)}
      d = [px-c(i,1)-rr(i) 0];
      case {px<c(i,1)-rr(i) && py>=c(i,2)-rr(i)&& py<=
      c(i,2)+rr(i)}
      d = [c(i,1)-rr(i)-px 0];
      case {py>c(i,2)+rr(i) && px>=c(i,1)-rr(i)&& px<=
      c(i,1)+rr(i)}
```

```
        d = [0 py-c(i,2)-rr(i)];
        case {py<c(i,2)-rr(i) && px>=c(i,1)-rr(i)&& px<=
        c(i,1)+rr(i)}
        d = [0 c(i,2)-rr(i)-py];
        case {px>c(i,1)+rr(i) && py>c(i,2)+rr(i)}
        d = [px py]-[c(i,1)+rr(i) c(i,2)+rr(i)];
        case {px<c(i,1)-rr(i) && py>c(i,2)+rr(i)}
        d = [px py]-[c(i,1)-rr(i) c(i,2)+rr(i)];
        case {px>c(i,1)+rr(i) && py<c(i,2)-rr(i)}
        d = [px py]-[c(i,1)+rr(i) c(i,2)-rr(i)];
        case {px<c(i,1)-rr(i) && py<c(i,2)-rr(i)}
        d = [px py]-[c(i,1)-rr(i) c(i,2)-rr(i)];
        otherwise
        d = [0 0];
      end

      %calculate the Euclidean distance f1 with a1...an
      f1 = f1 + norm(d);
  end

  %start value
  disp(['k x y f: ', num2str(1),' ', num2str(x1,7),' ',
  num2str(y1,    7),'   f: ',num2str(f1,8)])

for j=1:iterations
     g = [0 0];

  for i=1:num_squares

    switch logical(true)
      %Euclidean Norm subgradient projection to squares
          case {px>c(i,1)+rr(i) && py>=c(i,2)-rr(i)&& py<=
```

```
        c(i,2)+rr(i)}
            d = [1 0];
        case {px<c(i,1)-rr(i) && py>=c(i,2)-rr(i)&& py<=
        c(i,2)+rr(i)}
            d = [-1 0];
        case {py>c(i,2)+rr(i) && px>=c(i,1)-rr(i)&& px<=
        c(i,1)+rr(i)}
            d = [0 1];
        case {py<c(i,2)-rr(i) && px>=c(i,1)-rr(i)&& px<=
        c(i,1)+rr(i)}
            d = [0 -1];
        case {px>c(i,1)+rr(i) && py>c(i,2)+rr(i)}
            d = [px py]-[c(i,1)+rr(i) c(i,2)+rr(i)];
        case {px<c(i,1)-rr(i) && py>c(i,2)+rr(i)}
            d = [px py]-[c(i,1)-rr(i) c(i,2)+rr(i)];
        case {px>c(i,1)+rr(i) && py<c(i,2)-rr(i)}
            d = [px py]-[c(i,1)+rr(i) c(i,2)-rr(i)];
        case {px<c(i,1)-rr(i) && py<c(i,2)-rr(i)}
            d = [px py]-[c(i,1)-rr(i) c(i,2)-rr(i)];
    otherwise
        d = [0 0];
  end
        if d == 0
            %g = g + 0;
            %comment to save computing time
        else
            g = g + d/norm(d);
        end
end
   %new point to test
    x1 = x1 - (1/j)*(g(1));
    y1 = y1 - (1/j)*(g(2));
```

```
  %find the projection to constraint
  if ((x1-a)^2+(y1-b)^2)^.5>r
   d = [x1,y1]-[a,b];
   v=r*d/norm(d);
   px = v(1)+a;
   py = v(2)+b;
  else
   px = x1;
   py = y1;
  end

  f2 = 0;
  %calculate the distance f2 with c1...cn - r
for i=1:num_squares
   switch logical(true)
       %Euclidean Norm distance to squares
       case {px>c(i,1)+rr(i) && py>=c(i,2)-rr(i)&& py<=
       c(i,2)+rr(i)}
           d = [px-c(i,1)-rr(i) 0];
       case {px<c(i,1)-rr(i) && py>=c(i,2)-rr(i)&& py<=
       c(i,2)+rr(i)}
           d = [c(i,1)-rr(i)-px 0];
       case {py>c(i,2)+rr(i) && px>=c(i,1)-rr(i)&& px<=
       c(i,1)+rr(i)}
           d = [0 py-c(i,2)-rr(i)];
       case {py<c(i,2)-rr(i) && px>=c(i,1)-rr(i)&& px<=
       c(i,1)+rr(i)}
           d = [0 c(i,2)-rr(i)-py];
       case {px>c(i,1)+rr(i) && py>c(i,2)+rr(i)}
           d = [px py]-[c(i,1)+rr(i) c(i,2)+rr(i)];
       case {px<c(i,1)-rr(i) && py>c(i,2)+rr(i)}
```

```
            d = [px py]-[c(i,1)-rr(i) c(i,2)+rr(i)];
        case {px>c(i,1)+rr(i) && py<c(i,2)-rr(i)}
            d = [px py]-[c(i,1)+rr(i) c(i,2)-rr(i)];
        case {px<c(i,1)-rr(i) && py<c(i,2)-rr(i)}
            d = [px py]-[c(i,1)-rr(i) c(i,2)-rr(i)];
        otherwise
            d = [0 0];
    end

    %calculate the distance f1 with a1...an
    f2 = f2 + norm(d);
 end

   %find f1 = fmin(f1,f2) and x1 = min(x1,x2)
   if f2 <= f1
      f1 = f2;
   end

   if j+1==10 || j+1==10^2 || j+1==10^3 || j+1==10^4 ||
    j+1==10^5 ||j+1==2*10^5 || j+1==4*10^5 || j+1==6*10^5
    ||j+1==8*10^5 ||j+1==10*10^5 || j+1==12*10^5 ||
    j+1==14*10^5 ||j+1==16*10^5
    disp(['k x y f: ', num2str(j+1),' ', num2str(px,7),' ',
    num2str(py,7),'   f: ',num2str(f1,8)])
   end
   x1=px;
   y1=py;
end

  %plot xbar = x1
  disp(['x_',num2str(j+1),' = (', num2str(x1,7),',',num2str
  (y1,7),')   f_'
```

```
,num2str(j+1),' = ',num2str(f1,8)])
hold all;
plot(x1,y1,'r+')

%plot a1...an and connect them with lines
for j=1:num_squares

  switch logical(true)
  case {px>c(j,1)+rr(j) && py>=c(j,2)-rr(j)&& py<=c(j,2)+
  rr(j)}
      line([px c(j,1)+rr(j)],[py py],'marker','o')
  case {px<c(j,1)-rr(j) && py>=c(j,2)-rr(j)&& py<=c(j,2)+
  rr(j)}
      line([px c(j,1)-rr(j)],[py py],'marker','o')
  case {py>c(j,2)+rr(i) && px>=c(j,1)-rr(j)&& px<=c(j,1)+
  rr(j)}
      line([px px],[py c(j,2)+rr(j)],'marker','o')
  case {py<c(j,2)-rr(j) && px>=c(j,1)-rr(j)&& px<=c(j,1)+
  rr(j)}
      line([px px],[py c(j,2)-rr(j)],'marker','o')

  case {px>c(j,1)+rr(j) && py>c(j,2)+rr(j)}
      line([px c(j,1)+rr(j)],[py c(j,2)+rr(j)],'marker',
      'o')
  case {px<c(j,1)-rr(j) && py>c(j,2)+rr(j)}
      line([px c(j,1)-rr(j)],[py c(j,2)+rr(j)],'marker',
      'o')
  case {px>c(j,1)+rr(i) && py<c(j,2)-rr(j)}
      line([px c(j,1)+rr(j)],[py c(j,2)-rr(j)],'marker',
      'o')
  case {px<c(j,1)-rr(j) && py<c(j,2)-rr(j)}
      line([px c(j,1)-rr(j)],[py c(j,2)-rr(j)],'marker',
```

```
            'o')

            otherwise
                %no line necessary
        end
      end

      %adjust plot properties
      axis([-10 10 -10 10]);
      xlabel('x');
      ylabel('y');
      axis on;
      grid on;
end
cont = input('Do you wish to continue (Y or N)','s');
end
end
```

BIOGRAPHICAL SKETCH

Juan Salinas, Jr. received his B.S. in Electrical Engineering in 1999 and completed his M.S. in Mathematics in 2011, both, from the University of Texas-Pan American(UTPA). He worked at Alps Automotive, Inc. from 1999 to 2003 as an Electrical Engineer in the design and manufacturing of electronic automotive components. He became a Texas Certified High School Teacher since 2004 and has been teaching secondary mathematics to grade school, secondary, and post secondary students. His current interests are in the area of applied mathematics and mathematics education.

He had the opportunity to participate in the Summer Research Initiative for Graduate Students at UTPA during the Summer of 2010. His work received first prize in the 2010 Mathematics HESTEC Reserach Poster Competition at UTPA where he presented his research for the first time. In January 2011 he was invited to give a joint presentation with his Graduate Advisor, Dr. Nguyen, Mau Nam at the 2011 Joint Meeting of the American Mathematical Society (AMS). He wrote three joint research articles, [14, 15, 16], to appear in three important mathematics journals. Part of his results for his research are presented in this thesis.

He was offered a full time position as a High School Teacher at the Science Academy of the South Texas Independent School District for the 2011-2012 school year, which he aceepted. He will start the National Board Certification for Teacher Leaders in 2011 and would like to pursue a Ph.D. with emphasis in Mathematics Education in the near future. His permanent mailing address is 4031 Camino Real Viejo, Weslaco, Texas 78596.