6-3-2022

Monotonicity properties of functionals under Ricci flow on manifolds without and with boundary

Paul Bracken
The University of Texas Rio Grande Valley, paul.bracken@utrgv.edu

Recommended Citation

Bracken, Paul. "Monotonicity properties of functionals under Ricci flow on manifolds without and with boundary." International Journal of Geometric Methods in Modern Physics 19.9 (2022): 2250137-262. https://doi.org/10.1142/S0219887822501377

Monotonicity Properties of Functionals Under Ricci Flow on Manifolds Without and With Boundary

Paul Bracken

Department of Mathematics,

University of Texas,

Edinburg, TX

78541-2999

paul.bracken@utrgv.edu

Abstract

The idea of the Ricci flow is introduced and its significance and importance to related problems in mathematics is discussed. Several functionals are defined and their behavior is studied under Ricci flow. A unique minimizer is shown to exist for one of the functionals. This functional evaluated at the minimizer is strictly increasing. The results for the first functional considered are extended to manifolds with boundary. Finally two physically motivated examples are presented.

MSCs: 53A45, 53C20, 53C21

Keywords: boundary, Ricci flow, curvature, eigenvalue, evolution, Laplacian, nonlinear

IJGMMP, **19, 9**, 2250137 (1-21), 2022.

1 Introduction

Since the fundamental work of Hamilton and Perelman, the study of Ricci flow on differentiable manifolds has taken on a great deal of importance for many reasons. This is due in no small measure to its application to such problems as proving the Poincaré conjecture and as a tool to prove the Uniformization Theorem. The conjecture of Thurston states that every closed 3-manifold admits a parametric decomposition. The subject is of great geometric appeal and significance. In fact the study of the Ricci flow on surfaces or two-dimensional manifolds is related to the study of the logarithmic diffusion equation. There are many other deep applications that are mentioned later and others that are not.

In its basic form, given a 1-parameter family of metrics $g(t)$ on a Riemannian manifold M defined on a time interval $I \subset \mathbb{R}$, Hamilton's Ricci flow is defined by

$$\frac{\partial}{\partial t} g_{ij}(t) = -R_{ij}. \tag{1.1}$$

The manifolds here are n-dimensional unless stated otherwise. For any C^∞ metric g_0 on a closed manifold M, there exists a unique solution $g(t)$, $t \in [0, \epsilon)$ to the Ricci flow equation for some $\epsilon > 0$ with $g(0) = g_0$. This was proved initially by Hamilton **[1-4]** and shortly thereafter, a simpler proof was given by de Turek [**5**]. This maximal solution is referred to as the Ricci flow with initial condition g_0 [**6**]. If $T < \infty$ then we call T the blow up time. An example is the shrinking round 3-sphere with $g_0 = r_0^2 g_{S^2}$ and $g(t) = (r_0^2 - 4t) g_{S^2}$, in which case $T = r_0^2/4$ [**7**].

This can also be applied to classify and study various kinds of solutions of soliton nature that arise in the study of partial differential equations. For example, a steady breather is a Ricci flow solution on an interval $[t_1, t_2]$ that satisfies $g(t_2) = \varphi^* g(t_1)$ for some $\varphi \in \mathrm{Diff}\,(M)$. An expanding breather is a Ricci flow solution on $[t_1, t_2]$ that satisfies $g(t_2) = c\,\varphi^*\, g(t_1)$ for some $c > 1$ and $\varphi \in \mathrm{Diff}\,(M)$. Expanding soliton solutions are expanding breathers. Suppose that M is simply-connected. Based on the round 3-sphere case, one might say every Ricci flow blows up in finite time becomes round while shrinking to a point as t approaches T. By rescaling and $t \to T$ one would show that M admits a metric of constant positive curvature, so by a classical theorem, is diffeomorphic to S^3 **[8-11]**. An analogous argument does work in two dimensions. Further if g_0 has positive Ricci curvature then Hamilton showed that this is basically correct. The manifold shrinks to a point in finite time and becomes round as it shrinks. If M is simply connected but g_0 does not have positive Ricci curvature, a new phenomenon may occur. The Ricci flow solution may become singular before it has time to shrink to a point on account of a possible neck-pinch. Such an object can be modeled by a region $(-c, c) \times S^2$ in which one or many S^2 fibres separately shrink to a point at time T, due to positive curvature of S^2. There are very important applications of geometric flows to proving the geometrization conjecture: Every closed 3-manifold admits a geometric decomposition [**12**]. It also permits the study of such things as evolution of eigenvalues of geometric operators on a manifold as well as other quantities with respect to the parameter [**13**].

The study presented here looks first at the monotonicity properties of two different functionals of the curvature. These monotonicity properties can have important consequences, for example, the first functional has an application in physics where it plays the role of an entropy functional. By computing the variation of these functionals it can be shown that the Ricci flow is a gradient flow on the space of metrics. This is done first for the case where the manifold is closed hence has no boundary, the first case. The calculation is extended to determine the variation of the same functional in the case where the manifold has a boundary, the second case. This is done by applyng integral formulas which include boundary terms to the integrals that appear [**14**]. The boundary terms leads to results which are more complicated than the first case. Some interesting consequencess are presented that are restricted to the case of two-dimensional manifolds or surfaces. At the end, two applications from physics are presented which give support to the study of this subject [**6,15**].

2 Functionals on the Space of Metrics

Functionals are introduced which allow the study of the Ricci flow on manifolds of finite dimension. The case in which the manifold is closed is looked at first and after, the case of manifolds with boundary. then extended to manifolds with boundary. The first functional has a formal variation that may be used to show that the corresponding gradient flow is modified Ricci flow. Let M be a differentiable manifold of dimension n unless otherwise stated and let $\mathcal{M}$ be the space of smooth Riemannian metrics on a manifold of dimension n. The space $\mathcal{M}$ can be formally thought of as an infinite dimensional manifold. The tangent space $T_g\mathcal{M}$ consists of the symmetric covariant 2-tensors v_{ij} on M. As well $C^\infty(M)$ is an infinite-dimensional manifold with $T_f C^\infty(M) = C^\infty(M)$. The diffeomorphism group $\text{Diff}(M)$ acts on $\mathcal{M}$ and $C^\infty(M)$ by pullback. Let $d\mu(g) = d\mu_g$ be the Riemannian volume form on M associated to the metric g.

Definition 2.1. A functional F which depends on the scalar curvature R of M with $F : \mathcal{M} \times C^\infty(M) \to \mathbb{R}$ is defined as

$$F(g,f) = \int_M (|\nabla f|^2 + R) \cdot e^{-f} \, d\mu_g. \tag{2.1}$$

For $v_{ij} \in T_g \mathcal{M}$ and $h \in T_f C^\infty(M)$ the evolution of the differential dF on (v_{ij}, h) is written as $\delta F(v_{ij}, h)$ and we call $v = g^{ij} v_{ij}$.

To work with (2.1) and other functionals, additional results will be required. Suppose $g(t)$ is a one-parameter family of metrics and $\partial g_{ij}/\partial t = v_{ij}$, then the Γ_{ij}^k can be differentiated with respect to the parameter

$$\frac{\partial}{\partial t}\Gamma_{ij}^k = \frac{1}{2} g^{km} (\nabla_i v_{jm} + \nabla_j v_{im} - \nabla_m v_{ij}). \tag{2.2}$$

The components of the Riemann curvature tensor which is defined by $R(\partial_i, \partial_j)\partial_k = R_{ijk}^m \partial_m$ are given by

$$R_{ijk}^m = \partial_i \Gamma_{jk}^m - \partial_j \Gamma_{ik}^m + \Gamma_{jk}^p \Gamma_{ip}^m - \Gamma_{ik}^p \Gamma_{jp}^m. \tag{2.3}$$

The Ricci tensor follows from this by tracing $R_{ij} = R_{pij}^p$. The variation of the Ricci tensor in terms of the connection is known to be

$$\frac{\partial}{\partial t} R_{ij} = \nabla_p \left(\frac{\partial}{\partial t}\Gamma_{ij}^p\right) - \nabla_i \left(\frac{\partial}{\partial t}\Gamma_{pj}^p\right). \tag{2.4}$$

The calculations can be done efficiently using a normal coordinate system. Using (2.2) for the variation of Γ_{ij}^k

$$\begin{aligned}
\frac{\partial}{\partial t} R_{ij} &= \nabla_p \left(\frac{1}{2} g^{pm} (\nabla_i v_{jm} + \nabla_j v_{im} - \nabla_m v_{ij})\right) - \nabla_i \left(\frac{1}{2} g^{pm} (\nabla_p v_{jm} + \nabla_j v_{pm} - \nabla_m v_{pj})\right) \\
&= \frac{1}{2}\big(\nabla^m \nabla_i v_{jm} + \nabla^m \nabla_j v_{im} - \nabla^m \nabla_m v_{ij} - \nabla_i \nabla_j v + \nabla_i \nabla^m v_{mj}\big)
\end{aligned}$$

$$= \frac{1}{2}\nabla^m \left(\nabla_j v_{im} + \nabla_i\, v_{jm} - \nabla_m v_{ij}\right) - \frac{1}{2}\nabla_i \nabla_j v. \tag{2.5}$$

To evaluate the trace, the required variation formula for R is obtained

$$\frac{\partial R}{\partial t} = g^{ij}\frac{\partial R_{ij}}{\partial t} - \frac{\partial g^{ij}}{\partial t}\, R_{ij} = \nabla^m \nabla^i\, v_{im} - \frac{1}{2}\Delta\, v - v^{ij} R_{ij}. \tag{2.6}$$

Proposition 2.1. The variation of the F-functional in (2.1) has the form

$$\delta\, F(v_{ij}, h) = \int_M e^{-f}\big[- v_{ij}(R_{ij} + \nabla_i \nabla_j f) + (\frac{v}{2} - h)(2\Delta\, f - |\nabla\, f|^2 + R)\big]\, d\mu_g \tag{2.7}$$

Proof: Suppose $\delta f = h$ then the variation of (2.1) is given by

$$\delta F(v_{ij}, h) = \int_M \left(\delta R + \delta |\nabla f|^2\right) e^{-f}\, d\mu_g + \int_M (R + |\nabla f|^2)\, \delta(e^{-f} d\mu_g). \tag{2.8}$$

Using (2.4) and $|\nabla f|^2 = g^{ij}\, \nabla_i f\, \nabla_j f$ we compute

$$\delta |\nabla f|^2 = \delta\, g^{ij} \nabla_i f \nabla_j f + 2 g^{ij} \nabla_i f \nabla_j h = -v^{ij} \nabla_i f\, \nabla_j f + 2\langle \nabla f, \nabla h\rangle.$$

For the second term, since $\delta(d\mu_g) = (v/2)\, d\mu_g$, it is found that

$$\delta(e^{-f}\, d\mu_g) = (\frac{v}{2} - h)\, e^{-f}\, d\mu_g. \tag{2.9}$$

Substitute these results into (2.6) and we get the form we want,

$$\delta F = \int_M e^{-f}[-\Delta\, v + \nabla_i \nabla_j v^{ij} - R_{ij} v^{ij} - v^{ij} \nabla_i f \nabla_j f + 2\langle \nabla f, \nabla h\rangle + (R + |\nabla f|^2)(\frac{v}{2} - h)]\, d\mu_g \tag{2.10}$$

The next step is to express this in such a way that v_{ij} and h appear in just an algebraic way and thus no derivative terms. The following identity comes up frequently

$$\Delta e^{-f} = (|\nabla f|^2 - \Delta f)\, e^{-f},$$

Let us integrate by parts and use the fact M is closed. The following results in which only derivatives of f appear are required to transform (2.10),

$$\int_M e^{-f}(-\Delta v)\, d\mu_g = -\int_M (\Delta e^{-f})\, v d\mu = \int_M e^{-f}(\Delta f - |\nabla f|^2) v\, d\mu_g,$$

$$\int_M e^{-f} \nabla_i \nabla_j v_{ij}\, d\mu_g = -\int_M \nabla_i (e^{-f} \nabla_j f) v^{ij}\, d\mu_g = \int_M e^{-f}(\nabla_i f \nabla_j f - \nabla_i \nabla_j f)\, v\, d\mu_g \tag{2.11}$$

$$2\int_M e^{-f}\langle\nabla f,\nabla h\rangle\, d\mu_g = -2\int_M \langle\nabla e^{-f},\nabla h\rangle\, d\mu_g = 2\int_M \Delta e^{-f} h d\mu_g = 2\int_M (|\nabla f|^2-\Delta f)\, e^{-f}\cdot h d\mu_g.$$

Substitute the results in (2.11) into (2.10) for the variation of the F-functional,

$$\delta F(g,h) = \int_M \big[(\Delta f - |\nabla f|^2)v + (\nabla_i f\nabla_j f - \nabla_i\nabla_j f)v^{ij} - R_{ij}v^{ij} - v^{ij}\nabla_i f\nabla_j f$$

$$+2(|\nabla f|^2 - \Delta f)\, h + (R + |\nabla\, f|^2)(\frac{v}{2} - h)\big]\, e^{-f}\, d\mu_g$$

$$= \int_M \big[- v^{ij}(R_{ij} + \nabla_i\nabla_j f) + (\frac{v}{2} - h)(2\Delta f - |\nabla f|^2 + R)\big] e^{-f}\, d\mu_g \tag{2.12}$$

□

The last term in the integrand of (2.12) can be eliminated if we are allowed to restrict the variation so that $v/2 - h = 0$, or recalling (2.9)

$$\delta(e^{-f}\, d\mu_g) = 0. \tag{2.13}$$

which amounts to saying we assume that $e^{-f}\, d\mu_g$ is fixed. A smooth measure $d\nu$ can be established on M and used to relate f to g by requiring that $e^{-f}\, d\mu_g = d\nu$. Equivalently, define a section $s : \mathcal{M} \to \mathcal{M} \times C^\infty(M)$ by

$$s(g) = (g, \ln(\frac{d\mu_g}{d\nu})).$$

The composition $F^m = F \circ s$ is a function on $\mathcal{M}$ and its differential is then given by

$$\delta F^m(v_{ij}) = \int_M e^{-f}(-v^{ij}(R_{ij} + \nabla_i\nabla_j f))\, d\mu_g.$$

A formal Riemannian metric on the space of metrics can also be defined

$$\langle v_{ij}, v_{ij}\rangle = \frac{1}{2}\int_M v^{ij}v_{ij}\, d\mu_g.$$

Consequently, the gradient flow of F^m on the space of metrics is given by

$$(g_{ij})_t = -2(R_{ij} + \nabla_i\nabla_j\, f). \tag{2.14}$$

This allows the derivative f_t to be written as

$$f_t = \frac{1}{2}g^{ij}(g_{ij})_t = -g^{ij}(R_{ij} + \nabla_i\nabla_j f). \tag{2.15}$$

Then the induced flow equation for f is

$$f_t = \frac{\partial}{\partial t} \ln(\frac{\partial \mu_g}{\partial \nu}) = -\Delta f - R.$$

As with any gradient flow, the function F^m is nondecreasing along the flow line with its derivative being given by the squared length of the gradient flow

$$F^m_t = 2 \int_M [R_{ij} + \nabla_i \nabla_j f]^2 \, d\mu_g \tag{2.16}$$

as follows from (2.12) and (2.14).

Now (2.14) can be transformed back into the Ricci flow equation by means of time-dependent diffeomorphisms. If $Y(t)$ is the time-dependent generating vector field of the diffeomorphisms and L_Y denotes Lie derivative, the new equations for f and g are

$$(g_{ij})_t = -2(R_{ij} + \nabla_i \nabla_j f) + L_Y g, \qquad f_t = -\Delta f - R + L_Y \, f. \tag{2.17}$$

The Lie derivative of the metric, for example, is given by $(L_X g)(Y_1, Y_2) = g(\nabla_{Y_1} X, Y_2) + g(Y_1, \nabla_{Y_2}, X)$. In local coordinates, this implies that

$$(L_X g)_{ij} = \nabla_i X_j - \nabla_j X_i.$$

Suppose that f is a function such that $(L_{\nabla f} \, g)_{ij} = 2\nabla_i \nabla_j f$. If we define $Y = \nabla f$, then (2.12) reduces to

$$(g_{ij})_t = -2R_{ij}, \qquad f_t = -\Delta f - R + |\nabla f|^2. \tag{2.18}$$

As the functional $F(g, f)$ is left unchanged by a simultaneous pullback of g and f, and the right-hand side of F^m_t in (2.16) is also unchanged under a simultaneous pullback. It follows that under the new evolution equations (2.18)

$$\frac{d}{dt} F(g(t), f(t)) = 2 \int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} \, d\mu_g. \tag{2.19}$$

The diffeomorphisms that have been used have altered the relationship between g and f. They no longer satisfy $e^{-f} \, d\mu_g = d\nu$. In the case that $\int_M e^{-f} \, d\mu_g$ is constant in t as $e^{-f} \, d\mu_g$ is related to $d\nu$ by a diffeomorphism. To understand how to relate g and f now, assume we have a solution for the first equation in (2.18) under some initial metric. Then given a solution $g(t)$, require that f satisfy the second equation in (2.18) which is in terms of $g(t)$.

Functional (2.1) has a unique minimizer which can be used to show that a steady breather on a compact manifold is a gradient steady soliton.

Theorem 2.1: Given a metric g, there is a unique minimizer ζ of $F(g, f)$ subject to the constraint $\int_M e^{-f}\, d\mu_g = 1$.

Proof: Set $\Phi = e^{-f/2}$ which has derivatives $\nabla\Phi = -\frac{1}{2}\nabla f\Phi$ and $|\nabla\Phi|^2 = \frac{1}{4}|\nabla f|^2\Phi^2$ and the functional is transformed into the form

$$F = \int_M (4|\nabla\Phi|^2 + R\Phi^2)\, d\mu_g = \int_M (-4\Phi\Delta\Phi + R_g\Phi^2)\, d\mu_g = \int_M \Phi(-4\Delta\Phi + R_g\Phi)\, d\mu_g. \quad (2.20)$$

The constraint equation is then $\int_M \Phi^2 d\mu_g = 1$. Then ζ is the smallest eigenvalue of the operator $-4\Delta + R_g$ and $e^{-\zeta}$ is the corresponding normalized positive eigenvector. This operator is a Schrödinger operator, so it has a unique normalized positive eigenvector. □

A new functional can be defined using the fact there is a smallest eigenvalue. The related functional χ is defined by $\chi(g) = F(g, \zeta)$. If $g(t)$ is a smooth family of metrics, then it follows from eigenvalue perturbation theory that $\chi(g(t))$ and $\zeta(t)$ are smooth in t.

Theorem 2.2 If g is a Ricci flow solution, then $\chi(g(t))$ is nondecreasing in the parameter t.

Proof: Consider a time interval $[t_1, t_2]$ and the minimizer $\zeta(t_2)$ where

$$\chi(t_2) = F(g(t_2), \zeta(t_2)). \quad (2.21)$$

Put $u(t_2) = e^{-\zeta(t_2)}$ and solve the backward heat equation on $[t_1, t_2]$ which is

$$\frac{\partial u}{\partial t} = -\Delta u + R\, u. \quad (2.22)$$

The claim is that $u(x', t) > 0$ for all $x' \in M$ and $t' \in [t_1, t_2]$. To see this, take $t' \in [t_1, t_2]$ and let h be the solution to the forward heat equation on (t_1, t_2) with $\lim_{t\to t'} h(t) = \delta_{x'}$. We have

$$\frac{d}{dt}\int_M u(t)\cdot h(t)\, d\mu_g = \int_M [(\partial_t u + \Delta u - R_g\, u)\, u + u\, (\partial_t h - \Delta h)]\, d\mu_g = 0. \quad (2.23)$$

As it is known that $h(t) > 0$ for all $t \in (t', t_2]$ and since $u(t_2)$ is defined as a real exponential, we find that

$$u(x', t') = \int_M u(x, t')\delta_{x'}(x)\, d\mu_g = \lim_{r\to t'} \int_M u(t)h(t), d\mu_g = \int_M u(t_2)h(t_2)\, d\mu_g > 0. \quad (2.24)$$

Given $t \in [t_1, t_2]$ define $f(t)$ by $u(t) = e^{-f(t)}$, by using (2.20) it is concluded that

$$F(g(t_1), f(t_2)) \leq F(g(t_2), f(t_2)). \tag{2.25}$$

From the definition of ζ, it must be that

$$\chi(t_1) = F(g(t_1), \zeta(t_1)) \leq F(g(t_1), f(t_1)). \tag{2.26}$$

Using the normalization condition $\int_M e^{-f(t)}\, d\mu_g = 1$ for $t = t_1$ or t_2, we conclude that $\chi(t_1) \leq \chi(t_2)$. $\Box$

The monotonicity properties can be very useful as the following results show.

Lemma 2.1: A steady breather is a gradient is a gradient steady soliton.

Proof: Since $\chi(g(t_2)) = \chi(\varphi^* g(t_1)) = \chi(g(t_1))$. Thus equality holds in the previous result. From the proof, this implies that $F(g(t), \zeta(t))$ must be constant in time t. Since $R_{ij} + \nabla_i \nabla_j \zeta = 0$, $R + \Delta\, \zeta = 0$ and so (2.18) becomes $\zeta_t = |\nabla \zeta|^2$. $\Box$

A theorem related to Theorem 2.2, but perhaps improved, follows next.

Theorem 2.3

$$\frac{d\chi}{dt} \geq \frac{2}{n}\, \chi^2(t). \tag{2.27}$$

Proof: Given an interval $[t_1, t_2]$ and using the notation of the previous theorem, we write

$$\begin{aligned} \chi(t_1) &\leq F(g(t_1), f(t_1)) \\ &= F(g(t_2), f(t_2)) - 2 \int_{t_1}^{t_2} \int_M |R_{ij} + \nabla_i \nabla_j f|^2\, e^{-f}\, d\mu_g\, dt \\ &= \chi(t_2) - 2 \int_{t_1}^{t_2} \int_M |R_{ij} + \nabla_i \nabla_j\, f|^2 e^{-f}\, d\mu_g\, dt \end{aligned} \tag{2.28}$$

Collecting $\chi(t_2) - \chi(t_1)$ on the left, (2.28) implies

$$\chi(t_2) - \chi(t_1) \geq 2 \int_{t_1}^{t_2} \int_M |R_{ij} + \nabla_i \nabla_j f|^2\, e^{-f}\, d\mu_g dt$$

Dividing both sides by $t_2 - t_1$ and using the mean value theorem in t, the derivative with respect to t is obtained,

$$\frac{d\chi}{dt}|_{t_1 = t_2} = \lim_{t_1 \to t_2} \frac{\chi(t_2) - \chi(t_1)}{t_2 - t_1} \geq 2 \int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f}\, d\mu_g. \tag{2.29}$$

Of course, the right-hand side is evaluated at t_2. Hence for all t,

$$\frac{d\chi}{dt} \geq 2 \int_M |R_{ij} + \nabla_i \nabla_j \zeta|^2 e^{-\zeta}\, d\mu_g. \tag{2.30}$$

Since norm dominates trace, (2.30) implies that

$$\frac{d\chi}{dt} \geq \frac{2}{n} \int_M (R + \Delta\zeta)^2 e^{-\zeta}\, d\mu_g. \tag{2.31}$$

Using the normalization conditions as well as the Cauchy-Schwarz inequality,

$$\Big(\int_M (R + \Delta\zeta) e^{-\zeta}\, d\mu_g\Big)^2 \leq \int_M (R + \Delta\zeta)^2\, e^{-\zeta}\, d\mu_g. \tag{2.32}$$

Using the formula for $\Delta e^{-\zeta}$ gives

$$\begin{aligned}\int_M (R + \Delta\zeta) e^{-\zeta}\, d\mu_g = \int_M (R e^{-\zeta} + |\nabla\zeta|^2 e^{-\zeta} - \Delta e^{-\zeta})\, d\mu_g = \int_M (R + |\nabla\zeta|^2)\, e^{-\zeta}\, d\mu_g \\ = F(g(t), \zeta(t)) = \chi(t).\end{aligned} \tag{2.33}$$

3 The W Functional

Another functional which has interesting properties which include monotonicity is the W-functional.

Definition 3.1. The W-functional $W : \mathcal{M} \times C^\infty(M) \times \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$W(g, f, \tau) = \int_M \big[\tau(|\nabla f|^2 + R) + f - n\big] \cdot (4\pi\tau)^{-n/2} e^{-f}\, d\mu. \tag{3.1}$$

The W-functional is a scale-invariant version of F, and it has the following symmetries for $\phi \in$ Diff(M) and $c > 0$

$$W(\phi^* g, \phi^* f, \tau) = W(g, f, \tau), \qquad W(cg, f, c\tau) = W(g, f, \tau). \tag{3.2}$$

It is constant in $t = -\tau$ along a gradient shrinking soliton defined for $t \in (-\infty, 0)$. In this sense, W is a constant on gradient shrinking solitons, just as F is constant on all gradient steady solitons.

Examples of a gradient shrinking soliton can be found. Consider $\mathbb{R}^n$ with the flat metric, constant in time $t \in (-\infty, 0)$. Putting $\tau = -t$ and take f to be

$$f(t, \mathbf{x}) = \frac{|\mathbf{x}|^2}{4\tau}, \quad e^{-f} = e^{-|\mathbf{x}|^2/4\tau}.$$

Differentiating this f, it is clear that $f_t = -|\mathbf{x}|^2/4\tau^2$ and $\nabla_i f = x_i/2\tau$, so $\Delta f = n/2$. Therefore, the equation $f_t = -\Delta f + |\nabla f|^2 - R + n/2\tau$ holds. The integrand of W is given by $\tau(|\nabla f|^2 + R) + f - n = \tau|\mathbf{x}|^2/4\tau^2 + |\mathbf{x}|^2/4\tau - n = |\mathbf{x}|^2/4\tau - n$. Under the normalization conditions

$$\int_{\mathbb{R}^n} e^{-|\mathbf{x}|^2/4\tau}\, d\mu_g = (4\pi\tau)^{n/2}, \qquad \int_{\mathbb{R}^n} \frac{|\mathbf{x}|^2}{4\tau^2}\, e^{-|\mathbf{x}|^2/4\tau}\, d\mu_g = (4\pi\tau)^{n/2} \cdot \frac{n}{2\tau}, \tag{3.3}$$

we find that

$$\int_{\mathbb{R}^n} \big[\tau(|\nabla f|^2 + R) + f + n\big](4\pi\tau)^{n/2} \cdot e^{-f}\, d\mu_g$$
$$= \int_{\mathbb{R}^n} \frac{|\mathbf{x}|^2}{2\tau}(2\pi\tau)^{-n/2} \cdot e^{-f}\, d\mu_g - n\int_{\mathbb{R}^n} (4\pi\tau)^{-n/2} e^{-f}\, d\mu_g = 0.$$

From this it follows that $W(f) = 0$. The W-functional also has interesting monotonicity properties. To show this it is required to compute the variation of the functional in analogy with what was done with F. It is shown that a shrinking breather on a compact manifold is a gradient shrinking soliton. To do so, the following information is needed for the calculation: $\delta g_{ij} = v_{ij}$, $\delta f = h$ and take $\sigma = d\tau$.

Proposition 3.1 The variation of functional W in (3.1) is given by

$$\delta W(v_{ij}, h, \sigma) = \int_M \big[\sigma(R + |\nabla f|^2) - \tau v^{ij}(R_{ij} + \nabla_i \nabla_j f) + h$$
$$+ \big\{\tau(2\Delta f - |\nabla f|^2 + R) + f - n\big\}(\frac{n}{2} - h - \frac{n\sigma}{2\tau})\big](4\pi\tau)^{-n/2} \cdot e^{-f}\, d\mu_g. \tag{3.4}$$

Proof: The proof is similar to that for F, so a brief discussion is given. The variation of the integration measure in the functional is needed and it is found to be

$$\delta\big((4\pi\tau)^{-n/2} \cdot e^{-f}\, d\mu_g\big) = \big(\frac{v}{2} - h - \frac{n\sigma}{2\tau}\big)(4\pi\tau)^{n/2} \cdot e^{-f} d\mu_g \tag{3.5}$$

as well as the following results

$$\delta R = -\Delta\, v + \nabla_i \nabla_j v^{ij} - R_{ij} - v^{ij}, \qquad \delta|\nabla f|^2 = -v^{ij}\nabla_i f \nabla_j f + 2\langle \nabla f, \nabla h\rangle. \tag{3.6}$$

Using (3.6) in (3.4) results in,

$$\delta W(v_{ij}, h, \sigma) = \int_M \sigma(R + |\nabla f|^2)(4\pi\tau)^{-n/2} e^{-f}\, d\mu_g$$

$$+\int_M [\tau(-\Delta v+\nabla_i\nabla_j v^{ij}-R_{ij}v^{ij}-v^{ij}\nabla_i f\nabla_j f+2\langle\nabla f,\nabla h\rangle+h](4\pi\tau)^{-n/2}e^{-f}\,d\mu_g \qquad (3.7)$$
$$+\int_M [\tau(R+|\nabla f|^2)+f-n](\frac{v}{2}-1-\frac{n\sigma}{2\tau})(4\pi\tau)^{-n/2}\,e^{-f}\,d\mu_g$$

Use has been made of (2.6) in the last part of (3.7). Using (2.11) in (3.7), we get

$$\delta W=\int_M [\sigma(R+|\nabla f|^2)+\tau(\Delta f-|\nabla f|^2)v+\tau(\nabla_i f\nabla_j f-\nabla_i\nabla_j f)v^{ij}-\tau R_{ij}v^{ij}$$
$$-\tau v^{ij}\nabla_i f\nabla_j f+2\tau(|\nabla f|^2-\Delta f)h+h+\{\tau(R+|\nabla f|^2)+f-n\}(\frac{v}{2}-h-\frac{n\sigma}{2\tau}](4\pi\tau)^{-n/2}\cdot e^{-f}\,d\mu_g$$
$$=\int_M [\sigma(R+|\nabla f|^2)+\tau(\frac{v}{2}-h)(2\Delta f-2|\nabla f|^2)+\tau(\nabla_i f\nabla_j f-\nabla_i\nabla_j f)v^{ij}-\tau R_{ij}v^{ij}$$
$$-\tau v^{ij}\nabla_i f\nabla_j f+h+\{\tau(R+|\nabla f|^2)+f-n\}(\frac{v}{2}-h+\frac{n\sigma}{2\tau})](4\pi\tau)^{-n/2}e^{-f}\,d\mu_g. \qquad (3.8)$$

Using the identity for Δe^{-f} and the fact that the integral of the Laplacian of a function over a closed manifold vanishes, we obtain

$$\int_M \tau(\frac{v}{2}-h)(2\Delta f-2|\nabla f|^2)(4\pi\tau)^{-n/2}e^{-f}\,d\mu_g=\int_M \tau(\frac{v}{2}-h-\frac{\sigma n}{2\tau})(2\Delta f-2|\nabla f|^2)\,(4\pi\tau)^{-n/2}\cdot e^{-f}\,d\mu_g. \qquad (3.9)$$

Using (3.9) in (3.8), we arrive at (3.4) after some simplification

$$\delta W=\int_M [\sigma(R+|\nabla f|^2)-\tau(R_{ij}+\nabla_i\nabla_j f)v^{ij}+h+\{\tau(R+|\nabla f|^2)+f-n$$
$$+\tau(2\Delta f-2|\nabla f|^2)\}(\frac{v}{2}-h-\frac{n\sigma}{2\tau})](4\pi\tau)^{-n/2}\cdot e^{-f}\,d\mu_g. \qquad (3.10)$$

□

Fix a smooth measure $d\nu$ on M normalized to one and then relate f to τ and g by requiring that $(4\pi\tau)^{-n/2}\,e^{-f}\,d\mu_g=d\nu$ so we have $v/2-h-n\sigma/2\tau=0$. Thus (3.4) reduces to the form,

$$\delta W=\int_M [\sigma(R+|\nabla f|^2)-v^{ij}\,(R_{ij}+\nabla_i\nabla_j f)+h](4\pi\tau)^{-n/2}\cdot e^{-f}\,d\mu_g. \qquad (3.11)$$

Consider subjecting this to the set of constraints

$$(g_{ij})_t=-2(R_{ij}+\nabla_i\nabla_j f), \qquad f_t=-\Delta f-R+\frac{n}{2\tau}, \qquad \tau_t=-1. \qquad (3.12)$$

To apply the variation equation, we get

$$v_{ij}=-2(R_{ij}+\nabla_i\nabla_j f), \qquad h=-\Delta f-R+\frac{n}{2\tau}, \qquad \sigma=-1.$$

Using the constraint $v/2 - h - n\sigma/2\tau = 0$ and the equations above in (3.4), we obtain

$$\delta W = \int_M [-(R_{ij} + |\nabla f|^2) + 2\tau|R_{ij} + \nabla_i\nabla_j f|^2 - \Delta f - R + \frac{n}{2\tau}](4\pi\tau)^{-n/2}e^{-f}\, d\mu_g$$
$$= \int_M 2\tau\Big|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}\, g_{ij}\Big|^2 (4\pi\tau)^{-n/2} \cdot e^{-f}\, d\mu_g. \tag{3.13}$$

Adding a Lie derivative to the right side of (3.12) gives the new flow equations

$$(g_{ij})_t = -2R_{ij}, \qquad f_t = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \qquad \tau_t = -1.$$

where the integral constraint holds. At this point we look at the variational problem of minimizing $W(g, f, \tau)$ subject to this integral constraint. Make the change of variable $\Phi = e^{-f/2}$, we minimize the functional

$$(4\pi\tau)^{-n/2} \int_M [\tau(4|\nabla\Phi|^2 + R\Phi^2) - 2\Phi^2 \log\, \Phi - n\Phi^2]\, d\mu_g, \quad (4\pi\tau)^{-n/2} \int_M \Phi^2\, d\mu_g = 1. \tag{3.14}$$

The infimum is finite and there is a positive continuous minimizer Φ_m which will be a weak solution of the variational equation,

$$\tau(-4\Delta + R)\, \Phi = 2\Phi \log\, \Phi + (\varpi(g,\tau) + n)\, \Phi, \tag{3.15}$$

where $\varpi(g,\tau) = \inf_f \{W(g, f, \tau) : \int_M (4\pi\tau)^{-n/2}\, e^{-f}\, d\mu_g = 1\}$. Theory of elliptic equations says Φ is smooth which means $f = -2\log\, \Phi$ is also smooth.

4 Monotonicity of the F Functional on Manifolds with Boundary

The results obtained for the F-functional can be generalized to manifolds with non-empty boundary. All quantities such as the curvatures scalar products and operators are dependent on the metric $g(t)$. The functional of interest in (2.1), where $d\mu_g$ represents the volume form and $d\sigma_g$ the corresponding boundary integration measure. The following result is proved for the first variation of F in (2.1) for a manifold with boundary.

Proposition 4.1 Let the quantities $\delta\, g_{ij} = v_{ij}$, $\delta f = h$ and $g^{ij}v_{ij} = v$ be given quantities The first variation of the functional in (2.1) on a compact manifold M such that ∂M is nonempty is

$$\delta\, F = \int_M e^{-f}[-v^{ij}(R_{ij} + \nabla_i\nabla_j f) + (\frac{v}{2} - h)(2\Delta_g f - |\nabla f|^2 + R_g)]\, d\mu_g$$

$$-\int_{\partial M}[\frac{\partial v}{\partial \eta_g}+(v-2h)\frac{\partial f}{\partial \eta_g}]\cdot e^f\,d\sigma_g+\int_{\partial M} e^{-f}\nabla_i v^{ij}\eta_j\,d\sigma_g-\int_{\partial M}\nabla_j e^{-f}v^{ij}\eta_i\,d\sigma_g. \tag{4.1}$$

The Ricci tensor is R_{ij} and R_g the scalar curvature of M. In local coordinates $\partial/\partial\eta_g=\eta^i\,\partial_i$ is the outward unit normal at the boundary with respect to g, and the volume form on the boundary is $d\sigma_g$.

Proof: Start by obtaining δF as in the case in which there is no boundary

$$\delta F=\int_M\{-\Delta_g\,v+\nabla_i\nabla_j v^{ij}-R_{ij}v^{ij}-v^{ij}\nabla_i f\nabla_j f+2\langle\nabla f,\nabla h\rangle+(R_g+|\nabla f|^2)(\frac{v}{2}-h)\}e^{-f}\,d\mu_g. \tag{4.2}$$

Equation (4.2) has to be written so that both v_{ij} and h appear in such a way that they appear algebraicly. No derivative terms on the right side should appear. This can be done by using integration formulas such as parts which contain boundary terms.

On a compact manifold M, this can be stated as a relationship between the integrals

$$\int_M(u\Delta\,w-w\Delta\,u)\,d\mu_g=\int_{\partial M}(u\frac{\partial w}{\partial\nu}-\nu\frac{\partial u}{\partial\nu})\,d\sigma_g. \tag{4.3}$$

Some cases which are used to transform (4.3) to the required form will be discussed. Consider the case in which $u=-v$ are $w=e^{-f}$, (4.3) implies

$$\int_M(-v\Delta_g\,e^{-f}-e^{-f}\Delta_g\,(-v))\,d\mu_g=\int_{\partial M}(-v\frac{\partial e^{-f}}{\partial\nu}-e^{-f}\frac{\partial(-v)}{\partial\nu})\,d\sigma_g. \tag{4.4}$$

Solve (4.4) for the second term

$$\int_M(\Delta_g v)e^{-f}\,d\mu_g=\int_M v\Delta_g e^{-f}\,d\mu_g+\int_{\partial M}(-v\frac{\partial e^{-f}}{\partial\nu}+e^{-f}\frac{\partial v}{\partial\nu})\,d\sigma_g. \tag{4.5}$$

Solving again we get

$$\int_M(-\Delta_g v)e^{-f}d\mu_g=-\int_M v\Delta_g e^{-f}\,d\mu_g-\int_{\partial M}(-v\frac{\partial e^{-f}}{\partial\nu}+e^{-f}\,\frac{\partial v}{\partial\nu})\,d\sigma_g. \tag{4.6}$$

The divergence theorem when applied to a vector field X with $\div X=\nabla_i X^i$ will be useful

$$\int_M f\,\div(X)\,d\mu_g=-\int_M g(\nabla f,X)\,d\mu_g+\int_{\partial M} f\,g(X,\nu)\,d\sigma_g. \tag{4.7}$$

Apply (4.7) to the vector field $\nabla_j\,v^{ij}$, it follows that

$$\int_M e^{-f}\,\nabla_i\nabla_j vij\,d\mu_g=-\int_M g(\nabla e^{-f},\nabla_j v^{ij})\,d\mu_g+\int_{\partial M} g(\nabla_j v^{ij},\nu)\,e^{-f}\,d\mu_g$$

$$= - \int_M \nabla_i e^{-f} \nabla_j v^{ij} \, d\mu_g + \int_{\partial M} \nabla_j \, v^{ij} \nu_i \, d\sigma_g. \tag{4.8}$$

In the divergence theorem let the function equal one so it reduces to

$$\int_M \nabla_j (v^{ij} \nabla_i e^{-f}) \, d\mu_g = \int_{\partial M} v^{ij} \nabla_i e^{-f} \, \nu_j \, d\sigma_g. \tag{4.9}$$

It is the case that

$$\nabla_j (v^{ij} \nabla_i e^{-f}) = \nabla_i \nabla_j e^{-f} \, v^{ij} + \nabla_i e^{-f} \nabla_j v^{ij}.$$

Solve this for the last term then integrate on both sides with respect to the volume form on M then substitute (4.9) to obtain

$$\int_M \nabla_i e^{-f} \nabla_j v^{ij} \, d\mu_g = - \int_M \nabla_i \nabla_j e^{-f} \, v^{ij} \nu_j \, d\mu_g + \int_{\partial M} v^{ij} \nabla_i e^{-f} \, \nu_j \, d\mu_g. \tag{4.10}$$

Consequently, there is the useful integral formula

$$\int_M e^{-f} \nabla_i \nabla_j \, v^{ij} \, d\mu_g = \int_M \nabla_i \nabla_j \, e^{-f} v^{ij} \, d\mu_g - \int_{\partial M} \nabla_i e^{-f} \, v^{ij} \nu_j \, d\sigma_g + \int_{\partial M} e^{-f} \nabla_j v^{ij} \, \nu_i \, d\sigma_g. \tag{4.11}$$

There is one final integral to be developed and it is the following

$$\begin{aligned} 2 \int_M e^{-f} g(\nabla f, \nabla h) \, d\mu_g &= -2 \int_M g(\nabla e^{-f}, \nabla \, h) \, d\mu_g = -2 \int_M g^{ij} \nabla_i e^{-f} \nabla_j h \, d\mu_g \\ = -2 \int_M g^{ij} \, \nabla_i e^{-f} \nabla_j h \, d\mu_g &= -2 \int_M h \, \Delta_g e^{-f} \, d\mu_g + 2 \int_{\partial M} h \nabla_i e^{-f} \nu^i \, d\sigma_g \\ &= -2 \int_M h \Delta_g e^{-f} \, d\mu_g + 2 \int_{\partial M} \frac{\partial e^{-f}}{\partial \nu} \, h \, d\sigma_g. \end{aligned} \tag{4.12}$$

Finally combine all of these integral results and put them in (4.2) to obtain an expression that simplifies to (4.1)

$$\begin{aligned} \delta F = \int_M [-v^{ij} (R_{ij} + \nabla_i \nabla_j f) + (\frac{v}{2} - h)(2 \Delta_g f - |\nabla f|^2 + R_g)] \, d\mu_g \\ - \int_{\partial M} (\nu \frac{\partial f}{\partial \nu} + \frac{\partial v}{\partial \nu}) e^{-f} \, d\sigma_g + \int_{\partial M} \nabla_i f \, v^{ij} \nu_j e^{-f} \, d\sigma_g + \int_{\partial M} \nabla_j v^{ij} \nu_j e^{-f} \, d\sigma_g + 2 \int_{\partial M} h \frac{\partial f}{\partial \nu} e^{-f} \, d\sigma_g \\ - \int_{\partial M} (\frac{\partial v}{\partial \nu} + (v - 2h) \frac{\partial f}{\partial \nu}) \, e^{-f} \, d\sigma_g + \int_{\partial M} (e^{-f} \nabla_i v^{ij} \nu_j - \nabla_j e^{-f} \, v^{ij} \nu_i) \, d\sigma_g. \end{aligned}$$

After some further manipulation, this becomes (4.1). □

5 Ricci Flow on Two-Manifolds with Boundary

A formulation of the Ricci flow behavior with respect to manifolds of dimension two with boundary is presented. Let M be a compact manifold with boundary and smooth metric g. The system of equations to be studied can be presented as follows

$$\frac{\partial g}{\partial t} = -R_g\, g, \quad M \times (0,T),$$

$$k_g(\cdot,t) = \psi(\cdot,t), \qquad \partial M \times (0,T), \quad g(x,0) = g_0(x), \quad x \in M. \tag{5.1}$$

In (5.1) R_g represents the scalar curvature of M and k_g is the geodesic curvature of ∂M with respect to the time evolving metric g and ψ is a smooth real-valued function defined on $\partial M \times [0,\infty)$ which satisfies the compatibility condition.

Proposition 5.1 Let $(M, g(t))$ be a solution to (5.1). The scalar curvature satisfies the evolution equation

$$\frac{\partial R_g}{\partial t} = \Delta_g R_g + R_g^2, \qquad M \times (0,T),$$

$$\frac{\partial R_g}{\partial t} = k_g R_g - 2k_g' = \psi\, R_g - 2\psi', \quad \partial M \times (0,T). \tag{5.2}$$

Here ν is the outward directed unit normal with respect to the metric g and ψ' represents differentiation with respect to the parameter t.

Proof: The evolution equation for the scalar curvature R_g in the interior of M is known. The derivative of R_g with respect to the outward normal has to be worked out with respect to the boundary. To do this, choose local coordinates (x_1, x_2) at point $p \in M$ such that the equation $x_2 = 0$ defines the boundary. The corresponding coordinate frame (∂_1, ∂_2) is orthogonal at $p \in \partial M$ and $t = t_0$. This is the point and instant where the normal derivatives are computed. This means ∂_2 coincides with the outward unit normal to the boundary in the whole coordinate patch at $t = t_0$.

Since the deformation is conformal, ∂_t means normal to the boundary. The geodesic curvature is given when the flow is defined for $t = t_0$ by means of the equation

$$k_g\, g_{11} = -\frac{\Gamma_{11}^2}{(g_{22})^{1/2}} = -(g_{22})^{1/2}\Gamma_{11}^2. \tag{5.3}$$

Differentiate both sides of (5.3) with respect to t using (5.1)

$$(k_g g_{11})' = k_g' g_{11} + k_g g_{11}' = k_g' g_{11} - k_g R_g \, g_{11}. \tag{5.4}$$

Differentiate both sides of (5.3) to generate the following relation

$$(k_g g_{11})' = -\frac{1}{2(g_{22}^{1/2})}(g_{12})'\Gamma_{11}^2 - (g_{22})^{1/2} \cdot (\Gamma_{11}^2)'. \tag{5.5}$$

Replace the derivative $(g_{22})'$ in (5.5) using (5.1) to get

$$(k_g g_{11})' = \frac{1}{2} R_g (g_{22})^{1/2} \Gamma_{11}^2 - (g_{22})^{1/2} (\Gamma_{11}^2)'. \tag{5.6}$$

To evaluate $(\Gamma_{11}^2)'$ in this coordinate system, let ∇_j denote covariant differentiation with respect to ∂_i and note $g_{12} = 0$, $g_{ii} = 1$, hence

$$(\Gamma_{11}^2)' = \frac{1}{2} g^{2j} (\nabla_1 \, g_{1j}' + \nabla_1 \, g_{1j}' - \nabla_j g_{11}') = \frac{1}{2} g^{22} (-2\nabla_1 (R_g g_{12}) + \nabla_2 (R_g \, g_{11})). \tag{5.7}$$

Since $g_{12} = 0$, (5.7) reduces to the form

$$(\Gamma_{11}^2)' = \frac{1}{2} g^{22} (\nabla_2 \, R_g) g_{11} = \frac{1}{2} g^{22} (\partial_2 R_g). \tag{5.8}$$

Substitute (5.8) into (5.6), there results

$$(k_g g_{11})' = \frac{1}{2} R_g \, (g_{22})^{1/2} \Gamma_{11}^2 - \frac{1}{2} (g_{22})^{1/2} g^{22} (\partial_2 R_g). \tag{5.9}$$

Substitute (5.9) into (5.4) and an expression containing the normal derivative follows

$$k_g' \, g_{11} - k_g \, R_g g_{11} = -\frac{1}{2} R_g k_g - \frac{1}{2} \frac{1}{(g_{22})^{1/2}} \partial_2 R_g = -\frac{1}{2} R_g k_g - \frac{1}{2(g_{22})^{1/2}} \frac{\partial R_g}{\partial \nu_g}.$$

Solve this result for the normal derivative and (5.2) is obtained.

Proposition 5.2 Let $(M, g(t))$ where M is compact be a solution of (5.1). Assume that ψ, the boundary data, satisfies $\psi' = 0$. Then if $R_g \geq 0$ at $t = 0$, it remains so as long as the solution exists. Furthermore, if the initial data has positive scalar curvature, and boundary data ψ is non-negative, then R_g remains strictly positive and blows up in finite time.

Proof: The maximum principle for the heat equation states that if $g(t)$ is a family of metrics on a closed manifold M and $u : M \times (0, T) \to \mathbb{R}$ satisfies $u_t \geq \Delta u$, then if $u \geq 0$ at $t = 0$ for some

$c \in \mathbb{R}$, then $u \geq 0$ for all $t \geq 0$. Let u be the scalar curvature function R where $R \geq 0$ at $t = 0$. Then on $M \setminus \partial M$, the evolution equation for R implies that, since $R^2 \geq 0$,

$$\frac{\partial R}{\partial t} - \Delta R + R^2 \geq \Delta R, \qquad t > 0. \tag{5.10}$$

So if $R \geq 0$ at $t = 0$, then $R \geq 0$ for all $t \geq 0$ by the maximum principle.

Let us show that the solution to (5.1) must blow up in finite time. By Hopf's maximum principle on account of the hypothesis at the boundary ∂M, it holds that

$$\frac{\partial R}{\partial \eta} \geq 0, \tag{5.11}$$

the minimum of $R_{min}(t)$ of R at time t occurs in the interior $M \setminus \partial M)$ of M. Hence R_{min} satisfies the differential inequality

$$\frac{dR_{min}}{dt} \geq R_{min}^2.$$

Therefore comparing with the solution of the equation $u_t = u^2$, $u(0) = R_{min}(0)$, which is

$$u = \frac{1}{R_{min}(t)^{-1} - t}, \quad t \geq 0,$$

it follows that $R_{min}(t) \geq u(t)$ and $u > 0$ on $[0, R_{min}(0)^{-1})$, u must blow up in finite time and will take $R_{min}(t)$ with it. □

It may be pointed out that if $R \geq 0$ at time $t = 0$ and is strictly positive at a point under the assumption $\psi \geq 0$, $\psi' \geq 0$, it becomes strictly positive instantaneously, so the hypothesis in the previous proposition may be relaxed somewhat.

Let $(0, T)$ be the maximal interval of existence of a solution to (5.1) with $0 < T < \infty$, then

$$\limsup_{t \to T} \big(\sup_{x \in M} R_g(p, T) \big) = \infty. \tag{5.12}$$

First of all, if g_0 is the initial metric, then as the Ricci curvature flow preserves conformal structure, we have that the evolving metric can be represented as $g = e^u g_0$. Hence if R_g is the scalar curvature of the initial metric, at a fixed but arbitrary time, u satisfies the elliptic boundary value problem

$$\begin{aligned} &\Delta_{g_0} u + R_{g_0} = R_g e^u, \qquad x \in M, \\ &\frac{\partial u}{\partial \eta_g} + 2k_{g_0} = 2k_{g_0} e^{u/2}, \quad x \in \partial M. \end{aligned} \tag{5.13}$$

To reach a contradiction, assume that R_g remains uniformly bounded on $(0,T)$. A consequence of this assumption is that e^u remains bounded away from zero and uniformly bounded above on $(0,T)$. Now from bounds on the curvature and also of the geodesic curvature of the boundary on ψ and its derivative, we can obtain bounds on the derivative of u. Thus u and its derivative are uniformly bounded on $(0,T)$, and consequently they converge as $t \to T$ to a smooth function $\bar{u}$. If the Ricci flow is started at $t = T$ with initial information $e^{\bar{u}}\, g_0$ and the same boundary data, then the original solution could be continued past T, which contradicts the hypothesis. Therefore, if the Ricci flow (5.1) cannot be extended past $T < \infty$, the curvature has to blow up. There is an interesting case when the solutions to (5.1) blow up, as shown in the next proposition.

Proposition 5.3 Assume that $\int_M R_g d\sigma_{g_0} + \int_{\partial M} 2k_{g_0}\, ds_g > 0$ and assume that $\psi \leq 0$. Then the solution to (5.1) with initial condition g_0 and boundary condition ψ blows up in finite time.

Proof: Let $g(t)$ be the solution to (5.1) with initial g_0 and boundary condition ψ. If $a(t)$ represents the area of M with respect to $g(t)$ and $\chi(M)$ is the Euler characteristic of M, we calculate that

$$\frac{da}{dt} = -\int_M R_g\, d\sigma_g = -4\pi\chi(M) + 2\int_{\partial M} k_g\, ds_g \leq -4\pi\chi(M). \tag{5.14}$$

Therefore, the area cannot be positive for all time, hence a singularity must occur within finite time.

□

6 Physical Applications

It is important to note that the Ricci flow has already appeared in many physical applications to date. Two will be briefly described now. One of the interesting features of these cases is the parameter is not necessarily a time but another variable.

There is a nice application of the Ricci flow which appears in the study of the renormalization group. One model where this is relevant has the action

$$S[\phi,\alpha] = \alpha^{-1} \int_{\Sigma} [\mathrm{Tr}_{\gamma(x)}(\phi^* g)]\, d\mu_\gamma = \alpha^{-1} \int_{\Sigma} \gamma^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j\, g_{ij}\, d\mu_\gamma. \tag{6.1}$$

The critical points of this are harmonic maps of the Riemann surface (Σ, γ) into (M, g). Assume Σ is the flat torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, and $\gamma_{\mu\nu} = \delta_{\mu\nu}$. At each $x \in \Sigma$, the metric $\alpha^{-1} g(\phi(x))$ plays the role of the coupling constant for the fields $\phi(x)$ The action together with possible deformations describe a family of 2-dimensional quantum field theories known as non-linear sigma models. The discussion is limited is restricted to the quantum deformation of the theory involving only the metrical coupling $\alpha = a^{-1}\, g$. This is the situation generating the Ricci flow.

A technical difficulty in implementing a renormalization group procedure for the quantum field theory associated with (6.1) is that the space of maps (Σ, M) is a nonlinear functional space. In the weak-coupling limit, where the size of the surface (Σ, γ) is much smaller than the physical length scale of (M, g_{ab}), only fields fluctuating around a constant value $\phi_0 \in M$ play a role. One can then work in a geodesic ball centered at ϕ_0 with radius $r < \min\{\frac{1}{3}\mathrm{inj}(\phi_0), \pi/6\sqrt{K_u}\}$ where K_u is an upper bound to the sectional curvature of (M, g) and $\mathrm{inj}(\phi_0)$ denotes the injectivity radius of (M, g) at ϕ_0. In quantum field theory the standard procedure minimal subtraction, now consists in regarding the metric g in the effective action as formally infinite and extracting from it a divergent part to cancel the 1-loop singularity

$$g_{ij}(\psi) = g_0(\Lambda/\Lambda') - 2a\ln(\Lambda/\Lambda')R_{ij}(\psi) + O(a^2). \tag{6.2}$$

The metric g on the left is the bare metric and $g(\Lambda/\Lambda')$ is the normalized metric $R_{ij}(\psi)$ is the Ricci tensor of the bare metric $R_{ij}(\psi) = R_{ij}[g(\Lambda/\Lambda')]$, since the two metrics are equal to order zero and if $\mathcal{T}_{fin}$ represents terms which are finite the effective action assumes the form,

$$\Gamma_{(0)}(\psi) = \int_\Sigma g_{ij}(\Lambda/\Lambda')\, \partial^\mu \psi^i \partial_\mu\, \psi^j\, d\mu_\gamma + a\mathcal{T}_{fin} + O(a^2). \tag{6.3}$$

This procedure does not depend explicitly on the point ϕ_0 in the geodesic neighborhood $B(\phi_0, 2r)$ we work in. The splitting can be extended to all M. The result can be extended to all M, that is, to background fields ψ taking values in a geodesic neighborhood at any ϕ_0.

The renormalizability of the theory depends on the behavior of $g(\Lambda/\Lambda')$ when $\Lambda/\Lambda' \to \infty$ and is described by the beta function. By defining $\lambda = \log(\Lambda/\Lambda')$, we immediately have

$$\frac{\partial}{\partial\lambda} g_{ij} = \frac{\partial}{\partial\lambda} g_{ij}(\lambda) - 2aR_{ij}(g(\lambda)) + O(a^2) = 0. \tag{6.4}$$

Introduce the parameter $t = -a\lambda$, so that $\partial_t g$ has the same dimension as Ric, the renormalization group flow of the nonlinear sigma model at one loop is

$$\frac{\partial}{\partial t} g(t) = -2\text{Ric}(g(t)) + O(a). \tag{6.5}$$

A two loop calculation modifies this only at order a, so these results in the weak coupling limit $a \to 0$ yield the Ricci flow

$$\frac{\partial}{\partial t} g_{ij}(t) = -2\, R_{ij}(t), \qquad g_{ij}(0) = g_{ij}. \tag{6.6}$$

The second example comes from to statistical mechanics where the W functional is in a sense analogous to minus entropy. The partition function for the canonical ensemble at temperature β^{-1} is $Z = \int \exp(-\beta\, E)\, d\omega(E)$, where $\omega(E)$ is a density of states measure independent of β. From this we calculate the average energy $\langle E\rangle = -\partial \log\, Z/\partial\beta$, entropy $S = \beta\langle E\rangle + \log\, Z$ and the fluctuation $\sigma_f = \langle (E - \langle E\rangle)^2\rangle$.

Take a closed manifold M with probability measure ν and suppose now T is the parameter. The system is described by a metric which depends on the parameter, $g_{ij}(T)$, the temperature according to $(g_{ij})_T = 2(R_{ij} + \nabla_i\nabla_j f)$ where $d\nu = ud\mu_g$ where $u = (4\pi T)^{-n/2}e^{-f}$, so $\log\, Z = \int(-f + n/2)d\mu$. Then we can write

$$\langle E\rangle = -T^2 \int_M \big(R + |\nabla f|^2 - \frac{n}{2\tau}\big) d\nu, \qquad S = -\int_M (T(R + |\nabla f|^2) + f - n)\, d\nu,$$

$$\sigma = 2T^4 \int_M |R_{ij} + \nabla_i\nabla_j f - \frac{1}{2T}\, g_{ij}|^2\, d\nu. \tag{6.7}$$

Obviously σ is nonnegative and vanishes only on a gradient shrinking soliton. As well $\langle E\rangle$ is non-negative, whenever the flow exists for all sufficiently small $T > 0$. If u tends to a δ as $T \to 0$ or u is a limit of a sequence of functions u_i with each u_i tending to a δ -function as $T \to T_i > 0$ and $T_i \to 0$, then S is also nonnegative . In the first case all the quantities in (6.7) tend to zero as $T \to 0$, in the second, the entropy may tend to a positive limit..

7 References

[1] R. S. Hamilton, The Ricci flow on surfaces, Mathematics and General Relativity (Santa Cruz), CA (1986) 237-262, Contemp. Math. 71, AMS, Providence, RI (1988).

[2] R. S. Hamilton, A compactness property for solutions of the Ricci flow, Amer. J. Math. **117**, (1995), 545-572.

[3] R. S. Hamilton, The formation of singularities in the Ricci flow, Surveys in differential geometry, Vol. II, Cambridge MA, 1993.

[4] R. S. Hamiltonian, Three-Manifolds with Positive Curvature, J. Diff. Geom. **17**, (1982), 255-306.

[5] D. De Turek, Existence of Metrics with Positive Ricci Curvature: Local Theory, Invent. Math. **65**, (1981/82), 179-207.

[6] G. Perelman, The entropy formula for the Ricci flow and its geometric applications. (arXiv.math/0211159v1).

[7] B. Kleiner and J. Lott, Notes on Perelman's Papers, Geometry and Topology, **12**, (2008), 2557-2858.

[8] B. Chow, P. Lu and L. Ni, Hamilton's Ricci Flow, Graduate Studies in Mathematics, vol 77, AMS, Providence, RI, (2006).

[9] P. Bracken, An introduction to Ricci flow for two-dimensional manifolds, Advances in Mathematics Research, Nova Science Publishers, Hauppauge, NY, 155-193, (2017).

[10] B. Chow and D. Knopf, The Ricci flow: an introduction, Math. Surveys and Monographs 110, AMS, Providence, RI, (2004).

[11] H-D Cao and B. Chow, Recent developments on the Ricci flow, Bull. Amer. Math. Soc. **36**, (1999), 59-74.

[12] W. P. Thurston, Three dimensional geometry and topology, Vol 1, Ed. S. Levy, Princeton Math. Series, 35, Princeton University Press, Princeton, NJ, (1997).

[13] P. Bracken, Evolution of eigenvalues of a geometric operator under Ricci flow on a Riemannian manifold, J. Math. Anal. Appl. **509**, (2022), 125990.

[14] P. Bracken, On a generalization of a theorem of Lichnerowicz to manifolds with boundary, Int. J. Geom. Methods Mod. Phys, **8**, (2011), 639-646.

[15] M. Carfora, The Renormalization Group and the Ricci Flow, Milan J. Math, **78**, (2010), 319-353.